The Solution Set of the n-Player Scalar Feedback Nash Algebraic Riccati Equations

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by

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Abstract
In this paper we analyse the set of scalar algebraic Riccati equations (ARE) that play an important role in finding feedback Nash equilibria of the scalar N-player linear-quadratic differential game. We show that in general there exist maximal $2^N - 1$ solutions of the (ARE) that give rise to a Nash equilibrium. In particular we analyse the number of equilibria as a function of the state-feedback parameter and present both necessary and sufficient conditions for existence of a unique solution of the (ARE). Furthermore, we derive conditions under which the set of state-feedback parameters for which there is a unique solution grows with the number of players in the game.

Keywords: Linear quadratic games, feedback Nash equilibrium, solvability conditions, Riccati equations

Jelcodes: 110, 210

I. Introduction

During the last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. Engwerda et al. (1999-a) for references). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with eachother. One of these scenarios is the so-called feedback Nash scenario (see Başar and Olsder (1999) for a precise definition and survey of relevant literature).
Note that, since according this scenario the participating parties can react to eachother's policies, its economic relevance is mostly larger than that of the open-loop Nash scenario. In particular the feedback Nash scenario is very popular in studying problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives, the parties are striving for, can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first analyzed by Starr and Ho in (1969) (see also Lukes (1971) for a result on uniqueness within the class of affine memoryless strategies).
In this paper we study the infinite-planning horizon case and concentrate here on solving the with this problem associated algebraic Riccati equations. In Weeren et al. (1999) it was shown that in the two-player scalar case these equations have either one or three solutions which solve the optimization problem (see also Engwerda (1999-b) for a detailed study under which conditions on the system parameters these different situations occur).
In this paper we study the general N-player scalar case. We show that for any number N of players there exists a positive number such that if the state-feedback parameter is larger than this number, there exist (in general) $2^N - 1$ solutions for the (ARE) equations yielding a Nash equilibrium. Furthermore, we give both necessary and sufficient conditions under which there is exact one solution for the (ARE) equations. We also show that this situation is more likely to occur in case the number of players grows.
The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper. Section three analyzes the solutions of the algebraic Riccati equations. These results are used in section four to find necessary and sufficient conditions for existence of a unique solution. Section five presents some results on the effect on the uniqueness conditions of an increase of the number of players in the game. The paper ends with some concluding remarks.

II. Problem statement

In this paper we consider the problem where N parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x} = ax + \sum_{i=1}^{N} b_i u_i, \;\; x(0) = x_0. \tag{1}$$

Here x is the state of the system, u_i is a (control) variable player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, a (the state feedback parameter) and b_i, $i = 1,..,N$ are constant system parameters, and $\dot{x}$ denotes the time derivative of x. The performance criterion player $i = 1,..,N$ aims to minimize is:

$$J_i(u_1,..,u_N) := \frac{1}{2}\int_0^{\infty} \{x(t)^T q_i x(t) + u_i(t)^T r_{ii} u_i(t)\} dt.$$

We assume that both q_i and r_{ii} are positive and b_i differs from zero.
In this paper we consider the existence of limiting stationary feedback Nash equilibria of this differential game.
To that end we consider the following set of coupled algebraic Riccati equations (ARE):

$$(a - \sum_{j=1}^{N} k_j s_j)k_i + k_i(a - \sum_{j=1}^{N} s_j k_j) + q_i + k_i s_i k_i = 0, \;\; i = 1,..,N, \tag{2}$$

where $s_i := b_i r_{ii}^{-1} b_i$.
Given our assumptions on the system parameters one can immediately deduce from Başar and Olsder (1999, proposition 6.8) that:

<u>Theorem 1:</u>
Let $\bar{k}_i \geq 0$ solve the set of Riccati equations.
Then the stationary feedback policies

$$u_i = -r_{ii}^{-1} b_i \bar{k}_i x \tag{3}$$

$i = 1,..,N$, provide a Nash equilibrium, leading to the cost $J_i(u_1,..,u_N) := x_0 \bar{k}_i x_0$, for player i.
Moreover, the resulting system dynamics described by $\dot{x} = a_{cl}x$; $x(0) = x_0$, with $a_{cl} := a - \sum_{i=1}^{N} s_i \bar{k}_i$, is asymptotically stable. □

In fact, we conclude from Weeren et al (1999, corollary 3.1) that when the players are restricted at the outset to memoryless strategies (cf. Lukes (1971)) then existence of a positive solution to the above scalar Riccati equations is a both necessary and sufficient condition for existence of a feedback Nash equilibrium.
A natural question which arises is how many solutions the above set of algebraic Riccati equations (ARE) have. To analyze this question we introduce (for notational convenience) the variables:

$$\sigma_i := s_i q_i \text{ and } \kappa_i := s_i k_i, \;\; i = 1,..,N, \text{ and } \kappa_{N+1} = -a_{cl}.$$

Using this notation (2) can be rewritten as

$$\kappa_i^2 - 2\kappa_{N+1}\kappa_i + \sigma_i = 0,\ \ i = 1,..,N. \tag{4}$$

The above question can therefore be reformulated as under which conditions the above N quadratic equations and the equation

$$\kappa_{N+1} = -a + \sum_{j=1}^{N} \kappa_j \tag{5}$$

have a positive solution $\kappa_i,\ i = 1,..,N+1$.
In the next section we will study this problem in detail.

