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WELL-POSEDNESS OF SOME EVOLUTION PROBLEMS IN THE THEORY OF AUTOMATIC FEED-BACK CONTROL FOR SYSTEMS WITH DISTRIBUTED PARAMETERS*

AART VAN HARTEN† AND HANS SCHUMACHER‡

Abstract. The subject of this paper is the evolution, in time, of systems of diffusion type controlled by an automatic feed-back mechanism using a finite number of observators and control-inputs. For the initial-boundary value problem for a functional partial differential equation of parabolic type, which the state variable has to satisfy, the following types of questions are considered: existence, uniqueness, regularity and continuous dependence on data of a solution. Answers to these questions are given in function spaces of Hölder and Sobolev type.

1. Introduction.

1.1. Purpose of this paper. In this paper we shall consider a system for which, in the uncontrolled situation, the behavior is described by an evolution equation of diffusion type ((1.1.1) with $\Pi \equiv 0$) together with initial and boundary conditions (1.1.2–3). In the controlled situation an automatic feed-back control mechanism is applied to this system. The effect of this control is accounted for by the term Πu in (1.1.1). Π is called the feed-back control operator.

$$\frac{\partial u}{\partial t} = (L+\Pi)u + f, \qquad \text{FPDE}, \tag{1.1.1}$$

$$Bu = \phi, \qquad \text{BC}, \tag{1.1.2}$$

$$u(\cdot, 0) = \psi, \qquad \text{IC}. \tag{1.1.3}$$

In the uncontrolled case a rather well-developed theory for existence, uniqueness, regularity and continuous dependence on (f, ϕ, ψ) of the state u is available.

In Ladyzenskaja, Solonnikov, Uraltseva (1967) such a theory is given using function spaces of Hölder type. We shall refer to this type of theory as "C-theory".

A theory using function spaces of Sobolev type can be found in Lions, Magenes (1972). To the latter type of theory we shall refer as "H-theory".

In the controlled case, the structure of the operator Π will be such that (1.1.1) is no longer a PDE but a FPDE, where "F" abbreviates "functional".

The purpose of this paper is to give a "C-theory" as well as a "H-theory" for the existence, uniqueness, regularity and continuous dependence on (f, ϕ, ψ) of a solution of (1.1.1-2-3) in the controlled case.

As for the structure of the operator Π we shall take an approach, in which, right from the beginning, we shall deal with control in feed-back form using only a finite number of observators and control inputs. This in contrast with the work of Lions (1971), where much more of an open loop approach for the control term is taken.

Now we shall first demonstrate, with an example, what type of automatic feed-back control mechanisms we have in mind in § 1.2. In § 1.3 we shall explain, in full generality, the setting in which we shall consider the problem (1.1.1-2-3). In § 1.4 we shall discuss the strategy to attack (1.1.1-2-3). Next we shall look at the concept of compatibility of the data in § 2. In § 3 we shall deal with a theorem that appears to be very useful for the derivation of our main results. Then these main results will be discussed in §§ 4 and 5. In

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† Mathematical Institute, University of Utrecht, Budapestlaan 6, Utrecht, The Netherlands.

‡ Dept. of Mathematics, Free University of Amsterdam, De Boelelaan 1081, Amsterdam, The Netherlands.

§ 6 the developed theory is applied to some examples. There we shall also come back to the example of § 1.2 for explicit calculation of some solutions.

1.2. An example taken from physics. Consider the distribution of temperature u in a plate D with an isolating boundary ∂D. Let g be the autonomous production/absorption of heat and let ψ be the initial distribution of temperature. Suppose, that in order to control the system, temperature is permanently observed in the points: $y_1 \in \bar{D}, \cdots, y_p \in \bar{D}$, $\bar{D} = D \cup \partial D$. This information is fed-back to a heating/cooling apparatus characterised by control input functions on $D: c_1, \cdots, c_q$ (see Fig. 1).

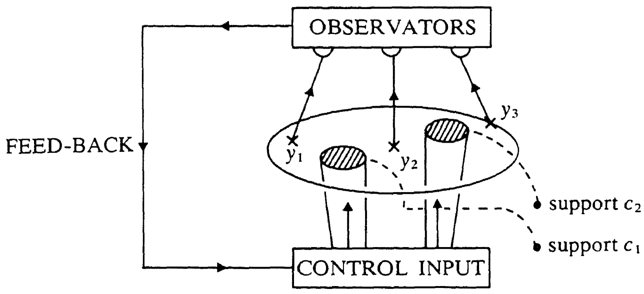

FIG. 1

Suppose that ideal values for the temperature in $y_1, \cdots, y_p$ at time $t \geqq 0$ are known to be $I_1(t), \cdots, I_p(t)$.

As for the feed-back we consider the following examples.

(i) Instantaneous feed-back: the control action producing/absorbing heat in a volume element dx at $x \in D$ during a time interval $(t, t+dt)$ is given by

$$\left[\sum_{i=1}^{q} \sum_{j=1}^{p} c_i(x, t) h_{ij}(t)(u(y_j, t) - I_j(t))\right] dx\, dt.$$

(ii) Feed-back with (simple) memory: the control action at x in dx during $(t, t+dt)$ is now given by

$$\left[\sum_{i=1}^{q} \sum_{j=1}^{p} c_i(x, t) \bar{h}_{ij}(t) \int_0^t \mu\, e^{-\mu(t-\tau)}(u(y_j, \tau) - I_j(\tau))\, d\tau\right] dx\, dt.$$

Here $\mu^{-1} > 0$ is the characteristic time for loss of memory.

A model for this controlled system is given by

$$\frac{\partial u}{\partial t} = (\Delta + \Pi) u + f, \qquad \text{FPDE,} \tag{1.2.1}$$

$$\frac{\partial u}{\partial \vec{n}} = 0, \qquad \text{BC,} \tag{1.2.2}$$

$$u(\cdot, 0) = \psi, \qquad \text{IC} \tag{1.2.3}$$

with Δ the Laplacian, $\partial/\partial \vec{n}$ the normal derivative.

For the feed-back control operator we get

$$\Pi = \sum_{i=1}^{q} \sum_{j=1}^{p} c_i F_{ij} \delta_{y_j} \tag{1.2.4}$$

with observators δ_{y_j} defined by

$$(\delta_{y_j}u)(t) = u(y_j, t), \tag{1.2.5}$$

and with feed-back coupling operators F_{ij} defined by

$$(F_{ij}\chi)(t) = h_{ij}(t)\chi(t) \qquad \text{in case (i)}$$
$$= \bar{h}_{ij}(t)\int_0^t \mu\, e^{-\mu(t-\tau)}\chi(\tau)\, d\tau \quad \text{in case (ii)}. \tag{1.2.6}$$

The inhomogeneous term f in (1.2.1) is given by

$$f = g - \sum_{i=1}^{q}\sum_{j=1}^{p} c_i F_{ij} I_j. \tag{1.2.7}$$

Note, that the operator Π of (1.2.4) has a nonlocal character in space. In case (ii), Π also has a delayed character in time. These facts cause (1.2.1) to be an equation of more complex type than a PDE.

1.3. Description of the general problem. Let us now discuss the general setting in which we consider the problem (1.1.1-2-3).

$u(x, t)$ will be the state of the system; x, the vector of space variables; $x \in \bar{D}$ and t, the time variable, $t \in [0, T]$. D will be a bounded, open domain $\subset \mathbb{R}^n$; $\bar{D}$ denotes the closure of D, $\partial D = \bar{D}\backslash D$. For simplicity we suppose, that ∂D is smooth. Let us introduce the notation: $Q = D \times (0, T)$, $\Gamma = \partial D \times (0, T)$. L will be a linear, second order, uniformly elliptic PDO:

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}D_{ij} + \sum_{i=1}^{n} a_i D_i + a_0 \tag{1.3.1}$$

with $D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$ and $\exists E > 0, \forall (x, t) \in \bar{Q}, \forall \xi \in \mathbb{R}^n: \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x, t)\xi_i\xi_j \geqq E \sum_{i=1}^n \xi_i^2$. All coefficients of L are allowed to depend on x and t. For simplicity we suppose: $a_{ij}, a_i, a_0 \in C^\infty(\bar{Q})$, $1 \leqq i, j \leqq n$.

The operator B will be linear and of order ν with $\nu = 0$ or $\nu = 1$.

$$B = 1, \qquad \text{Dirichlet BC}, \quad \nu = 0,$$
$$B = \sum_{i=1}^{n} b_i D_i + b_0, \quad \text{Neumann BC}, \quad \nu = 1. \tag{1.3.2}$$

The coefficients b_i, b_0, $1 \leqq i \leqq n$ are allowed to depend on x and t. For simplicity we suppose b_i, $b_0 \in C^\infty(\bar{\Gamma})$, $1 \leqq i \leqq n$. Furthermore we assume, that $\forall (x, t) \in \bar{\Gamma}$, $\sum_{i=1}^n b_i(x, t)\vec{n}_i(x) > 0$ with $\vec{n}(x)$ the outward directed normal on ∂D at x.

As for the regularity of the data f, ϕ, ψ we suppose

$$\begin{array}{l|l} f \in C^{\beta,\beta/2}(\bar{Q}), & f \in H^{\beta,\beta/2}(Q), \\ \phi \in C^{\hat{\beta},\hat{\beta}/2}(\bar{\Gamma}), & \phi \in H^{\hat{\beta},\hat{\beta}/2}(\Gamma), \\ \psi \in C^{\beta_0}(\bar{D}), & \psi \in H^{\beta_0}(D), \\ \beta, \hat{\beta}, \beta_0 \geqq 0 & \beta, \hat{\beta}, \beta_0 \geqq 0. \end{array} \tag{1.3.3}$$

Here we used for the first time a form of specification which will often be used afterwards in the same way: at the left of the vertical bar the specification refers to

"C-theory", at the right of the vertical bar the specification refers to "H-theory", i.e.,

"C-theory" | "H-theory".

$C^{\beta,\beta/2}(\bar{Q})$ denotes the Hölder space of order β in the space directions and of order $\beta/2$ in the time direction. For the definition of $C^{\beta,\beta/2}(\bar{Q})$, $C^{\hat{\beta},\hat{\beta}/2}(\bar{\Gamma})$, $C^{\beta_0}(\bar{D})$ and their usual norms $|\cdot|_{\beta,\beta/2}$, $|\cdot|_{\hat{\beta},\beta/2}$, $|\cdot|_{\beta_0}$ we refer to Ladyzenskaja, Solonnikov, Uraltseva (1967) with the important remark that these spaces are indicated there with C replaced by H.

For the definition of the Sobolev spaces $H^{\beta,\beta/2}(Q)$, $H^{\hat{\beta},\hat{\beta}/2}(\Gamma)$ and $H^{\beta_0}(D)$, and their usual norms $\|\cdot\|_{\beta,\beta/2}$, $\|\cdot\|_{\hat{\beta},\beta/2}$, $\|\cdot\|_{\beta_0}$, we refer to Lions, Magenes (1972, vol. II, p. 6) and Adams (1975, p. 208).

From now on the notation β, $\hat{\beta}$, β_0 will be reserved for and will always refer to the indices indicating the regularity of the data as introduced in (1.3.3).

In addition to the regularity of the data it is important to consider the compatibility of f, ϕ, ψ. This is done in § 2.

Let us now describe the general form of the feed-back control operator Π. We shall take

$$\Pi = \sum_{i=1}^{q} \sum_{j=1}^{p} c_i F_{ij} P_j \tag{1.3.4}$$

with

control input functions c_i, $1 \leqq i \leqq q$;

observators P_j, $1 \leqq j \leqq p$;

feed-back coupling operators F_{ij}, $1 \leqq i \leqq q$, $1 \leqq j \leqq p$.

Further we suppose

(1.3.5)

C-theory	H-theory
$\Pi \in L(C^{\check{\gamma},\check{\gamma}/2}(\bar{Q}) \to C^{\gamma,\gamma/2}(\bar{Q}))$, $\check{\gamma} \geqq \gamma \geqq 0$	$\Pi \in L(H^{\check{\gamma},\check{\gamma}/2}(Q) \to H^{\gamma,\gamma/2}(Q))$, $\check{\gamma} \geqq \gamma \geqq 0$.

For Banach spaces X, Y we denote by $L(X \to Y)$ the space of bounded, linear operators from X into Y.

An important role will be played by

$$\alpha = \check{\gamma} - \gamma. \tag{1.3.6}$$

α is called the order of the feed-back control operator.

It will not be surprising that for our theory of well-posedness of the problem (1.1.1-2-3), we have to distinguish between the cases where the order of the feed-back control operator Π is smaller than, or larger than/equal to, the order of the diffusion operator L. In § 4 we shall deal with the case $0 \leqq \alpha < 2$ and in § 5 we consider the case $\alpha \geqq 2$.

Our requirements for P_j, F_{ij}, c_i, $1 \leqq i \leqq q$, $1 \leqq j \leqq p$ are specified in (1.3.7-8-9).

(1.3.7)

C-theory	H-theory
$P_j \in L(C^{\check{\gamma},\check{\gamma}/2}(\bar{Q}) \to C^{\gamma/2}[0, T])$	$P_j \in L(H^{\check{\gamma},\check{\gamma}/2}(Q) \to H^{\gamma/2}(0, T))$
and:	and:
$\exists \tilde{P}_j \in C^{\gamma/2}([0, T] \to C^{\alpha}(\bar{D})')$ such that	$\exists \tilde{P}_j \in H^{\gamma/2}((0, T) \to H^{\alpha}(D)')$ such that
$\forall u \in C^{\check{\gamma},\check{\gamma}/2}(\bar{Q})$, $\forall t \in [0, T]$,	$\forall u \in H^{\check{\gamma},\check{\gamma}/2}(Q)$ a.e. in $t \in (0, T)$,
$(P_j u)(t) = \tilde{P}_j(t) u(\cdot, t)$.	$(P_j u)(t) = \tilde{P}_j(t) u(\cdot, t)$.

