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CONSISTENCY OF THE EQUAL SPLIT-OFF SET

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Consistency of the equal split-off set

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Abstract

This paper axiomatically studies the equal split-off set (cf. Branzei et al. (2006)) as a solution for cooperative games with transferable utility. This solution extends the well-known Dutta and Ray (1989) solution for convex games to arbitrary games. By deriving several characterizations, we explore the relation of the equal split-off set with various consistency notions.

Keywords: transferable utility games; egalitarianism; equal split-off set; consistency
JEL classification: C71

1 Introduction

A solution for transferable utility games prescribes how to allocate joint revenues among cooperating players while taking into account their economic opportunities in coalitions. A well-known egalitarian solution for convex transferable utility games was defined in the seminal paper of Dutta and Ray (1989). This solution can be considered as a compromise between egalitarianism and stability, where egalitarianism is formalized by Lorenz domination and stability is modeled by core-like participation constraints. As a consequence, the Dutta and Ray (1989) solution assigns to any convex game the unique Lorenz dominating core allocation. Hougaard et al. (2001), Arin and Iñarra (2001), and Arin et al. (2008) extended this approach to balanced games and analyzed Lorenz undominated core allocations.

In line with the ideas of Dutta and Ray (1989), and inspired by the computational algorithm of their solution, Branzei et al. (2006) introduced the equal split-off set as an extension to all transferable utility games. Imagine a group of cooperating players believing in equality as a desirable social goal and facing the problem of sharing their joint revenues. The arbitrator proposes equal division. However, some coalitions may complain since the members can obtain more by equal division of the corresponding coalitional worth. The arbitrator selects one coalition with the highest complaint, equally divides the worth among its members, and lets them leave with their assigned payoffs. The remaining players now face a reduced game in which the worth of a coalition equals the joint marginal contribution of its members to the leaving players in the previous game. Again, the arbitrator proposes equal division and the process is repeated until all players have left. The equal split-off set consists of all allocations which can be generated by this procedure. Sanchez-Soriano et al. (2014) showed that all equal split-off set allocations Lorenz dominate all core allocations.

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Dutta (1990) obtained two characterizations of the Dutta and Ray (1989) solution for convex games. The main axioms in these characterizations are based on the consistency principle. Imagine a group of cooperating players agreeing on applying a certain solution concept for the allocation of their joint revenues. Suppose that some players leave with their assigned payoffs and that the remaining players reevaluate their payoffs by applying the solution to a reduced game. The solution is consistent if it prescribes for this reduced game the same allocation as for the original game. Thomson (2011) provides a general introduction to the consistency principle.

The exact interpretation of the consistency principle in cooperative games depends on the axiomatic formulation, which is mainly determined by the specific definition of reduced games. The results of Dutta (1990) involve the formulations proposed by Davis and Maschler (1965) and Hart and Mas-Colell (1989), to which we refer as max-consistency and self-consistency, respectively. Klijn et al. (2000) obtained similar results using weak variants of these axioms which only require consistent allocations in situations where the richest players leave with their assigned payoffs, to which we refer as rich-restricted consistency. Moreover, they derived a third characterization based on an alternative consistency formulation which closely resembles the definition of the equal split-off set, to which we refer as rich-restricted marginal-consistency.

Recently, Llerena and Mauri (2017) introduced the class of exact partition games and wondered whether the characterizations of Klijn et al. (2000) on convex games can be extended to this larger class. We show that the class of exact partition games is exactly the class of games for which the equal split-off set intersects the core and we provide a full answer to this open question. In particular, we show that the characterizations based on max-consistency and marginal-consistency can be extended to exact partition games, but the characterization based on self-consistency cannot. Moreover, we weaken the rather specific rich-restricted marginal-consistency to the rich-restricted version of the well-known consistency notion introduced by Moulin (1985), to which we refer as complement-consistency. Both max-consistency and complement-consistency have been used in axiomatizations of the core, respectively by Peleg (1986) and Tadenuma (1992). We show that rich-restricted complement-consistency can also play a similar role as rich-restricted max-consistency in the extended characterization of Klijn et al. (2000) on the class of exact partition games. As a by-product, we provide a new characterization of the Dutta and Ray (1989) solution on the class of convex games.

Next to consistency notions, the properties equal division stability and feasible richness play a key role in the results of Klijn et al. (2000) on convex games and in our results on exact partition games. On the class of all transferable utility games, the equal split-off set violates equal division stability. Moreover, it violates rich-restricted max-consistency. However, the equal split-off set does satisfy a weak form of equal division stability and rich-restricted self-, complement-, and marginal-consistency. We show that all solutions satisfying feasible richness, weak equal division stability, and rich-restricted self-, complement-, or marginal-consistency on the class of all games necessarily prescribe equal split-off set allocations.

This paper is organized as follows. Section 2 provides preliminary notions and notations. Section 3 formally introduces the equal split-off set and presents some elementary results. Section 4 axiomatically characterizes the equal split-off set on the class of exact partition games and Section 5 derives maximality results on the class of all games.

2 Preliminaries

Let N be a nonempty and finite set. Denote $2^N = \{S \mid S \subseteq N\}$. A *partition* of N is a collection $\{T_1, \ldots, T_m\} \subseteq 2^N \setminus \{\emptyset\}$ such that $\bigcup_{k=1}^m T_k = N$ and $T_k \cap T_\ell = \emptyset$ for each pair $k, \ell \in \{1, \ldots, m\}$ with $k \neq \ell$. An *order* of N is a bijection $\pi : \{1, \ldots, |N|\} \to N$. Let Π^N denote the set of all orders of N. An allocation $x \in \mathbb{R}^N$ *Lorenz dominates* $y \in \mathbb{R}^N$, denoted by $x \succ_L y$, if $\min_{S \in 2^N : |S|=k} \sum_{i \in S} x_i \geq \min_{S \in 2^N : |S|=k} \sum_{i \in S} y_i$ for each $k \in \{1, \ldots, |N|\}$, with at least one strict inequality. For all $x \in \mathbb{R}^N$, we define $R_0^x = \emptyset$ and for each $k \in \{1, \ldots, |N|\}$,

$$R_k^x = \left\{ i \in N \;\middle|\; \forall j \in N \setminus R_{k-1}^x : x_j \leq x_i \right\} \quad \text{and} \quad a_k^x = x_i \text{ for all } i \in R_k^x \setminus R_{k-1}^x.$$

A *transferable utility game* is a pair (N, v) where N is a nonempty and finite set of *players* and $v : 2^N \to \mathbb{R}$ assigns to each *coalition* $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let Γ_{all} denote the class of all games. The *subgame* $(T, v|_T)$ of $(N, v) \in \Gamma_{all}$ on $T \in 2^N \setminus \{\emptyset\}$ is defined by $v|_T(S) = v(S)$ for all $S \in 2^T$.

A *solution* σ on a class of games $\Gamma \subseteq \Gamma_{all}$ assigns to each $(N, v) \in \Gamma$ a set of payoff allocations $\sigma(N, v) \subseteq \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ for all $x \in \sigma(N, v)$. A solution σ on a class of games $\Gamma \subseteq \Gamma_{all}$ is a *maximal* solution on Γ satisfying certain properties if for each solution σ' on Γ satisfying these properties, $\sigma'(N, v) \subseteq \sigma(N, v)$ for all $(N, v) \in \Gamma$.