<u>III. The solution set</u>

We will assume, without loss of generality, that the σ_i's satisfy $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq ... \geq \sigma_N$. Provided that $\sqrt{\sigma_1} \leq \kappa_{N+1}$ we have that the first N equations always have two positive solutions. From (4) we have that either $\kappa_i = \kappa_{N+1} + \sqrt{\kappa_{N+1}^2 - \sigma_i}$ or $\kappa_i = \kappa_{N+1} - \sqrt{\kappa_{N+1}^2 - \sigma_i},\ i = 1,..,N$. Substitution of this into (5) shows that $\kappa_{N+1} > \sqrt{\sigma_1}$ must satisfy the following equation

$$(N-1)\kappa_{N+1} \pm \sqrt{\kappa_{N+1}^2 - \sigma_1} ... \pm \sqrt{\kappa_{N+1}^2 - \sigma_N} = a. \tag{6}$$

To study the number of solutions to these equations, we introduce the next recursively defined functions for $n = 1,..,N-1$:

$$f_i^{n+1}(x) := f_i^n(x) + x - \sqrt{x^2 - \sigma_{n+1}}, i = 1,..,2^n \tag{7}$$

$$f_{i+2^n}^{n+1}(x) := f_i^n(x) + x + \sqrt{x^2 - \sigma_{n+1}}, i = 1,..,2^n \tag{8}$$

with

$$f_1^1(x) := -\sqrt{x^2 - \sigma_1} \text{ and } f_2^1(x) := \sqrt{x^2 - \sigma_1}. \tag{9}$$

It is easily verified (by induction) that due to this construction the functions f_i^N satisfy the following monotonicity property

$$f_i^N(\sqrt{\sigma_1}) \leq f_{i+1}^N(\sqrt{\sigma_1}), i = 1,..,2^N - 1. \tag{10}$$

In particular, the next three functions will play an important role in the subsequent analysis

$$f_1^N(x) = (N-1)x - \sum_{i=1}^{N} \sqrt{x^2 - \sigma_i} \tag{11}$$

$$f_2^N(x) = (N-1)x + \sqrt{x^2 - \sigma_1} - \sum_{i=2}^{N} \sqrt{x^2 - \sigma_i} \tag{12}$$

and

$$f_3^N(x) = (N-1)x - \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2} - \sum_{i=3}^{N} \sqrt{x^2 - \sigma_i}. \tag{13}$$

Now, (6) has a solution if and only if $f_i^N = a$ has a solution for some $i \in 1,..,2^N$. Furthermore, it is obvious from this relationship that the number of solutions to (ARE) coincides with the number of solutions to the equation

$$\Pi_{i=1}^{2^N}(f_i^N - a) = 0. \tag{14}$$

For the moment, concentrate on the 2-player case. It is easily verified that the above equation (14) has then the following algebraic structure

$$f(a_0, a_1, a_2) := (a_0 - a_1 - a_2)(a_0 + a_1 - a_2)(a_0 - a_1 + a_2)(a_0 + a_1 + a_2) = 0. \tag{15}$$

The structure of f for the general N-player case is similar and is omitted in order to avoid unnecessary cumbersome notation. From this it is not difficult to see that f only has quadratic entries. That is, more precisely,

Lemma 2:
$f(a_0,..,a_N)$ is a sum of terms, where each term consists of $\Pi_{i=0}^N a_i^{2k_i}$ for some nonnegative integers k_i satisfying $\sum_{i=0}^N 2k_i = 2^N$.

Proof:
It is easily verified that $f(-a_0, a_1,..,a_N) = (-1)^{2^N} f(a_0,..,a_N) = f(a_0,..,a_N)$ and, also, $f(a_0,..,-a_i,..,a_N) = f(a_0,..,a_i,..,a_N)$, for any $i \in 1,..,N$.
Now, assume that f has a term in which, e.g., a_0 has an odd exponent. Then, collect all terms of f containing odd exponents in a_0. As a consequence $f = a_0 g(a_0,..,a_N) + h(a_0,..,a_N)$, where in all terms of both g and h a_0 appears with an even exponent. Since $f(-a_0, a_1,..,a_N) = f(a_0, a_1,..,a_N)$ we conclude immediately from this that g must be zero. The rest of the proof follows then straightforwardly. □

Using this lemma we can then easily derive the following result on the number of solutions to the (ARE) equations

Theorem 3:
(ARE) always has at least one and at most $2^N - 1$ positive solutions.