By $C^\alpha(\bar{D})'$, $H^\alpha(D)'$ we denote the topological dual spaces of $C^\alpha(\bar{D})$, $H^\alpha(D)$ equipped with their usual norms.

Note that the contents of (1.3.7) are that P_j acts instantaneously with its value on u at time t given by $\tilde{P}_j(t)u(\cdot, t)$, where $\tilde{P}_j(t)$ possesses a prescribed regularity in t.

If $P_j : C^\infty(\bar{Q}) \to C[0, T]$ (this is not necessarily true in the case of H-theory if $0 \leqq \gamma \leqq 1$, but is implied by (1.3.7) in the other cases) then a consequence of the fact that P_j acts instantaneously as defined in (1.3.7) is: $\forall u, v \in C^\infty(\bar{Q})$, $\forall t \in [0, T]$, $u(\cdot, t) = v(\cdot, t) \Rightarrow (P_j u)(t) = (P_j v)(t)$.

(1.3.8)
$$F_{ij} \in L(C^{\gamma/2}[0, T]),\ F_{ij} \text{ nonanticipative} \quad \Big| \quad F_{ij} \in L(H^{\gamma/2}(0, T)),\ F_{ij} \text{ nonanticipative.}$$

For a Banach space X we abbreviate $L(X \to X)$ to $L(X)$.

Nonanticipativity of F_{ij} means, that for all elements ξ, η of $C^{\gamma/2}[0, T]$, $H^{\gamma/2}(0, T)$ and $\forall t \in (0, T)$:

$$\xi|_{(0,t)} = \eta|_{(0,t)} \Rightarrow F_{ij}\xi|_{(0,t)} = F_{ij}\eta|_{(0,t)}.$$

(1.3.9)
$$c_i \in C^{\gamma,\gamma/2}(\bar{Q}) \quad \Big| \quad c_i \in H^{\gamma,\gamma/2}(Q).$$

The operator Π will be called of type (γ, α), if (1.3.5-6-7-8-9) are satisfied.

From now on the notation γ, α will be reserved for, and will always refer to, the type of the operator Π as introduced above. A natural question is of course whether the requirements (1.3.6-7-8-9) already imply (1.3.5).

In the case of "C-theory" this is indeed true. In the case of "H-theory" it is true if $\gamma > 1$ but not necessarily if $0 \leqq \gamma \leqq 1$! One of the reasons is, that in general, for $c_i \in H^{\gamma,\gamma/2}(Q)$ and $\xi \in H^{\gamma/2}(0, T)$ with $0 \leqq \gamma \leqq 1$, the function defined by $c_i(x, t)\xi(t)$ is not an element of $H^{\gamma,\gamma/2}(Q)$.

A sufficient supplement to (1.3.6-7-8-9) in the case of "H-theory" with $0 \leqq \gamma \leqq 1$, in order to imply (1.3.5), is

(1.3.10)
$$\operatorname*{ess\,sup}_{0<t<T} \{\|c_i(\cdot, t)\|_\gamma + \|\tilde{P}(t)\|_{H^\alpha(D)'}\} < \infty \quad \text{if } \gamma \in [0, 1],$$
and furthermore, if $\gamma > 0$, $\exists \bar{\delta} > \gamma/2$ such that
$$\operatorname*{ess\,sup}_{0<t<\tau<T} |t-\tau|^{-\bar{\delta}} \{\|c(\cdot, t) - c(\cdot, \tau)\|_0 + \|\tilde{P}(t) - \tilde{P}(\tau)\|_{H^\alpha(D)'}\} < \infty.$$

Note, that if in the example of § 1.2 we take $c_i \in C^\infty(\bar{Q})$, h_{ij}, $\bar{h}_{ij} \in C^\infty[0, T]$, then (1.3.5-6-7-8-9) are satisfied for any choice of $\gamma \in [0, \infty)$ with

(1.3.11)
$$\alpha = 0 \quad \Big| \quad \alpha = \frac{n}{2} + \varepsilon, \quad \varepsilon > 0 \text{ arbitrarily small}$$

The choice of α in the case of "H-theory" is a consequence of the fact that for $s > n/2$, $H^s(D)$ is continuously imbedded in $C(\bar{D})$, see Adams (1975, p. 97).

It will be clear that the setting (1.3.5-6-7-8-9) for the control term is rather rich: it admits many examples quite different from the ones of § 1.2. The following additional examples may serve to demonstrate this.

Firstly it is allowed to take observators which use derivatives in space-directions and/or integration, averaging in space.

In the model of § 1.2 one can, for example, think of an observator $\tilde{P}$ that observes the flow of heat through the smooth boundary S of a sub-domain Ω of D. This leads to

$(\tilde{P}u)(t) = \int_S (\partial u/\partial \vec{n})(x, t)\, dx$, where $\partial/\partial \vec{n}$ denotes the normal derivative in outward direction on S. Such an observator would fit in (1.3.7) with

$$\gamma \geqq 0, \quad \alpha = 1 \;\Big|\; \gamma > 0, \quad \alpha = \tfrac{3}{2}. \tag{1.3.12}$$

In the case of "H-theory" the choice of α is here a consequence of the theorem on traces on the boundary, see Lions, Magenes (1972, vol. II, p. 9).

Secondly it is allowed that the feed-back coupling uses time-derivatives in an essential way.

In the following example the feed-back coupling operator F maps the function on which it operates to a smoothed piecewise linear approximation: $(F\xi)(t) = \sum_{0 \leqq n < t} [\xi(n) + \dot{\xi}(n)(t-n)]H(t-n)$.

Here $\dot{\xi}$ denotes the time derivative of ξ. The function H is supposed to be $\in C^\infty[0, \infty)$ and further $H \geqq 0$, $H(0) = 0$, $H(t) = 1$ for $\varepsilon \leqq t \leqq 1$, $H(t) = 0$ for $t \geqq 1+\varepsilon$, ε a positive constant $\ll 1$.

Such a feed-back coupling would fit in (1.3.8) if

$$\gamma \geqq 1 \;\Big|\; \gamma > \tfrac{3}{2}. \tag{1.3.13}$$

The choice of γ in (1.3.13), in the case of "H-theory", is a consequence of the imbedding theorem (see Adams (1975, p. 97)).

Finally it is useful to remark that certain feed-back operators Π which are not a priori in the form specified here above can be rewritten to that form.

For example, $\Pi = cFP$ with $c \in C^\infty(\bar{Q})$, F the identity operator and $(Pu)(t) = \delta_y(u + s(\partial u/\partial t))$. Here s denotes a constant >0 and δ_y is as in (1.2.5).

Note that this P predicts in a simple way the value of $\delta_y u$ at the time $t+s$ from the data at time t.

Certainly P doesn't satisfy the conditions given in (1.3.7), because of the $\delta_y(\partial/\partial t)$ operation. But this operation can be eliminated!

From (1.1.1) we derive

$$\delta_y \frac{\partial u}{\partial t} = \delta_y(Lu + f + c\delta_y u) + s\delta_y c\delta_y \frac{\partial u}{\partial t}.$$

In other words: $\delta_y(\partial u/\partial t) = z\delta_y(Lu + f + c\delta_y u)$ if we suppose $z = (1 - s\delta_y c)^{-1}$ to be nonsingular.

Now we can rewrite $\Pi u = \hat{\Pi} u + \hat{f}$ with $\hat{\Pi} = c_1\delta_y(1 + sL)$.

Here we denote $c_1 = zc$ and $\hat{f} = scz\,\delta_y f$.

Equation (1.1.1) can be rewritten as $\partial u/\partial t = (L + \hat{\Pi})u + g$ with $g = f + \hat{f}$. This equation is of the same type as (1.1.1) with the difference that Π has been replaced by $\hat{\Pi}$.

Now $\hat{\Pi}$ satisfies the conditions in (1.3.5-6-7-8-9) for any $\gamma \in [0, \infty)$ with

$$\alpha = 2 \;\Big|\; \alpha = 2 + \frac{n}{2} + \varepsilon, \quad \varepsilon > 0 \text{ arbitrarily small.} \tag{1.3.14}$$

The verification that the choice for α in the case of "H-theory" suffices is analogous to (1.3.11).

1.4. On the strategy to attack the problem. Using the compatibility of the data the problem (1.1.1-2-3) will first be rewritten as

$$(1.4.1)\qquad \begin{aligned} &\frac{\partial u}{\partial t} = (L+\Pi)w + f_0,\\ &Bw = \phi_0,\\ &w(\cdot, 0) = 0, \end{aligned}$$

where f_0 and ϕ_0 vanish up to a certain order for $t \downarrow 0$.

The advantage of (1.4.1) is that the solution W of the corresponding uncontrolled problem is a "nice" function, where originally this is not necessarily true. Now (1.4.1) is equivalent to

$$(1.4.2)\qquad w = S\Pi w + W$$

with S the solution operator of the uncontrolled problem with homogeneous initial— and boundary conditions.

In the case that the order of Π is less than 2, it is possible to show directly from the compactness and nonanticipativity of $S\Pi$ the invertibility of $I - S\Pi$. This is done by the important lemma given in § 3. The solution of (1.4.2) is given by $(I - S\Pi)^{-1} W$ and can be analyzed.

This method fails, however, if the order of Π is larger than 2, since $S\Pi$ is not compact then. In this case we deduce from (1.4.2) a Volterra integral equation for the functions $P_i w$, $i = 1, \cdots, p$. This is done by operating on (1.4.2) with $\tilde{P}_j(t)$. This Volterra integral equation is then solved by using the lemma of § 3. The properties of w are next found from (1.4.2), since we have $w = S \sum_{i=1}^{q} \sum_{j=1}^{p} c_i F_{ij} P_j w + W$, where the $P_j w$'s are now known.

2. On the compatibility of the data. In order to have a solution of (1.1.1-2-3) which is sufficiently regular at $t = 0$ near the boundary of D, it is necessary that the data f, ϕ, ψ are compatible there in a certain sense. This is well-known in the uncontrolled case. Let us now investigate compatibility in the controlled situation.

In order to do so it is very useful to introduce the following function spaces:

$$(2.1)\qquad \begin{array}{l|l} \underset{0}{C}^{\beta,\beta/2}(\bar{Q}), & \underset{0}{H}^{\beta,\beta/2}(Q), \\[2ex] \underset{0}{C}^{\hat{\beta},\hat{\beta}/2}(\bar{\Gamma}), & \underset{0}{H}^{\hat{\beta},\hat{\beta}/2}(\Gamma), \\[2ex] \underset{0}{C}^{\delta}[0, T] & \underset{0}{H}^{\delta}(0, T). \end{array}$$

They are defined as the closure in the spaces without the subscript $_0$ of the ∞-differentiable functions with supports that have no point in common with $\{t = 0\}$. It can be shown (see Ladyzenskaja, Solonnikov, Uraltseva (1967) and Lions, Magenes (1972)), that usually the function spaces introduced in (2.1) can also be described as the subspaces of the function spaces without the subscript $_0$ consisting of functions for which the time derivatives at $t = 0$ vanish up to some number. This number is given below together with an indication when the characterization is valid.

$$
(2.2)\qquad
\begin{array}{ll|lll}
[\beta/2] & \text{if } \beta \geqq 0, & [\frac{1}{2}(\beta-1)] & \text{if } \beta \geqq 0, & \frac{1}{2}(\beta+1) \notin \mathbb{N}, \\
[\hat\beta/2] & \text{if } \hat\beta \geqq 0, & [\frac{1}{2}(\hat\beta-1)] & \text{if } \hat\beta \geqq 0, & \frac{1}{2}(\hat\beta+1) \notin \mathbb{N}, \\
[\delta] & \text{if } \delta \geqq 0 & [\delta-\frac{1}{2}] & \text{if } \delta \geqq 0, & \delta+\frac{1}{2} \notin \mathbb{N}.
\end{array}
$$

As usual $[\delta]$ denotes the largest integer $\leqq \delta$.

If the number given in (2.2) is less than 0 the function space with subscript $_0$ coincides with the space without subscript $_0$.

Now we use the function spaces introduced in (2.1) to define the concept of compatibility of the data f, ϕ, ψ.

The data will be called compatible of type β_2, β_1, $\hat\beta_1$, if $\beta_2 \geqq \max(2, \gamma)$, $\beta_1 \geqq 0$ and

$$
(2.3)\qquad
\begin{array}{l|l}
\exists v \in C^{\beta_2,\beta_2/2}(\bar Q) \text{ such that} & \exists v \in H^{\beta_2,\beta_2/2}(Q) \text{ such that} \\
v(\cdot, 0) = \psi, & v(\cdot, 0) = \psi, \\
\phi - Bv \in \underset{0}{C}^{\hat\beta_1,\hat\beta_1/2}(\bar\Gamma), & \phi - Bv \in \underset{0}{H}^{\hat\beta_1,\hat\beta_1/2}(\Gamma), \\
f - \dfrac{\partial v}{\partial t} + (L+\Pi)v \in \underset{0}{C}^{\beta_1,\beta_1/2}(\bar Q) & f - \dfrac{\partial v}{\partial t} + (L+\Pi)v \in \underset{0}{H}^{\beta_1,\beta_1/2}(Q).
\end{array}
$$

From now on the notation β_2, β_1, $\hat\beta_1$ will be reserved for, and will always refer to, the indices indicating the type of compatibility as introduced in (2.3). Let us introduce

$$
(2.4)\qquad
\begin{array}{l|l}
|(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1} = & \|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} = \\
\quad \inf\{|v|_{\beta_2,\beta_2/2} \mid v \text{ satisfies (2.3)}\} & \quad \inf\{\|v\|_{\beta_2,\beta_2/2} \mid v \text{ satisfies (2.3)}\}.
\end{array}
$$

Here we use the convention $\inf \varnothing = \infty$. An equivalent formulation of (2.3) is

$$
(2.5)\qquad |(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1} < \infty \;\Big|\; \|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} < \infty.
$$

However, it would be preferable if under certain conditions we would be able to derive a less abstract, better verifiable criterion for the compatibility of the data than (2.3).

