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**Abstract.** This paper deals with a specific aspect of the problem of coalition formation in a situation described by a TU-game. First, we define a very simple normal form game which models the process of coalition formation. To define the payoff functions of the players we use an allocation rule for TU-games. The main objective of this paper is ascertain what conditions of the allocation rule lead to the grand coalition being a strong equilibrium of the normal form game, when the original TU-game is convex.

**Key words:** Coalition formation, normal form game, convex TV-game, strong equilibrium

## 1. Introduction

In recent years, the interest of game theorists in the problem of coalition formation has increased considerably. The usual approach to such a problem is to consider a cooperative situation modeled by a characteristic function and to define, from it, a non-cooperative game which describes the process of coalition formation in the initial cooperative situation.

When defining such a non-cooperative negotiation game, two points of view can be adopted. In the first one, an allocation rule is supposed to be given (for instance, because it has been agreed by the players). This rule is then used to compute the payoffs in the negotiation game, whereas the strategies of the players are claims for coalitions. The first contributions to this

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approach were proposals of allocation rules for different types of cooperation patterns. Aumann and Drèze (1974), Owen (1977) and Myerson (1977) are some of such proposals. Non-cooperative models to analyze the strategic formation of a cooperation structure when an allocation rule is given are described in Hart and Kurz (1983), Aumann and Myerson (1988) and Dutta et al. (1995).

In the second view, the coalition structure as well as the distribution of payoffs are claimed by the players, in the sense that their strategies in the negotiation game are claims for a coalition and for a payoff. This point of view was adopted, for instance, in Chatterjee et al. (1993) and Okada (1996). Sometimes this approach has been used to implement certain cooperative solutions for TU-games, as in Selten (1981), Binmore (1985), Gul (1989), Borm and Tijs (1992), Perry and Reny (1994) and Hart and Mas-Colell (1996).

In this paper we take the first point of view and analyze a specific aspect of the extremely complex problem of coalition formation. The classical approach to TU-games aims to define allocation rules to predict or prescribe how rational players will distribute the gains they obtain from cooperation, assuming in general that the grand coalition will be formed. One interesting issue is to find conditions under which this assumption is really appropriate. It seems reasonable that, in a cooperative situation that can be described by a convex TU-game, the grand coalition emerges from the coalition formation process. To investigate to what extent this is true, we propose a simple and natural non-cooperative model of coalition formation. To do so, we use an allocation rule to define the payoff functions of the players. The main aim of this paper is to investigate under what conditions on the allocation rule the grand coalition will be a stable agreement for the players (i.e. a strong equilibrium).

Some of the results of this paper are similar to those proved in Dutta et al. (1995). However, the approach of the latter paper is different. There, it is investigated under what conditions on an allocation rule for graph-restricted cooperative games (see Myerson, 1977), the complete graph cooperation structure will be formed in a superadditive TU-game. Here, we study what properties an allocation rule for TU-games should satisfy in order to be able to assume that the grand coalition will be formed in a convex TU-game.

The organization of the paper is as follows. In Section 2 we set up the notation for cooperative games and state some preliminary results. In Section 3 we describe the non-cooperative model of coalition formation that we use. Finally, in Section 4 we state and prove the main results of this work.

## 2. Allocation rules for TU-games

A cooperative game with transferable utility (TU-game) is a pair $(N, v)$, where $N$ is a finite set of natural numbers (denoting the players) and $v$ is the characteristic function, defined from $2^N = \{S|S \subset N\}$ to $\mathbb{R}$ and satisfying $v(\emptyset) = 0$. We often identify a TU-game with its characteristic function $v$. We use $G^N$ to denote the set of TU-games with set of players $N$ and $G$ to denote the set of all TU-games with any finite set of players. If $v \in G^N$ and $S \subset N$, we write $v_S$ for the restriction of $v$ to $2^S$. Obviously, $v_S \in G^S$.

Now let us recall some basic definitions which are important in this paper.

