

## Estimation and testing in models containing both jumps and conditional heteroskedasticity

Authors	Drost,F.C.; Nijman,T.E.; Werker,B.J.M.
Published in	CentER Discussion Paper
Publication Date	1994
Link	<a href="https://research.tilburguniversity.edu/en/publications/4a81702c-3af7-4b6c-99b6-5b00ce83cab">https://research.tilburguniversity.edu/en/publications/4a81702c-3af7-4b6c-99b6-5b00ce83cab</a>
Citation	Drost, F C, Nijman, T E & Werker, B J M 1994 'Estimation and testing in models containing both jumps and conditional heteroskedasticity' CentER Discussion Paper, vol. 1994-105, Unknown Publisher.
Download Date	2026-05-17 12:51:45
Rights	<p>General rights</p> <p>Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.</p> <ul style="list-style-type: none"> <li>- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.</li> <li>- You may not further distribute the material or use it for any profit-making activity or commercial gain</li> <li>- You may freely distribute the URL identifying the publication in the public portal"</li> </ul> <p>Take down policy</p> <p>If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.</p>

# Estimation and Testing in Models containing both Jumps and Conditional Heteroskedasticity

FEIKE C. DROST, THEO E. NIJMAN, AND BAS J.M. WERKER \*  
*Tilburg University*

## Abstract

In this paper we develop tests for the hypothesis that a series (observed in discrete time) is generated by a diffusion process and discuss the results of these tests for several exchange rates and stock market indices. The tests of this hypothesis that have been proposed up to now in literature are all based on arbitrary and non-testable assumptions on the conditional distribution of the smooth component of the series. Instead, our tests are based on the weaker assumption that the series is weak GARCH; this hypothesis can easily be tested. To investigate the presence of jumps, we propose a test that is based on an overidentifying relation between variance and kurtosis parameters at an arbitrary frequency, which holds for GARCH diffusions. Monte Carlo evidence suggests that this test is slightly conservative, but nevertheless it has good power properties. The empirical results clearly indicate the presence of jumps in dollar exchange rates. For stock market indices the results are mixed. The finding of jumps has important consequences for derivative pricing as well as for modeling the distribution of future spot prices. In order to assess the size and intensity of the jumps, we estimate a model containing both jumps and conditional heteroskedasticity.

Keywords: Continuous time modeling, GARCH, conditional heteroskedasticity, jumps.

## 1 Introduction

Many models in financial economics are specified in continuous time. Continuous time models of, for example, interest rates, stock prices, and exchange rates have been used

---

\*CentER and Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. The first author is research fellow of the Royal Netherlands Academy of Arts and Sciences (K.N.A.W.). The first and second author gratefully acknowledge financial support within the Human Capital and Mobility program of the EU. Part of this research was carried out when the first author was visiting CREST, Paris.

to price derivative instruments, to determine optimal hedging strategies, and to derive the distribution of the future value of a portfolio. Usually it is assumed that the underlying sample paths are continuous, i.e. that the series under consideration do not contain jumps. It is intuitively clear that such models may severely overestimate the effectiveness of short term hedging strategies based on dynamic portfolio adjustment, if jumps are present. Similarly, the prices of out-of-the-money options that are close to maturity will be underestimated if jumps are present that are not taken into account. Not only option prices are sensitive to jumps, but also the optimal early exercise decision for American options alters significantly when jumps are present, see Amin (1993). This even holds for jumps occurring with small probability.

Merton (1976) has suggested a model in which stock prices are generated by a geometric Brownian motion with Poisson jumps superimposed on them. In this model he derived implied option prices, assuming that the jump component represents idiosyncratic risk only. In the Merton model tests for the presence of jumps and estimation of the parameters is fairly straightforward, because all deviations from log-normality of observed stock returns at any frequency can be attributed to the jumps. However, a major drawback of the model is that it implies that returns are independent and identically distributed at all frequencies, which is in conflict with the overwhelming evidence of conditional heteroskedasticity in returns at high frequencies [see, e.g., the survey by Bollerslev et al. (1992) and the many references cited there].

Recently a number of authors have addressed the issue of testing for jumps and estimation of frequency and size of jumps within models that also allow for conditional heteroskedasticity [see, e.g., Jorion (1988), Nieuwland, Verschoor, and Wolff (1991), and Vlaar and Palm (1993)]. The estimators and tests that have been proposed are all based on the assumption that the conditional distribution of the non-jump component of the return, at the available data frequency, is known a priori up to a finite number of parameters. Then, similar to the identification of jumps in the Merton model, the jump component can be identified from deviations of the estimated conditional distribution from the assumed one.

The assumption of a known conditional distribution of the non-jump component is an heroic one and is not part of the null hypothesis to be tested. Recently, Drost and Werker (1994) derived a number of properties of discrete time data that are generated by underlying continuous time processes which accommodate both conditional heteroskedasticity and jumps. In this paper their results are used to develop tests for the presence of jumps and to construct estimators of the parameters of the jump component, assuming that the series under consideration is weak GARCH(1,1) as introduced by Drost and Nijman (1993), which is in line with the stylized facts at daily and lower frequencies. Assumptions on the conditional distribution are avoided. Because our results are explicitly based on an underlying model in continuous time, it can be used to derive hedging strategies and option prices in the familiar way [see, e.g., Amin and Ng (1993)].

The plan of this paper is as follows. In Section 2 we discuss tests for the presence of jumps and the associated estimators of parameters of the jump component which have been proposed in the literature. In Section 3 we summarize some key results of Drost and

Werker (1994) and show how they can be used to test for the presence of jumps. Section 4 describes Monte Carlo results that illustrate the size and power of the tests in moderate sample sizes. In Section 5 we present testing results for the presence of jumps in several exchange rates as well as in three stock market indices. All series, except the UK's stock market, contain evidence of jumps which are more predominant, however, for the exchange rates. Estimation of a continuous time model that generates both jumps and conditional heteroskedasticity is considered in Section 6. Finally, Section 7 contains the concluding remarks.

## 2 Models based on a priori assumptions on the conditional distribution of the smooth component

To the best of our knowledge, all tests for the presence of jumps that are available in the literature are based on ad hoc assumptions on the conditional distribution of the smooth component. In this section we briefly summarize these tests and the associated model for the jump component. Subsequently we argue that the empirical results will crucially depend on the validity of the assumed conditional distribution for the smooth component and that, if a general class of conditional error distributions is considered, no jump component will be identified at all.