In order to do so let us first give some easy consequences of the nonanticipativity of the feed-back coupling.

$$
(2.6)\qquad
\begin{array}{l|l}
F_{ij}\underset{0}{C}^{\gamma/2}[0,T] \subset \underset{0}{C}^{\gamma/2}[0,T] & F_{ij}\underset{0}{H}^{\gamma/2}(0,T) \subset \underset{0}{H}^{\gamma/2}(0,T) \\
\text{for } 1 \leqq i \leqq q, \quad 1 \leqq i \leqq p, & \text{for } 1 \leqq i \leqq q, \quad 1 \leqq j \leqq p, \\
\Pi\, \underset{0}{C}^{\tilde\gamma,\tilde\gamma/2}(\bar Q) \subset \underset{0}{C}^{\gamma,\gamma/2}(\bar Q) & \Pi\, \underset{0}{H}^{\tilde\gamma,\tilde\gamma/2}(Q) \subset \underset{0}{H}^{\gamma,\gamma/2}(Q).
\end{array}
$$

It appears to be possible to calculate the time-derivatives at $t=0$ of $F_{ij}\xi$, in terms of the time-derivatives at $t=0$ of ξ itself, by a simple matrix multiplication. More precisely suppose:

$$
(2.7)\qquad \xi \in C^{\gamma/2}[0,T] \;\Big|\; \xi \in H^{\gamma/2}(0,T), \quad \gamma > 1, \quad \tfrac{1}{2}(\gamma+1) \notin \mathbb{N}.
$$

Define

$$d(\gamma)=[\gamma/2] \quad \Big| \quad d(\gamma)=[\tfrac{1}{2}(\gamma-1)].$$

If we define Der ξ as the vector in $\mathbb{R}^{d(\gamma)+1}$ with

$$(\text{Der } \xi)_l=\frac{1}{l!}\frac{d^l\xi}{dt^l}(0), \qquad 0\leqq l\leqq d(\gamma) \tag{2.8}$$

then we have the relation

$$\text{Der } F_{ij}\xi=A^{ij} \text{ Der } \xi \tag{2.9}$$

with A^{ij} the $(d(\gamma)+1)\times(d(\gamma)+1)$ matrix with the matrix elements

$$A^{ij}_{kl}=\left\{\frac{1}{k!}\left(\frac{d}{dt}\right)^k F_{ij}(t^l)\right\}\Big|_{t=0}, \qquad 0\leqq k,\, l\leqq d(\gamma) \tag{2.10}$$

(where with (t^l) we mean the function $t\to t^l$).

In order to prove (2.10) we observe that $\xi(t)-\sum_{l=0}^{d(\gamma)}(\text{Der } \xi)_l t^l$ defines an element of $\underset{0}{C}^{\gamma/2}[0, T]$, $\underset{0}{H}^{\gamma/2}(0, T)$. Using (2.6) we find that $F_{ij}\xi-\sum_{l=0}^{d(\gamma)}(\text{Der } \xi)_l F_{ij}(t^l)$ is in $\underset{0}{C}^{\gamma/2}[0, T]$, $\underset{0}{H}^{\gamma/2}(0, T)$, and from this fact (2.9–10) are straightforwardly found.

A special situation arises if the matrices A^{ij} have a lower triangle form:

$$k<l\Rightarrow A^{ij}_{kl}=0. \tag{2.11}$$

This will be the case for example, if in addition to (1.3.8), we suppose that

$$F_{ij}\in L(C[0, T]) \quad \Big| \quad F_{ij}\in L(L_2(0, T)). \tag{2.12}$$

This statement is proven in the following way.

Suppose $A^{ij}_{kl}\neq 0$ with $k<l$. Then $F_{ij}(t^{l-1})$ behaves as $A^{ij}_{kl}t^k$ for $t\downarrow 0$. Consider $\xi_\mu(t)=t^l\bar{H}(\mu t)$ with $\bar{H}\in C^\infty[0, \infty)$, $\bar{H}(\tau)=1$ for $\tau\in[0, 1]$, $\bar{H}(\tau)=0$ for $\tau\geqq 2$. A simple calculation shows that F_{ij} cannot be bounded in the sense of (2.12) on the functions ξ_μ for $\mu\to\infty$. From this contradiction we see that (2.11) follows from (2.12).

In the example of § 1.2 with h_{ij}, $\bar{h}_{ij}\in C^\infty[0, T]$, (2.12) is certainly fulfilled and consequently we are in the situation of (2.11) there.

The following notation will be useful:

$$c_{i,l}=\left\{\frac{1}{l!}\left(\frac{\partial}{\partial t}\right)^l c_i\right\}\Big|_{t=0}, \qquad \tilde{P}_{j,l}=\left\{\frac{1}{l!}\left(\frac{\partial}{\partial t}\right)^l \tilde{P}_j\right\}\Big|_{t=0},$$

(2.13) $a_{ij,l}$, $a_{i,l}$ and $a_{0,l}$ are defined analogously to $c_{i,l}$

$$L_l=\sum_{i=1}^n\sum_{j=1}^n a_{ij,l}D_{ij}+\sum_{i=1}^n a_{i,l}D_i+a_{0,l}.$$

Note that

(2.14)

$c_{i,l}\in C^{\gamma-2l}(\bar{D})$,	$c_{i,l}\in H^{\gamma-2l+1}(D)$,
$P_{j,l}\in C^\alpha(\bar{D})'$	$P_{j,l}\in H^\alpha(D)'$
for $0\leqq l\leqq d(\gamma)$	if $\gamma>1, \tfrac{1}{2}(\gamma-1)\notin\mathbb{N}$
	and $0\leqq l\leqq d(\gamma)$.

Our first result for a more direct characterization of compatibility concerns the case $0 \leqq \alpha < 2$.

LEMMA 2.1. *Suppose that*:

ASSUMPTION 2.1.1. *The indices indicating regularity of the data* (*see* (1.3.3)), *compatibility of the data* (*see* (2.3)), *and the type of the operator* (*see* (1.3-5-6-7-8-9)) *satisfy*:

$$\beta_2 \geqq 2, \quad \beta_1 = \beta = \gamma = \beta_2 - 2, \quad 0 \leqq \alpha < 2,$$

$$(2.15) \qquad \begin{array}{l|l} \beta_0 = \beta_2, & \beta_0 = \beta_2 - 1, \\ \hat{\beta} = \hat{\beta}_1 = \beta_2 - \nu, & \hat{\beta} = \hat{\beta}_1 = \beta_2 - \frac{1}{2} - \nu, \\ \beta_2 \notin \mathbb{N} & \frac{1}{2}(\beta_2 - 1) \notin \mathbb{N}, \ \beta_2 - \frac{1}{2} \notin \mathbb{N}. \end{array}$$

ASSUMPTION 2.1.2. *If $d(\gamma) \geqq 0$ with $d(\gamma)$ defined as in* (2.7), *then the following linear algebraic system of equations in the variables $z_{s,s',m}$, $1 \leqq s \leqq p$, $0 \leqq m \leqq d(\gamma)$, $0 \leqq s' \leqq d(\gamma) - m$ possesses only the trivial solution*

$$z_{s,s',0} = 0, \qquad 1 \leqq s \leqq p, \quad 0 \leqq s' \leqq d(\gamma),$$

$$(2.16) \quad \textit{if } d(\gamma) \geqq 1 \textit{ then for } 1 \leqq m+1 \leqq d(\gamma),\ 1 \leqq s \leqq p,\ 0 \leqq s' \leqq d(\gamma) - m - 1:$$

$$(m+1) z_{s,s',m+1} = \sum_{j=1}^{p} \sum_{l=0}^{d(\gamma)} \sum_{j'=0}^{l} J_{j,l}^{s,s',m} z_{j,j',l-j'}.$$

Here we define:

$$J_{j,l}^{s,s',m} = \sum_{i=1}^{q} \sum_{m'=0}^{m} \sum_{k=0}^{m'} \tilde{P}_{s,s'} b_{i,k}^{m,m'} A_{kl}^{ij}$$

$$b_{i,k}^{m,m'} = \begin{cases} c_{i,m-k} & \text{if } m' = m \\ \displaystyle\sum_{\substack{l_1 \geqq 0, \cdots, l_n \geqq 0 \\ n \geqq 1 \\ m' + l_1 + \cdots + l_n = m - n}} \left(\prod_{r=1}^{n} \left(m - \sum_{j=1}^{r} l_j \right)^{-1} L_{l_r} \right) c_{i,m'-k} & \text{if } m' < m, \end{cases}$$

$$\prod_{r=1}^{n} \mathrm{term}(r) = \mathrm{term}(1) \cdot \cdots \cdot \mathrm{term}(n).$$

Under these conditions there are explicitly determinable operators $M_k \in$

$$(2.17) \qquad \begin{array}{r|l} L(C^{\beta,\beta/2}(\bar{Q}) + C^{\beta_0}(\bar{D}) & L(H^{\beta,\beta/2}(Q) \times H^{\beta_0}(D) \\ \to C^{\beta_0 - 2k - \nu}(\partial D)), & \to H^{\beta_0 - 2k - \nu - 1/2}(\partial D)), \\ 0 \leqq k \leqq \left[\dfrac{\beta_0 - \nu}{2} \right] = k_0, & 0 \leqq k \leqq \left[\dfrac{\beta_0 - \nu - 1/2}{2} \right] = k_0 \end{array}$$

only dependent on the operators L, Π, B, such that:

(2.18)

$$|(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1\hat{\beta}_1}<\infty \Leftrightarrow M_k(f,\psi)=\left\{\left(\frac{\partial}{\partial t}\right)^k\phi\right\}\Big|_{t=0},\quad 0\leqq k\leqq k_0.$$

If $\beta_2>\frac{3}{2}+\nu$ *then:*

$$\|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1\hat{\beta}_1}<\infty \Leftrightarrow M_k(f,\psi)=\left\{\left(\frac{\partial}{\partial t}\right)^k\phi\right\}\Big|_{t=0},\quad 0\leqq k\leqq k_0.$$

If $\beta_2<\frac{3}{2}+\nu$ *then always* $\|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1\hat{\beta}_1}<\infty$.

Moreover, there is a constant $K_{cp}>0$ *such that for all compatible triples* (f,ϕ,ψ):

$$(2.19)\qquad |(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1}\leqq K_{cp}(|f|_{\beta,\beta/2}+|\psi|_{\beta_0})\quad\Big|\quad \|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1}\leqq K_{cp}(\|f\|_{\beta,\beta/2}+\|\psi\|_{\beta_0}).$$

Proof of Lemma 2.1. "⇒" Suppose that v satisfies (2.3). Define

$$(2.20)\qquad v_l=\left\{\frac{1}{l!}\left(\frac{\partial}{\partial t}\right)^l v\right\}\Big|_{t=0},\qquad f_l=\left\{\frac{1}{l!}\left(\frac{\partial}{\partial t}\right)^l f\right\}\Big|_{t=0}.$$

Note, that:

$$(2.21)\qquad v_l\in C^{\beta_2-2l}(\bar{D}),\quad f_l\in C^{\beta-2l}(\bar{D})\quad\Big|\quad v_l\in H^{\beta_2-2l-1}(D),\quad f_l\in H^{\beta-2l-1}(D),$$

and

$$(2.22)\qquad v_0=\psi.$$

If $d(\gamma)\geqq 0$ then we find for $0\leqq m\leqq d(\gamma)$ by expansion in powers of t of $\partial v/\partial t-(L+\Pi)v-f$:

$$(2.23)\qquad \begin{aligned}(m+1)v_{m+1}&=\sum_{l=0}^{m}L_l v_{m-l}+f_m\\&+\sum_{i=1}^{q}\sum_{j=1}^{p}\sum_{k=0}^{m}\sum_{l=0}^{d(\gamma)}\sum_{j'=0}^{l}c_{i,m-k}A^{ij}_{kl}\tilde{P}_{j,j'}v_{l-j'}.\end{aligned}$$

We eliminate $\sum_{l=0}^{m}L_l v_{m-l}$ from (2.23). This leads to

$$(2.24)\qquad \begin{aligned}(m+1)v_{m+1}&=\bar{N}_{m+1}(f,\psi)\\&+\sum_{i=1}^{q}\sum_{j=1}^{p}\sum_{m'=0}^{m}\sum_{k=0}^{m'}\sum_{l=0}^{d(\gamma)}\sum_{j'=0}^{l}b^{m,m'}_{i,k}A^{ij}_{kl}\tilde{P}_{j,j'}v_{l-j'}\end{aligned}$$

with $b_{i,k}^{m,m'}$ as introduced in Assumption 2.1.2 and:

$$\bar{N}_{m+1}(f, \psi) = \Big(L_m + \sum_{\substack{l_1 \geqq 0, \cdots, l_n \geqq 0 \\ n \geqq 2 \\ l_1 + \cdots + l_n = m+1-n}} \prod_{i=1}^{n} \Big(m - \sum_{j=1}^{i} l_j\Big)^{-1} L_{l_i}\Big)\psi$$

$$+ f_m + \Big(\sum_{\substack{l_0 \geqq 0, \cdots, l_n \geqq 0 \\ n \geqq 1 \\ l_0 + \cdots + l_n = m-n}} \prod_{i=1}^{n} \Big(m - \sum_{j=1}^{i} l_j\Big)^{-1} L_{l_i}\Big) f_{l_0}.$$