Throughout this paper, Γ is the generic notation for a class of games, (N, v) is the generic notation for a game in Γ, and σ is the generic notation for a solution on Γ.

Let (N, v) be a game. The *core* is given by

$$C(N, v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall S \in 2^N : \sum_{i \in S} x_i \geq v(S) \right\}.$$

The *Weber set* is given by

$$W(N, v) = \text{conv}\left(\left\{ \mu^\pi(N, v) \;\middle|\; \pi \in \Pi^N \right\} \right),$$

where $\mu^\pi(N, v) \in \mathbb{R}^N$ corresponding to $\pi \in \Pi^N$ is for each $k \in \{1, \ldots, |N|\}$ given by

$$\mu^\pi_{\pi(k)}(N, v) = v(\{\pi(\ell) \in N \mid \ell \leq k\}) - v(\{\pi(\ell) \in N \mid \ell < k\}).$$

Note that $W(N, v) \neq \emptyset$. Weber (1988) and Derks (1992) showed that $C(N, v) \subseteq W(N, v)$. A game (N, v) is *convex* (cf. Shapley (1971) and Ichiishi (1981)) if $C(N, v) = W(N, v)$. Let Γ_{conv} denote the class of all convex games.

3 The equal split-off set

In this section, we formally introduce the equal split-off set as a solution for all transferable utility games and present some elementary results. The equal split-off set is based on the computational algorithm of the egalitarian Dutta and Ray (1989) solution for convex games. Consider an arbitrary cooperative game for which we face the problem of sharing the worth of the grand coalition among the players. One of the coalitions with highest average worth is selected and the members equally divide this worth and leave. The remaining players determine the joint marginal contribution of each subgroup to the departed players in the game. One of the subgroups with highest average contribution is selected and the members equally divide this contribution and leave. This process continues and results in a payoff allocation for the players. The equal split-off set consists of all allocations which can be generated by this procedure.

Definition 1 (cf. Branzei et al. (2006))
Let (N, v) be a game. Define $N_0 = N$, $v_0 = v$, and $T_0 = \emptyset$. The *equal split-off set* $ESOS(N, v)$ consists of all payoff allocations $x \in \mathbb{R}^N$ for which there exists a partition $\{T_1, \ldots, T_m\}$ of N such that for each $k \in \{1, \ldots, m\}$,

$$T_k \in \operatorname*{argmax}_{S \in 2^{N_k} \setminus \{\emptyset\}} \frac{v_k(S)}{|S|} \quad \text{and} \quad x_i = \max_{S \in 2^{N_k} \setminus \{\emptyset\}} \frac{v_k(S)}{|S|} \text{ for all } i \in T_k,$$

where (N_k, v_k) is the game defined by

$$N_k = N_{k-1} \setminus T_{k-1} \quad \text{and} \quad v_k(S) = v_{k-1}(S \cup T_{k-1}) - v_{k-1}(T_{k-1}) \text{ for all } S \in 2^{N_k}.$$

Example 1
Let (N, v) with $N = \{1, 2, 3\}$ be the game given by

$$v(S) = \begin{cases} 9 & \text{if } S = N; \\ 8 & \text{if } S = \{1, 2\}; \\ 5 & \text{if } S = \{1\}; \\ 0 & \text{otherwise.} \end{cases}$$

The equal split-off set is $ESOS(N, v) = \{(5, 3, 1)\}$ corresponding to partition $\{\{1\}, \{2\}, \{3\}\}$. Note that $ESOS(N, v) \subseteq C(N, v)$. △

Example 2
Let (N, v) with $N = \{1, 2, 3\}$ be the game given by

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1, 2\}, \{1, 3\}, N\}; \\ 0 & \text{otherwise.} \end{cases}$$

The equal split-off set is $ESOS(N, v) = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2})\}$ corresponding to partitions $\{\{1, 2\}, \{3\}\}$ and $\{\{1, 3\}, \{2\}\}$. The core is $C(N, v) = \{(1, 0, 0)\}$. △

Example 3
Let (N, v) with $N = \{1, 2, 3\}$ be the game given by

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

The equal split-off set is $ESOS(N, v) = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$ corresponding to partitions $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, and $\{\{2, 3\}, \{1\}\}$. The core is $C(N, v) = \emptyset$. △

Note that for each $x \in ESOS(N, v)$ with corresponding $\{T_1, \ldots, T_m\}$ the following holds:

- $x_i = x_j$ for all $i, j \in T_k$ with $k \in \{1, \ldots, m\}$;
- $x_i \geq x_j$ for all $i \in T_k$ and $j \in T_\ell$ with $k, \ell \in \{1, \ldots, m\}$ and $k \leq \ell$;
- $\sum_{\ell=1}^{k} \sum_{i \in T_\ell} x_i = v(\bigcup_{\ell=1}^{k} T_\ell)$ for each $k \in \{1, \ldots, m\}$.

This means that for each $x \in ESOS(N, v)$,

- $\sum_{i \in R_k^x} x_i = v(R_k^x)$ for each $k \in \{1, \ldots, |N|\}$;
- $a_k^x = \frac{v(R_k^x) - v(R_{k-1}^x)}{|R_k^x \setminus R_{k-1}^x|}$ for each $k \in \{1, \ldots, |N|\}$ with $R_{k-1}^x \neq N$.

The following two punctual properties for solutions play a central role throughout this paper.

Nonemptiness
$\sigma(N,v) \neq \emptyset$.

Feasible richness
for each $x \in \sigma(N,v)$, $\sum_{i \in R_1^x} x_i \leq v(R_1^x)$.

Nonemptiness requires that a solution assigns to any game at least one allocation. Feasible richness requires that the richest players should be able to obtain their payoffs by themselves.[1] Clearly, the equal split-off set satisfies nonemptiness and feasible richness. The core violates these properties in general. Sanchez-Soriano et al. (2014) showed that each equal split-off set allocation Lorenz dominates each (other) core allocation.

Theorem (cf. Sanchez-Soriano et al. (2014))
Let (N,v) be a game. For each $x \in ESOS(N,v)$ and each $y \in C(N,v) \setminus \{x\}$, $x \succ_L y$.

This result implies that the intersection of the equal split-off set and the core consists of at most one allocation. This also follows from the results of Llerena and Mauri (2016). We show that the equal split-off set consists of one allocation if it intersects the core.

Lemma 1
Let (N,v) be a game. If $ESOS(N,v) \cap C(N,v) \neq \emptyset$, then $|ESOS(N,v)| = 1$.