Suppose that a stationary discrete time-series of returns  $y_t$  ( $t = 1, \dots, T$ ) is observed and that one is interested in determining the conditional distribution of these returns. Obviously, knowledge of this conditional distribution is of great practical importance when studying stock returns or exchange rates. Beckers (1981), Ball and Torous (1985), Jorion (1988), Vlaar and Palm (1993), and others start off by assuming that the series  $y_t$  can be written as the sum of a smooth component  $s_t$  and a jump component  $n_t$ . Consider, e.g., the following model for logarithmic returns  $y_t$ ,

$$\begin{cases} y_t = s_t + n_t, \\ s_t = \sigma_t \xi_t, \text{ with } \xi_t \sim N(0, 1), \\ \sigma_t^2 = \psi + \beta \sigma_{t-1}^2 + \alpha s_{t-1}^2, \\ n_t = \sum_{s=1}^{N_t} e_s, \text{ with } e_s \sim N(0, \delta^2), N_t \sim P(1/\zeta), \end{cases} \quad (2.1)$$

where the innovations in the smooth component  $\xi_t$ , the jump frequencies  $N_t$ , and the jump sizes  $e_t$  are all independent.  $N(0, 1)$  denotes a normal distribution with zero mean and unit variance. Beckers (1981) and Ball and Torous (1985) do not consider conditional heteroskedasticity ( $\alpha = \beta = 0$ ). Jorion (1988) also allows for first order ARCH effects (i.e.  $\beta = 0$ ). The model of Vlaar and Palm (1993) is slightly more general than (2.1) in that they allow for autocorrelation in the mean, higher order GARCH effects and Bernoulli jumps. For ease of presentation we focus on (2.1) and avoid the straightforward extension of the argument to such more general cases.

If (2.1) is valid, the parameters can easily be estimated consistently using either moment estimators as in Beckers (1981) or maximum likelihood estimation as proposed by the other

authors. Their empirical results clearly suggest the presence of jumps in currency returns, while the results for stock returns are mixed. Note however that these results fully depend on the fact that fat-tailedness of the conditional distribution in excess of the assumed normal distribution will be attributed (by assumption) to the jump component of the model. The same phenomenon will occur if one introduces other error distributions. This implies that the more restrictive the assumption on the smooth component is, the more jumps will be found. Furthermore, the distributional assumption on  $\xi_t$  cannot be tested if deviations are automatically attributed to jumps. Moreover, if model (2.1) holds at one frequency, say the daily frequency, it will not hold at every other frequency, say the weekly frequency. Another problem with the model (2.1) is the specification of the smooth component  $s_t$ . No relevant continuous time model, either with continuous or discontinuous sample paths, is known that generates  $s_t$  at discrete frequencies.

To overcome the problems sketched above we make the weaker assumption that the processes are weak GARCH (definitions and precise statements will be given in the next section). A large class of continuous time models generates weak GARCH at all discrete frequencies. This class contains both diffusions and jump-diffusions. In the next section we discuss procedures to distinguish between smooth and non-smooth models in this class.

### 3 Testing for the presence of jumps in the presence of conditional heteroskedasticity

In this section we will consider continuous time processes  $Y_t$  and the implied discrete time processes, at frequency  $h$ ,  $y_{(h)t} = Y_t - Y_{t-h}$ ,  $t \in h\mathbb{N}$ . We will propose tests for the presence of jumps which are based on the assumption that the series under consideration is weak GARCH(1,1) at all discrete frequencies. Drost and Nijman (1993) have defined a stationary symmetric discrete time process  $y_{(h)t} - \mu_h$ , with finite fourth moments, to be weak GARCH with parameters  $(\mu_h, \alpha_h, \beta_h, \psi_h, \kappa_h)$  if there exists a stationary process  $\sigma_{(h)t}^2$ , with

$$\sigma_{(h)t}^2 = \psi_h + \beta_h \sigma_{(h)t-h}^2 + \alpha_h y_{(h)t-h}^2, \quad t \in h\mathbb{N}, \quad (3.1)$$

such that, for  $t \in h\mathbb{N}$ ,  $\sigma_{(h)t}^2$  is the best linear predictor of  $y_{(h)t}^2$  in terms of a constant and lagged values of  $y_{(h)t}$  and  $y_{(h)t}^2$ . The parameter  $\kappa_h = Ey_{(h)t}^4 / (Ey_{(h)t}^2)^2$  denotes the kurtosis of the process. According to Kolmogorov's criterion [see, e.g., Theorem I.1.8 in Revuz and Yor (1991)] a continuous time process  $Y_t$  for which  $Ey_{(h)t}^4/h^2$  is bounded (as a function of  $h$ ) has continuous sample paths. Continuous time models generating weak GARCH(1,1) processes at every frequency and having smooth sample paths will be referred to as GARCH diffusions. If sample paths are not smooth we will call them GARCH jump-diffusions.

Drost and Werker (1994) derive explicit relations expressing the dependence of the weak GARCH parameters in arbitrary GARCH diffusions. They show that parameters  $\eta \in \mathbb{R}$ ,  $\theta > 0$ ,  $\omega > 0$ , and  $\lambda \in (0, 1)$  exist such that

$$(\mu_h, \alpha_h, \beta_h, \psi_h, \kappa_h) = f(h\eta, h\theta, h\omega, \lambda). \quad (3.2)$$

An explicit expression for the function  $f$  is repeated in Appendix A of this paper for ease of reference. Note that the discrete time models are weak GARCH rather than strong GARCH which was assumed in (2.1). An example of a GARCH diffusion is the model considered by Nelson (1990), for logarithmic spot prices,

$$dY_t = \eta dt + \sigma_t dW_t^{(1)}, \quad (3.3)$$

$$d\sigma_t^2 = \theta(\omega - \sigma_t^2)dt + \sqrt{2\lambda\theta} \sigma_t^2 dW_t^{(2)}, \quad (3.4)$$

where  $\eta \in \mathbb{R}$ ,  $\theta > 0$ ,  $\omega > 0$ ,  $\lambda \in (0, 1)$ , and  $W_t^{(1)}$  and  $W_t^{(2)}$  are independent Brownian motions. This model reduces to the model proposed by Merton (1976) if conditional homoskedasticity is imposed (i.e.  $\theta = \lambda = 0$ ). Another example of a GARCH diffusion is the sum of the process in (3.3)–(3.4) and an independent Brownian motion, which according to Nijman and Sentana (1993) also yields weak GARCH(1,1) at all discrete frequencies and obviously has continuous sample paths. Yet another example of a GARCH diffusion is obtained when  $W_t^{(2)}$  is replaced by a Lévy process [a precise definition of Lévy processes can be found in, e.g., Protter (1990)]. Such processes still have continuous sample paths, but allow for jumps in the volatility process. Naik (1993) derived option prices in these kind of models.