**Definition 1.** *Let $v \in G^N$ be a TU-game.*

- *We say that $v$ is convex if $v(S \cup i) - v(S) \geq v(T \cup i) - v(T)$ for all $T$ and $S$ with $T \subset S \subset N \backslash i$ and for all $i \in N$.*
- *We say that $v$ is strictly convex if $v(S \cup i) - v(S) > v(T \cup i) - v(T)$ for all $T$ and $S$ with $T \neq S$ and $T \subset S \subset N \backslash i$ and for all $i \in N$.*
- *We say that $j \in N$ is a dummy for $v$ if $v(S \cup j) = v(S) + v(j)$ for all $S \subset N \backslash j$.*

An allocation rule $\phi$ is a map which assigns to every $(N, v) \in G$ an element of $\mathbb{R}^N$. For all $(N, v) \in G$ and all $i \in N$, we use $\phi_i(v)$ to denote the allocation for $i$ proposed by $\phi$ in $(N, v)$, and $\phi(v)$ to denote the vector $(\phi_i(v))_{i \in N}$. A specially important allocation rule is the Shapley value (Shapley, 1953). For any $(N, v) \in G$, $\Phi(v)$ denotes its Shapley value.

Next we introduce some properties for an allocation rule $\phi$.

1. Efficiency (EFF). We say that $\phi$ is efficient if, for all $(N, v) \in G$, $\sum_{i \in N} \phi_i(v) = v(N)$.
2. Weak Monotonicity (WMON). We say that $\phi$ satisfies weak monotonicity if, for all $(N, v)$, $(N, w) \in G$ such that

$$v(S \cup i) - v(S) \geq w(S \cup i) - w(S)$$

for all $S \subset N$ and all $i \in N$, then $\phi(v) \geq \phi(w)$.

This property means that, if $(N, v)$ and $(N, w)$ belong to $G$ and the marginal contributions of all players are greater or equal in $v$ that in $w$, then $\phi(v) \geq \phi(w)$. Note that it is weaker than the strong monotonicity property (see Young, 1985).

3. Dummy Out (DUMOUT). We say that $\phi$ satisfies the dummy out property if, for all $(N, v) \in G$ and all $D \subset N$ such that $D$ is a set of dummies for $v$, $\phi_i(v) = \phi_i(v_{N \backslash D})$ for all $i \in N \backslash D$.

This property (see Tijs and Otten, 1993) means that, if the dummies of a game abandon it, the other players are not affected.

These three properties together are not too strong. For instance, it is easy to verify that the Shapley value and the weighted Shapley values (see Kalai and Samet, 1987) satisfy all of them. The Shapley semivalues (see Dubey et al., 1981) also satisfy WMON and DUMOUT.

## 3. A non-cooperative model of coalition formation

A normal form game is given by $\Gamma = (N, (X_i, \pi_i)_{i \in N})$ where, for all $i \in N$ (the set of players), $X_i$ is the set of strategies of player $i$ and $\pi_i$ is player $i$'s payoff function. Each $\pi_i$ is a map from $X = \prod_{i \in N} X_i$ to $\mathbb{R}$, assigning to every strategy profile $(x_i)_{i \in N} \in X$ a real number $\pi_i(x)$ which represents the payoff to player $i$ if $x$ is played. For any $S \subset N$, we use $X_S$ to denote the set $\prod_{i \in S} X_i$ and $x_S$ to denote the restriction of an $x \in X$ to $X_S$.

In this paper the following equilibrium concept will be used.

**Definition 2.** *We say that $x \in X$ is a strong equilibrium of $\Gamma$ if there do not exist a non-empty $S \subset N$ and $y_S \in X_S$, such that $\pi_j(x_{N \backslash S}, y_S) \geq \pi_j(x)$ for all $j \in S$ and $\pi_k(x_{N \backslash S}, y_S) > \pi_k(x)$ for some $k \in S$.*

This concept is considered, for instance, in Borm and Tijs (1992). An alternative notion of strong equilibrium is provided in Aumann (1959). In Aumann's paper a strategy profile is said to be a strong equilibrium if it is impossible for all agents in some $S$ to strictly increase their utilities by changing their strategies. Note that Aumann's strongness concept is weaker than ours. Hence, Theorem 1 below would still be true with Aumann's definition. However, it is easy to see that Theorems 2 and 3 below would not be true with Aumann's definition.