Proof. Assume that $ESOS(N,v) \cap C(N,v) \neq \emptyset$. Let $x \in ESOS(N,v) \cap C(N,v)$ and let $y \in ESOS(N,v)$. We show by induction that $R_k^x = R_k^y$ for all $k \in \{1,\ldots,|N|\}$. Clearly, $R_0^x = R_0^y$. Let $k \in \{1,\ldots,|N|\}$ and assume that $R_\ell^x = R_\ell^y$ for all $\ell < k$. Suppose that $R_{k-1}^x \neq N$. Since $x, y \in ESOS(N,v)$,

$$(R_k^x \setminus R_{k-1}^x), (R_k^y \setminus R_{k-1}^y) \in \operatorname*{argmax}_{S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}} \frac{v(S \cup R_{k-1}^x) - v(R_{k-1}^x)}{|S|}.$$

Since $x \in C(N,v)$, for all $T \in \operatorname{argmax}_{S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}} \frac{v(S \cup R_{k-1}^x) - v(R_{k-1}^x)}{|S|}$,

$$\sum_{i \in T} x_i = \sum_{i \in T \cup R_{k-1}^x} x_i - \sum_{i \in R_{k-1}^x} x_i \geq v(T \cup R_{k-1}^x) - v(R_{k-1}^x) = |T| a_k^x = \sum_{i \in T} a_k^x \geq \sum_{i \in T} x_i.$$

This means that

$$R_k^x \setminus R_{k-1}^x = \bigcup \operatorname*{argmax}_{S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}} \frac{v(S \cup R_{k-1}^x) - v(R_{k-1}^x)}{|S|}.$$

Suppose that $R_k^x \neq R_k^y$. Then

$$a_{k+1}^y = \max_{S \in 2^{N \setminus R_k^y} \setminus \{\emptyset\}} \frac{v(S \cup R_k^y) - v(R_k^y)}{|S|} \geq \frac{v(R_k^x) - v(R_k^y)}{|R_k^x \setminus R_k^y|} = \frac{|R_k^x \setminus R_k^y| a_k^y}{|R_k^x \setminus R_k^y|} = a_k^y.$$

This is a contradiction. Hence, $R_k^x = R_k^y$ for all $k \in \{1,\ldots,|N|\}$. Since $x, y \in ESOS(N,v)$, this implies that $x = y$. $\square$

[1]Klijn et al. (2000) called this the *bounded maximum payoff property* and Llerena and Mauri (2017) called this property *rich player feasibility*.

The following example shows that the converse of Lemma 1 does not hold, i.e. the equal split-off set does not necessarily intersect the core if it consists of one allocation.

Example 4
Let (N, v) with $N = \{1, 2, 3, 4\}$ be the game given by

$$v(S) = \begin{cases} 8 & \text{if } S = N; \\ 6 & \text{if } S = \{1, 2\}; \\ 5 & \text{if } S = \{1, 3\}; \\ 0 & \text{otherwise.} \end{cases}$$

The equal split-off set is $ESOS(N, v) = \{(3, 3, 1, 1)\}$ with corresponding $\{\{1, 2\}, \{3, 4\}\}$. Clearly, $|ESOS(N, v)| = 1$ and $ESOS(N, v) \nsubseteq C(N, v)$. We know that $(3, 3, 1, 1)$ Lorenz dominates each allocation in $C(N, v)$. $\triangle$

If the equal split-off set intersects the core, then it consists of the Lorenz dominating core allocation. Moreover, it is contained in the Weber set since the core is contained in the Weber set. We show that the equal split-off set is contained in the Weber set in general.

Theorem 1
Let (N, v) be a game. Then $ESOS(N, v) \subseteq W(N, v)$.

Proof. Let $x \in ESOS(N, v)$ with corresponding partition $\{T_1, \ldots, T_m\}$. Let $k \in \{1, \ldots, m\}$. Then $T_k \in \text{argmax}_{S \in 2^{N_k} \setminus \{\emptyset\}} \frac{v_k(S)}{|S|}$ and $x_i = \frac{v_k(T_k)}{|T_k|}$ for all $i \in T_k$. This means that

$$\sum_{i \in T_k} x_i = \sum_{i \in T_k} \frac{v_k(T_k)}{|T_k|} = |T_k| \frac{v_k(T_k)}{|T_k|} = v_k(T_k) = v_k|_{T_k}(T_k)$$

and for all $S \in 2^{T_k} \setminus \{\emptyset\}$,

$$\sum_{i \in S} x_i = \sum_{i \in S} \frac{v_k(T_k)}{|T_k|} \geq \sum_{i \in S} \frac{v_k(S)}{|S|} = |S| \frac{v_k(S)}{|S|} = v_k(S) = v_k|_{T_k}(S),$$

so $(x_i)_{i \in T_k} \in C(T_k, v_k|_{T_k})$. This implies that $(x_i)_{i \in T_k} \in W(T_k, v_k|_{T_k})$, i.e. there is $w^k \in \mathbb{R}_+^{\Pi^{T_k}}$ with $\sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} = 1$ such that for all $i \in T_k$, denoting $P_i^{\pi_k} = \{j \in T_k \mid \pi_k^{-1}(j) < \pi_k^{-1}(i)\}$,

$$\begin{aligned} x_i &= \sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} \mu_i^{\pi_k}(T_k, v_k|_{T_k}) \\ &= \sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} \left(v_k|_{T_k}(P_i^{\pi_k} \cup \{i\}) - v_k|_{T_k}(P_i^{\pi_k})\right) \\ &= \sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} \left(v_k(P_i^{\pi_k} \cup \{i\}) - v_k(P_i^{\pi_k})\right) \\ &= \sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} \left(\left(v(\bigcup_{\ell=1}^{k-1} T_\ell \cup P_i^{\pi_k} \cup \{i\}) - v(\bigcup_{\ell=1}^{k-1} T_\ell) \right) - \left(v(\bigcup_{\ell=1}^{k-1} T_\ell \cup P_i^{\pi_k}) - v(\bigcup_{\ell=1}^{k-1} T_\ell) \right) \right) \\ &= \sum_{\pi_k \in \Pi^{T_k}} w^k_{\pi_k} \left(v(\bigcup_{\ell=1}^{k-1} T_\ell \cup P_i^{\pi_k} \cup \{i\}) - v(\bigcup_{\ell=1}^{k-1} T_\ell \cup P_i^{\pi_k}) \right). \end{aligned}$$

Define $w \in \mathbb{R}_+^{\Pi^N}$ by

$$w_\pi = \begin{cases} \prod_{k=1}^m w_{\pi_k}^k & \text{if } \pi = (\pi_1, \ldots, \pi_m) \text{ with } \pi_k \in \Pi^{T_k} \text{ for each } k \in \{1, \ldots, m\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{\pi \in \Pi^N} w_\pi &= \sum_{\pi_1 \in \Pi^{T_1}} \cdots \sum_{\pi_m \in \Pi^{T_m}} \prod_{k=1}^m w_{\pi_k}^k \\ &= \sum_{\pi_1 \in \Pi^{T_1}} \cdots \sum_{\pi_m \in \Pi^{T_m}} w_{\pi_1}^1 \cdots w_{\pi_m}^m \\ &= \sum_{\pi_1 \in \Pi^{T_1}} w_{\pi_1}^1 \cdots \sum_{\pi_m \in \Pi^{T_m}} w_{\pi_m}^m \\ &= 1. \end{aligned}$$