Proof:
Consider equation (14). Let $a_0 := (n-1)x - a$ and $a_i := \sqrt{x^2 - \sigma_i}$. With this notation, equation (14) coincides with (15). According lemma 2 this equation is a polynomial in x of degree 2^N.
Next, we show that this polynomial has at most $2^N - 1$ roots larger than $\sqrt{\sigma_1}$. To that end we first note that f can be rewritten as

$$f = \Pi_{i=1}^{2^{N-2}}(a_0 - (a_1 + g_i))(a_0 + (a_1 + g_i))(a_0 - (a_1 - g_i))(a_0 + (a_1 - g_i)), \tag{16}$$

where g_i is a linear combination (with coefficients +1 or −1) of $a_2, .., a_N$.
From (16) we immediately have that $f = \Pi_{i=1}^{2^{N-2}}(a_0^2 - (a_1 + g_i)^2)(a_0^2 - (a_1 - g_i)^2)$. Now at $x = \sqrt{\sigma_1}$, $a_1 = 0$. Therefore we conclude that at $x = \sqrt{\sigma_1}$, $f = \Pi_{i=1}^{2^{N-2}}(a_0^2 - (g_i)^2)^2 > 0$. Furthermore, it is easily verified that except for the term $a_0 - \sum_{i=1}^N a_i$, all terms $a_0 \pm a_1 \pm g_i$ in (16) are positive if $x \to \infty$. Therefore, the leading term x^{2^N} of the polynomial has a negative sign. So, we conclude that the polynomial has always a root located at the lefthandside of $\sqrt{\sigma_1}$. Or stated differently, (ARE) has at most $2^N - 1$ positive solutions. To see that (ARE) always has at least one solution, we study the equations $f_1^N(x) = a$ and $f_2^N(x) = a$ (see (11,12)). Obviously, $f_1^N(\sqrt{\sigma_1}) = f_2^N(\sqrt{\sigma_1})$. Since both functions are continuous with $\lim_{x\to\infty} f_1^N = -\infty$ and $\lim_{x\to\infty} f_2^N = \infty$, it is clear that either the equation $f_1^N(x) = a$ or $f_2^N(x) = a$ will have a solution $x \geq \sqrt{\sigma_1}$, which completes the proof. □

To get an impression how the number of solutions of (ARE) varies with the state parameter a, we sketched in figure 1 for the three player case the curves f_i^3.

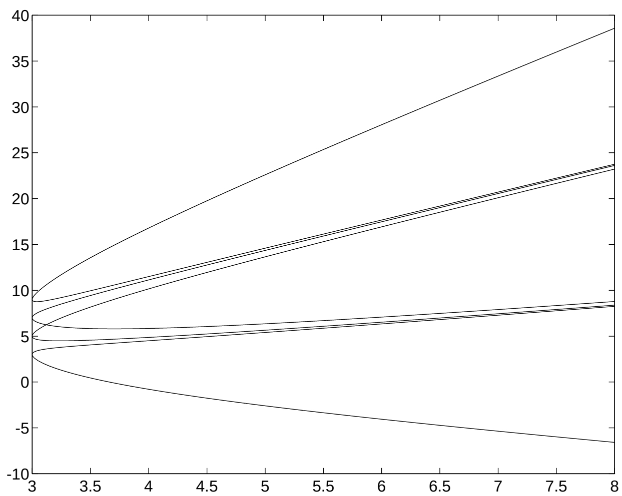

Figure 1: The curves f_i^3 for $\sigma_1 = 9$; $\sigma_2 = 8$; $\sigma_3 = 5$.

From this figure we see, by counting the number of points of the different curves f_i^3 which have level a, that the number of solutions of (ARE) increases monotonically from 1 to 7 as a function of a. That this monotonicity does in general not hold is illustrated by the next figure 2, where we plotted for different parameter values f_2^3 and f_3^3. Since f_1^3 is a monotonically decreasing function and $f_i^3(x) \geq f_3^3(x)$ for $i > 3$ (as we will show later on (see lemma 7)), we see that the number of solutions first increases from 1 to 3 and then drops back to 1 before it increases again.

In particular note from these examples that an even number of solutions occurs only at isolated points for a, whereas an uneven number of solutions occurs at intervals for a. We will not elaborate this subject further here, but it seems that this property holds in general.

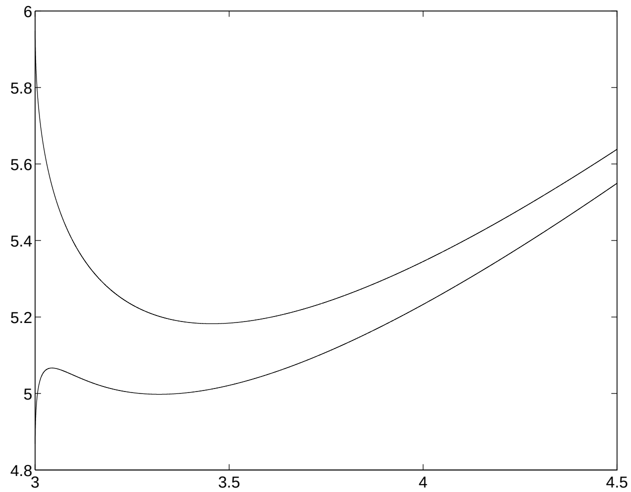

Figure 2: The curves f_2^3 and f_3^3 for $\sigma_1 = 9;\ \sigma_2 = 8.7;\ \sigma_3 = 8.65$.