Let us introduce now our model of coalition formation. Take $v \in G^N$ and an allocation rule $\phi$ agreed by all players. Consider the claim game in normal form $\Gamma(v, \phi) = (N, (X_i, \pi_i)_{i \in N})$ where, for all $i \in N$,

- $X_i = \{x_i \in 2^N | i \in x_i\}$
- For any $x \in X$, $\pi_i(x) = \phi_i(v_{x_i})$ if $x_j = x_i$ for all $j \in x_i$, and $\pi_i(x) = v(i)$ in any other case.

This claim game corresponds to the following situation. Initially there is a TU-game $v$ and an agreement among players on what allocation rule should be used. Then, each player proposes the coalition to which he would like to belong. These proposals are made simultaneously and independently by all players. A coalition is eventually formed if all its members have proposed that such a coalition is formed. Player $i$ not belonging to any of such coalitions actually formed belongs to coalition $\{i\}$. Finally, for any coalition $S$ resulting from this process, $v(S)$ is allocated among its members according to $\phi$.

Note that, in view of the process described above, any combination of strategies $x$ produces a coalition structure $P^x$ (a partition of $N$) defined by the following equivalence relation:

$$i, j \in N, i \sim j :\Leftrightarrow i = j \text{ or } [j \in x_i \text{ and } x_i = x_k \text{ for all } k \in x_i].$$

In the next section we focus our attention on the problem of ascertaining under what conditions on $\phi$, the grand coalition appears as the coalition structure associated to a strong equilibrium of $\Gamma(v, \phi)$ for a convex game $v$. We use $S(v, \phi)$ to denote the set of strong equilibria of $\Gamma(v, \phi)$ and $P(v, \phi)$ to denote the set:

$$\{P \text{ partition of } N | \exists x \in S(v, \phi) \text{ such that } P^x = P\}.$$

## 4. Main results

In this section we state and prove the main results of this paper.

**Lemma 1.** *Let $v \in G^N$ be a convex game and take an allocation rule $\phi$ satisfying WMON and DUMOUT. Then, for every non-empty $S \subset N$, $\phi_i(v) \geq \phi_i(v_s)$ for all $i \in S$.*

*Proof:* Define the games $v^S$, $\hat{v}^S \in G^N$ by

- $v^S(T) = v(S \cap T)\ \forall T \subset N$
- $\hat{v}^S(T) = v^S(T) + \sum_{i \in T \setminus S} v(i)\ \forall T \subset N.$

Now, since $v$ is convex,

$$v(T \cup i) - v(T) \geq v^S(T \cup i) - v^S(T) = \hat{v}^S(T \cup i) - \hat{v}^S(T) \tag{1}$$

for all $i \in S$ and all $T \subset N$. Moreover, for all $i \in N \backslash S$ and all $T \subset N \backslash i$, in view of the convexity of $v$,

$$v(T \cup i) - v(T) \geq v(i) \tag{2}$$

and therefore, since $v(i) = \hat{v}^S(T \cup i) - \hat{v}^S(T)$, we conclude from (1) and (2) that

$$v(T \cup i) - v(T) \geq \hat{v}^S(T \cup i) - \hat{v}^S(T)$$

for all $i \in N$ and all $T \subset N$. Then, since $\phi$ satisfies WMON, $\phi(v) \geq \phi(\hat{v}^S)$. Note that, for all $j \in N \backslash S$, $j$ is a dummy in $\hat{v}^S$ and therefore, taking into account that $\phi$ satisfies DUMOUT, $\phi_i(\hat{v}^S) = \phi_i(\hat{v}^S_S)$ for all $i \in S$. Now, since $\hat{v}^S_S = v_S$, we conclude that $\phi_i(v) \geq \phi_i(v_S)$ for all $i \in S$. □

This Lemma 1 is essential to prove Theorems 1 and 2 below. It shows that, if a cooperative situation can be described by a convex TU-game and an allocation rule satisfying WMON and DUMOUT is going to be used, then no player would benefit if some other players left the game.