If it is assumed that the series under consideration is weak GARCH(1,1) at all discrete frequencies, absence of a jump component implies that five parameters at every discrete frequency depend only on four parameters in the continuous time model, as is formalized in (3.2). Obviously these restrictions can be tested. A first implication of (3.2) is, for example, that the mean and variance parameters at some (arbitrary) frequency uniquely determine  $\eta$ ,  $\theta$ ,  $\omega$  and  $\lambda$  and thereby determine the mean and variance parameters at every other frequency. Equation (3.2) implies that, under the assumption of a GARCH diffusion, the parameters  $\theta$  and  $\lambda$  can be identified from the weak GARCH parameters at an arbitrary frequency  $h$  because [see (A.1)–(A.5)]

$$\theta = g_\theta(h; \alpha_h, \beta_h) = -\frac{1}{h} \ln(\alpha_h + \beta_h), \quad (3.5)$$

$$\begin{aligned} \lambda &= g_\lambda(h; \alpha_h, \beta_h) \\ &= 2 \ln^2(\alpha_h + \beta_h) \left\{ \frac{\{1 - (\alpha_h + \beta_h)^2\}(1 - \beta_h)^2}{\alpha_h - \alpha_h \beta_h (\alpha_h + \beta_h)} \right. \\ &\quad \left. + 6 \ln(\alpha_h + \beta_h) + 2 \ln^2(\alpha_h + \beta_h) + 4(1 - \alpha_h - \beta_h) \right\}^{-1}. \end{aligned} \quad (3.6)$$

These properties can be tested by comparing estimates of the weak GARCH parameters at two different frequencies. From Drost and Nijman (1993) it follows that the variance parameters of a weak GARCH(1,1) model at two different frequencies will yield the same value of the parameter  $\theta$ , also without assuming the validity of an underlying GARCH diffusion. A Wald test of the hypothesis  $g_\theta(h; \alpha_h, \beta_h) = g_\theta(r; \alpha_r, \beta_r)$  for  $r \neq h$  will therefore be referred to as the weak GARCH test or  $\theta$ -test. This yields a simple test for the validity of our weak GARCH assumption at both frequencies. If the DGP is a GARCH

diffusion, then also  $g_\lambda(h; \alpha_h, \beta_h) = g_\lambda(r; \alpha_r, \beta_r)$ . Wald tests of this hypothesis will be referred to as  $\lambda$ -tests for the presence of an underlying GARCH diffusion.

Due to the overidentifying restriction in (3.2), the kurtosis  $\kappa_h$  at an arbitrary frequency is uniquely determined by the variance parameters at that frequency. This relation can be tested directly. Substitution of (3.5) and (3.6) in (A.5) yields a function  $K$  such that

$$K(\alpha_h, \beta_h, \kappa_h) = 0, \quad (3.7)$$

where the precise functional form of  $K$  is obvious from (A.5) and avoided for brevity. Drost and Werker (1994) show that  $K(\alpha_h, \beta_h, \kappa_h)$  will be strictly larger than zero for GARCH jump-diffusions [for the model (6.1)–(6.2) this follows easily from (A.5) and (A.7)]. Therefore the test of the hypothesis in (3.7), which will be referred to as the  $\kappa$ -test, is a one-sided test.

In any attempt to derive the asymptotic distribution of the tests proposed above two problems arise: not much is known about the properties of estimators in weak GARCH models at a given frequency and in order to derive properties of the  $\theta$ - and  $\lambda$ -tests we face the additional problem that we need the joint distribution of estimators at different frequencies which are based on the same underlying data. Quasi maximum likelihood estimators (QMLE) as if the series  $y_t$  ( $t = 1, \dots, T$ ) under consideration is strong GARCH with a conditionally normal distribution is the most frequently used estimator for the variance parameters. Recently, Lee and Hansen (1994) have shown that this estimator is consistent and asymptotically normal under mild regularity conditions on the rescaled residuals even if the data generating process is only semi-strong GARCH [see also Bollerslev and Wooldridge (1992) and Lumsdaine (1991)]. Subsequently, the kurtosis of rescaled innovations,  $\kappa_{\xi_h}$ , is estimated by the fourth moment of the rescaled residuals. This yields an estimate of  $\varphi_h = (\alpha_h, \beta_h, \psi_h, \kappa_{\xi_h})$ . In our case however the series will presumably not be semi-strong GARCH. In the remainder high and low frequency parameter estimates are obtained with QMLE. If high frequency observations  $y_t$  ( $t = 1, \dots, T$ ) are available, estimates of the low frequency parameters after aggregation over  $m$  periods can be obtained from  $m$  different low frequency samples,  $\bar{y}_{t+k} = \sum_{i=0}^{m-1} y_{t+k-i}$ , ( $t = m, 2m, \dots, T$ ) for  $k = 0, \dots, m-1$ . The corresponding QMLE estimators will be denoted by  $\hat{\varphi}_h(k)$ . In the sequel we will use the estimator  $\bar{\varphi}_{h,m} = \frac{1}{m} \sum_{k=0}^{m-1} \hat{\varphi}_h(k)$  which is computationally slightly more demanding but probably more efficient than each individual estimator  $\hat{\varphi}_h(k)$ . In Appendix B we determine the (multivariate normal) asymptotic distribution of  $\tilde{\varphi}_h = (\hat{\varphi}_h, \bar{\varphi}_h)$  as if sufficient regularity conditions are satisfied. The  $\theta$ -,  $\lambda$ - and  $\kappa$ -tests will then be carried out as straightforward Wald tests. The simulations in Section 4 show that this distribution yields a proper approximation to the true size of the tests.

## 4 Size and power of the tests for the presence of jumps in models containing conditional heteroskedasticity

In this section we present several Monte Carlo results. We compare the true finite sample size of our tests with the approximation suggested by the derivations in Appendix B. We also present several results on the power of the tests.

The size of the tests has been determined for five different diffusions. In the first two cases (3.3)–(3.4) is the data generating process and the parameters in the DGP have been chosen in such a way that the variance parameters that Baillie and Bollerslev (1989) find for daily returns in the Pound-Dollar and DMark-Dollar rates are matched. In the third case the series under consideration is the sum of the independent series generated in the first two cases. The results in Nijman and Sentana (1993) imply that it is also weak GARCH at every frequency and obviously it has smooth sample paths. In the fourth and fifth case conditionally homoskedastic white noise is added to the series generated in the first two cases, where the variance of the white noise equals the unconditional variance of the simulated exchange rate. The latter three simulations are to assess the validity of the tests if the DGP is not of the form (3.3), but still a GARCH diffusion. In all cases the low frequency in the test procedure is weekly.

The daily variance parameters for the Pound-Dollar rate found by Baillie and Bollerslev (1989) are  $\beta = 0.910$  and  $\alpha = 0.061$ . According to Drost and Werker (1994) these numbers are compatible with a GARCH diffusion if  $\kappa_\xi = 3.45$  and correspond to the values  $\theta = 0.029$  and  $\lambda = 0.272$  for the diffusion parameters<sup>1</sup>. The value of  $\eta$  is put equal to zero and  $\omega$  is an irrelevant scaling parameter. It is obvious that, strictly speaking, one cannot simulate from a diffusion. We approximate the drawings from the diffusion (3.3) by drawings from a strong GARCH process at the frequency of one-tenth of a day. The convergence results in Nelson (1990) show the validity of this procedure. The variance parameters corresponding to this frequency of one-tenth of a day are  $\beta = 0.972$  and  $\alpha = 0.025$ . The conditional distribution in this approximating strong GARCH process is formally irrelevant for the convergence results. However, we decided to take a t-distribution with degrees of freedom such that the (by the variance parameters) implied value of  $\kappa_\xi = 3.16$  at this frequency is matched. We also simulated at an even five times higher frequency choosing the parameters along the same lines, which did not affect the results. The daily variance parameters for the DMark-Dollar rate that we used, in line with Baillie and Bollerslev (1989), are  $\beta = 0.881$  and  $\alpha = 0.085$ . The implied values for the diffusion parameters are  $\theta = 0.035$  and  $\lambda = 0.427$ , while the GARCH parameters at the frequency where we actually simulated are  $\beta = 0.962$ ,  $\alpha = 0.034$ , and  $\kappa_\xi = 3.23$ . The estimated sizes of the various tests using 5,000 daily observations (approximately 20 years) and 1,000 replications are reported in Table 1.