Operating at (2.24) with $\tilde{P}_{ss'}$, we find the following set of linear algebraic equations for $\tilde{P}_{ss'}v_m$, $1 \leqq s \leqq q$, $0 \leqq m \leqq d(\gamma)$, $0 \leqq s' \leqq d(\gamma) - m$, if $d(\gamma) \geqq 0$:

(2.25) $$P_{s,s'}v_0 = P_{s,s'}\psi, \qquad 1 \leqq s \leqq p, \qquad 0 \leqq s' \leqq d(\gamma);$$

if $d(\gamma) \geqq 1$, then for $1 \leqq m+1 \leqq d(\gamma)$, $1 \leqq s \leqq p$, $0 \leqq s' \leqq d(\gamma)$,

$$(m+1)\tilde{P}_{s,s'}v_{m+1} = \tilde{P}_{s,s'}\bar{N}_{m+1}(f, \psi) + \sum_{j=1}^{p} \sum_{l=0}^{d(\gamma)} \sum_{j'=0}^{l} J_{j,l}^{s,s',m} \tilde{P}_{j,j'} v_{l-j'}.$$

Because of Assumption 2.1.2, we know that (2.25) possesses a unique solution. This solution is denoted by

(2.26) $$\tilde{P}_{s,s'}v_m = E_{s,s',m}(f, \psi), \qquad 1 \leqq s \leqq p, \quad 0 \leqq m \leqq d(\gamma), \quad 0 \leqq s' \leqq d(\gamma) - m.$$

Substitution of the result of (2.26) in (2.24) provides us with explicit expressions of the following type:

(2.27) $$v_m = N_m(f, \psi), \qquad 0 \leqq m \leqq d(\gamma) + 1,$$

where, by definition, $N_0(f, \psi) = \psi$. It is not difficult to verify that N_m is an operator $\in$

(2.28) $$L(C^{\beta,\beta/2}(\bar{Q}) \times C^{\beta_0}(\bar{D}) \to C^{\beta_0 - 2m}(\bar{D})) \;\Big|\; L(H^{\beta,\beta/2}(Q) \times H^{\beta_0}(D) \to H^{\beta_0 - 2m}(D)).$$

It will now be clear how the operators M_k are defined: in the case of Dirichlet BC:

(2.29) $$M_k(f, \psi) = \text{restriction to } \partial D \text{ of } N_k(f, \psi);$$

in the case of Neumann BC:

(2.30)
$$M_k = k! \sum_{l=0}^{k} B_l N_{k-l} \quad \text{with}$$
$$B_l = \sum_{i=1}^{n} b_{il} D_i + b_{0,l}, \qquad b_{i,l} = \Big\{\frac{1}{l!}\Big(\frac{\partial}{\partial t}\Big)^l b_i\Big\}\Big|_{t=0}.$$

An expansion of $\phi - Bv$ in powers of t yields exactly the contents of (2.18) "$\Rightarrow$".

"$\Leftarrow$" Given that if $k_0 \geqq 0$, the relations $M_k(f, \psi) = \{(\partial/\partial t)^k \phi\}|_{t=0}$ are satisfied for $0 \leqq k \leqq k_0$ with M_k's as introduced in (2.29–30) we shall show the existence of a function v as required in (2.3). Define:

(2.31) $$v_k = N_k(f, \psi), \qquad 0 \leqq k \leqq d(\gamma) + 1$$

with N_k as introduced in (2.27).

Using the theory as developed in the Theorems 4.1-2-3-4 of Ladyzenskaja, Solonnikov, Uraltseva (1967, pp. 298–301), we find the existence of a function v such that:

$$\left\{\frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k v\right\}\Big|_{t=0} = v_k \quad \text{for } 0 \leqq k \leqq d(\gamma)+1 \tag{2.32}$$

and moreover:

$$\begin{array}{l|l} |v|_{\beta_2,\beta_2/2} \leqq C \sum_{k=0}^{d(\gamma)+1} |v_k|_{\beta_0-2k} & \|v\|_{\beta_2,\beta_2/2} \leqq C \sum_{k=0}^{d(\gamma)+1} \|v_k\|_{\beta_0-2k} \\ \leqq K_{cp}(|f|_{\beta,\beta/2}+|\psi|_{\beta_0}) & \leqq K_{cp}(\|f\|_{\beta,\beta/2}+\|\psi\|_{\beta_0}) \end{array} \tag{2.33}$$

with certain constants C, $K_{cp}>0$ only dependent of L, Π.

By construction we now have $v(\cdot,0)=\psi$, and it is easy to show that $\partial v/\partial t-(L+\Pi)v-f\in$

$$\underset{0}{C}^{\beta_1,\beta_1/2}(\bar{Q}) \;\Big|\; \underset{0}{H}^{\beta_1,\beta_1/2}(Q). \tag{2.34}$$

Namely, if $d(\gamma)\geqq 0$, then expansion in powers of t yields for $1\leqq m+1\leqq d(\gamma)+1$:

$$\left\{\frac{1}{m!}\left(\frac{\partial}{\partial t}\right)^m\left(\frac{\partial v}{\partial t}-(L+\Pi)v-f\right)\right\}\Big|_{t=0} = 2.23^* = 2.24^* = 2.25^* = 0$$

because of the definition of the operators N_m. The superscript * means: put the right-hand side of the indicated equation in parentheses, change the equality sign of the indicated equation into a minus sign and take v_k's as introduced in (2.31–32) in this expression.

It is also completely straightforward to show that $\phi - Bv \in$

$$\underset{0}{C}^{\hat{\beta}_1,\hat{\beta}_1/2}(\bar{\Gamma}) \;\Big|\; \underset{0}{H}^{\hat{\beta}_1,\hat{\beta}_1/2}(\Gamma). \tag{2.35}$$

Let us give the calculation in the case of Neumann BC:

$$\begin{aligned}\left\{\left(\frac{\partial}{\partial t}\right)^k(\phi-Bv)\right\}\Big|_{t=0} &= \left\{\left(\frac{\partial}{\partial t}\right)^k\phi\right\}\Big|_{t=0} - k!\sum_{l=0}^{k} B_l v_{k-l} \\ &= \left\{\left(\frac{\partial}{\partial t}\right)^k\phi\right\}\Big|_{t=0} - k!\sum_{l=0}^{k} B_l N_{k-l}(f,\psi) = 0\end{aligned}$$

by the definition of M_k and the given relations.

So (2.18) "$\Leftarrow$" and (2.19) have also been demonstrated. □

It is interesting to notice the following

COROLLARY (to Lemma 2.1). *Assumption* 2.1.2 *is certainly satisfied if the matrices* A^{ij} *have a lower triangle form* (*see* (2.11–12)).

This is true since in this case $l>m \Rightarrow J_{j,l}^{s,s',m}=0$. As a consequence the system of (2.16) (and also (2.23)) possesses a recursive structure with respect to m, i.e., in order to calculate $z_{s,s',m+1}(v_{m+1})$ we only use $z_{s,s',k}(v_k)$ with $0\leqq k\leqq m$. So indeed the only solution of (2.16) is then the trivial one.

In the following lemma we shall give an analogue of the previous results in the case $\alpha\geqq 2$.

LEMMA 2.2. *Suppose that*:

ASSUMPTION 2.2.1. *The indices indicating regularity of the data* (*see* (1.3.3)), *compatibility of the data* (*see* (2.3)) *and the type of the operator* Π (*see* (1.3.5-6-7-8-9)) *satisfy*:

$$\beta_2 \geqq \alpha \geqq 2, \quad \beta_1 = \gamma = \beta_2 - \alpha, \quad \beta = \beta_2 - 2,$$

$$(2.36)\qquad \begin{array}{l|l} \beta_0 = \beta_2, & \beta_0 = \beta_2 - 1, \\ \hat\beta = \hat\beta_1 = \beta_1 + 2 - \nu, & \hat\beta = \hat\beta_1 = \beta_1 + \frac{3}{2} - \nu, \\ c_i \in C^{\beta,\beta/2}(\bar Q),\ 1 \leqq i \leqq q, & c_i \in H^{\beta,\beta/2}(Q),\ 1 \leqq i \leqq q, \\ \beta_2 \notin \mathbb{N} & \frac{1}{2}(\beta_2 - 1) \notin \mathbb{N},\ \beta_2 - \frac{1}{2} \notin \mathbb{N}. \end{array}$$

ASSUMPTION 2.2.2. *It is assumed, that the contents of Assumption* 2.1.2 *are valid with* γ as defined in Assumption 2.2.1.

Under these conditions there are explicitly determinable operators $M_k \in$

$$(2.37)\qquad \begin{array}{r|r} L(C^{\beta,\beta/2}(\bar Q) \times C^{\beta_0}(\bar D) \to & L(H^{\beta,\beta/2}(Q) \times H^{\beta_0}(D) \to \\ C^{\beta_0 - 2k - \nu}(\partial D)) & H^{\beta_0 - 2k - \nu - (1/2)}(\partial D)) \\ \textit{for } 0 \leqq k \leqq \bar k_0, & \textit{for } 0 \leqq k \leqq \bar k_0, \\ \bar k_0 = \min\left(\left[\frac{\beta_0 - \nu}{2}\right], \left[\frac{\beta_1 + 2}{2}\right]\right) & \bar k_0 = \min\left(\left[\frac{\beta_0 - (1/2) - \nu}{2}\right], \left[\frac{\beta_1 + 1}{2}\right]\right) \end{array}$$

only dependent on the operators L, Π, B *such that*:

$$(2.38)\qquad \begin{array}{r|l} & \textit{If } \beta_2 > \frac{3}{2} + \nu \textit{ then}: \\ |(f, \phi, \psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1} < \infty & \|(f, \phi, \psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} < \infty \\ \Leftrightarrow M_k(f, \psi) = \left\{\left(\frac{\partial}{\partial t}\right)^k \phi\right\}\Big|_{t=0}, & \Leftrightarrow M_k(f, \psi) = \left\{\left(\frac{\partial}{\partial t}\right)^k \phi\right\}\Big|_{t=0}, \\ 0 \leqq k \leqq \bar k_0. & \qquad 0 \leqq k \leqq \bar k_0. \\ & \textit{If } \beta_2 < \frac{3}{2} + \nu \textit{ then always} \\ & \|(f, \phi, \psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} < \infty. \end{array}$$

Moreover, there is a constant $K_{cp} > 0$ *such that for all compatible triples* (f, ϕ, ψ):

$$(2.39)\qquad \begin{array}{r|r} |(f, \phi, \psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1} & \|(f, \phi, \psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} \\ \leqq K_{cp}(|f|_{\beta,\beta/2} + |\psi|_{\beta_0}) & \leqq K_{cp}(\|f\|_{\beta,\beta/2} + \|\psi\|_{\beta_0}). \end{array}$$

Proof of Lemma 2.2. "$\Rightarrow$" This part of the proof is completely analogous to the corresponding part of the proof of Lemma 2.1. One can even use the same text as before with, in the final conclusion, (2.18) replaced by (2.38). However, a few remarks should be made.

It is now used, that $P_{s,s'} v_m$ is well-defined for $1 \leqq s \leqq p$, $0 \leqq m \leqq d(\gamma)$, $0 \leqq s' \leqq d(\gamma) - m$ if $d(\gamma) \geqq 0$. This is the case since $\beta_0 - 2d(\gamma) \geqq \alpha$.

Further we note that v_m is given by the formula in (2.20) for $0 \leqq m \leqq d(\beta_2)$, but that the formula of (2.27) is only valid for $0 \leqq m \leqq d(\gamma) + 1$ and it is certainly possible that $d(\beta_2) > d(\gamma) + 1$!

"$\Leftarrow$" Given that if $\bar k_0 \geqq 0$, the relations $M_k(f, \psi) = \{(\partial/\partial t)^k \phi\}|_{t=0}$ are satisfied for $0 \leqq k \leqq \bar k_0$ we now construct a function v as required in (2.3) as follows. Define:

$$(2.40)\qquad \begin{array}{ll} v_k = N_k(f, \psi), & 0 \leqq k \leqq d(\gamma) + 1 \\ v_k = 0, & d(\gamma) + 2 \leqq k \leqq d(\beta_2) \quad \text{if } d(\beta_2) > d(\gamma) + 1. \end{array}$$

As before the theory of Ladyzenskaja, Solonnikov, Uraltseva (1967) provides us with a function v such that

$$\left\{\frac{1}{k!}\left(\frac{\partial}{\partial t}\right)^k v\right\}\bigg|_{t=0} = v_k, \qquad 0 \leqq k \leqq d(\beta_2) \tag{2.41}$$

for which (2.33) holds.

The proof is concluded in the same way as the proof of Lemma 2.1. □

COROLLARY (to Lemma 2.2). *Assumption* 2.2.2 *is certainly satisfied if the matrices* A^{ij} *have a lower triangle form* (*see* (2.11–12)).

This is in fact the same remark as the Corollary to Lemma 2.1.