**Theorem 1.** *Let $v \in G^N$ be a convex game and take an allocation rule $\phi$ satisfying WMON and DUMOUT. Then $\{N\} \in P(v, \phi)$.*

*Proof:* If $\{N\} \notin P(v, \phi)$, then there must exist a non-empty $S \subset N$ such that $\phi_i(v_S) \geq \phi_i(v)$ for all $i \in S$ and $\phi_j(v_S) > \phi_j(v)$ for some $j \in S$. This contradicts Lemma 1. □

From Theorem 1 we know that, in our model of coalition formation, if $v$ is convex and $\phi$ satisfies WMON and DUMOUT, the grand coalition can be supported by a strong equilibrium. In Theorems 2 and 3 below, we consider the unicity: will the grand coalition be the only coalition structure supported by a strong equilibrium? From Theorem 2 we know that the answer is, in general, no. However, in any other coalition structure supported by a strong equilibrium, players obtain exactly the same payoff as in the grand coalition. Theorem 3 shows that, if we impose some extra conditions, the answer to the question is yes.

**Theorem 2.** *Let $v \in G^N$ be a convex game and take an allocation rule $\phi$ satisfying WMON and DUMOUT. Under these conditions, if $x \in S(v, \phi)$, then $\pi_i(x) = \phi_i(v)$ for all $i \in N$.*

*Proof:* In view of Theorem 1 we know that, in these conditions, $\{N\} \in P(v, \phi)$. Take $x \in S(v, \phi)$ and consider the partition associated to it: $P^x = \{P^x_1, \dots, P^x_r\}$. Clearly, $\pi_i(x) = \phi_i(v_{P^x_k})$ for $i \in P^x_k$. From Lemma 1 we know that $\phi_i(v) \geq \phi_i(v_{P^x_k})$ for all $i \in P^x_k$, and therefore $\phi_i(v) \geq \pi_i(x)$ for all $i \in N$. Hence,

$\pi_j(x) \neq \phi_j(v)$ for some $j \in N$ implies that $\pi_j(x) < \phi_j(v)$ and $\pi_i(x) \leq \phi_i(v)$ for all $i \in N \setminus \{j\}$. In such a case, $x$ would not belong to $S(v, \phi)$. □

**Theorem 3.** *Let $v \in G^N$ be a strictly convex game and take an allocation rule $\phi$ satisfying WMON, DUMOUT and EFF. Then, $P(v, \phi) = \{\{N\}\}$.*

*Proof:* From Theorem 1 we know that $\{N\} \in P(v, \phi)$. Suppose that there exists a strong equilibrium $x$ with $P^x = \{P_1^x, \dots, P_k^x\}$ $(k \geq 2)$. Since $v$ is strictly convex and $\phi$ satisfies EFF,

$$\sum_{i=1}^{n} \phi_i(v) = v(N) > \sum_{j=1}^{k} v(P_j^x) = \sum_{j=1}^{k} \sum_{i \in P_j^x} \pi_i(x) = \sum_{i=1}^{n} \pi_i(x).$$

This contradicts Theorem 2. □

It is easy to find examples which show that Theorems 1 and 2 are not true when the WMON or the DUMOUT assumptions for $\phi$ are dropped. It is also easy to see that Theorem 3 is not true if we do not assume that the allocation rule is efficient. Finally, the following example shows that Theorem 1 is false if we drop the assumption of convexity for $v$.

*Example:* We take the 3-person game $v$ given by $v(i) = 0$ for all $i \in N = \{1, 2, 3\}$, $v(1, 2) = 10$, $v(1, 3) = 1$, $v(2, 3) = 1$, $v(N) = 10$. It is easy to see that $v$ is a superadditive and balanced game ($(5, 5, 0)$ belongs to its core). However, it is not convex. Recall that the Shapley value satisfies WMON and DUMOUT. However, $\{N\} \notin P(v, \Phi)$ because $\Phi(v) = (29/6, 29/6, 2/6)$ and $(\Phi(v_{\{1,2\}}), \Phi(v_{\{3\}})) = (5, 5, 0)$. This shows that Theorem 1 is not true if we drop the assumption of convexity for $v$. It even fails to be true for superadditive and balanced games.

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