Table 1 clearly shows that the sizes of all tests are certainly not above the 5% level

---

<sup>1</sup>A software package that computes weak GARCH parameters at all frequencies and GARCH diffusion/jump-diffusion parameters in any of these models is available on request.

	$\theta$ -test	$\lambda$ -test	$\kappa$ -test (daily)	$\kappa$ -test (weekly)
Simulated "Pound-Dollar" Diffusion	3.7%	3.8%	2.3%	2.6%
Simulated "DMark-Dollar" Diffusion	3.2%	2.8%	1.8%	2.4%
Simulated "P-D" + "D-DM" Diffusion	4.7%	4.4%	2.3%	3.0%
Sim. "P-D" + hom. white noise	3.7%	2.7%	3.1%	0.9%
Sim "DM-D" + hom. white noise	3.0%	2.3%	2.8%	1.2%

Table 1: Size of the weak GARCH test and the tests for the presence of jumps. Sample size is  $T = 5,000$  and results are based on 1,000 replications. See text for the assumed DGP's.

	$\theta$ -test	$\lambda$ -test	$\kappa$ -test (daily)	$\kappa$ -test (weekly)
Simulated "Pound-Dollar" Diff.+jumps	3.2%	4.4%	88.8%	3.2%
Simulated "DMark-Dollar" Diff.+jumps	3.5%	5.6%	84.9%	4.1%
Simulated "P-D" + "D-DM" Diff.+jumps	4.3%	5.2%	88.7%	5.4%
Sim. "P-D" + white noise+jumps	3.7%	4.0%	57.7%	1.0%
Sim. "DM-D" + white noise+jumps	3.9%	3.2%	52.0%	1.2%

Table 2: Power of the tests for the presence of jumps. Mean arrival time of jumps is one day. Sample size is  $T = 5,000$  and results are based on 1,000 replications. See text for the assumed DGP's.

suggested by the approximation to the finite sample distribution in all five cases. If one takes into account that the standard errors of the rejection frequencies are approximately 0.7%, we see that the tests are even somewhat conservative. In any case Table 1 clearly shows that tests may be based on the critical values derived in Appendix B. Of course, this simulation study does not imply the validity of the tests for all DGP's within the null hypothesis. Table 1, however, indicates that the size of the tests is in line with the value suggested by the asymptotic theory. This is also confirmed by several other simulations whose results are not reported here for conciseness.

In order to assess the power of the tests we superimposed a jump component on the simulated series. The arrival time of the jumps is assumed to be a Poisson process with mean arrival time either one day or one week. The jump size is Gaussian with zero mean and a variance chosen such that the variance of the jump component equals half that of the smooth component. Thus one third of the total unconditional variance is caused by jumps. The power results are reported in Tables 2 and 3. As before the high and low frequencies in the tests are chosen to be the daily and weekly frequency respectively.

Recall that the  $\theta$ -test is used to test whether the process is weak GARCH at both frequencies. Hence it should have no power against the alternatives considered here. This is confirmed by the numbers in Table 2 and 3. However, the tables also clearly show that the  $\lambda$ -test, based on the comparison of the variance estimates at the two frequencies, has very

limited power compared to the size reported in Table 1. The daily  $\kappa$ -test, a comparison of the daily estimated kurtosis with the value implied by the daily variance parameters as if an underlying diffusion model applies, has very good power properties even though this test is slightly conservative. The power is somewhat smaller for the fourth and fifth simulation if on average each day a jump occurs. The  $\kappa$ -test on weekly data has low power if the mean arrival time of the jumps is one day, i.e. if the DGP contains many small jumps. If the mean arrival time is one week, i.e. if the number of jumps is smaller given the same variance, the weekly  $\kappa$ -test has some power as well. From a different point of view one might say that the  $\kappa$ -test has best power if jumps occur less frequent than the data frequency used in the test. This conforms the intuitively appealing idea that one should use the data at the highest frequency available.

## 5 Empirical results on testing for the presence of jumps

In this section we present empirical results on the presence of jumps in several exchange rates (British Pound, Dutch Guilder, French Franc, German Mark, Italian Lire, Japanese Yen, and Swiss Franc) with respect to the US Dollar, as well as in the return on stock market indices for France, the UK and the US. As indicated in the introduction it is important to know whether spot prices of financial assets contain discontinuities or not. This section shows that they do, and the next section is concerned with estimation of the size of these jumps. All data are taken from Datastream. First we discuss results on the exchange rates and subsequently we address stock market indices.

The sample for the exchange rates<sup>2</sup>, runs from January 1980 until April 1994. This yields approximately 3,700 daily observations and 740 weekly observations per series. The estimated daily and weekly GARCH parameters for the exchange rates, which are presented in Table 4, are roughly in line with those found in the literature, e.g. in Baillie and Bollerslev (1989).

<sup>2</sup>We also considered the Australian Dollar, Austrian Shilling, Belgian Franc, Canadian Dollar, Danish Kroner, Norwegian Kroner, and Spanish Peseta and obtained similar results.

	$\theta$ -test	$\lambda$ -test	$\kappa$ -test (daily)	$\kappa$ -test (weekly)
Simulated "Pound-Dollar" Diff.+jumps	3.1%	7.5%	99.9%	29.6%
Simulated "DMark-Dollar" Diff.+jumps	2.8%	9.3%	100.0%	24.2%
Simulated "P-D" + "D-DM" Diff.+jumps	2.8%	6.3%	99.9%	35.8%
Sim. "P-D" + white noise+jumps	3.8%	5.2%	99.7%	10.1%
Sim. "DM-D" + white noise+jumps	3.6%	4.9%	99.6%	12.5%

Table 3: Power of the tests for the presence of jumps. Mean arrival time of jumps is one week. Sample size is  $T = 5,000$  and results are based on 1,000 replications. See text for the assumed DGP's.

	daily estimates			weekly estimates		
	$\alpha$	$\beta$	$\kappa_\xi$	$\alpha$	$\beta$	$\kappa_\xi$
Exchange rates						
British Pound	0.05	0.93	4.7	0.08	0.86	5.5
Dutch Guilder	0.08	0.89	4.6	0.12	0.73	3.7
French Franc	0.09	0.87	7.0	0.13	0.68	4.6
German Mark	0.07	0.89	4.5	0.12	0.72	3.7
Italian Lire	0.09	0.87	5.3	0.12	0.74	8.7
Japanese Yen	0.06	0.89	5.7	0.11	0.67	5.2
Swiss Franc	0.06	0.90	4.2	0.10	0.75	3.4
Stock market indices						
France	0.12	0.82	6.8	0.15	0.72	5.6
United Kingdom	0.10	0.86	7.5	0.10	0.77	12.0
United States	0.06	0.92	9.2	0.13	0.81	8.8

Table 4: Daily and weekly estimates for seven US\$-exchange rates (1/1/1980 – 1/4/1994) and three stock market indices (1/1/1976 – 1/9/1994).