3. Nilpotency of nonanticipative compact linear operators. Here we give and prove a lemma with contents as indicated in the title of this section, which will be very useful in the next sections.

Let B be a nontrivial Banach space with norm $\|\cdot\|_B$. We suppose, that on B we have a strongly continuous semi-group of bounded linear operators $\{U(\tau); \tau \geqq 0\}$ with the following properties:

(a) *existence of a finite time-horizon* $T > 0$, *i.e.*,

$$\begin{aligned} &\ker U(\tau) \neq B \quad \textit{for } \tau \in [0, T), \\ &\ker U(\tau) = B \quad \textit{for } \tau \geqq T. \end{aligned} \tag{3.1}$$

(b)

$$\forall \tau \in [0, T] \quad \ker U(\tau) = \operatorname{ran} U(T - \tau). \tag{3.2}$$

The $U(\tau)$'s are called time shifts.

Examples of Banach spaces with time shifts and a finite time horizon $T > 0$ are $\underset{0}{C}^{\mu}[0, T]$, $\underset{0}{H}^{\mu}(0, T)$, $\underset{0}{C}^{2\mu,\mu}(\bar{Q})$, $\underset{0}{H}^{2\mu,\mu}(Q)$ with $Q = D \times (0, T)$.

On these spaces we define time shifts in the obvious way:

$$(U(\tau)f)(\cdot, t) = \begin{cases} 0 & \text{for } t \in [0, \min(\tau, T)] \\ f(\cdot, t - \tau) & \text{for } t \in (\tau, T] \text{ if } \tau < T \end{cases} \tag{3.3}$$

(the $\cdot$ in $(\cdot, t)$ indicates possible other variables).

The properties a, b are easily verified in these examples. An operator $A \in L(B)$ will be called nonanticipative if:

$$\forall \tau \in [0, \infty) \quad A \ker U(\tau) \subset \ker U(\tau). \tag{3.4}$$

Note that the nonanticipativity of the feed-back coupling as defined in (1.3.8) induces nonanticipativity of the F_{ij}'s in the sense of (3.4) on $\underset{0}{C}^{\gamma/2}[0, T]$ in the case of "C-theory" and on $\underset{0}{H}^{\gamma/2}(0, T)$ in the case of H-theory.

Now we shall prove the following result:

LEMMA 3.1. *Let B be a nontrivial Banach space with time shift with a finite time horizon $T > 0$ for which* (a) *and* (b) *are satisfied.*

If $A \in L(B)$ is nonanticipative and compact then A is nilpotent, i.e., $\sigma(A) = \{0\}$.

Proof of Lemma 3.1. This lemma is a straightforward application of Corollary 4.3.11 of Ringrose (1971, p. 177).

The continuous chain of closed subspaces of B mentioned in that corollary is here given by

(3.5) $$\tilde{\mathscr{F}} = \{\ker U(\tau) | \tau \in [0, T]\}.$$

Of course $\ker U(\tau)$ is closed, for $U(\tau)$ is a bounded operator on B.

The system $\tilde{\mathscr{F}}$ is totally ordered by inclusion:

(3.6) $$\ker U(\tau_1) \supset \ker U(\tau_2) \quad \text{for } \tau_1 \geqq \tau_2.$$

The chain $\tilde{\mathscr{F}}$ also satisfies

(3.7) $$0 = \ker U(0) \in \tilde{\mathscr{F}}, \qquad B = \ker U(T) \in \tilde{\mathscr{F}}.$$

Further we have the property that for any subset $I \subset [0, T]$,

(3.8) $$\bigcap_{\tau \in I} \ker U(\tau) = \ker U(\inf I),$$

(3.9) $$\overline{\bigcup_{\tau \in I} \ker U(\tau)} = \ker U(\sup I).$$

As for (3.8) it is clear, that $\bigcap_{\tau \in I} \ker U(\tau) \supset \ker U\ (\inf I)$.

Let τ_n now be a sequence in I such that $\tau_n \downarrow \inf I$ for $n \uparrow \infty$ and let f satisfy $U(\tau_n)f = 0$ for $n \in N$.

By the strong continuity of the semi-group we have $U(\lim_{n\uparrow\infty} \tau_n)f = \lim_{n\uparrow\infty} U(\tau_n)f = 0$. Consequently $\bigcap_{\tau \in I} \ker U(\tau) \subset \ker U(\inf I)$.

As for (3.9) it is clear, that $\overline{\bigcap_{\tau \in I} \ker U(\tau)} \subset \ker U(\sup I)$. If $f \in \ker U(\tau_0)$, $\tau_0 = \sup I$ then $f = U(T - \tau_n)\ g$ because of (b).

Let τ_n now be a sequence in I such that $\tau_n \uparrow \tau_0$ for $n \uparrow \infty$ and define $f_n = U(T - \tau_n)g$.

By the strong continuity of the semi-group we have $U(T - \tau_0)g = \lim_{n\uparrow\infty} U(T - \tau_n)g$; so $f = \lim_{n\uparrow\infty} f_n$. Consequently $\overline{\bigcup_{\tau \in I} \ker U(\tau)} \supset \ker U(\sup I)$.

The properties of $\tilde{\mathscr{F}}$ given in (3.7–8–9) imply that conditions (i) and (ii) of Ringrose (1971, pp. 166–167) are satisfied.

Now it remains to show that in Ringrose's notation $M_- = M$ for each $M \in \tilde{\mathscr{F}}$.

If $M = \ker U(\tau_0)$ then we have in this situation: $M_- = \mathrm{cl}\,\{\bigcup L | L \in \tilde{\mathscr{F}},\ L \subsetneqq M\} = \overline{\bigcup_{\tau < \tau_0} \ker U(\tau)}$.

Indeed $\ker U(\tau) \neq \ker U(\tau_0)$ for $\tau < \tau_0$ since if we suppose $\ker U(\tau) = \ker U(\tau_0)$, then we would have $\operatorname{ran} U(T - \tau) = \operatorname{ran} U(T - \tau_0)$,

$$\Rightarrow U(\tau) \operatorname{ran} U(T - \tau) = U(\tau) \operatorname{ran} U(T - \tau_0)$$

$$\Rightarrow \{0\} = \operatorname{ran} U(T - \tau_0 + \tau_1) = \ker U(\tau_0 - \tau_1)$$

$$\Rightarrow \ker U(\varepsilon) = \{0\} \text{ for some } \varepsilon > 0$$

$$\Rightarrow \ker U(t) = \{0\}\ \forall t \geqq 0$$

which would contradict (a). Now $M_- = M$ follows from (3.9). Herewith all conditions of the corollary have been verified. □

A direct consequence of Lemma 3.1 is that for such an operator A the equation for $w \in B$,

(3.10) $$w = Aw + g \quad \text{with } g \in B$$

possesses a unique solution $\in B$:

$$w=(I-A)^{-1}g \tag{3.11}$$

with $(I-A)^{-1}\in L(B)$, $\|w\|_B \leqq \|(I-A)^{-1}\|_{L(B)}\|g\|_B$.

This is the way Lemma 3.1 will be used further on.

4. Well-posedness if the order of the feed-back control operator is less than the order of the diffusion operator. Here we consider the case $0\leqq\alpha<2$. We are now ready to prove the following result on existence, uniqueness, regularity, and continuous dependence on the data of a solution of (1.1.1–2–3).

THEOREM 4.1. *Let us suppose*:

ASSUMPTION 4.1. *The indices indicating the regularity of the data (see* (1.3.3)), *and the type of the operator* Π *(see* (1.3.5–6–7–8–9) *satisfy*

$$\begin{array}{l|l} \beta>0,\quad \beta\notin\mathbb{N}, & \beta\geqq 0,\quad \tfrac{1}{2}(\beta+1)\notin\mathbb{N},\quad \beta+\tfrac{1}{2}\notin\mathbb{N},\\ \beta_0=\beta+2, & \beta_0=\beta+1,\\ \hat\beta=\beta+2-\nu, & \hat\beta=\beta+\tfrac{3}{2}-\nu,\\ \gamma=\beta,\quad 0\leqq\alpha<2 & \gamma=\beta,\quad 0\leqq\alpha<2. \end{array} \tag{4.1}$$

ASSUMPTION 4.2. *The data are compatible in the following sense*:

$$|(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1}<\infty \;\Big|\; \|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1}<\infty \tag{4.2}$$

with $\beta_2=\beta+2$, $\beta_1=\beta$, $\hat\beta_1=\hat\beta$.

Then there exists a unique solution u of (1.1.1–2–3) *in the space*:

$$C^{2s,s}(\bar Q)\;\Big|\;H^{2s,s}(Q) \tag{4.3}$$

with $2s=\beta+2$. *This solution depends continuously on the data in the following sense*:

$$\begin{array}{l|l} |u|_{2s,s}\leqq K(|f|_{\beta,\beta/2} & \|u\|_{2s,s}\leqq K(\|f\|_{\beta,\beta/2}\\ \quad +|\phi|_{\hat\beta,\hat\beta/2} & \quad +\|\phi\|_{\hat\beta,\hat\beta/2}\\ \quad +|(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1}) & \quad +\|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1}) \end{array} \tag{4.4}$$

with a constant $K>0$ *independent of* f, ϕ, ψ.

Note that in Assumption 4.1 the exceptional cases for β are exactly as in the uncontrolled problem (see Ladyzenskaja, et al. (1967, Thms. 5.2–3, p. 320), and Lions, Magenes (1972, Thm. 6.2, p. 37)). Further it will be clear that the compatibility condition of Assumption 4.2 is nececessary in order to have (4.3).

Note, that theorem 4.1 is applicable to the example of § 1.2 with h_{ij}, $\bar h_{ij}\in C^\infty[0,T]$ in the case of "C-theory". In the case of "H-theory", this is also true if $D\subset\mathbb{R}^n$ with $n\leqq 3$, see (1.3.11).

Proof of Theorem 4.1. Let v be a function for which (2.3) is satisfied with

$$|v|_{\beta_2,\beta_2/2}\leqq|(f,\phi,\psi)|^{cp}_{\beta_2,\beta_1,\hat\beta_1}+\varepsilon \;\Big|\; \|v\|_{\beta_2,\beta_2/2}\leqq\|(f,\phi,\psi)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1}+\varepsilon \tag{4.5}$$

with $\varepsilon>0$ arbitrarily small.

Put $w=u-v$; then

$$
(4.6)\qquad \begin{aligned}
&\frac{\partial w}{\partial t}=(L+\Pi)w+f_0, && \text{FPDE},\\
&Bw=\phi_0, && \text{BC},\\
&w(\cdot,0)=0, && \text{IC}
\end{aligned}
$$

with

$f_0\in \underset{0}{C}^{\beta,\beta/2}(\bar{Q})$,	$f_0\in \underset{0}{H}^{\beta,\beta/2}(Q)$,
$\phi_0\in \underset{0}{C}^{\hat{\beta},\hat{\beta}/2}(\bar{\Gamma})$,	$\phi_0\in \underset{0}{H}^{\hat{\beta},\hat{\beta}/2}(\Gamma)$,
(4.7) $\lvert f_0\rvert_{\beta,\beta/2}\leqq \lvert f\rvert_{\beta,\beta/2}+C\lvert v\rvert_{\beta_2,\beta_2/2}$,	$\lVert f_0\rVert_{\beta,\beta/2}\leqq \lVert f\rVert_{\beta,\beta/2}+C\lVert v\rVert_{\beta_2,\beta_2/2}$,
$\lvert\phi_0\rvert_{\hat{\beta},\hat{\beta}/2}\leqq \lvert\phi\rvert_{\hat{\beta},\hat{\beta}/2}+C\lvert v\rvert_{\beta_2,\beta_2/2}$	$\lVert\phi_0\rVert_{\hat{\beta},\hat{\beta}/2}\leqq \lVert\phi\rVert_{\hat{\beta},\hat{\beta}/2}+C\lVert v\rVert_{\beta_2,\beta_2/2}$.

In this proof $C>0$ will always denote some constant only dependent of L, B, $\dot{\Pi}$, β, $\hat{\beta}$ β_2, β_1, $\hat{\beta}_1$, γ, α, D.

We shall now solve (4.6). In order to do so we shall rewrite (4.6) in the form (4 below.

Using the solution theory for the parabolic problem as given in Ladyzensk Solonnikov, Uraltseva (1967) and Lions, Magenes (1972, Thm. 6.2, p. 37), we see tl

(i) the uncontrolled problem corresponding to (4.6) possesses a unique solutioı with

$W\in \underset{0}{C}^{\beta+2,(\beta+2)/2}(\bar{Q})$,	$W\in \underset{0}{H}^{\beta+2,(\beta+2)/2}(Q)$,
(4.8) $\lvert W\rvert_{\beta+2,(\beta+2)/2}\leqq C(\lvert f_0\rvert_{\beta,\beta/2}+\lvert\phi_0\rvert_{\hat{\beta},\hat{\beta}/2})$	$\lVert W\rVert_{\beta+2,(\beta+2)/2}\leqq C(\lVert f_0\rVert_{\beta,\beta/2}+\lVert\phi_0\rVert_{\hat{\beta},\hat{\beta}/2})$;

(ii) the solution operator S for the uncontrolled problem with homogeneous and BC is an element of:

(4.9) $L(\underset{0}{C}^{\beta,\beta/2}(\bar{Q})\to \underset{0}{C}^{\beta+2,(\beta+2)/2}(\bar{Q}))$	$L(\underset{0}{H}^{\beta,\beta/2}(Q)\to \underset{0}{H}^{\beta+2,(\beta+2)/2}(Q))$.