Let $i \in N$ and let $k \in \{1, \ldots, m\}$ such that $i \in T_k$. Then

$$\begin{aligned} \sum_{\pi \in \Pi^N} w_\pi \mu_i^\pi(N, v) &= \sum_{\pi_1 \in \Pi^{T_1}} \cdots \sum_{\pi_m \in \Pi^{T_m}} w_{\pi_1}^1 \cdots w_{\pi_m}^m \mu_i^\pi(N, v) \\ &= \sum_{\pi_1 \in \Pi^{T_1}} w_{\pi_1}^1 \cdots \sum_{\pi_k \in \Pi^{T_k}} w_{\pi_k}^k \mu_i^{\pi_k}(T_k, v_k|_{T_k}) \cdots \sum_{\pi_m \in \Pi^{T_m}} w_{\pi_m}^m \\ &= \sum_{\pi_k \in \Pi^{T_k}} w_{\pi_k}^k \mu_i^{\pi_k}(T_k, v_k|_{T_k}) \\ &= x_i. \end{aligned}$$

Hence, $x \in W(N, v)$. □

A class of games for which the equal split-off set intersects the core is the class of convex games. For convex games, Branzei et al. (2006) showed that the equal split-off set coincides with the Dutta and Ray (1989) solution. In the next section, we study the equal split-off set on the full class of games for which it intersects the core.

4 Exact partition games

In this section, we axiomatically study the equal split-off set on the class of exact partition games and derive several characterizations based on various consistency notions. A game is an exact partition game if there is a core allocation for which all richest players whose assigned payoffs belong to the k highest ones can obtain their payoffs by themselves.

Definition 2 (cf. Llerena and Mauri (2017))
A game (N, v) is an *exact partition game* if there exists $x \in C(N, v)$ such that for each $k \in \{1, \ldots, |N|\}$, $\sum_{i \in R_k^x} x_i = v(R_k^x)$.

Let Γ_{exp} denote the class of all exact partition games. Clearly, $\Gamma_{conv} \subseteq \Gamma_{exp} \subseteq \Gamma_{all}$. We show that the equal split-off set intersects the core of a game if and only if it is an exact partition game.

Lemma 2
Let (N, v) be a game. Then $ESOS(N, v) \cap C(N, v) \neq \emptyset$ if and only if $(N, v) \in \Gamma_{exp}$.

Proof. Assume that $ESOS(N, v) \cap C(N, v) \neq \emptyset$. Let $x \in ESOS(N, v) \cap C(N, v)$. Since $x \in ESOS(N, v)$, $\sum_{i \in R_k^x} x_i = v(R_k^x)$ for each $k \in \{1, \dots, |N|\}$. Hence, (N, v) is an exact partition game.

Assume that (N, v) is an exact partition game. Let $x \in C(N, v)$ such that for each $k \in \{1, \dots, |N|\}$, $\sum_{i \in R_k^x} x_i = v(R_k^x)$. Let $k \in \{1, \dots, |N|\}$ such that $R_{k-1}^x \neq N$. Then $a_k^x = \frac{v(R_k^x) - v(R_{k-1}^x)}{|R_k^x \setminus R_{k-1}^x|}$. Since $x \in C(N, v)$, for all $S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}$,

$$a_k^x = \frac{\sum_{i \in S} a_k^x}{|S|} \geq \frac{\sum_{i \in S} x_i}{|S|} = \frac{\sum_{i \in S \cup R_{k-1}^x} x_i - \sum_{i \in R_{k-1}^x} x_i}{|S|} \geq \frac{v(S \cup R_{k-1}^x) - v(R_{k-1}^x)}{|S|}.$$

This means that

$$R_k^x \setminus R_{k-1}^x \in \operatorname*{argmax}_{S \in 2^{N \setminus R_{k-1}^x} \setminus \{\emptyset\}} \frac{v(S \cup R_{k-1}^x) - v(R_{k-1}^x)}{|S|}.$$

Hence, $x \in ESOS(N, v)$. □

The following egalitarian stability property plays a central role throughout this section.

Equal division stability
for each $x \in \sigma(N, v)$ and each $S \in 2^N \setminus \{\emptyset\}$, there is $i \in S$ such that $x_i \geq \frac{v(S)}{|S|}$.

Equal division stability states that no coalition should be better off by equally dividing the worth among its members. By Lemma 1 and Lemma 2, the equal split-off set satisfies equal division stability on the class of exact partition games since it is contained in the core.

Llerena and Mauri (2017) formulated the open question whether the characterizations of the Dutta and Ray (1989) solution for convex games obtained by Klijn et al. (2000) can be extended to the class of exact partition games. Klijn et al. (2000) reformulated the characterizations of Dutta (1990) based on the consistency principle. Consider a cooperative game for which we apply a certain solution to allocate the worth of the grand coalition among the players. Some players leave with their assigned payoffs and the remaining players reevaluate their payoffs on the basis of a reduced game. The solution is consistent if it prescribes the same allocation for the reduced game as for the original game. Davis and Maschler (1965) defined the worth of a coalition in such a reduced game as the maximal joint surplus in cooperation with any subgroup of departed players when these departed players are assigned their solution payoffs. Peleg (1986) used the corresponding consistency axiom, to which we refer as max-consistency, in a characterization of the core. Klijn et al. (2000) used a weaker version which only requires consistent allocations when all richest players leave, to which we refer as rich-restricted max-consistency, in combination with feasible richness and equal division stability to characterize the Dutta and Ray (1989) solution.

Rich-restricted max-consistency
for each $x \in \sigma(N, v)$ with $R_1^x \neq N$, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{max}^x)$, where

$$v_{max}^x(S) = \begin{cases} v(N) - \sum_{i \in R_1^x} x_i & \text{if } S = N \setminus R_1^x; \\ \max_{Q \subseteq R_1^x} \{v(S \cup Q) - \sum_{i \in Q} x_i\} & \text{if } \emptyset \subset S \subset N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Theorem (cf. Klijn et al. (2000))
The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted max-consistency.

Llerena and Mauri (2017) showed that the equal split-off set satisfies rich-restricted max-consistency on the class of exact partition games. We show that this characterization in terms of nonemptiness, feasible richness, equal division stability, and rich-restricted max-consistency can be extended to the class of exact partition games. This result is stronger than the result of Llerena and Mauri (2017) in which the axiom *core selection* is used instead of the weaker equal division stability axiom. The proof is provided in the Appendix.

Theorem 2
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted max-consistency.

The empty solution, which assigns to any exact partition game the empty set, satisfies feasible richness, equal division stability, and rich-restricted max-consistency, but violates nonemptiness. The solution which coincides with the core for any two-player game, and coincides with the equal split-off set for any other exact partition game, satisfies nonemptiness, equal division stability, and rich-restricted max-consistency, but violates feasible richness. The equal division solution, which divides the worth of the grand coalition in any exact partition game equally among the players, satisfies nonemptiness, feasible richness, and rich-restricted max-consistency, but violates equal division stability. The solution which assigns $(5, 4, 0)$ to the game in Example 1, and coincides with the equal split-off set for any other exact partition game, satisfies nonemptiness, feasible richness, and equal division stability, but violates rich-restricted max-consistency. This means that the properties in Theorem 2 are independent.

