TILBURG UNIVERSITY

TILEC

TILEC Discussion Paper

DP 2004-002

Compensating Losses and Sharing Surpluses in Project-allocation Situations

By
Yuan Ju, Pieter H.M. Ruys and Peter Borm

ISSN 1572-4042

Compensating Losses and Sharing Surpluses in Project-allocation Situations[1]

Yuan Ju[2] Pieter H.M. Ruys[3] Peter Borm[4]

January 2004

[1]The authors thank Robert P. Gilles and Youwei Li for stimulating conversations and valuable comments. For helpful discussions on early drafts, we also thank the seminar participants at Tilburg University, Université de Toulouse 1, the 14th International Conference on Game Theory at Stony Brook, the EEA-ESEM Conference in Stockholm and the 30th EARIE Conference in Helsinki. Of course, the authors are responsible for the remaining deficiencies of the paper. The financial support from TWM is gratefully acknowledged.

[2]CentER for Economic Research and Department of Econometrics and Operations Research, Tilburg University, Warandelaan 2, P.O.Box 90153, 5000 LE, Tilburg, the Netherlands. E-mail: Y.Ju@uvt.nl

[3]TILEC, Department of Econometrics and Operations Research, and Tias Business School, Tilburg University. E-mail: ruys@uvt.nl

[4]CentER for Economic Research and Department of Econometrics and Operations Research, Tilburg University, E-mail: P.E.M.Borm@uvt.nl

Abstract

By introducing the notions of projects and shares, this paper studies a class of economic environments, the so-called project-allocation situations, in which society may profit from cooperation, i.e., by reallocating the initial shares of projects among agents. This paper mainly focuses on the associated issues of compensation of losses and surplus sharing arising from the reallocation of projects. For this purpose, we construct and analyze an associated project-allocation game and a related system of games that explicitly models the underlying cooperative process. Specific solution concepts are proposed.

JEL classification codes: C71, H70
Keywords: projects-allocation situations/games; loss compensation; surplus sharing.

1 Introduction

This paper has two aims. Firstly, it develops a general framework for studying a class of economic environments in which a coalition of agents cooperates in a combination of projects: project-allocation situations. Secondly, within this framework, we analyze the problems associated with a possible reshuffling of projects among agents, which asks for a compensation of losses due to reshuffling and a proposal on sharing the surpluses arising from enhanced efficiency.

In an economy characterized by changing capabilities and preferences of agents and changing technology embodied in projects, people need to continuously adapt their positions to obtain efficiency. That is how our society has evolved into nowadays prosperity. Every change in the production structure requires a reshuffling of responsibilities, which is hard to implement if possible "losers" are not sufficiently compensated to cooperate. When all parties gain from the change in the end, a win-win situation can be achieved. Examples are abundant. Similarly, when extra profit is generated simply by cooperation after reshuffling, a surplus sharing problem occurs. Obviously, solving the problems of compensation and surplus sharing is essential for creating and maintaining flexibility and creating efficiency in an economy. However, generally speaking, the two concepts are not well distinguished in theoretical research so that the corresponding practical problems can not be treated adequately. In a strict sense, compensation refers to a financial remuneration to an agent for the loss caused by her being removed from some project. On the other hand, surplus sharing deals with the extra benefits created by cooperation among agents assigned to some combination of projects, which benefits are in excess of the sum of individual payoffs. One may observe that trade unions have obtained generic rules for laborers that include some compensation for lay-offs in a firm, as well as labor laws and other safety nets on the macro-level. Our paper focuses on the micro-level. We assume that gains and losses for every particular situation can be endogenously specified and may serve as a basis for the issues of compensation of losses and surplus sharing.

Consider, for example, a restaurant and a boat company, working independently, both situated on the shore of the same beautiful lake. The restaurant is operated by agent 1 who can be understood as a group of managers, waiters and kitchen staff. Agent 2, a group of people as well, manages the boat company. They are considering collaboration and have two proposals. The first one is simply setting up a joint lunch-sightseeing program that will benefit both parties. The second proposal is more involved and induces a reshuffling of the two projects, i.e. the restaurant and the boat company. Since agent 2 has excellent expertise in both travelling and restaurant management, much more profit will be generated if the restaurant is also managed by her. Whereas the first proposal only entails a surplus sharing problem, the second one is further complicated by the problem of compensating agent 1 for giving up her user rights of the restaurant project.