Now we can rewrite (4.6) in the following way:

$$
(4.10)\qquad w=Aw+W\quad \text{with } A=S\Pi.
$$

Equations (4.6) and (4.10) are equivalent for

$$
w\in \underset{0}{C}^{2s,s}(\bar{Q})\ \Big|\ w\in \underset{0}{H}^{2s,s}(Q).
$$

We shall show that

$$A \in L(\underset{0}{C}^{2s,s}(\bar{Q})) \;\Big|\; A \in L(\underset{0}{H}^{2s,s}(Q)), \tag{4.11}$$

and A is compact and nonanticipative.

Then Lemma 3.1 yields (see (3.15–16)) that (4.10) possesses a unique solution

$$A \in L(\underset{0}{C}^{2s,s}(\bar{Q})) \;\Big|\; A \in L(\underset{0}{H}^{2s,s}(Q)), \tag{4.11}$$

and

$$|w|_{2s,s} \leqq C|W|_{2s,s} \;\Big|\; \|w\|_{2s,s} \leqq C\|W\|_{2s,s}. \tag{4.13}$$

Herewith the first part of theorem 4.1 has then been proven. The estimate (4.4) is a trivial consequence of $u = w + v$ and (4.5–7–8–13) if we take the limit $\varepsilon \downarrow 0$.

So it remains to prove (4.11).

(a) Using (2.6) and (4.9) we find $\Pi \in$

$$L(\underset{0}{C}^{\check{\gamma},\check{\gamma}/2}(\bar{Q}) \to \underset{0}{C}^{2\gamma,\gamma}(\bar{Q})) \;\Big|\; L(\underset{0}{H}^{\check{\gamma},\check{\gamma}/2}(Q) \to \underset{0}{H}^{2\gamma,\gamma}(Q)) \tag{4.14}$$

with $\check{\gamma} = \gamma + \alpha = \beta + \alpha < \beta + 2 = 2s$.

Because of the compact imbedding of

$$\underset{0}{C}^{2s,s}(\bar{Q}) \quad \text{in } \underset{0}{C}^{\check{\gamma},\check{\gamma}/2}(\bar{Q}) \;\Big|\; \underset{0}{H}^{2s,s}(Q) \quad \text{in } \underset{0}{H}^{\check{\gamma},\check{\gamma}/2}(Q), \tag{4.15}$$

we obtain that A is an element of the space indicated in (4.11) and that A is compact. Note that we used the assumption $0 \leqq \alpha < 2$ here!

(b) The operator S is nonanticipative:

$$\forall \tau \in (0, T],\ u(\cdot, t) = 0 \text{ for } 0 < t < \tau \quad \Rightarrow \quad (Su)(\cdot, t) = 0 \text{ for } 0 < t < \tau. \tag{4.16}$$

For $u \in C^\infty(\bar{Q})$ this is a consequence of the causal type of maximum principle valid for equations of parabolic type (see Protter, Weinberger (1967) or Friedman (1975)). By continuity of S, the property is then valid on the whole of the domain of S.

The operator Π is nonanticipative because of the nonanticipativity of the feed-back coupling, see (1.3.8). This immediately implies the nonanticipativity of A. □

COROLLARY (to Theorem 4.1). *If Assumption* 2.1.2 *is satisfied then the characterization of compatibility of the data given in Lemma* 2.1, (2.18) *is valid and* (4.4) *can be replaced by the estimate*

$$\begin{aligned} |u|_{2s,s} \leqq K(&|f|_{\beta,\beta/2} \\ &+ |\phi|_{\hat{\beta},\hat{\beta}/2} + |\psi|_{\beta_0}) \end{aligned} \;\Bigg|\; \begin{aligned} \|u\|_{2s,s} \leqq K(&\|f\|_{\beta,\beta/2} \\ &+ \|\phi\|_{\hat{\beta},\hat{\beta}/2} + \|\psi\|_{\beta_0} \end{aligned} \tag{4.17}$$

with a constant $K > 0$ independent of f, ϕ, ψ. This implies continuous dependence of the data in the usual sense.

This corollary follows from the fact, that Assumptions 4.1 and 4.2 imply Assumption 2.1.1. As a consequence we can apply Lemma 2.1. Then (4.17) is found by inserting (2.19) into (4.4).

5. The case where the order of the feed-back control operator equals or exceeds the order of the diffusion operator

5.1. A result on well-posedness. Now we shall consider the case $\alpha \geqq 2$. In order to derive a well-posedness result we have to make a number of assumptions.

ASSUMPTION 5.1.1. *The indices indicating the regularity of the data (see* (1.3. *and the type of the operator* Π *(see* (1.3.5–6–7–8–9)) *satisfy*:

	$\beta \geqq 2\alpha - 4 \geqq 0,$	$\beta \geqq 2\alpha - 4 \geqq 0, \frac{1}{2}(\beta+1) \notin \mathbb{Z}, \beta + \frac{1}{2} \notin \mathbb{Z},$
	$\beta \notin \mathbb{Z}, \beta - \alpha \notin \mathbb{Z},$	$\frac{1}{2}(\beta - \alpha + 1) \notin \mathbb{Z}, \beta + \dfrac{1}{2} - \dfrac{\alpha}{2} \notin \mathbb{Z},$
	$\beta_0 - \beta + \alpha,$	$\beta_0 = \beta + \alpha - 1,$
(5.1.1)	$\hat{\beta} = \beta + 2 - \nu,$	$\hat{\beta} = \beta + \frac{3}{2} - \nu,$
	$\gamma = \beta$ *and also the choice* $\gamma = 2\delta$ *with* $\delta = \frac{1}{2}(\beta + 2 - \alpha)$ *is allowed.*	$\gamma = \beta$ *and also the choice* $\gamma = 2\delta$ *with* $\delta = \frac{1}{2}(\beta + 2 - \alpha)$ *is allowed.*
(5.1.2)		$\operatorname*{ess\,sup}_{0<t<T} \lVert c_i(\cdot, t)\rVert_\gamma < \infty$ *if* $0 \leqq \delta < \frac{1}{2}$; $\exists \bar{\delta} > \delta$ *such that*: $\operatorname*{ess\,sup}_{0<t<\tau<T} \lvert t - \tau\rvert^{-\bar{\delta}} \lVert c_i(\cdot, t) - c_i(\cdot, \tau)\rVert_0 < \infty$ *if* $0 < \delta < \frac{1}{2}$.

ASSUMPTION 5.1.2. *The data are compatible in the following sense*:

$$|(f, \phi, \psi)|^{cp}_{\beta_2, \beta_1, \hat{\beta}_1} < \infty \quad \Big| \quad \lVert (f, \phi, \psi) \rVert^{cp}_{\beta_2, \beta_1, \hat{\beta}_1} < \infty \tag{5.1.3}$$

with $\beta_2 = \beta + \alpha$, $\beta_1 = \beta$, $\hat{\beta}_1 = \hat{\beta}$.

Assumptions 5.1.1 and 5.1.2 are rather analogous to Assumptions 4.1 and 4.2 However, in this case we have to assume more.

Introduce $\mathscr{C}_i(x, t; \tau)$, $1 \leqq i \leqq q$ as the solution of the following problem:

$$\frac{\partial \mathscr{C}_i}{\partial t} = L\mathscr{C}_i, \qquad \text{PDE,} \tag{5.1.4}$$

$$B\mathscr{C}_i = 0, \qquad \text{BC,} \tag{5.1.5}$$

$$\mathscr{C}_i(\cdot, \tau; \tau) = c_i(\cdot, \tau), \qquad \text{IC.} \tag{5.1.6}$$

Note that the $\mathscr{C}_i$'s are well-defined.

Using a perturbation argument and Theorem XIV of Dunford, Schwartz (1963, § 8.1), we see that $\mathscr{C}_i(\cdot, t; \tau)$ is a continuous function of t for $t \in [\tau, T]$ in $L_2(D)$ and moreover that $\mathscr{C}_i(\cdot, \cdot; \tau) \in C^\infty(\bar{D}x(\tau, T])$ for all $\tau \in [0, T)$.

Next we introduce, for $1 \leqq i \leqq q$, $1 \leqq k \leqq p$:

$$Z_{ki}(t, \tau) = \tilde{P}_k(t)\mathscr{C}_i(\cdot, t; \tau).$$

Note, that Z_{ki} is well-defined for $t > \tau$.

Our following assumption requires that the Z_{ki}'s are sufficiently regular for $t \downarrow \tau$.

ASSUMPTION 5.1.3. *The functions* Z_{ki}, $1 \leqq k \leqq p$, $1 \leqq i \leqq q$ *have the following property*:

$$(5.1.7) \qquad Z_{ki} \in C^{\delta}\{0 \leqq \tau \leqq t \leqq T\} \;\Big|\; Z_{ki} \in L^{\delta}_{\infty}\{0 < \tau < t < T\}.$$

Here for a domain Ω we define $L^{\delta}_{\infty}(\Omega)$ as the space $\tilde{W}^{\delta,\infty}(\Omega)$ defined in Adams (1975, 7.49).

Now the following result will be proven.

THEOREM 5.1. *Under the Assumptions* 5.1.1, 5.1.2, 5.1.3 *there exists a unique solution u of* (1.1.1–2–3) *with the following properties*:

$$(5.1.8) \qquad \begin{array}{ll|ll} u \in C^{2s,s}(\bar{Q}), & & u \in H^{2s,s}(Q), & \\ P_j u \in C^{\delta}[0, T], & 1 \leqq j \leqq p & P_j u \in H^{\delta}(0, T), & 1 \leqq j \leqq p \end{array}$$

with $2s = \beta + 4 - \alpha$. *Moreover this solution depends continuously on the data in the following sense*:

$$(5.1.9) \qquad \begin{array}{l|l} |u|_{2s,s} + \sum_{j=1}^{p} |P_j u|_{\delta} & \|u\|_{2s,s} + \sum_{j=1}^{p} \|P_j u\|_{\delta} \\ \quad \leqq K(|f|_{\beta,\beta/2} + |\phi|_{\hat{\beta},\hat{\beta}/2} & \quad \leqq K(\|f\|_{\beta,\beta/2} + \|\phi\|_{\hat{\beta},\hat{\beta}/2} \\ \quad + |(f, \phi, \psi)|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1}) & \quad + \|(f, \phi, \psi)\|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1}) \end{array}$$

with a constant $K > 0$ *independent of* f, ϕ, ψ.

Note, that for $\alpha > 2$ in this theorem, a loss of regularity in the space directions takes place from $t = 0$ to $t > 0$, namely:

$$\begin{array}{r|r} \psi \in C^{\beta_0}(\bar{D}),\ u(\cdot, t) \in C^{\beta_2}(\bar{D}) & \psi \in H^{\beta_0}(D),\ u(\cdot, t) \in H^{\beta_2 - 1}(D) \\ \text{for } t > 0 & \text{for } t > 0 \\ \text{with } \beta_2 < \beta_0 & \text{with } \beta_2 - 1 < \beta_0. \end{array}$$

The Assumption 5.1.3 is difficult to understand at a first glance. However, it is used in a very essential way in our proof of Theorem 5.1. In the case of time-independent operators L, B and time independent control input functions c_i and observators $\tilde{P}_j$, this assumption can be relaxed substantially, see Corollary 5.1.c. In Corollary 5.1.b. we shall give some concrete conditions under which Assumption 5.1.3 is certainly fulfilled.

Proof of Theorem 5.1. In exactly the same way as in the proof of Theorem 4.1, (1.1.1–2–3) can be rewritten in the equivalent form:

$$(5.1.10) \qquad w = S\Pi w + W$$

with

$$W \in \underset{0}{C}{}^{\beta+2,(\beta+2)/2}(\bar{Q}) \;\Big|\; W \in \underset{0}{H}{}^{\beta+2,(\beta+2)/2}(Q).$$

However, now we cannot prove that the operator $S\Pi$ is compact on a suitable space and from here on we have to proceed differently. In this case we shall exploit, in an essential way, that the control uses a finite number of observators and control inputs. This is done by deriving a system of equations for the observation functions $P_k w$ and analyzing existence, uniqueness, and regularity of solutions of this system of equations.

In order to do so we observe that (5.1.10) can be written as

$$w(\cdot, t) = W(\cdot, t) + \sum_{i=1}^{q} \sum_{j=1}^{p} \int_0^t \mathscr{C}_i(\cdot, t; \tau)\{F_{ij}P_j w\}(\tau)\, d\tau. \tag{5.1.11}$$

Suppose that w satisfies (5.1.11) and that

$$\begin{array}{l|l} w \in C^{2s,s}(\bar{Q}), & w \in H^{2s,s}(Q), \\ P_j w \in C^{s}[0, T], \quad 1 \leqq j \leqq p & P_j w \in H^{s}(0, T), \quad 1 \leqq j \leqq p. \end{array} \tag{5.1.12}$$

We shall show that the functions $P_k w$, $1 \leqq k \leqq p$ satisfy the following equation:

$$(P_k w)(t) = (P_k W)(t) + \sum_{i=1}^{q} \sum_{j=1}^{p} \int_0^t Z_{ki}(t, \tau)\{F_{ij}P_j w\}(\tau)\, d\tau. \tag{5.1.13}$$

In order to show (5.1.13) we operate on both sides of (5.1.11) with $\tilde{P}_k(t)$. It will be clear that we find (5.1.13) if for $1 \leqq k$, $j \leqq p$, $1 \leqq i \leqq q$:

$$\tilde{P}_k(t) \int_0^t \mathscr{C}_i(\cdot, t; \tau)\chi_{ij}(\tau)\, d\tau = \int_0^t \tilde{P}_k(t)\mathscr{C}_i(\cdot, t; \tau)\chi_{ij}(\tau)\, d\tau$$

with $\chi_{ij} = F_{ij}w$.