Hart and Mas-Colell (1989) proposed a reduced game in which the worth of a coalition is defined as the total payoff to the members in the joint subgame with all the departed players. Dutta (1990) defined the corresponding consistency axiom for general multi-valued solutions, to which we refer as self-consistency. Klijn et al. (2000) showed that the corresponding rich-restricted version can replace rich-restricted max-consistency in their characterization of the Dutta and Ray (1989) solution.

Rich-restricted self-consistency
for each $x \in \sigma(N, v)$ with $R_1^x \neq N$, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{self}^{f^x})$ for a selector f^x of σ, where

$$v_{self}^{f^x}(S) = \sum_{i \in S} f_i^x(S \cup R_1^x, v|_{S \cup R_1^x}) \text{ for all } S \subseteq N \setminus R_1^x.$$

Theorem (cf. Klijn et al. (2000))
The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted self-consistency.

As the following example shows, the equal split-off set violates rich-restricted self-consistency on the class of exact partition games, since subgames of exact partition games are not necessarily exact partition games. This means that the characterization of Klijn et al. (2000) involving rich-restricted self-consistency cannot be extended to this domain.

Example 5
Let $(N,v) \in \Gamma_{exp}$ with $N = \{1,2,3\}$ be the exact partition game from Example 1. Then $ESOS(N,v) = \{(5,3,1)\}$, $R_1^{(5,3,1)} = \{1\}$, and for any selector $f^{(5,3,1)}$ of $ESOS$,

$$v_{self}^{f^{(5,3,1)}}(\{3\}) = f_3^{(5,3,1)}(\{1,3\}, v|_{\{1,3\}}),$$

where

$$v|_{\{1,3\}}(S) = \begin{cases} 5 & \text{if } S = \{1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Since $C(\{1,3\}, v|_{\{1,3\}}) = \emptyset$, we have $(\{1,3\}, v|_{\{1,3\}}) \notin \Gamma_{exp}$. This means that the equal split-off set violates rich-restricted self-consistency on the class of exact partition games. △

Klijn et al. (2000) proposed a third type of reduced game in which the worth of a coalition is defined as the joint marginal contribution to the departed players. We refer to the corresponding consistency axiom as marginal-consistency. Klijn et al. (2000) showed that rich-restricted marginal-consistency can replace feasible richness and rich-restricted max- or self-consistency in the characterization of the Dutta and Ray (1989) solution.

Rich-restricted marginal-consistency
for each $x \in \sigma(N,v)$ with $R_1^x \neq N$, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{marg}^x)$, where

$$v_{marg}^x(S) = v(S \cup R_1^x) - v(R_1^x) \text{ for all } S \subseteq N \setminus R_1^x.$$

Theorem (cf. Klijn et al. (2000))
The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying nonemptiness, equal division stability, and rich-restricted marginal-consistency.

We show that this characterization can be extended to the class of exact partition games.

Theorem 3
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, equal division stability, and rich-restricted marginal-consistency.

The empty solution satisfies equal division stability and rich-restricted marginal-consistency, but violates nonemptiness. The equal division solution satisfies nonemptiness and rich-restricted marginal-consistency, but violates equal division stability. The core satisfies nonemptiness and equal division stability, but violates rich-restricted marginal-consistency. This means that the properties in Theorem 3 are independent.

To our knowledge, the rather specific marginal-consistency has never been applied elsewhere in the literature. We show that rich-restricted marginal-consistency is in fact stronger than the rich-restricted variant of the well-known consistency axiom proposed by Moulin (1985), to which we refer as complement-consistency. In the corresponding reduced game, the worth of a coalition is defined as the joint surplus in cooperation with all departed players when they are assigned their solution payoffs.

Rich-restricted complement-consistency
for each $x \in \sigma(N,v)$ with $R_1^x \neq N$, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x)$, where

$$v_{comp}^x(S) = \begin{cases} v(S \cup R_1^x) - \sum_{i \in R_1^x} x_i & \text{if } \emptyset \subset S \subseteq N \setminus R_1^x; \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Lemma 3
If a solution satisfies rich-restricted marginal-consistency, then it satisfies rich-restricted complement-consistency.

Proof. Let $\Gamma \subseteq \Gamma_{all}$ be a class of games and let σ be a solution on Γ. Assume that σ satisfies rich-restricted marginal-consistency. Let $(N, v) \in \Gamma$ and let $x \in \sigma(N, v)$ such that $R_1^x \neq N$. By rich-restricted marginal-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{marg}^x)$. For all $S \subseteq N \setminus R_1^x$,

$$\begin{aligned} v_{comp}^x(S) &= v(S \cup R_1^x) - \sum_{i \in R_1^x} x_i &&= v(S \cup R_1^x) - \left(\sum_{i \in N} x_i - \sum_{i \in N \setminus R_1^x} x_i \right) \\ &= v(S \cup R_1^x) - \big(v(N) - v_{marg}^x(N \setminus R_1^x)\big) &&= v(S \cup R_1^x) - (v(N) - (v(N) - v(R_1^x))) \\ &= v(S \cup R_1^x) - v(R_1^x) &&= v_{marg}^x(S). \end{aligned}$$

This means that $v_{comp}^x = v_{marg}^x$ and $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x)$. Hence, σ satisfies rich-restricted complement-consistency. □

The core violates rich-restricted marginal-consistency, but satisfies complement-consistency. Tadenuma (1992) used complement-consistency in a characterization of the core. We derive an alternative characterization of the equal split-off set on the class of exact partition games in terms of nonemptiness, feasible richness, equal division stability, and rich-restricted complement-consistency.

Theorem 4
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted complement-consistency.

The empty solution satisfies feasible richness, equal division stability, and rich-restricted complement-consistency, but violates nonemptiness. The solution which coincides with the core for any two-player game, and coincides with the equal split-off set for any other exact partition game, satisfies nonemptiness, equal division stability, and rich-restricted complement-consistency, but violates feasible richness. The equal division solution satisfies nonemptiness, feasible richness, and rich-restricted complement-consistency, but violates equal division stability. The solution which assigns $(5, 4, 0)$ to the game in Example 1, and coincides with the equal split-off set for any other exact partition game, satisfies nonemptiness, feasible richness, and equal division stability, but violates rich-restricted complement-consistency. This means that the properties in Theorem 4 are independent.

Since complement-reduced convex games are convex games, the result of Theorem 4 is also valid on the class of convex games, providing a new characterization of the Dutta and Ray (1989) solution.

Theorem 5
The Dutta and Ray (1989) solution is the unique solution on Γ_{conv} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted complement-consistency.

5 Arbitrary games

In this section, we study the equal split-off set on the class of all transferable utility games. On this domain, equal division stability is incompatible with nonemptiness, which implies that the equal split-off set violates equal division stability. However, the equal split-off set generally does satisfy a weak form of equal division stability.

Weak equal division stability

for each $x \in \sigma(N,v)$, there are $T \in \text{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$ and $i \in T$ such that $x_i \geq \frac{v(T)}{|T|}$.

Weak equal division stability states that there should be a coalition with maximal average worth which is not better off by equally dividing the worth among its members. Clearly, the equal split-off set satisfies weak equal division stability on the class of all games.