Next, we show that the functions $f_i^N(x)$ do not intersect if x becomes large. To prove this property we first concentrate on the case that all σ_i differ. So, we assume from now on that $\sigma_1 > \sigma_2 > ...\sigma_N$.
The next lemma is a preliminary result used to show the correctness of theorem 5.

Lemma 4:
Assume that all σ_i differ. Then, there exists a constant x_1 such that the functions $f_i^N(x),\ i = 2, .., 2^N$ do not intersect on the interval (x_1, ∞).

Proof:
We show that any two functions f_i^N and f_j^N only have a finite number of intersection points, from which the conclusion is then obvious.
So, assume $f_i^N(x) = f_j^N(x)$. Since all σ's differ the equation $f_i^N - f_j^N = 0$ can be rewritten as

$$2 * (b_1\sqrt{x^2 - \sigma_1} + ... + b_N\sqrt{x^2 - \sigma_N}) = 0, \tag{17}$$

where $b_i \in \{-1, 0, 1\}$ and not all b_i are simultaneously zero. Now, denote $b_i\sqrt{x^2 - \sigma_i}$ by $a_i(x)$. Then the question as whether (17) has a finite number of zero's can be rephrased whether $\sum_{i=1}^N a_i(x) = 0$ has a finite number of zero's. To prove that this is the case, we restrict for the moment to the case $N = 3$. How the general case can be proved will be clear from this.
So, we have to prove that $a_1 + a_2 + a_3 = 0$ has only a finite number of zero's. Like in (15) we consider now the following function

$$f(a_1, a_2, a_3) := (a_1 - a_2 - a_3)(a_1 + a_2 - a_3)(a_1 - a_2 + a_3)(a_1 + a_2 + a_3) = 0.$$

Obviously, $a_1 + a_2 + a_3 = 0$ has a finite number of zero's, if f has a finite number of zero's. However, from lemma 2 we know that f is a polynomial which degree is at most 8. So, f has at most 8 zero's, which proves the claim. $\square$

Next, consider $f_2^N(x)$. By differentiating $f_2^N(x)$ it is easily verified that $f_2^N(x)$ will be monotonically increasing for all $x \geq x_1^*$ for some number $x_1^* > \sqrt{\sigma_1}$. Furthermore, since $\lim_{x\to\infty} f_2^N(x) = \infty$ and $f_2^N(x)$ is bounded from above on the interval $(\sqrt{\sigma_1}, x_1^*)$, it follows that there exists a positive number a_1^{**} such that for all $a \geq a_1^{**}$ the equation $f_2^N(x) = a$ has exactly one solution. A similar reasoning holds for all the other $f_i^N(x)$, $i = 2, .., 2^N$ (see also figure 1 for a visualization in case $N = 3$). Next, take the maximum over all a_i^{**}. Since according lemma 4 for a fixed a the solutions for $f_i^N(x) = a$ differ for all i if a is chosen large enough, it is easily verified that the corresponding solutions $(\kappa_1, ..., \kappa_N)$ to (4,5) will also differ. So, it is clear then that the next conclusion holds

<u>Theorem 5:</u>
Assume that σ_i differ. Then, there exists a positive number $\hat{a}$ such that for every state feedback parameter $a \geq \hat{a}$ the set of algebraic Riccati equations (2) has $2^N - 1$ (positive) solutions. $\square$

<u>Remark 6:</u>
In case the σ_i do not differ, it is easily verified from the above analysis that a similar conclusion holds. That is, there exists a number $\hat{a}$ such that for all $a > \hat{a}$ the number of solutions to (ARE) does not increase anymore. This number equals the number of distinct (ultimately) monotonically increasing functions f_i^N. Without providing a formal proof we note that if one denotes by s the number of σ_i's that coincide, some carefull counting shows that this number of solutions is

$$2^N - 2^{N-s} \sum_{i=1}^{s-1} \begin{pmatrix} s \\ s-i \end{pmatrix} + (s-1)2^{N-s} - 1,$$

for $N > s$. Here the term $2^N - 2^{N-s} \sum_{i=1}^{s-1} \begin{pmatrix} s \\ s-i \end{pmatrix}$ counts the number of solutions that do not coincide with any other solution; $(s-1)2^{N-s}$ counts the number of solutions that occur multiple times and -1 comes from the number of monotonically decreasing functions. Furthermore, it is easily verified that if $N = s$, the number of solutions equals $[\frac{N}{2}] + 1$. Here $[\frac{N}{2}]$ denotes the largest integer smaller than $\frac{N}{2}$ (e.g. $[\frac{3}{2}] = 1$). So, e.g. if $N = 5$ and $\sigma_1 = \sigma_2 = \sigma_3 > \sigma_4 > \sigma_5$, $s = 3$ and the maximum number of solutions will be 15. $\square$

IV. Uniqueness conditions

In this section we will give both necessary and sufficient conditions under which (ARE) will have a unique solution. To solve this problem, we study the functions f_i^N as defined in (7,8) in some more detail. First we note that

Lemma 7:
For every $N \geq 2$ the following inequalities hold: $f_1^N \leq f_2^N \leq f_3^N \leq f_i^N$ for any $i \geq 4$.