The results in Drost and Nijman (1993) can be used to compute the weekly variance parameters that are implied by the estimated daily variance and kurtosis parameters if the process is weak GARCH. Moreover the results in Drost and Werker (1994), as summarized in Section 3, can be used to compute the weekly variance parameters that are implied by the estimated daily variance parameters if the underlying process in continuous time is a GARCH diffusion. These implied weekly variance parameters are reported in the first four columns of Table 5. A comparison of the implied weekly variance parameters with the estimated parameters from weekly data in Table 4 shows that the two sets of parameters are usually not far apart given the size of the standard errors of the weekly variance parameter estimates which is around 0.03 for the  $\alpha$ 's and around 0.06 for the  $\beta$ 's. Such a direct comparison obviously does not yield a joint test and moreover does not correct for the fact that the daily parameters are subject to estimation error as well. The formal  $\theta$ - and  $\lambda$ -tests for equality of the direct and implied variance parameters are presented in the first two columns of Table 6. Not surprisingly given the information in Tables 4 and 5, neither the hypothesis that the discrete time process is weak GARCH nor the hypothesis that the continuous time process is a GARCH diffusion is rejected. Non-rejection of the  $\theta$ -test confirms the validity of the model we use. The fact that the  $\lambda$ -test does not reject is not surprising given the Monte Carlo evidence in Section 4. The  $\kappa$ -test is more suited to test for the presence of jumps and we will turn to it now.

As discussed in Section 3, under the assumption that the continuous time process is a GARCH diffusion, the kurtosis parameter is determined by the variance parameters at every frequency and this yields another possibility to test for the presence of jumps. In the last two columns of Table 5 we present the daily and weekly pseudo-kurtosis of the innovations as given in (B.1), where  $\kappa_h$  is calculated from the variance parameters under

	weekly parameters implied by daily estimates if weak GARCH		weekly parameters implied by daily estimates if GARCH diffusion		daily $\kappa_\xi$ implied by $(\alpha_d, \beta_d)$ if no jumps	weekly $\kappa_\xi$ implied by $(\alpha_w, \beta_w)$ if no jumps
	$\alpha$	$\beta$	$\alpha$	$\beta$	$\kappa_D$	$\kappa_D$
Exchange rates						
British Pound	0.10	0.82	0.08	0.84	3.4	3.6
Dutch Guilder	0.12	0.72	0.11	0.73	3.6	4.2
French Franc	0.16	0.66	0.12	0.70	3.7	4.5
German Mark	0.12	0.73	0.10	0.75	3.6	4.3
Italian Lire	0.15	0.67	0.13	0.70	3.8	4.3
Japanese Yen	0.09	0.68	0.07	0.71	3.4	4.2
Swiss Franc	0.09	0.74	0.07	0.75	3.4	3.9
Stock market indices						
France	0.18	0.54	0.15	0.58	4.2	4.9
United Kingdom	0.18	0.63	0.13	0.68	3.8	3.9
United States	0.15	0.75	0.10	0.81	3.5	4.3

Table 5: Parameter values implied by estimated parameters in Table 4 under the assumption that the series is weak GARCH and under the assumption that jumps are absent.

the assumption of continuous sample paths, see (A.5). The direct daily estimates, reported in Table 4, are substantially larger for all currencies. Formal tests of the equality of the implied and estimated values are presented in Table 6. The  $\kappa$ -test rejects for all currencies on the daily frequency. At the weekly level, the estimated  $\kappa_\xi$  parameter is larger than the implied values for four out of seven currencies. The deviation is significant for the British Pound only. A comparison with the simulation results in Section 4 suggests that the mean arrival time of the jumps is closer to the daily frequency than to the weekly frequency, because otherwise more rejections of the weekly  $\kappa$ -test would have been observed.

We now turn to the analysis for the stock markets in France, the United Kingdom, and the USA<sup>3</sup>. The stock market indices we used are constructed by Datastream, which makes comparison between the stock markets in different countries feasible. For the stock market indices the sample runs from January 1976 up to September 1994. This amounts to 4,900 daily observations and almost 1,000 weekly observations.

Comparing Tables 4 and 5 shows that there is, as is the case for exchange rates, very little difference between the estimated weekly variance parameters and the implied weekly variance parameters from the daily ones under the hypothesis of a diffusion. This suggests once more that the  $\lambda$ -test will not reject the hypothesis of continuous sample paths. The

<sup>3</sup>We also considered stock market indices for Germany and Japan, but the estimated GARCH coefficients in these cases are incompatible with a continuous time GARCH process. Of course this could be due to estimation error.

Exchange rates	$\theta$ -test	$\lambda$ -test	$\kappa$ -test weekly	$\kappa$ -test daily
British Pound	0.1	0.1	2.5	3.8
Dutch Guilder	0.0	0.1	-1.1	4.4
French Franc	0.0	0.0	0.1	1.7
German Mark	0.1	0.5	-1.2	4.6
Italian Lire	0.4	0.0	1.4	3.9
Japanese Yen	0.0	1.5	1.2	5.5
Swiss Franc	0.1	0.9	-1.2	4.3
Stock market indices				
France	0.6	0.0	0.2	1.6
United Kingdom	0.7	2.2	1.5	1.1
United States	0.3	1.2	1.8	3.2

Table 6: Realizations of the statistics for testing weak GARCH and the absence of jumps for seven US\$-exchange rates (1/1/1980 – 1/4/1994) and three stock market indices (1/1/1976 – 1/9/1994). Critical values are taken from a  $\chi_1^2$ -distribution for the  $\theta$ - and  $\lambda$ -test and from a normal distribution (one-sided) for the  $\kappa$ -tests.

formal test results in Table 6 confirm this. Moreover, the  $\theta$ -test confirms the weak GARCH hypothesis, at both the daily and weekly frequency.

Examining the results concerning the  $\kappa$ -tests one concludes that there is much less evidence for the presence of jumps, in addition to conditional heteroskedasticity, in the case of stock market indices compared to the foreign exchange markets. This is also found by others, e.g. Jorion (1988). For the UK's stock market there is no evidence at all for the presence of jumps. Jumps are clearly present in the US's stock market, but for France the situation is not quite clear. Given the conservativeness of the  $\kappa$ -test, it is probably valid to conclude that there are jumps in that market as well.

At first sight it may seem surprising that the realized  $\kappa$ -test statistics for the stock markets are so low, given that the differences between the estimated kurtosis and the implied kurtosis under the assumption of a diffusion is larger than in the case of exchange rates. But, of course, such a comparison is not valid since one does not take into account joint estimation error.

The testing results in this section clearly indicate the presence of jumps in all series under consideration, except the UK's stock market. In the next section we will estimate the relative importance of jumps.