In fact, all characterizations on the class of exact partition games in the previous section can be strengthened by weakening equal division stability to weak equal division stability. This does not imply that these stronger characterizations are also valid on the class of all games. As the following example shows, the equal split-off set violates rich-restricted max-consistency on the class of all games.

Example 6

Let $(N,v) \in \Gamma_{all}$ with $N = \{1,2,3,4\}$ be the game from Example 4. Then $ESOS(N,v) = \{(3,3,1,1)\}$, $R_1^{(3,3,1,1)} = \{1,2\}$, and

$$v_{max}^{(3,3,1,1)}(S) = \begin{cases} 2 & \text{if } S \in \{\{3\},\{3,4\}\}; \\ 0 & \text{otherwise.} \end{cases}$$

This means that $ESOS(\{3,4\}, v_{max}^{(3,3,1,1)}) = \{(2,0)\}$, so $(1,1) \notin ESOS(\{3,4\}, v_{max}^{(3,3,1,1)})$. Hence, the equal split-off set violates rich-restricted max-consistency on the class of all games. △

Surprisingly, in contrast to the class of exact partition games, the equal split-off set does satisfy rich-restricted self-consistency on the class of all games. Moreover, we show that all solutions satisfying feasible richness, weak equal division stability, and rich-restricted self-consistency necessarily prescribe equal split-off set allocations. To our knowledge, this is the first axiomatic result of a multi-valued solution involving self-consistency.

Lemma 4

The equal split-off set satisfies rich-restricted self-consistency on Γ_{all}.

Proof. Let $(N,v) \in \Gamma_{all}$ and let $x \in ESOS(N,v)$ such that $R_1^x \neq N$. Then $\sum_{i \in R_1^x} x_i = v(R_1^x)$ and $R_1^x \in \text{argmax}_{T \in 2^N \setminus \{\emptyset\}} \frac{v(T)}{|T|}$. Let $S \subseteq N \setminus R_1^x$. Then

$$R_1^x \in \underset{T \in 2^{S \cup R_1^x} \setminus \{\emptyset\}}{\text{argmax}} \frac{v|_{S \cup R_1^x}(T)}{|T|}.$$

This means that there is a selector f^x of $ESOS$ such that $f_i^x(S \cup R_1^x, v|_{S \cup R_1^x}) = x_i$ for all $i \in R_1^x$. Let f^x be such a selector of $ESOS$. Then

$$\begin{aligned} v_{self}^{f^x}(S) &= \sum_{i \in S} f_i^x(S \cup R_1^x, v|_{S \cup R_1^x}) \\ &= \sum_{i \in S \cup R_1^x} f_i^x(S \cup R_1^x, v|_{S \cup R_1^x}) - \sum_{i \in R_1^x} f_i^x(S \cup R_1^x, v|_{S \cup R_1^x}) \\ &= v(S \cup R_1^x) - \sum_{i \in R_1^x} x_i \\ &= v(S \cup R_1^x) - v(R_1^x). \end{aligned}$$

This means that $x_{N \setminus R_1^x} \in ESOS(N \setminus R_1^x, v_{self}^{f^x})$. Hence, the equal split-off set satisfies rich-restricted self-consistency on Γ_{all}. □

Theorem 6
The equal split-off set is the maximal solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted self-consistency.

The solution which coincides with the core for any game with at most two players, and coincides with the empty set for any other game, satisfies equal division stability and rich-restricted self-consistency, but violates feasible richness. The equal division solution satisfies feasible richness and rich-restricted self-consistency, but violates weak equal division stability. The solution which assigns $(5, 4, 0)$ to the game in Example 1, and coincides with the equal split-off set for any other game, satisfies feasible richness and weak equal division stability, but violates rich-restricted self-consistency. This means that the properties in Theorem 6 are independent.

Whereas max-consistency and complement-consistency played symmetric roles in the characterizations on the class of exact partition games, the equal split-off set satisfies rich-restricted complement-consistency on the class of all games although it violates rich-restricted max-consistency. Moreover, all solutions satisfying feasible richness, weak equal division stability, and rich-restricted complement-consistency necessarily select from the equal split-off set.

Theorem 7
The equal split-off set is the maximal solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted complement-consistency.

The core satisfies equal division stability and rich-restricted complement-consistency, but violates feasible richness. The equal division solution satisfies feasible richness and rich-restricted complement-consistency, but violates weak equal division stability. The solution which assigns $(5, 4, 0)$ to the game in Example 1, and coincides with the equal split-off set for any other game, satisfies feasible richness and weak equal division stability, but violates rich-restricted complement-consistency. This means that the properties in Theorem 7 are independent.

Similar to the results in the previous section, rich-restricted marginal-consistency can replace feasible richness and rich-restricted complement-consistency in the characterization of the equal split-off set.

Theorem 8
The equal split-off set is the maximal solution on Γ_{all} satisfying weak equal division stability and rich-restricted marginal-consistency.

The equal division solution satisfies rich-restricted marginal-consistency, but violates weak equal division stability. The core satisfies equal division stability, but violates rich-restricted marginal-consistency. This means that the properties in Theorem 8 are independent.

The equal split-off set of Branzei et al. (2006) can be considered as a marginal-reduced equal split-off set which coincides with the complement-reduced equal split-off set in the family of reduced equal split-off sets described by Llerena and Mauri (2016). Future research could characterize this full family or axiomatically compare the original equal split-off set with other equal split-off sets in this family, in particular the max-reduced equal split-off set.

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Appendix

Theorem 2
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted max-consistency.

Proof. Clearly, the equal split-off set satisfies nonemptiness and feasible richness on Γ_{exp}. By Lemma 1 and Lemma 2, the equal split-off set satisfies equal division stability on Γ_{exp}. Llerena and Mauri (2017) showed that the equal split-off set satisfies rich-restricted max-consistency on Γ_{exp}.

Let σ be a solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted max-consistency. We show by induction on the number of players that $\sigma(N,v)$ consists of one uniquely defined allocation for all $(N,v) \in \Gamma_{exp}$. By nonemptiness, $\sigma(N,v) = \{v(N)\}$ for all $(N,v) \in \Gamma_{exp}$ with $|N| = 1$. Let $k \in \mathbb{N}$ and assume that $\sigma(N,v)$ consists of one uniquely defined allocation for all $(N,v) \in \Gamma_{exp}$ with $|N| \leq k$. Let $(N,v) \in \Gamma_{exp}$ with $|N| = k+1$. By nonemptiness, there is $x \in \sigma(N,v)$. By equal division stability, $a_1^x \geq \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. By feasible richness,

$$a_1^x = \frac{|R_1^x| a_1^x}{|R_1^x|} = \frac{\sum_{i \in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i \in R_1^x} x_i}{|R_1^x|} \leq \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}.$$

This means that $a_1^x = \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \operatorname{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$.