Proof:
The proof is by induction.
For $N = 2$, $f_1^2(x) = x - \sqrt{x^2-\sigma_1} - \sqrt{x^2-\sigma_2}$, $f_2^2(x) = x + \sqrt{x^2-\sigma_1} - \sqrt{x^2-\sigma_2}$, $f_3^2(x) = x - \sqrt{x^2-\sigma_1} + \sqrt{x^2-\sigma_2}$ and $f_4^2(x) = x + \sqrt{x^2-\sigma_1} + \sqrt{x^2-\sigma_2}$. Since by assumption $\sigma_1 \geq \sigma_2$, the correctness of all inequalities follows by straightforward verification.
Now, assume the inequalities hold for $N = k$. Then, by definition, for $i = 1, 2, 3$ we have $f_i^{k+1}(x) = f_i^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} \leq f_{i+1}^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_{i+1}^{k+1}(x)$. In a similar way we have for $i = 5, .., 2^k$ that $f_i^{k+1}(x) = f_i^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} \geq f_4^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_4^{k+1}(x)$, and for $i = 2^k + 1, .., 2^{k+1}$ $f_{i+2^k}^{k+1}(x) = f_i^k(x) + x + \sqrt{x^2 - \sigma_{k+1}} \leq f_4^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_4^{k+1}(x)$. $\square$

By differentiating (11) it is obvious that $f_1^N(x)$ is a strict monotonically decreasing function. Furthermore, it is easily verified in the same way that all other functions $f_i^N(x)$ are strictly monotonically increasing for all $x > x_1$ for some x_1. In the next theorem we will use this together with the previous lemma to derive conditions under which the (ARE) will have only one poitive solution. But, first, we introduce a convention w.r.t. local versus global extrema. With a local extremum we mean an extremum which occurs somewhere on the open interval $(\sqrt{\sigma_1}, \infty)$; whereas for the definition of a global extremum we take the whole domain of definition $[\sqrt{\sigma_1}, \infty)$.
Furthermore, we need some technical results presented in the next lemma.

Lemma 8:
i) If $\sigma_1 > \sigma_2$, $f_3^N(x)$ has exact one local minimum.
ii) $f_2^N(x)$ has at most two local extrema.
iii) If $f_2^N(x)$ has a local minimum, then arg min $f_3^N(x) \geq$ arg local minimum $f_2^N(x)$.

Proof:
i) The first derivative of $f_3^N(x)$ (see (13)) is $N - 1 - \sum_{i\neq 2}^N \frac{x}{\sqrt{x^2-\sigma_i}} + \frac{x}{\sqrt{x^2-\sigma_2}}$. So, if $\sigma_1 > \sigma_2$, $\lim_{x\downarrow\sqrt{\sigma_1}} f_3^{N'}(x) = -\infty$ and $\lim_{x\to\infty} f_3^{N'}(x) = 1$. Furthermore, $f_3^{N"}(x) = \sum_{i\neq 2}^N \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{\sigma_2}{(x^2-\sigma_2)^{3/2}}$. Since $\sigma_1 \geq \sigma_2$ it is clear that $f_3^{N"}(x) > 0$. So, $f_3^{N'}(x)$ has exact one zero, from which the conclusion is obvious.
ii) Differentiation of $f_2^N(x)$ (see (12)) yields $f_2^{N'}(x) = N - 1 - \sum_{i\neq 1}^N \frac{x}{\sqrt{x^2-\sigma_i}} + \frac{x}{\sqrt{x^2-\sigma_1}}$ and $f_2^{N"}(x) = \sum_{i\neq 1}^N \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{\sigma_1}{(x^2-\sigma_1)^{3/2}}$. Now, assume $f_2^{N"}(x)$ has a zero at p. Some rewriting of $f_2^{N"}(p) = 0$ shows then that $\sigma_1 = (p^2-\sigma_1)^{3/2} \sum_{i\neq 1}^N \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}}$. Substitution of

this expression into $f_2^{N"}(x)$ yields:

$$\begin{aligned} f_2^{N"}(x) &= \sum_{i\neq 1}^{N} \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}} \sum_{i\neq 1}^{N} \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}} \\ &= \sum_{i\neq 1}^{N} \left(\frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}} \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}} \right) \\ &= \sum_{i\neq 1}^{N} \sigma_i \left(\frac{(x^2-\sigma_1)^{3/2}(p^2-\sigma_i)^{3/2} - (x^2-\sigma_i)^{3/2}(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}(p^2-\sigma_i)^{3/2}(x^2-\sigma_i)^{3/2}} \right). \end{aligned}$$