This paper analyzes both the loss compensation and surplus sharing problem as illustrated by the second proposal in the above example from a cooperative game theoretic point of view. In our framework, the value of some coalition of players crucially depends on the involvement of the players in this coalition in a well-defined set of projects. The involvement is measured by the notion of a player's shares in projects. That defines a so-called *project-allocation situation* (in short, P-A situation) and an associated *project-allocation game* (in short, P-A game). The value function of the project-allocation game is derived from the underlying profit functions for every coalition given a specific share profile of the projects. So in particular, the value function of this game can be viewed as a generalization of the neoclassical profit function, with labor and capital as inputs and with prices given.

Naturally, any specific solution concept for a cooperative TU (transferable utility) game may of course be applied to solve project-allocation games, and implicitly solve the *combined* loss compensation and surplus sharing problems. We restrict our attention to two additive one-point solution concepts: the Shapley value (Shapley (1953)) and the *consensus value* (cf. Ju, Borm and Ruys (2004)). Arguments for the suitability of these rules in this specific context are provided.

However, since P-A games are just a partial abstraction of P-A situations, this traditional approach is incapable of disentangling all necessary details to adequately model the basic mechanisms concerning the physical reallocation of projects (loss compensation) and co-working in joint projects (surplus sharing). In fact, the process to realize the maximal gain of a coalition is a blackbox. Therefore, in order to make the framework operational for solving practical problems, one further step has to be made. By explicitly incorporating an underlying cooperative structure in terms of project reallocation and cooperation after reallocation, we devise different stages in such a way that the loss compensation due to project reallocation and the sharing of extra surplus from cooperation can be clearly and logically distinguished. For each of the stages, a game is constructed. Consistently, the same solution concept is applied to each of the stage games. Thus, following a general stage approach, also a solution concept for the combined problem is obtained. Finally, an example of public-private partnerships is provided, which indicates an interesting application of the framework into real economic issues.

Although there exists some fundamental work that is helpful for our research, it seems that the problem of compensation has largely been ignored in economic research. The analysis of cost sharing situations (cf. Moulin (1987), Tijs and Branzei (2002)) and linear production situations (Owen (1975)) is in the same spirit but does not explicitly discriminate between the problems of surplus sharing and loss compensation. An exception in a somewhat different context is the the work on sequencing games (Curiel et al (1989), Hamers (1995), Klijn (2000)). In this framework, time slots could be considered as projects. Agents change the initial order (shares or rights on time slots) into an optimal one so that the individual payoffs are changed and compensation is needed. Moreover, since joint total costs decrease as well, also the issue of surplus sharing becomes prominent.

In addition to this section introducing the problem and reviewing the literature briefly, the remaining part of the paper is structured as follows. In the next section, we present the main analytical framework by formally introducing project-allocation situations. Section 3 addresses the project-allocation games and possible solution concepts. Section 4 distinguishes stages to explicitly solve the problems of compensation and surplus sharing in project-allocation situations separately. The final section illustrates our analysis with an economic example.

2 Project-allocation situations

Let us consider a situation in which there is a finite set N of agents/players who can operate a finite set M of *projects*.

We use the word "project" in this paper in a very general sense. A project is a specific entity that can be exploited or operated for some purpose(s) (and mostly for value-creation). It can be a machine, a research project, a firm, or a public utility, etc. Generally, a project can either be *divisible* or *indivisible*. A project is divisible if it is capable of being separated into parts and can be partially operated or owned by some party, without loss of its original function. For instance, a tree farm as a project can be perfectly divided among players. Indivisibility means that for the purpose of value-creation, a project can only be completely owned or exploited as a whole. A truck is then an indivisible project because it will lose the basic function if divided into parts. Since divisibility is a context-dependent concept which may imply physical divisibility, operational divisibility or ownership divisibility, we have to point out that this paper focuses on the operational divisibility.

Players are characterized by *specialities* and *shares* in projects. Specialities of a player are her competence or productivity in exploiting projects. Different players have different specialities. With some speciality(-ies), one player may have a relatively favorable position, or some advantage in working on some project compared with the others. For example, two researchers co-work on a joint research report that consists of two projects: the theoretical modelling and the empirical work. Researcher 1 may be good in the first project but not acquainted with the latter. Researcher 2 has sound experience in empirical studies but is not as good as 1 at the theoretical level. Then, 1 and 2 have different specialities in those two projects. As a result, 1 has advantage in the former while 2 has advantage in the latter.