Denote $U^v = \bigcup \operatorname{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. Since (N,v) is an exact partition game, Lemma 1 and Lemma 2 imply that $U^v \in \operatorname{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. Suppose that $R_1^x \neq U^v$. By rich-restricted max-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{max}^x)$. By equal division stability, there is $i \in U^v \setminus R_1^x$ such that

$$x_i \geq \frac{v_{max}^x(U^v \setminus R_1^x)}{|U^v \setminus R_1^x|} \geq \frac{v(U^v) - \sum_{i \in R_1^x} x_i}{|U^v \setminus R_1^x|} = \frac{|U^v| a_1^x - |R_1^x| a_1^x}{|U^v \setminus R_1^x|} = \frac{|U^v \setminus R_1^x| a_1^x}{|U^v \setminus R_1^x|} = a_1^x.$$

This is a contradiction, so $R_1^x = U^v$. If $R_1^x = U^v = N$, then $\sigma(N,v)$ consists of one uniquely defined allocation x. Suppose that $R_1^x = U^v \neq N$. By rich-restricted max-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{max}^x)$, where $\sigma(N \setminus R_1^x, v_{max}^x)$ consists of one uniquely defined allocation since $|N \setminus R_1^x| \leq k$. Hence, $\sigma(N,v)$ consists of one uniquely defined allocation. □

Theorem 3
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, equal division stability, and rich-restricted marginal-consistency.

Proof. Clearly, the equal split-off set satisfies nonemptiness and rich-restricted marginal-consistency on Γ_{exp}. By Lemma 1 and Lemma 2, the equal split-off set satisfies equal division stability on Γ_{exp}.

Let σ be a solution on Γ_{exp} satisfying nonemptiness, equal division stability, and rich-restricted marginal-consistency. We show by induction on the number of players that $\sigma(N,v)$ consists of one uniquely defined allocation for all $(N,v) \in \Gamma_{exp}$. By nonemptiness, $\sigma(N,v) = \{v(N)\}$ for all $(N,v) \in \Gamma_{exp}$ with $|N| = 1$. Let $k \in \mathbb{N}$ and assume that $\sigma(N,v)$ consists of one uniquely defined allocation for all $(N,v) \in \Gamma_{exp}$ with $|N| \leq k$. Let $(N,v) \in \Gamma_{exp}$ with $|N| = k+1$. By nonemptiness, there is $x \in \sigma(N,v)$.

By equal division stability, $a_1^x \geq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. If $R_1^x = N$, then

$$a_1^x = \frac{|N|a_1^x}{|N|} = \frac{\sum_{i\in N} a_1^x}{|N|} = \frac{\sum_{i\in N} x_i}{|N|} = \frac{v(N)}{|N|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}.$$

By rich-restricted marginal-consistency, if $R_1^x \neq N$, then

$$\begin{aligned} a_1^x &= \frac{|R_1^x|a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} x_i}{|R_1^x|} = \frac{\sum_{i\in N} x_i - \sum_{i\in N\setminus R_1^x} x_i}{|R_1^x|} \\ &= \frac{v(N) - v_{marg}^x(N\setminus R_1^x)}{|R_1^x|} = \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}. \end{aligned}$$

This means that $a_1^x = \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$.

Denote $U^v = \bigcup \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. Since (N, v) is an exact partition game, Lemma 1 and Lemma 2 imply that $U^v \in \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. Suppose that $R_1^x \neq U^v$. By rich-restricted marginal-consistency, $x_{N\setminus R_1^x} \in \sigma(N\setminus R_1^x, v_{marg}^x)$. By equal division stability, there is $i \in U^v \setminus R_1^x$ such that

$$x_i \geq \frac{v_{marg}^x(U^v\setminus R_1^x)}{|U^v\setminus R_1^x|} = \frac{v(U^v) - v(R_1^x)}{|U^v\setminus R_1^x|} = \frac{|U^v|a_1^x - |R_1^x|a_1^x}{|U^v\setminus R_1^x|} = \frac{|U^v\setminus R_1^x|a_1^x}{|U^v\setminus R_1^x|} = a_1^x.$$

This is a contradiction, so $R_1^x = U^v$. If $R_1^x = U^v = N$, then $\sigma(N, v)$ consists of one uniquely defined allocation x. Suppose that $R_1^x = U^v \neq N$. By rich-restricted marginal-consistency, $x_{N\setminus R_1^x} \in \sigma(N\setminus R_1^x, v_{marg}^x)$, where $\sigma(N\setminus R_1^x, v_{marg}^x)$ consists of one uniquely defined allocation since $|N\setminus R_1^x| \leq k$. Hence, $\sigma(N, v)$ consists of one uniquely defined allocation. □

Theorem 4
The equal split-off set is the unique solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted complement-consistency.

Proof. Clearly, the equal split-off set satisfies nonemptiness and feasible richness. By Lemma 1 and Lemma 2, the equal split-off set satisfies equal division stability on Γ_{exp}. By Lemma 3, the equal split-off set satisfies rich-restricted complement-consistency on Γ_{exp} since it satisfies rich-restricted marginal-consistency.

Let σ be a solution on Γ_{exp} satisfying nonemptiness, feasible richness, equal division stability, and rich-restricted complement-consistency. We show by induction on the number of players that $\sigma(N, v)$ consists of one uniquely defined allocation for all $(N, v) \in \Gamma_{exp}$. By nonemptiness, $\sigma(N, v) = \{v(N)\}$ for all $(N, v) \in \Gamma_{exp}$ with $|N| = 1$. Let $k \in \mathbb{N}$ and assume that $\sigma(N, v)$ consists of one uniquely defined allocation for all $(N, v) \in \Gamma_{exp}$ with $|N| \leq k$. Let $(N, v) \in \Gamma_{exp}$ with $|N| = k+1$. By nonemptiness, there is $x \in \sigma(N, v)$. By equal division stability, $a_1^x \geq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. By feasible richness,

$$a_1^x = \frac{|R_1^x|a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} x_i}{|R_1^x|} \leq \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}.$$

This means that $a_1^x = \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$.

Denote $U^v = \bigcup \text{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. Since (N, v) is an exact partition game, Lemma 1 and Lemma 2 imply that $U^v \in \text{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. Suppose that $R_1^x \neq U^v$. By rich-restricted complement-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x)$. By equal division stability, there is $i \in U^v \setminus R_1^x$ such that

$$x_i \geq \frac{v_{comp}^x(U^v \setminus R_1^x)}{|U^v \setminus R_1^x|} = \frac{v(U^v) - \sum_{i \in R_1^x} x_i}{|U^v \setminus R_1^x|} = \frac{|U^v| a_1^x - |R_1^x| a_1^x}{|U^v \setminus R_1^x|} = \frac{|U^v \setminus R_1^x| a_1^x}{|U^v \setminus R_1^x|} = a_1^x.$$

This is a contradiction, so $R_1^x = U^v$. If $R_1^x = U^v = N$, then $\sigma(N, v)$ consists of one uniquely defined allocation x. Suppose that $R_1^x = U^v \neq N$. By rich-restricted complement-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{comp}^x)$, where $\sigma(N \setminus R_1^x, v_{comp}^x)$ consists of one uniquely defined allocation since $|N \setminus R_1^x| \leq k$. Hence, $\sigma(N, v)$ consists of one uniquely defined allocation. □

Theorem 6
The equal split-off set is the maximal solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted self-consistency.