Now, $\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)} - \sqrt{(x^2-\sigma_i)(p^2-\sigma_1)} = \frac{(x^2-\sigma_1)(p^2-\sigma_i)-(x^2-\sigma_i)(p^2-\sigma_1)}{\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)}+\sqrt{(x^2-\sigma_i)(p^2-\sigma_1)}} = \frac{(\sigma_1-\sigma_i)(x^2-p^2)}{\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)}+\sqrt{(x^2-\sigma_i)(p^2-\sigma_1)}} > 0$, if and only if $x > p$. From this it follows then easily that $f_2^{N"}(x)$ will have only one root and that $f_2^{N'}(x)$ will have a local minimum at p. The stated result follows then directly.
iii) Assume $f_3^N(x)$ has a local minimum at p, so $f_3^{N'}(p) = 0$. From this we have that $N-1 = \sum_{i\neq 2}^{N} \frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p}{\sqrt{p^2-\sigma_2}}$. Substitution of this expression into $f_2^{N'}(p+\delta)$ yields then for positive δ

$$\begin{aligned} f_2^{N'}(p+\delta) &= N-1-\sum_{i\neq 1}^{N} \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} + \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} \\ &= \sum_{i\neq 2}^{N} \frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p}{\sqrt{p^2-\sigma_2}} - \sum_{i\neq 1}^{N} \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} + \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} \\ &= \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_2}} + \frac{p}{\sqrt{p^2-\sigma_1}} - \frac{p}{\sqrt{p^2-\sigma_2}} + \\ &\quad \sum_{i\neq 3}^{N} \left(\frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} \right) \\ &> 0, \end{aligned}$$

where the last inequality follows from the facts that $\sigma_1 \geq \sigma_2$ and, according the mean value theorem, $\frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} = \delta \frac{\sigma_i}{(\xi^2-\sigma_i)^{3/2}}$, for some $p < \xi < p+\delta$.
So, the derivative of $f_2^{N'}(x)$ is always positive at the righthandside of the local minimum of $f_2^{N'}(x)$, which proves the claim. □

Theorem 9:
Assume that $\sigma_1 > \sigma_2$. Then, (ARE) has exactly one positive solution if and only if either one of the next conditions is satisfied
i) if f_2^N is monotonically increasing and $a < \min f_3^N$.
ii) if f_2^N is not monotonically increasing and a satisfies either I. $a <$ local minimum $f_2^N(x)$ or II. local maximum $f_2^N(x) < a < \min f_3^N(x)$.

Proof:
First consider the case that $f_2^N(x)$ is monotonically increasing. Since $f_1^N(x)$ is strict monotonically decreasing and $f_1^N(\sqrt{\sigma_1}) = f_2^N(\sqrt{\sigma_1})$, it is obvious from the fact that $f_i^N(x) \geq f_3^N(x), i = 4,..,2^N$ (see lemma 7) that for a fixed a there will be only one intersection point with the functions $f_i^N(x)$ if and only if a is smaller than the global minimum of $f_3^N(x)$ (see e.g. figure 1).
Next, consider the case that $f_2^N(x)$ is not monotonically increasing. According lemma 8.ii, $f_2^N(x)$ has then a local maximum and a local minimum. Furthermore, (see lemma 8.i and iii) this local minimum is located at the lefthandside of the local minimum of $f_3^N(x)$ (see figure 2 for an illustration of this situation). Since $f_3^N(x) \geq f_2^N(x)$ it is clear that for all a smaller than the local minimum of $f_2^N(x)$, there will be only one intersection point with the different f_i^N. Obviously, when a is located between the local minimum and the local maximum of $f_2^N(x)$ there will be three solutions. In case the local minimum of $f_3^N(x)$ is larger than the local maximum value of $f_2^N(x)$, the number of solutions drops, again, to 1. If a is larger than this local minimum of $f_3^N(x)$, there will always be at least one intersection point with $f_2^N(x)$ and one with $f_3^N(x)$, which concludes the proof. $\square$

Remark 10:
In case $\sigma_1 = \sigma_2$, $f_2^N(x)$ and $f_3^N(x)$ coincide. Moreover, at $\sqrt{\sigma_1}$, $f_i^N(x), i = 1,..,4$ coincide. From this it is easily seen that there will be exact one intersection point of a with all these functions if and only if a is smaller than the global minimum of $f_2^N(x)$ (in fact this inequality has to be strict in case $f_2^N(x)$ has a local minimum (which is then also the global one)). $\square$

In the following figure we illustrate, for fixed σ_i, the two possibilities that can occur for the set of parameters a for which there is a unique equilibrium

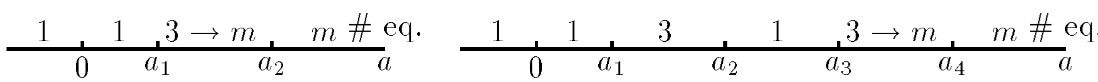

Figure 3: Structure of sets where (ARE) has a unique positive solution. Here $m \leq 2^N - 1$ denotes the maximum number of solutions.