A player's share[1] in a project, denoted by $\rho_{i,k}$ (i.e. player i's share in project k), is a number between 0 and 1 representing the fraction of the project owned by that player. If a divisible project is fully owned by one player, then her share in that project is 1; if half of it, the share is then 0.5. In the case of indivisibility, the share in a project can only be 0 or 1. But if two or more players jointly own an indivisible project, what are the individual shares?

For analytical convenience, we define *fictitious divisibility* as below: for any indivisible project $k \in M$, suppose it is controlled by a coalition S, where $S \in 2^N$, then player i's share in project k is $\rho_{i,k} = 1/|S|$, for all $i \in S$. Thus, if an indivisible project is owned by a coalition, equal shares in the project can be "fictitiously" distributed among players. This allows for describing each player's share in any indivisible project and for solving the problems of compensation and surplus sharing in such cases mathematically. That is, despite that an indivisible project can not be divided in itself, the value generated from it can be equally shared among the players who jointly own it. Furthermore, although in principle indivisible projects are different from divisible projects, this definition helps to incorporate indivisible projects into the analytical framework of divisible projects.

Hence, all the players' shares in the projects at one time form a *share profile*, denoted by ρ, which is an $N \times M$ matrix[2] describing an assignment of projects among players. Then, we denote the set of share profiles by

$$R := \left\{ \rho \in [0,1]^{N\times M} | \sum_{i=1}^{n} \rho_{i,k} \leq 1, \forall k \in M \right\} \tag{1}$$

For any share profile $\rho \in R$, there exists a map $f^\rho : 2^N \longrightarrow \mathbb{R}$ assigning to any coalition $S \in 2^N$ a real number such that $f^\rho(\emptyset) = 0$. The function f^ρ is called the payoff function under share profile ρ. $f^\rho(S)$ is called the payoff of coalition S under share profile ρ.

[1] Here, we do not restrict the implications of *shares*, or in another sentence, we do not give a definite economic interpretation but only care about how the shares (in a general sense) in projects that players have will affect cooperation or even determine compensation and surplus sharing. It may have different meanings in different contexts. For example, it can represent the ownership/property rights or managerial rights.

[2] Given player set N of size n and project set M of size m, ρ is in fact an $n \times m$ matrix. We use $N \times M$ to emphasize that ρ is a matrix associated to player set and project set. The same explanation applies in cases of other matrices, for instance, when we say that ρ_S is an $S \times M$ matrix.

We assume that there is an initial share profile $\rho^0 \in R$ specifying the initial allocation scheme of projects to players in this situation.

Based on the above description, formally, a *project-allocation situation* is denoted by $(N, M, R, \rho^0, \{f^\rho\}_{\rho \in R})$, where N is the set of players, M is the set of projects, R is the set of share profiles, ρ^0 is the initial share profile, and $f^\rho : 2^N \longrightarrow \mathbb{R}$ is the payoff function under a share profile $\rho \in R$.

For analytical convenience, we make the following assumptions.

Assumption 1 (continuity)*:*
for any $S \in 2^N$, $f^\rho(S)$ is continuous with respect to parameter ρ.

Assumption 2 (no externality among coalitions)*:*
$f^{\rho^1}(S) = f^{\rho^2}(S)$ *for all* $S \in 2^N$ *whenever* $\rho^1_S = \rho^2_S$.

Here ρ_S is the $S \times M$ submatrix of ρ.

Assumption 3 (gains from cooperation)*:*
$f^\rho(S \cup T) \geq f^\rho(S) + f^\rho(T)$ *for all* $\rho \in R$ *and for all* $S, T \in 2^N$ *with* $S \cap T = \emptyset$.

Assumption 4 (ordinary cooperation)*:*
$f^{\rho^1}(S) \geq f^{\rho^2}(S) \Rightarrow \sum_{i \in S} f^{\rho^1}(\{i\}) \geq \sum_{i \in S} f^{\rho^2}(\{i\})$ *for all* $\rho^1, \rho^2 \in R$ *and for all* $S \in 2^N$.

The last assumption means that if a share profile is preferable for a coalition of players in cooperation, then the corresponding pre-cooperation situation is also preferable in terms of the sum of their individual payoffs.

The class of all project-allocation situations with player set N and project set M satisfying the above assumptions is denoted by $PAS^{N,M}$.

In a project-allocation situation, players may increase the joint payoff by reshuffling projects: the (maximal) payoffs of a coalition S depend on the set of feasible reallocation profiles. We call a profile $\rho \in R$ *feasible for S* with respect to ρ^0 if it satisfies the following two conditions:

(1) $\rho_{N \setminus S} = \rho^0_{N \setminus S}$;

(2) $\sum_{i \in S} \rho_{i,k} = \sum_{i \in S} \rho^0_{i,k}$ for all $k \in M$.