Let us show that the interchangement of the integral $\int_0^t$ and the operation of $\tilde{P}_k(t)$ is allowed here.

Using the remark following (5.1.6) and the Assumption 5.1.3 it is found that $\mathscr{C}_i(\cdot, t; \tau)$ is a continuous function of $\tau \in [0, t]$ in the sense of the (somewhat unusual) norm $\|\cdot\|_{k,t} = \|\cdot\|_0 + |\tilde{P}_k(t)|$. Consequently $\|\mathscr{C}_i(\cdot, t; \tau)\chi_{ij}(\tau)\|_{k,t}$ is integrable with respect to τ on $(0, t)$.

This means that the integral $\int_0^t \mathscr{C}_i(\cdot, t, \tau)\chi_{ij}(\tau)\, d\tau$ is well-defined in Bochner's sense with respect to $\|\cdot\|_{k,t}$.

An application of Corollary 2 of Yosida (1965, p. 134) shows that the interchangement of $\int_0^t$ and $\tilde{P}_k(t)$ is indeed allowed. □

On the other hand suppose, that ξ_k, $1 \leqq k \leqq p$ satisfies

$$\xi_k \in \underset{0}{C}^{s}[0, T] \;\Big|\; \xi_k \in \underset{0}{H}^{s}(0, T), \tag{5.1.14}$$

$$\xi_k(t) = (P_k W)(t) + \sum_{i=1}^{q} \sum_{j=1}^{p} \int_0^t Z_{k,i}(t, \tau)\{F_{ij}\xi_j\}(\tau)\, d\tau. \tag{5.1.15}$$

Then w defined by

$$w(\cdot, t) = W(\cdot, t) + \sum_{i=1}^{q} \sum_{j=1}^{p} \int_0^t \mathscr{C}_i(\cdot, t; \tau)\{F_{ij}\xi_j\}(\tau)\, d\tau \tag{5.1.16}$$

is an element of

$$\underset{0}{C}^{2s,s}(\bar{Q}) \;\Big|\; \underset{0}{H}^{2s,s}(Q), \tag{5.1.17}$$

and w satisfies (5.1.11). Let us demonstrate this.

Operating with $\tilde{P}_k(t)$ on both sides of (5.1.16) we find:

$$\xi_k = P_k w. \tag{5.1.18}$$

Substitution of (5.1.18) in (5.1.16) gives that w satisfies (5.1.11). In order to show the regularity given in (5.1.17) we note that

$$\sum_{i=1}^{q}\sum_{j=1}^{p}\int_0^t \mathscr{C}_i(\cdot, t; \tau)\{F_{ij}\xi_j\}(\tau)\, d\tau = \Big(\sum_{i=1}^{q} S\tilde{c}_i\Big)(\cdot, t)$$

with $\tilde{c}_i(\cdot, t) = c_i(\cdot, t) \sum_{j=1}^{p} (F_{ij}\xi_j)(t)$. Now we have

$$\begin{array}{r|l} \tilde{c}_i \in \underset{0}{C}^{2\delta,\delta}(\bar{Q}), & \tilde{c}_i \in \underset{0}{H}^{2\delta,\delta}(\bar{Q}), \\ S\tilde{c}_i \in \underset{0}{C}^{2s,s}(\bar{Q}) & S\tilde{c}_i \in \underset{0}{H}^{2s,s}(\bar{Q}). \end{array}$$

In the case of "H-theory" and $0 \leqq \delta < \frac{1}{2}$ we have used (5.1.2) here. □

We shall now show that (5.1.15) possesses a unique solution ξ in the following function space:

(5.1.19) $$\{L_2(0, T)\}^p \;\Big|\; \{L_2(0, T)\}^p.$$

As for the regularity of this solution ξ we shall show that (5.1.14) holds.

Let us introduce the following shorthand vector notation of (5.1.17).

(5.1.20) $$\xi = \eta + A\xi$$

with

$$\eta_k = P_k W,$$

$$(A\xi)_k = \sum_{i=1}^{q}\sum_{j=1}^{p}\int_0^t Z_{k,i}(t, \tau)\{F_{ij}\xi_j\}(\tau)\, d\tau.$$

Now A defines a nonanticipative, compact, linear operator on

(5.1.21) $$\begin{array}{l|l} \{L_2(0, T)\}^p, & \{L_2(0, T)\}^p, \\ \{\underset{0}{C}^{\delta}[0, T]\}^p & \{\underset{0}{H}^{\delta}(0, T)\}^p. \end{array}$$

The only nontrivial part of this statement is the compactness of A.

It is clear, that $A \in L(\{L_2(0, T)\}^p \to \{L_\infty(0, T)\}^p)$ and this implies the compactness of $A \in L(\{L_2(0, T)\}^p)$.

It is also easy to verify that $A \in L(\{\underset{0}{H}^{\delta}(0, T)\}^p \to \{\dot{L}^{\delta}_{\infty}(0, T)\}^p)$ which proves the compactness of $A \in L(\{\underset{0}{H}^{\delta}(0, T)\}^p)$.

Now consider A as an element of $L(\{\underset{0}{C}^{\delta}[0, T]\}^p)$. Define a sequence $A^{(n)}$ by replacing $Z_{k,i}$ in the definition of A by $Z_{k,i}^{(n)}$ with $Z_{k,i}^{(n)} \in C^\infty(\{0 \leqq \tau \leqq t \leqq T\})$ and $Z_{k,i}^{(n)} \to Z_{k,i}$ in $C^{\delta}(\{0 \leqq \tau \leqq t \leqq T\})$ for $n \to \infty$.

It is not difficult to show that $A^{(n)} \in L(\{\underset{0}{C}^{\delta}[0, T]\}^p \to \{\underset{0}{C}^{\delta+1}[0, T]\}^p)$, i.e., $A^{(n)}$ is a compact element of $L(\{\underset{0}{C}^{\delta}[0, T]\}^p)$ and that $A^{(n)} \to A$ in $L(\{\underset{0}{C}^{\delta}[0, T]\}^p)$ for $n \to \infty$. A well-known theorem (see Kato (1966, p. 158)) gives the desired compactness result for A in this case.

So we are in the situation that we can apply Lemma 3.1. We note that

(5.1.22)
$$\eta \in \{\underset{0}{C}^{\delta}[0, T]\}^p \quad \Big| \quad \eta \in \{\underset{0}{H}^{\delta}(0, T)\}^p$$

and the above mentioned result is found. □

We have now proved the existence of a unique solution u of (1.1.1–2–3) properties of (5.1.8).

It is not difficult to deduce the estimate (5.1.9), since all operations to solution u appear to be continuous in an obvious sense.

COROLLARY a (to Theorem 5.1). *If Assumption* 2.2.2 *is satisfied and if, n*

(5.1.23)
$$f, c_i \in C^{\bar\beta,\bar\beta/2}(\bar Q),\ 1 \leqq i \leqq q \quad \Big| \quad f, c_i \in H^{\bar\beta,\bar\beta/2}(Q),\ 1 \leqq i \leqq q,$$

with $\bar\beta = \beta + \alpha - 2$, then the characterization of compatibility given in Lemma 2.: *is valid and* (5.1.9) *can be replaced by*

(5.1.24)
$$|u|_{2s,s} + \sum_{j=1}^{p} |P_j u|_\delta \leqq K(|f|_{\bar\beta,\bar\beta/2} + |\phi|_{\hat\beta,\hat\beta/2} + |\psi|_{\beta_0}) \quad \Big| \quad \|u\|_{2s,s} + \sum_{j=1}^{p} \|P_j u\|_\delta \leqq K(\|f\|_{\bar\beta,\bar\beta/2} + \|\phi\|_{\hat\beta,\hat\beta/2} + \|\psi$$

with a constant K independent of f, ϕ, ψ.

This follows from the fact that Assumptions 5.1.1 and 5.1.2, supplement (5.1.23), imply Assumption 2.2.1 with β replaced by $\bar\beta$. As a consequence we ca Lemma 2.2. Then (5.1.24) is found by inserting (2.39) into (5.1.9).

COROLLARY b (to Theorem 5.1). *The Assumption* 5.1.3 *is certainly fulfill*

(i) *the control input functions satisfy*:

(5.1.25)
$$c_i \in C^\infty(\bar Q), \qquad \textit{distance (support } c_i, \bar\Gamma) > 0, \quad 1 \leqq i \leqq q;$$

or

(ii) *the coefficients of the operators L, B are time-independent and the contro functions have the following form*:

(5.1.26)
$$c_i = \sum_{l=1}^{d} \hat c_{il} e_l, \qquad d < \infty, \quad 1 \leqq i \leqq q$$

with $\{e_l; l \in \mathbb{N}\}$ the eigenfunctions of the stationary uncontrolled problem:

(5.1.27)
$$Le = \lambda e \quad PDE,\ \lambda \textit{ spectral parameter}, \qquad Be = 0 \quad BC,$$

and the $\hat c_{il}$'s functions of time only:

(5.1.28)
$$\hat c_{il} \in C^\infty[0, T], \qquad 1 \leqq l \leqq d, \quad 1 \leqq i \leqq q.$$

In the first case, we get that $\mathscr{C}_i \in C^\infty(\bar D \times \{0 \leqq \tau \leqq t \leqq T\})$, $1 \leqq i \leqq q$, since $c_i(\cdot, \tau)$ satisfies the compatibility conditions of the problem (5.1.4) up to any

In the second case, we can even specify the form of each $\mathscr{C}_i$, $1 \leqq i \leqq q$:

(5.1.29)
$$\mathscr{C}_i(\cdot, t; \tau) = \sum_{l=1}^{d} \hat c_{i,l}(\tau) e_l(\cdot) \operatorname{Pol}_l(t-\tau)\, e^{\lambda_l(t-\tau)}$$

with $e_l \in C^\infty(\bar D)$, λ_l the eigenvalue corresponding to e_1 and Pol_l some polynomi

It will be clear that, indeed, in both cases Assumption 5.1.3 is fulfilled.

COROLLARY c (to Theorem 5.1). *If the operators L, B have time-independent coefficients and if the observators $\tilde{P}_j$, and the control input functions c_i are time-independent, i.e.,*

$$\tilde{P}_j(t) = \tilde{P}_j^s, \qquad c_i(x, t) = c_i^s(x) \tag{5.1.30}$$

with

$$\begin{array}{l|l} \tilde{P}_j^s \in C^\alpha(\bar{D})', & \tilde{P}_j^s \in H^\alpha(D)', \\ c_i^s \in C^\gamma(\bar{D}), & c_i^s \in H^\gamma(D), \end{array}$$

then the Assumption 5.1.3 *can be relaxed. In this situation the $Z_{k,i}$'s are only dependent of the variable $t-\tau$, i.e.,*

$$Z_{k,i}(t, \tau) = Z_{k,i}^s(t-\tau), \qquad 1 \leqq k \leqq p, 1 \leqq i \leqq q. \tag{5.1.31}$$

It is sufficient to require:

$$\begin{array}{l|l} Z_{k,i}^s \in L_1(0, T), & Z_{k,i}^s \in L_1(0, T) \text{ if } \delta > \tfrac{1}{2} \\ 1 \leqq k \leqq p, \quad 1 \leqq i \leqq q & Z_{k,i}^s \in L_r(0, T) \text{ if } 0 \leqq \delta < \tfrac{1}{2} \\ & \quad \text{with } r = 2(1+2\delta)^{-1}, \\ & \qquad 1 \leqq k \leqq p, 1 \leqq i \leqq q. \end{array} \tag{5.1.32}$$

Proof of Corollary 5.1.c. The interchangement of $\tilde{P}_k(t)$ and $\int_0^t$ necessary to derive (5.1.13) is also allowed under the conditions of (5.1.32). This is easily derived by the same reasoning as before.

In the case of "H-theory" and $0 \leqq \delta < \frac{1}{2}$ we use that $H^\delta(0, T)$ is continuously inbedded in $L_{\bar{r}}(0, T)$ with $\bar{r} = 2(1-2\delta)^{-1}$ (see Adams (1975)) and that $r^{-1} + \bar{r}^{-1} = 1$.

Now we have for the operator A of (5.1.20),

$$\begin{aligned} (A\xi)_k(t) &= \int_0^t Z_{ki}^s(t-\tau)(F_{ij}\xi_j)(\tau)\, d\tau \\ &= \int_0^t Z_{ki}^s(\tau)(F_{ij}\xi_j)(t-\tau)\, d\tau \\ &= \left\{\int_0^T Z_{ki}^s(\tau) U(\tau)(F_{ij}\xi)\, d\tau\right\}(t) \end{aligned} \tag{5.1.33}$$

with $U(\tau)$ the shift operator introduced in (3.4).

So we have the following formula:

$$A = \left(\int_0^T Z^s(\tau) U(\tau)\, d\tau\right) F, \tag{5.1.34}$$

where Z^s, F are matrices with matrix elements Z_{ki}^s, F_{ij}. We also used Assumption 5.1.3 to show the compactness of the operator A. But this can now be shown only with the use of (5.1.32–34).

In order to prove the compactness of A in this case, we approximate A by a sequence $A^{(n)}$ obtained by replacing, in (5.1.34), Z^s by $Z^{s,n}$ with $Z^{s,n} \in \{C^\infty[0, T]\}^{p\times q}$, $n \in \mathbb{N}$ and $Z^{s,n} \to Z^s$ in the sense of $\{L_2(0, T)\}^{p\times q}$ for $n \to \infty$.