We conclude this section with three related issues.
First we like to mention that in case $\sigma_1 \geq \sigma_2 + ... + \sigma_N$, $f_2^N(x)$ is monotonically increasing. This can be shown by a direct evaluation of its derivative. We have

$$\begin{aligned} f_2^{N'}(x) &= N - 1 + \frac{x}{\sqrt{x^2 - \sigma_1}} - \sum_{i=2}^{N} \frac{x}{\sqrt{x^2 - \sigma_i}} \\ &= \frac{x}{\sqrt{x^2 - \sigma_1}} - \sum_{i=2}^{N} \frac{\sigma_i}{\sqrt{x^2 - \sigma_i}(x + \sqrt{x^2 - \sigma_i})} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\sqrt{x^2-\sigma_1}}(x-\sum_{i=2}^{N}\frac{\sigma_i}{x+\sqrt{x^2-\sigma_i}}) \\
&= \frac{1}{\sqrt{x^2-\sigma_1}}\frac{1}{x+\sqrt{x^2-\sigma_2}}(x^2+x\sqrt{x^2-\sigma_2}-\sigma_2-\sum_{i=3}^{N}\sigma_i\frac{x+\sqrt{x^2-\sigma_2}}{x+\sqrt{x^2-\sigma_i}}) \\
&\geq \frac{1}{\sqrt{x^2-\sigma_1}}\frac{1}{x+\sqrt{x^2-\sigma_2}}(x^2+x\sqrt{x^2-\sigma_2}-\sum_{i=2}^{N}\sigma_i) \\
&\geq \frac{1}{\sqrt{x^2-\sigma_1}}\frac{1}{x+\sqrt{x^2-\sigma_2}}(\sigma_1+x\sqrt{x^2-\sigma_2}-\sum_{i=2}^{N}\sigma_i) > 0.
\end{aligned}$$

So, under this condition we have that the set of a-parameters for which there is a unique solution to the (ARE) equations is given by a half line.
A second related issue is that for all $a < \sqrt{\sigma_1} - \sqrt{\sigma_1 - \sigma_2}$ there will always be a unique solution too. To show this, first note from theorem 8 that whenever $a <$ minimum f_2^N, there will be a unique solution to the (ARE). It is easily verified that f_2^2 is monotonically increasing and therefore its minimum is given by $f_2^2(\sqrt{\sigma_1}) = \sqrt{\sigma_1} - \sqrt{\sigma_1 - \sigma_2}$. Since $f_2^N(x) \leq f_2^{N+1}(x)$, the rest of the argument follows by induction.
Finally, the third issure we like to address is that in Engwerda (1999-b) it was shown, for the two player case, that the additional requirement that amongst all (ARE) solutions we look for a solution which minimizes aggregate performance always gives rise to a unique solution. Unfortunately this property does not hold for the general case, as we can see from figure 1. In this figure we see that the curves f_4^3 and f_5^3 intersect at some point (κ_4^*, a^*) (approximately (3.2,6.5)). From (5) we therefore conclude that at this point for both solutions we have that $\kappa_1 + \kappa_2 + \kappa_3 = \kappa_4^* + a^*$. Now, choose the parameters b_i and r_i such that $s_1 = s_2 = s_3 = 1$ (and consequently, $q_1 = 9; q_2 = 8$ and $q_3 = 5$). Then $k_i = \kappa_i$ and consequently the cost player i has at this equilibrium is $x_0^2\kappa_i$. So, the aggregate cost is $x_0^2(\kappa_1 + \kappa_2 + \kappa_3)$. Consequently, at $a = a^*$ two different solutions yield the same aggregate cost, which is obviously (see figure 1 again) also the minimum attainable aggregate cost in this case.

V. Uniqueness versus the number of players

Next we consider the influence of the number of players on the uniqueness conditions we derived in the previous section. We address the question whether the parameter set for a for which there is a unique solution to (ARE), increases if the number of players increases. This would sustain the intuition that in a noncooperative game it becomes more difficult to reach an agreement in case the number of players increases.
To analyze this problem we introduce for a fixed sequence $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_N \geq ...$, the set $U_N := \{a|$ (ARE) has a unique positive solution$\}$. The following lemma is an immediate consequence of an exhaustive analyses of theorem 9.

Lemma 11:

$U_N \subset U_{N+1}$ if and only if either one of the next three conditions holds:
1) f_2^{N+1} is monotonically increasing;
2) local minimum $f_2^{N+1} >$ minimum f_3^N;
3) minimum $f_3^N <$ local maximum f_2^N.

Theorem 12:
If $\sigma_N \to \sigma > 0$ then there exists an $\bar{N}$ such that for all $N > \bar{N}$ $U_N \subset U_{N+1}$.