## 6 Estimation of continuous time models generating both conditional heteroskedasticity and jumps

The testing results in Section 5 clearly indicate the presence of both jumps and conditional heteroskedasticity in exchange rates movements and stock market fluctuations. As outlined in the introduction, the jumps have impact on many results in derivative pricing, hedging strategies, early exercise decisions, future valuation of portfolios, etc. Therefore it is important to estimate the relevant continuous time parameters in the presence of both jumps and conditional heteroskedasticity.

We consider a class of continuous time models generating both conditional heteroskedasticity and jumps. Based on the estimates presented in Section 5 we construct estimates of the parameters describing the diffusion and jump part in the model. More precisely we consider the following model for logarithmic returns<sup>4</sup>

$$dY_t = \eta dt + \sigma_t dL_t, \quad (6.1)$$

$$d\sigma_t^2 = \theta(\omega - \sigma_t^2)dt + \sqrt{2\lambda\theta} \sigma_t^2 dW_t^{(2)}, \quad (6.2)$$

$$L_t = \sqrt{\gamma} W_t^{(1)} + \sqrt{1 - \gamma} N_t, \quad (6.3)$$

where  $\eta \in \mathbb{R}$ ,  $\theta > 0$ ,  $\omega > 0$ ,  $\lambda \in (0, 1)$ ,  $\gamma \in [0, 1]$ , and  $W_t^{(1)}$ ,  $W_t^{(2)}$  and  $N_t$  are independent. The first two processes are Brownian motions while the latter is a compound Poisson process, i.e. a jump process where jumps occur according to a Poisson process with intensity  $\zeta$  and with independent, normally distributed jumps. We impose zero mean on the jump distribution. The parameter  $\gamma$  measures the importance of jumps in the evolution of spot prices. More precisely,  $(1 - \gamma)$  is the proportion of the variance in the logarithmic spot prices that is caused by the pure jump part  $N_t$ , the remaining fraction  $\gamma$  originates from the smooth part  $W_t^{(1)}$ . In order to avoid underidentification the Poisson process  $N_t$  has to be standardized and, consequently, the variance of the jumps has to be  $1/\zeta$ . We take  $\eta = 0$  which seems reasonable given the observed logarithmic returns in all series. The model (6.1) reduces to the model proposed by Merton (1976) if conditional homoskedasticity is imposed (i.e.  $\theta = \lambda = 0$ ). The jump component is included such that jump sizes also depend on the conditional volatility. This seems reasonable since jumps in periods with high volatility will probably have a larger impact than jumps in periods of low volatility.

Drost and Werker (1994) show that a discrete time series generated by (6.1)–(6.2) is weak GARCH at every frequency and derive explicit formulae for the dependence of the discrete time weak GARCH parameters on the parameters in the continuous time model:

$$(\mu_h, \alpha_h, \beta_h, \psi_h, \kappa_h) = F(h\eta, h\theta, h\omega, \lambda, \gamma, h\zeta). \quad (6.4)$$

These formulae are given in Appendix A for ease of reference. This shows that it is very well possible for a continuous time model including jumps to generate pure weak GARCH

---

<sup>4</sup>The results in this section are still applicable when  $L$  is an arbitrary symmetric Levy process in which case we estimate the kurtosis of  $L_1$ . See Appendix A for more details.

at every discrete frequency. The model (6.1) contains six parameters  $\eta$ ,  $\theta$ ,  $\omega$ ,  $\lambda$ ,  $\gamma$ , and  $\zeta$ , while only five parameters have been estimated in Section 5. This implies that not all parameters can be identified from the GARCH parameters at an arbitrary frequency. In Appendix A it is shown that one can identify  $\eta$ ,  $\theta$ ,  $\omega$ ,  $\lambda$ , and  $(1 - \gamma)^2/\zeta$ . Observe that estimation at two frequencies will not solve this identification issue<sup>5</sup>.

---

<sup>5</sup>One way out would be to include sixth moments in the estimation, but we found that it is not possible to estimate this moment with acceptable precision. This is not surprising given the evidence on the non-existence of higher moments in financial data.

	$\theta$	$\omega$	$\lambda$	$(1 - \gamma)^2/\zeta$	$1 - \gamma$ if daily jumps	$1 - \gamma$ if weekly jumps
Exchange rates						
British Pound	0.016 (0.006)	0.49 (0.06)	0.43 (0.13)	0.31 (0.09)	0.56	0.25
Dutch Guilder	0.035 (0.009)	0.51 (0.05)	0.48 (0.11)	0.19 (0.05)	0.44	0.19
French Franc	0.040 (0.010)	0.53 (0.07)	0.74 (0.24)	0.85 (0.23)	0.92	0.41
German Mark	0.033 (0.008)	0.51 (0.05)	0.44 (0.10)	0.19 (0.05)	0.44	0.19
Italian Lire	0.039 (0.014)	0.51 (0.07)	0.64 (0.17)	0.26 (0.09)	0.51	0.23
Japanese Yen	0.051 (0.019)	0.44 (0.03)	0.36 (0.09)	0.48 (0.10)	0.69	0.31
Swiss Franc	0.038 (0.009)	0.61 (0.05)	0.29 (0.06)	0.18 (0.05)	0.42	0.19
Stock market indices						
France	0.063 (0.017)	1.10 (0.11)	0.84 (0.17)	0.31 (0.17)	0.56	0.25
United Kingdom	0.041 (0.012)	0.85 (0.11)	0.87 (0.35)	0.52 (0.34)	0.72	0.32
United States	0.019 (0.011)	0.77 (0.18)	0.93 (0.53)	1.00 (0.30)	1.00	0.45

Table 7: Estimation results for US\$-exchange rates (1/1/1980 – 1/4/1994) and stock market indices (1/1/1976 – 1/9/1994) in the model (6.1)–(6.3). Heteroskedasticity consistent asymptotic standard errors between parentheses.

Note that the total unconditional variance of logarithmic returns at frequency  $h$  equals  $h\omega$ . Table 7 contains the estimation results for the seven exchange rates and the three stock market indices considered before. In all cases logarithmic returns are measured in percentages. The estimated values for  $\theta$  of the exchange rates imply a persistence at the daily frequency, i.e.  $h = 1$ , of approximately  $\alpha_h + \beta_h = 0.96$ . Similar numbers are found by others [see, e.g., Baillie and Bollerslev (1989)]. Recall that  $h\omega$  is the unconditional variance of the logarithmic returns at frequency  $h$ . Generally the estimated unconditional variance is higher for the stock market indices. One can show that  $2n$ -th unconditional moment of  $Y_t$  exists if and only if  $\lambda \in (0, (n - 1)^{-1})$ . This shows that  $\lambda$  indicates how many unconditional moments exist. For most exchange rates  $\lambda$  is somewhat smaller than 0.5. This implies that sixth moments of  $Y_t$  exist, but they are probably very large. The estimated  $\lambda$ 's for the stock markets are considerably larger, but still smaller than one. This

implies that fourth moments of  $Y_t$  also exist in these markets, but sixth moments don't.