It is not difficult to verify, that:

$$A^{(n)} \in L(\{L_2(0, T)\}^p \Rightarrow \{\underset{0}{H}{}^1(0, T)\}^p) \quad \text{that is} \quad A^{(n)} \in L(\{L_2(0, T)\}^p)$$

is compact. $A^{(n)} \to A$ in the sense of $L(\{L_2(0, T)\}^p)$ for $n \to \infty$, so (see Kat $A \in L(\{L_2(0, T)\}^p)$ is compact.

In the same way, we can show that $A \in L(\{\underset{0}{H}^\delta(0, T)\}^p)$ is compact $A \in L(\{\underset{0}{C}^\delta[0, T]\}^p)$ is compact.

Now it is clear that the proof of Theorem 5.1 keeps holding in this situatic before. □

5.2. An example which shows how bad things can be. Consider the f problem:

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) - \frac{\partial^2 u}{\partial x^2}(0, t), \qquad \text{FPDE},$$

$$u(0, t) = u(1, t) = 0, \qquad \text{BC}, \tag{5.2.1}$$

$$u(\cdot, 0) = \psi, \qquad \text{IC}.$$

Note that $D = (0, 1) \subset \mathbb{R}$, $L = \partial^2/\partial x^2$, $\Pi = cF\delta_0''$ with $c \equiv -1$, F the identity, (δ_0'' $(\partial^2 u/\partial x^2)(0, t)$, i.e., $p = q = 1$.

So the operators L, B have time-independent coefficients and the control f and observator are time-independent. The feed-back control is instantaneous.

It is not difficult to give a family of solutions of (5.2.1) for special initial d $\lambda \in [1, \infty)$.

We take ψ_λ the solution of

$$(L - \lambda)\psi_\lambda = -1, \qquad \psi_\lambda(o) = \psi_\lambda(1) = 0. \tag{5.2.2}$$

Note that ψ_λ is well-defined and $\psi_\lambda \in C^\infty[0, 1]$.

Now consider u_λ defined by

$$u_\lambda(x, t) = e^{\lambda t}\psi_\lambda(x). \tag{5.2.3}$$

It is easy to verify that u_λ satisfies (5.2.1) with the initial data ψ_λ.

Take $T > 0$. Then we find:

$$|u_\lambda|_{0,0} \geqq K\lambda^{-1} \exp(\lambda T), \quad \Big| \quad \|u_\lambda\|_{0,0} \geqq K\lambda^{-1} \exp(\lambda T) \tag{5.2.4}$$

with some constant $K > 0$ independent of $\lambda \in [1, \infty)$.

But it is also clear that

$$\begin{array}{l|l} |(0, 0, \psi_\lambda)|^{cp}_{\beta_2,\beta_1,\hat\beta_1} & \|(0, 0, \psi_\lambda)\|^{cp}_{\beta_2,\beta_1,\hat\beta_1} \\ \leqq |v_\lambda|_{\beta_2,\beta_2/2} \leqq \tilde K\lambda^{(\beta_2-2)/2} & \leqq \|v_\lambda\|_{\beta_2,\beta_2/2} \leqq \tilde K\lambda^{(\beta_2-2)/2} \end{array} \tag{5.2.5}$$

with some constant $\tilde K > 0$ independent of $\lambda \in [1, \infty)$.

In (5.2.5) v_λ is defined by

$$v_\lambda(x, t) = u_\lambda(x, t)\bar H(\lambda t) \tag{5.2.6}$$

with $\bar H \in C^\infty[0, \infty)$,

$$\bar H(\tau) = 1 \quad \text{for } 0 \leqq \tau \leqq 1,$$

$$\bar H(\tau) = 0 \quad \text{for } \tau \geqq 2.$$

Comparing the behavior for $\lambda \to \infty$ in (5.2.4) and (5.2.5) we see that no matter how we choose β_2, β_1, $\hat{\beta}_1$ *it is not true* that (5.2.1) possesses a solution u for which

(5.2.7) $$|u|_{0,0} \leqq K_0 |(0, 0, \psi)|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1} \quad \Big| \quad \|u\|_{0,0} \leqq K_0 \|(0, 0, \psi)\|^{cp}_{\beta_2,\beta_1,\hat{\beta}_1}$$

for some $K_0 > 0$ independent of ψ.

So this problem (5.2.1) is very badly posed!

A result such as Theorem 5.1 clearly can not be true here. The reason is that Assumption 5.1.3 is certainly not fulfilled in this example.

6. Application of the theory to some examples. Let us consider the following examples:

(6.1) $$\frac{\partial u}{\partial t} = (\Delta - \bar{c}FP)u + f, \qquad \text{FPDE},$$
$$\frac{\partial u}{\partial \vec{n}} = 0, \qquad \text{BC},$$
$$u(\cdot, 0) = \Psi, \qquad \text{IC}.$$

Note that $p = q = 1$. The control input function is taken constant in space and time: $\bar{c} \in \mathbb{R}$, $\bar{c} > 0$.

As for F and P we shall consider the following cases.

(i) $P = \delta_y$, $(F\xi)(t) = \xi(t)$,

(ii) $P = \delta_y$, $(F\xi)(t) = \mu \int_0^t \exp(-\mu(t-\tau))\xi(\tau)\, d\tau$, $\mu \in \mathbb{R}$, $\mu > 0$,

(iii) $P = \delta_y(1 + s(\partial/\partial t))$, $(F\xi)(t) = \xi(t)$, $s \in \mathbb{R}$, $s > 0$.

Note that (i), (ii) are special cases of the examples of § 1.2, (iii) is a special case of the example considered at the end of § 1.3 (near (1.3.14)).

At the end of § 1.3 it was also shown that (iii) falls within the setting for the control operator, when the FPDE is rewritten as

(6.1.(iii)) $$\frac{\partial u}{\partial t} = (\Delta + \hat{\Pi})u + g,$$

where $\hat{\Pi} = -c_1\delta_y\{1 + s\Delta\}$, $c = \bar{c}(1 + s\bar{c})^{-1}$, $g = f - s\bar{c}(1 + s\bar{c})^{-1}\delta_y f$. Let us now apply our Theorems 4.1 and 5.1 to (6.1).

Case (i), (ii): C-theory. Note that the order of Π is $\alpha = 0$ (see (1.3.11)). So we have to apply the C-theory version of Theorem 4.1 here. This leads to the following result. If $f \in C^{\varepsilon,\varepsilon/2}(\bar{Q})$, $\Psi \in C^{2+\varepsilon}(\bar{D})$ with $0 < \varepsilon < 1$ and Ψ satisfies the compatibility relation $\partial\Psi/\partial n = 0$ on ∂D; then there exists a unique solution of (6.1) in $C^{2+\varepsilon,(2+\varepsilon)/2}(\bar{Q})$ which depends continuously on f and Ψ. The compatibility relation follows from Lemma 2.1.

Case (iii): C-theory. Note that the order of $\hat{\Pi}$ is $\alpha = 2$ (see (1.3.14)). We apply now the C-theory version of Theorem 5.1 to (6.1(iii)). The Assumption 5.1.3 is fulfilled here because of the Corollary b (1 is an eigenfunction of Δ with Neumann boundary conditions corresponding to the eigenvalue 0). Assumption 5.1.4 can be omitted because of Corollary c. We find, that if $f \in C^{\varepsilon,\varepsilon/2}(\bar{Q})$, $\Psi \in C^{2+\varepsilon}(\bar{D})$ with $0 < \varepsilon < 1$ and Ψ satisfies the compatibility relation $\partial\Psi/\partial n = 0$ on ∂D; then there exists a unique solution of (6.1) in $C^{2+\varepsilon,(2+\varepsilon)/2}(\bar{Q})$ which depends continuously on f and Ψ.

Case (i), (ii): H-theory. In this situation the order of Π is $\alpha = (n/2) + \varepsilon$, $\varepsilon > 0$ arbitrarily small (see (1.3.11)). So we have to distinguish between the cases: $n \leqq 3$ and $n \geqq 4$.

In the case $n \leqq 3$, an application of the H-theory version of Theorem 4.1 yields: $f \in L_2(Q)$ and $\Psi \in H^1(D)$, then there exists a unique solution of (6.1) in $H^{2,1}(Q)$, whi depends continuously on f and Ψ. Note that no further compatibility conditions on are necessary in accordance with Lemma 2.1.

In the case $n \geqq 4$, we have to apply the H-theory version of Theorem 5.1. Tl Assumption 5.1.3 is fulfilled for reasons completely analogous to the ones of Case (ii C-theory. We now use Corollary a and we obtain the result: if $f \in H^{3\alpha-6,(3\alpha-6)/2}$ (Q $\Psi \in H^{3\alpha-5}(D)$ with $\alpha = (n/2)+\varepsilon$, $\varepsilon > 0$ sufficiently small and the data are compatibl then there exists a unique solution of (6.1) in $H^{\alpha,\alpha/2}(Q)$ which depends continuously c f and Ψ.

The compatibility of the data is always satisfied for $n = 4$. For $n = 5$ the data ai compatible, if $\partial\Psi/\partial\vec{n} = 0$ on ∂D. For $n = 6, 7$ the data are compatible if $\partial\Psi/\partial\vec{n} = 0$ on ∂ and $\partial/\partial\vec{n}\{(\Delta+\Pi_0)\Psi + f|_{t=0}\} = 0$ on ∂D with $\Pi_0 = \Pi$ in Case (i), $\Pi_0 = 0$ in Case (ii). This easily checked using Lemma 2.2.

It is left to the reader to obtain explicit expressions for the compatibility relatior for general n.

Case (iii): H-theory can be dealt with in an analogous way. Details are left to th reader.

Till now we have exclusively considered the well-posedness of problems such a (1.1.1–2–3) without worrying about (i) the construction of the solution if such a solutioı exists (ii) the properties of the solution.

Let us now conclude this paper by showing that, at least for the examples (6.1(i) (ii), (iii)) it is easy to calculate the solution. This calculation is done by using th technique of expansion in eigenfunctions of the uncontrolled stationary problem

$$f = \sum_{n=0}^{\infty} f_n e_n, \quad \Psi = \sum_{n=0}^{\infty} \Psi_n e_n, \quad u = \sum_{n=0}^{\infty} u_n e_n. \tag{6.2}$$

Here $\{e_n;\, n \geqq 0\}$ denotes the sequence of, in $L_2(D)$-sense, orthonormal eigenfunction: of Δ with Neumann boundary conditions. e_0 corresponds to the eigenvalue 0, i.e., e_0 is constant $= \{\mathrm{vol}\, D\}^{1/2}$.

The eigenvalue corresponding with e_n is denoted by $-\lambda_n$. The numbering is such that $n \geqq m \Rightarrow \lambda_n \geqq \lambda_m$.

Of course the coefficients f_n, Ψ_n and u_n are defined by:

$$f_n(t) = \langle f(\cdot, t), e_n\rangle, \quad \Psi_n = \langle \Psi, e_n\rangle, \quad u_n(t) = \langle u(\cdot, t), e_n\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the innerproduct in $L_2(D)$.

Substitution in (6.1) leads to the following system of equations for the u_n's, $n \geqq 0$:

$$\left.\begin{aligned} &\frac{du_n}{dt} = -\lambda_n u_n + f_n, \\ &u_n(0) = \Psi_n \end{aligned}\right\} \text{ for } n \geqq 1, \tag{6.3}$$

$$\begin{aligned} &\frac{du_0}{dt} = \nu F u_0 + R_0, \\ &u_0(0) = \Psi_0. \end{aligned} \tag{6.4}$$

In (6.4), we have in case (i), (ii):

$$\nu = \bar{c}, \quad R_0 = f_0 - \bar{c} \sum_{n=1}^{\infty} q_n F u_n, \quad q_n = \frac{\delta_y e_n}{\delta_y e_0},$$

and in case (iii):

$$\nu = c_1(1 - sa_0), \quad R_0 = g_0 - \bar{c} \sum_{n=1}^{\infty} \hat{q}_n F u_n,$$

$$g_0 = f_0 - sc_1 \frac{\delta_y f}{\delta_y e_0}, \quad \hat{q}_n = (1 - s\lambda_n) \frac{\delta_y e_n}{\delta_y e_0}.$$

These systems of equations can be solved in an elementary way. To do this for general f is left to the reader. In the case $\Psi \equiv 1$, $f \equiv 0$ we obtain:

$u(\cdot, t) = u_0(t)e_0$, where $u_0(t)$ equals:

uncontrolled: 1;

case (i): $\exp(-\bar{c}t)$;

case (ii): $\dot{v}(t) + \mu v(t)$ with

$$v(t) = \frac{1}{\nu_1 - \nu_2}\{\exp(\nu_1 t) - \exp(\nu_2 t)\},$$

(6.5)
$$\nu_{1,2} = \tfrac{1}{2}\{-\mu \pm \sqrt{\mu^2 - 4\bar{c}}\} \quad \text{if } \nu_1 \neq \nu_2,$$

$$v(t) = \exp(-\tfrac{1}{2}\mu t)\{1 - \tfrac{1}{2}\mu t\} \quad \text{if } \nu_1 = \nu_2;$$

case (iii): $\exp\left(-\frac{\bar{c}}{1 + s\bar{c}} t\right)$.

Note that all these solutions decay for $t \to \infty$.

For a given $\bar{c}$, the most rapid decay for $t \to \infty$ which is possible under (i), (ii), (iii) arises if $\bar{c} \geqq 4$ in case (i) and if $0 < \bar{c} < 4$ in case (ii) with $\mu = 2\sqrt{\bar{c}}$.

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