Proof. Clearly, the equal split-off set satisfies feasible richness and weak equal stability on Γ_{all}. By Lemma 4, the equal split-off set satisfies self-consistency on Γ_{all}.

Let σ be a solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted self-consistency. We show by induction on the number of players that $\sigma(N, v) \subseteq ESOS(N, v)$ for all $(N, v) \in \Gamma_{all}$. For all $(N, v) \in \Gamma_{all}$ with $|N| = 1$, $\sigma(N, v) = \emptyset$ or $\sigma(N, v) = \{v(N)\}$, so $\sigma(N, v) \subseteq ESOS(N, v) = \{v(N)\}$. Let $k \in \mathbb{N}$ and assume that $\sigma(N, v) \subseteq ESOS(N, v)$ for all $(N, v) \in \Gamma_{all}$ with $|N| \leq k$. Let $(N, v) \in \Gamma_{all}$ with $|N| = k+1$. If $\sigma(N, v) = \emptyset$, then $\sigma(N, v) \subseteq ESOS(N, v)$. Suppose that $\sigma(N, v) \neq \emptyset$ and let $x \in \sigma(N, v)$. By equal division stability, $a_1^x \geq \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. By feasible richness,

$$a_1^x = \frac{|R_1^x| a_1^x}{|R_1^x|} = \frac{\sum_{i \in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i \in R_1^x} x_i}{|R_1^x|} \leq \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}.$$

This means that $a_1^x = \max_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \text{argmax}_{S \in 2^N \setminus \{\emptyset\}} \frac{v(S)}{|S|}$. If $R_1^x = N$, then $\sigma(N, v) \subseteq ESOS(N, v)$. Suppose that $R_1^x \neq N$. By rich-restricted self-consistency, $x_{N \setminus R_1^x} \in \sigma(N \setminus R_1^x, v_{self}^f)$ for a selector f of σ, where $\sigma(N \setminus R_1^x, v_{self}^f) \subseteq ESOS(N \setminus R_1^x, v_{self}^f)$ since $|N \setminus R_1^x| \leq k$. Hence, $\sigma(N, v) \subseteq ESOS(N, v)$. □

Theorem 7
The equal split-off set is the maximal solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted complement-consistency.

Proof. Clearly, the equal split-off set satisfies feasible richness and weak equal stability on Γ_{all}. By Lemma 3, the equal split-off set satisfies rich-restricted complement-consistency on Γ_{all} since it satisfies rich-restricted marginal-consistency.

Let σ be a solution on Γ_{all} satisfying feasible richness, weak equal division stability, and rich-restricted complement-consistency. We show by induction on the number of players that $\sigma(N, v) \subseteq ESOS(N, v)$ for all $(N, v) \in \Gamma_{all}$. For all $(N, v) \in \Gamma_{all}$ with $|N| = 1$, $\sigma(N, v) = \emptyset$ or $\sigma(N, v) = \{v(N)\}$, so $\sigma(N, v) \subseteq ESOS(N, v) = \{v(N)\}$. Let $k \in \mathbb{N}$ and assume that $\sigma(N, v) \subseteq ESOS(N, v)$ for all $(N, v) \in \Gamma_{all}$ with $|N| \leq k$. Let $(N, v) \in \Gamma_{all}$ with $|N| = k+1$. If $\sigma(N, v) = \emptyset$, then $\sigma(N, v) \subseteq ESOS(N, v)$. Suppose that $\sigma(N, v) \neq \emptyset$ and let $x \in \sigma(N, v)$.

By equal division stability, $a_1^x \geq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. By feasible richness,

$$a_1^x = \frac{|R_1^x|a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} x_i}{|R_1^x|} \leq \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}.$$

This means that $a_1^x = \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. If $R_1^x = N$, then $\sigma(N,v) \subseteq ESOS(N,v)$. Suppose that $R_1^x \neq N$. By rich-restricted complement-consistency, $x_{N\setminus R_1^x} \in \sigma(N\setminus R_1^x, v_{comp}^x)$, where $\sigma(N\setminus R_1^x, v_{comp}^x) \subseteq ESOS(N\setminus R_1^x, v_{comp}^x)$ since $|N\setminus R_1^x| \leq k$. Hence, $\sigma(N,v) \subseteq ESOS(N,v)$. □

Theorem 8
The equal split-off set is the maximal solution on Γ_{all} satisfying weak equal division stability and rich-restricted marginal-consistency.

Proof. Clearly, the equal split-off set satisfies weak equal division stability and rich-restricted marginal-consistency on Γ_{all}.

Let σ be a solution on Γ_{all} satisfying weak equal division stability and rich-restricted marginal-consistency. We show by induction on the number of players that $\sigma(N,v) \subseteq ESOS(N,v)$ for all $(N,v) \in \Gamma_{all}$. For all $(N,v) \in \Gamma_{all}$ with $|N| = 1$, $\sigma(N,v) = \emptyset$ or $\sigma(N,v) = \{v(N)\}$, so $\sigma(N,v) \subseteq ESOS(N,v) = \{v(N)\}$. Let $k \in \mathbb{N}$ and assume that $\sigma(N,v) \subseteq ESOS(N,v)$ for all $(N,v) \in \Gamma_{all}$ with $|N| \leq k$. Let $(N,v) \in \Gamma_{all}$ with $|N| = k+1$. If $\sigma(N,v) = \emptyset$, then $\sigma(N,v) \subseteq ESOS(N,v)$. Suppose that $\sigma(N,v) \neq \emptyset$ and let $x \in \sigma(N,v)$. By weak equal division stability, $a_1^x \geq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. If $R_1^x = N$, then

$$a_1^x = \frac{|N|a_1^x}{|N|} = \frac{\sum_{i\in N} a_1^x}{|N|} = \frac{\sum_{i\in N} x_i}{|N|} = \frac{v(N)}{|N|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}.$$

By rich-restricted marginal-consistency, if $R_1^x \neq N$, then

$$\begin{aligned} a_1^x &= \frac{|R_1^x|a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} a_1^x}{|R_1^x|} = \frac{\sum_{i\in R_1^x} x_i}{|R_1^x|} = \frac{\sum_{i\in N} x_i - \sum_{i\in N\setminus R_1^x} x_i}{|R_1^x|} \\ &= \frac{v(N) - v_{marg}^x(N\setminus R_1^x)}{|R_1^x|} = \frac{v(R_1^x)}{|R_1^x|} \leq \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}. \end{aligned}$$

This means that $a_1^x = \max_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$ and $R_1^x \in \text{argmax}_{S\in 2^N\setminus\{\emptyset\}} \frac{v(S)}{|S|}$. If $R_1^x = N$, then $\sigma(N,v) \subseteq ESOS(N,v)$. Suppose that $R_1^x \neq N$. By rich-restricted marginal-consistency, $x_{N\setminus R_1^x} \in \sigma(N\setminus R_1^x, v_{marg}^x)$, where $\sigma(N\setminus R_1^x, v_{marg}^x) \subseteq ESOS(N\setminus R_1^x, v_{marg}^x)$ since $|N\setminus R_1^x| \leq k$. Hence, $\sigma(N,v) \subseteq ESOS(N,v)$. □