Finally we investigate the behavior of the jump parameter  $(1 - \gamma)^2/\zeta$ . The  $\kappa$ -test at the weekly frequency hardly ever rejects the null hypothesis of no jumps in contrast to the daily  $\kappa$ -test. Hence, from the power investigations in Section 4 we conclude that mean interarrival times are probably smaller than one week. The last two columns in Table 7 show the fraction of the unconditional variance explained by jumps if the mean interarrival time of jumps is one day or one week, respectively. These numbers show that, in case of weekly jumps, roughly one quarter of the variance is explained by jumps. Exceptions are returns on the French Franc exchange rate and in the US's stock market. In both these cases almost half the variance is explained by jumps under the assumption that  $\zeta = 0.2$ . Jorion (1988) estimated that for the US\$/DM rate as much as 96% of the variance has to be contributed to jumps<sup>6</sup>. Given our estimated value of 0.19 for  $(1 - \gamma)^2/\zeta$  this would imply on average almost five jumps a day. This is in accordance with the remarks in Section 2 where it is argued that also misspecification in the smooth component of the model (2.1) will be attributed to the jump component. Similar conclusions hold if one uses the fraction of total variance caused by jumps that are estimated by Vlaar and Palm (1993). They find that for the US\$/DM exchange rate about 70% of the total variance is explained by the discontinuous movements in the exchange rate, which would, under the same condition, amount to an average of 2.5 jumps a day.

The estimated parameters show that jumps in the US's stock market are very predominant, while there is almost no evidence for the presence of jumps in the UK's stock market. In contrast to the  $\kappa$ -tests, the hypothesis  $(1 - \gamma)^2/\zeta = 0$  is rejected for the French stock market index (note that this is a one-sided test as well). Apart from randomness this may be explained by the conservativeness of the  $\kappa$ -test, see the discussion in Section 5.

## 7 Concluding remarks

In this paper we have developed tests for the hypothesis that a series (observed in discrete time) is generated by a diffusion process and we have discussed the results of these tests for seven currencies and three stock market indices. Several authors have proposed tests for this hypothesis. However, these tests are based on arbitrary and non-testable assumptions on the conditional distribution of the smooth component at the data frequency that has been chosen. Instead our tests are based on the assumption that the series is weak GARCH; this hypothesis can easily be tested and is never rejected for the examined financial series. Throughout Monte Carlo evidence suggests that the power of the proposed test, based on the relation between variance and kurtosis parameters implied by a GARCH diffusion, is good. It is important to note that, in contrast to previous tests based on a specific assumption at the available data frequency, our tests are valid at any frequency. The empirical results clearly indicate the presence of jumps in dollar exchange rates, while the results for the stock

---

<sup>6</sup>This number is however based on weekly observations for a different sample period and on the ad-hoc assumptions outlined in Section 2.

markets are mixed. These conclusions are in line with the results reported by others that imposed restrictions on the conditional distribution of the smooth component. However, we argue that they probably overestimate the size of the jumps. The finding of jumps has important consequences for derivative pricing, hedging strategies, modeling the distribution of future spot rates, etc. Also early exercise decisions can be severely influenced by the presence of jumps.

## A Relation between continuous time and discrete time parameters in GARCH processes

Proofs of all statements in this appendix can be found in Drost and Werker (1994). Let  $Y_t$  be generated by a GARCH diffusion, i.e. let  $Y_t$  have smooth sample paths and increments that are weak GARCH at every frequency. Denote the vector of weak GARCH parameters as  $(\mu_h, \alpha_h, \beta_h, \psi_h, \kappa_h)$ . Then there exist  $\eta \in \mathbb{R}$ ,  $\theta > 0$ ,  $\omega > 0$ , and  $\lambda \in (0, 1)$  such that

$$\mu_h = h\eta, \quad (\text{A.1})$$

$$\frac{\beta_h}{1 + \beta_h^2} = \frac{c_h \exp(-h\theta) - 1}{c_h \{1 + \exp(-2h\theta)\} - 2}, \quad (\text{A.2})$$

$$\alpha_h = \exp(-h\theta) - \beta_h, \quad (\text{A.3})$$

$$\psi_h = h\omega \{1 - \exp(-h\theta)\}, \quad (\text{A.4})$$

$$\kappa_h = 3 + 6 \frac{\lambda}{1 - \lambda} \frac{\exp(-h\theta) - 1 + h\theta}{h^2 \theta^2}, \quad (\text{A.5})$$

where

$$c_h = \frac{4\{\exp(-h\theta) - 1 + h\theta\} + 2h\theta(1 + h\theta \frac{1-\lambda}{\lambda})}{1 - \exp(-2h\theta)}. \quad (\text{A.6})$$

A similar result can be derived for arbitrary GARCH jump-diffusions, see Proposition 4.1 of Drost and Werker (1994). The specific model (6.1)–(6.2), with  $L_t$  an arbitrary symmetric Lévy process with finite fourth moments, defines a GARCH jump-diffusion [see Example 4.1 of Drost and Werker (1994)]. To identify  $\omega$  one has to set  $EL_1^2 = 1$ . From the GARCH parameters at any frequency one can identify the parameters  $\theta$ ,  $\omega$ ,  $\lambda$ , and  $\kappa_L$  where  $\kappa_L$  denotes the kurtosis of  $L_1$ . More precisely the functional relations (A.2)–(A.1) still hold but (A.5) and (A.6) should be replaced by

$$\kappa_h = 3 + \frac{\kappa_L - 3}{h(1 - \lambda)} + 6 \frac{\lambda}{1 - \lambda} \frac{\exp(-h\theta) - 1 + h\theta}{h^2 \theta^2}, \quad (\text{A.7})$$

$$c_h = \frac{4\{\exp(-h\theta) - 1 + h\theta\} + 2h\theta(1 + h\theta \frac{1-\lambda}{\lambda} + \frac{h\theta}{2\lambda} \frac{\kappa_L - 3}{h})}{1 - \exp(-2h\theta)}. \quad (\text{A.8})$$

Since the kurtosis of Brownian motion equals three, (A.7) and (A.8) reduce to (A.5) and (A.6) if  $L_t$  is a Brownian motion. In the specification (6.3) one readily verifies

$$\kappa_L = EL_1^4 = 3 + 3(1 - \gamma)^2 / \zeta.$$

Consequently, estimates for  $(1 - \gamma)^2/\zeta$  can easily be obtained from estimates of  $\kappa_L$ .