Proof:
We will show that under the above assumption, condition 3) of lemma 11 is satisfied if $N \to \infty$. To that end we first note that:

$$\begin{aligned} \text{local maximum} f_2^N(x) \geq f_2^N(\sqrt{\sigma_1}) &= (N-1)\sqrt{\sigma_1} - \sum_{i=2}^{N} \sqrt{\sigma_1 - \sigma_i} \\ &= \sum_{i=2}^{N} \frac{\sigma_i}{\sqrt{\sigma_1} + \sqrt{\sigma_1 - \sigma_i}} \\ &> \sum_{i=2}^{N} \frac{\sigma_i}{2\sqrt{\sigma_1}} > \frac{N-2}{2} \frac{\sigma}{\sqrt{\sigma_1}} \quad (1). \end{aligned}$$

On the other hand we have that the minimum of $f_3^N(x)$ is smaller than $f_3^N(\sqrt{N\sigma_1})$. Since $\sigma_i < \sigma_1$, $\sqrt{N\sigma_1 - \sigma_i} > \sqrt{(N-1)\sigma_1}$. Therefore,

$$\begin{aligned} \text{minimum } f_3^N(x) &\leq \sum_{i \neq 2}^{N} \frac{\sigma_i}{2\sqrt{(N-1)\sigma_1}} + \sqrt{N\sigma_1 - \sigma_2} \\ &\leq \sqrt{\sigma_1} 2 \sum_{i \neq 2}^{N} \frac{1}{\sqrt{N-1}} + \sqrt{N\sigma_1 - \sigma_2} \quad (2). \end{aligned}$$

Comparing (1) and (2), elementary analysis shows that for N large enough, condition 3) of lemma 11 will be satisfied. □

Remark 13:
Since $f_i^N(x) \leq f_i^{N+1}(x), i = 1, .., 2^N$, in principle the set U_N shifts to the right. Therefore, in the case illustrated in figure 2, we have that by taking here all σ_i for $i > 3$ approximately zero, U_N will never be included in U_{N+1}. We illustrated this phenomenon in figure 4. So, the assumption in theorem 12 that $\sigma > 0$ is essential.

IV. Concluding remarks

In this paper we studied the positive solutions of the algebraic Riccati equations that play an important role in the study of limiting stationary feedback Nash equilibria in the N-player linear quadratic scalar differential game. We showed that this set of equations

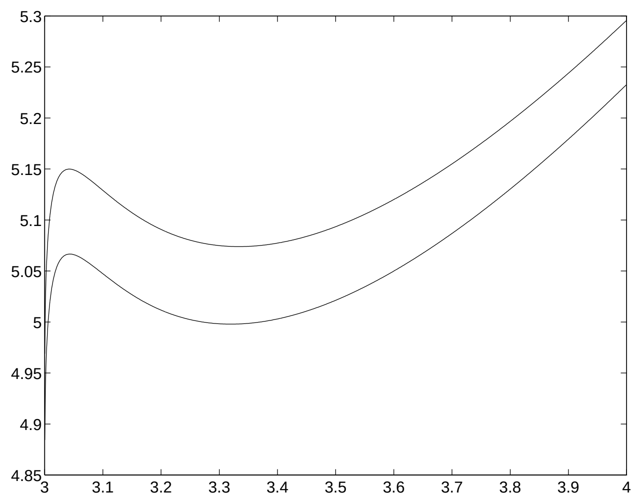

Figure 3: The curves f_2^3 and f_2^4 for $\sigma_1 = 9$; $\sigma_2 = 8.7$; $\sigma_3 = 8.65$; $\sigma_4 = 0.5$.

always has a finite number of different positive solutions and that this number is bounded by $2^N - 1$. In particular we analyzed the set of state parameters for which the (ARE) have a unique solution. Fixing all other system parameters, we saw that this set is either a half line or the union of a half line and an open (bounded) interval. We showed how this set can be determined from the analysis of two scalar functions. It turned out that for all stable systems there will always be a unique solution to the (ARE) equations. In this respect it is interesting to recall from the two-player case (see Engwerda (1999-b)) that whenever the system is not stable, there always exist combinations of the remaining system parameters such that the (ARE) have more than one positive solutions.
On the other hand we showed that there is always a threshold such that if the state feedback parameter exceeds this threshold (assuming all other system parameters again fixed), the number of positive solutions will not increase. In general this number of positive solutions is $2^N - 1$.
In between these two limiting cases, the number of solutions gradually increases from 1 to this maximum number if the system feedback parameter grows. However, this increase is (in general) not monotonically. So, roughly spoken, the conclusion is that the larger the instability of the system is, the more positive solutions the (ARE) equations will have.
The above outcomes raise a couple of new questions. Two of them are, first, whether aggregate efficiency can be used as an additional constraint to determine a unique equilibrium amongst all solutions of the (ARE). We showed in an example that this is not the case. Second, whether the set of parameters for which there will be a unique equilibrium will always increase if the number of players in the game increases. In general the answer to this second question is negative too. Only in case some parameter condition is satisfied, which can be interpreted as that the new players really have both an interest

and can influence the game, the assertion holds (in the end).
One of the main remaining topics is of course how things generalize for the multivariable case. We hope that the obtained results may be helpfull in analyzing this problem.

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