## B Asymptotic properties of tests

In this appendix we will briefly outline how to derive the limiting properties of the tests proposed as if sufficient regularity conditions are satisfied. All tests we consider are standard Wald-type tests of non-linear restrictions, and hence it suffices to derive the joint limiting distribution of the unrestricted QMLE-estimator applied at frequency 1 and  $m$ ,  $m \in \mathbb{N}$ . Note that although we are interested in their joint distribution, parameter estimates can be obtained by two separate likelihood optimizations. Let  $\varphi_{HF} = (\alpha_{HF}, \beta_{HF}, \psi_{HF}, \kappa_{\xi;HF})'$  denote the high frequency (i.e. at frequency 1) variance parameters and the pseudo-kurtosis of rescaled residuals, the latter parameter is related to the kurtosis of  $y_t$  by

$$\kappa_h = \kappa_{\xi h} \frac{1 - (\alpha_h + \beta_h)^2}{1 - (\alpha_h + \beta_h)^2 - (\kappa_{\xi h} - 1)\alpha_h^2}. \quad (\text{B.1})$$

Similarly, let  $\varphi_{LF}$  denote low frequency parameters in the same order and finally let  $\varphi = (\varphi'_{HF}, \varphi'_{LF})'$ . For conciseness, we only discuss estimation based on one specific low frequency sample and avoid the straightforward extension to the averaged low frequency estimator  $\bar{\varphi}_{LF}$  based on all possible low frequency aggregations. QMLE-Parameter estimates at any frequency are obtained by solving

$$\frac{1}{T} \sum_{t=1}^T d_t(\varphi; y) = 0,$$

where  $T$  denotes sample size and  $d_t$  is defined by

$$d_t(\alpha, \beta, \psi, \kappa_{\xi}; y) = \left[ \frac{1}{2} \frac{y_t^2 - \sigma_t^2}{\sigma_t^4} \frac{\partial}{\partial \alpha} \sigma_t^2, \frac{1}{2} \frac{y_t^2 - \sigma_t^2}{\sigma_t^4} \frac{\partial}{\partial \beta} \sigma_t^2, \frac{1}{2} \frac{y_t^2 - \sigma_t^2}{\sigma_t^4} \frac{\partial}{\partial \psi} \sigma_t^2, \frac{y_t^4}{\sigma_t^4} - \kappa_{\xi} \right]'$$

and  $\sigma_t^2 = \psi + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2$ . For expository reasons we want to distinguish between high and low frequency scores and write  $d_t^{HF}$  and  $d_t^{LF}$ , respectively.

Now suppose  $y_t$ ,  $t = 1, \dots, T$  are high frequency observations from some weak GARCH process. The low frequency observations are defined by

$$\bar{y}_t = \sum_{i=0}^{m-1} y_{t-i}, \quad t = m, \dots, T.$$

The estimates  $\hat{\varphi}_{HF}$  and  $\hat{\varphi}_{LF}$  are easily seen to satisfy

$$\frac{1}{T/m} \sum_{t=1}^{T/m} \tilde{d}_{mt}(\hat{\varphi}; y) = 0,$$

where  $\tilde{d}$  is defined by

$$\tilde{d}_{mt}(\varphi; y) = \left[ \begin{array}{c} \sum_{i=0}^{m-1} d_{mt-i}^{HF}(\varphi_{HF}; y) \\ d_{mt}^{LF}(\varphi_{LF}; \bar{y}) \end{array} \right].$$

Proceeding as usual Taylor's Theorem gives,

$$0 = \frac{1}{T/m} \sum_{t=1}^{T/m} \tilde{d}_{mt}(\varphi; y) + \frac{1}{T/m} \sum_{t=1}^{T/m} \frac{\partial \tilde{d}_{mt}}{\partial \varphi'}(\varphi; y)(\hat{\varphi} - \varphi) + o(\hat{\varphi} - \varphi).$$

Applying a Central Limit Theorem and a Weak Law of Large Numbers suggests that, under suitable regularity conditions, the joint estimator of the parameters at both frequencies, i.e.  $\sqrt{T/m}(\hat{\varphi} - \varphi)$ , has asymptotically a normal distribution. The variance of this distribution can easily be estimated using the averaged Hessian matrix and the averaged outer product of scores [see, e.g., Lee and Hansen (1994)]. In a similar manner one derives the expression for the joint distribution of  $\hat{\varphi}_{HF}$  and the averaged low-frequency estimators  $\bar{\varphi}_{LF}$ .

### References

- AMIN, K.I. (1993), Jump Diffusion Option Valuation in Discrete Time, *Journal of Finance*, **48** 1833–1863.
- AMIN, K.I. and NG, V.K. (1993), Option Valuation with Systematic Stochastic Volatility, *Journal of Finance*, **48** 881–910.
- BAILLIE, R.T. and BOLLERSLEV, T. (1989), The message in daily exchange rates: a conditional variance tale, *Journal of Business and Economic Statistics*, **7** 297–305.
- BALL, C. and TOROUS, W. (1985), On jumps in stock prices and their impact on call option pricing, *Journal of Finance*, **40** 155–173.
- BECKERS, S. (1981), A note on estimating the parameters of the diffusion-jump model of stock returns, *Journal of Financial and Quantitative Analysis*, **16** 127–140.
- BOLLERSLEV, T., CHOU, R.Y., and KRONER, K.F. (1992), ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics*, **52** 5–59.
- BOLLERSLEV, T. and WOOLDRIDGE, J.M. (1992), Quasi-maximum Likelihood Estimation and Inference in Dynamic Models with Time Varying Covariances, *Econometric Reviews*, **11** 143–172.
- DROST, F.C. and NIJMAN, T.E. (1993), Temporal aggregation of GARCH processes, *Econometrica*, **61** 909–927.
- DROST, F.C. and WERKER, B.J.M. (1994), Closing the GARCH gap: continuous time GARCH modeling, *Journal of Econometrics*, forthcoming.
- JORION, P. (1988), On jump processes in the foreign exchange and stock markets, *The Review of Financial Studies*, **1** 427–445.
- LEE, S-W. and HANSEN, B.E. (1994), Asymptotic Theory for the GARCH(1,1) Quasi-Maximum Likelihood Estimator, *Econometric Theory*, **10** 29–52.
- LUMSDAINE, R.L. (1991), Asymptotic Properties of the Maximum Likelihood Estimator in GARCH(1,1) and IGARCH(1,1) Models, *Princeton University*, mimeographed.

- MERTON, R. (1976), Option pricing when returns are discontinuous, *Journal of Financial Economics*, **3** 125–144.
- NAIK, V. (1993), Option Valuation and Hedging Strategies with Jumps in the Volatility of Asset Returns, *Journal of Finance*, **48** 1969–1984.
- NELSON, D.B. (1990), ARCH models as diffusion approximations, *Journal of Econometrics*, **45** 7–38.
- NIEUWLAND, F.G.M.C., VERSCHOOR, W.F.C., and WOLFF, C.C.P. (1991), EMS Exchange Rates, *Journal of International Financial Markets, Institutions and Money*, **2** 21–42.
- NIJMAN, T.E. and SENTANA, E. (1993), Marginalization and contemporaneous aggregation in multivariate GARCH processes, *CentER Discussion Paper* 9313, Tilburg University, forthcoming in *Journal of Econometrics*.
- PROTTER, P. (1990), *Stochastic Integration and Differential Equations: a new approach*, Springer Verlag, Berlin.
- REVUZ, D. and YOR, M. (1991), *Continuous Martingales and Brownian Motion*, Springer Verlag, Berlin.
- VLAAR, P.J.G. and PALM, F.C. (1993), The message in weekly exchange rates in the European Monetary System: Mean reversion, conditional heteroskedasticity and jumps, *Journal of Business and Economic Statistics*, **11** 351–360.