Given ρ^0, the set of feasible profiles for a coalition S is denoted by $\mathcal{F}(S, \rho^0)$. For notational simplicity, we use $\mathcal{F}(S)$ if there is no confusion about ρ^0. Thus, feasibility means that players can re-arrange their shares in projects subject to the capacity determined by the initial share profile within the coalition they participate in without affecting the allocations outside of the coalition.

3 Project-allocation games and cooperative solution concepts

In this section, loss compensation and surplus sharing problems will be investigated in a combined way within a cooperative game framework from a traditional (one-stage) point of view.

By defining the value of a coalition as the maximal payoff that a coalition can achieve by means of feasible share profiles, we obtain a standard cooperative game with transferable utility called a *project-allocation game*. Formally, for a project-allocation situation $\mathcal{P} = (N, M, R, \rho^0, \{f^\rho\}_{\rho \in R}) \in PAS^{N,M}$, the corresponding project-allocation game (N, v) is defined by

$$v(S) := \max_{\rho \in \mathcal{F}(S)} f^\rho(S) \tag{2}$$

for all coalitions $S \in 2^N \backslash \{\emptyset\}$ and $v(\emptyset) = 0$. A share profile $\rho \in \mathcal{F}(S)$ with $f^\rho(S) = v(S)$ is called an *optimal share profile* for coalition S, denoted by $\rho^*(S)$.

Proposition 3.1 *Project-allocation games are superadditive.*

Proof. Let $(N, M, R, \rho^0, \{f^\rho\}_{\rho \in R})$ be a project-allocation situation and let the corresponding P-A game be given by (N, v). We need to show $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.

Consider optimal share profiles $\rho^*(S \cup T)$, $\rho^*(S)$ and $\rho^*(T)$ for coalitions $S \cup T$, S, T, respectively. Since $S \cap T = \emptyset$, we can construct a new share profile $\tilde{\rho} \in \mathcal{F}(S \cup T)$ such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, by definition and Assumption 2 and 3, we have

$$\begin{aligned} & v(S \cup T) \\ = \; & f^{\rho^*(S \cup T)}(S \cup T) \\ \geq \; & f^{\tilde{\rho}}(S \cup T) \\ \geq \; & f^{\tilde{\rho}}(S) + f^{\tilde{\rho}}(T) \\ = \; & f^{\rho^*(S)}(S) + f^{\rho^*(T)}(T) \\ = \; & v(S) + v(T) \end{aligned}$$

■

We want to note that project-allocation games are not necessarily convex games. Such an example is easy to get.

We will consider two related solution concepts: the Shapley value Φ and *the consensus value* γ.

Recall that

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v)$$

for all $v \in TU^N$. Here $\Pi(N)$ denotes the set of all bijections $\sigma : \{1, 2, ..., |N|\} \longrightarrow N$ of N and the marginal vector $m^{\sigma}(v) \in \mathbb{R}^N$, for $\sigma \in \Pi(N)$, is defined by

$$m^{\sigma}_{\sigma(k)} := v(\{\sigma(1), ..., \sigma(k)\}) - v(\{\sigma(1), ..., \sigma(k-1)\})$$

for all $k \in \{1, ..., |N|\}$.

When we go over the definition and the properties of the Shapley value, we can find that it may not be entirely adequate to analyze the project-allocation situations mainly by the following two reasons.

Firstly, the Shapley value relies on the basic notion of marginal vectors. Here, given some ordering of players entering a game, the payoffs are determined by the marginal contributions, which is not satisfying in a constructive or bargaining type of physical setting since a later entrant gets the whole surplus. In a superadditive game, the incumbents will not accept such an arrangement as their contributions are not reflected. While if a game is subadditive, the entrant will not accept such a contract. Apparently, in the practice of a project-allocation situation, a marginal vector is even harder to implement as it may involve reshuffling of projects by current incumbents.

Secondly, the dummy property does not seem too imperative in P-A situations. The informational requirements in a P-A situation are rather hard: it is difficult to recognize dummies from the basic description of the situation.

We propose an alternative solution concept for TU games: the consensus value. This rule follows from a natural and simple idea to share coalition values. For more information including an axiomatic characterization of this solution concept, we refer to Ju, Borm and Ruys (2004).

Consider the following 3-player example. We first have two players: 1 and 2. They cooperate with each other and form a coalition $\{1, 2\}$. The coalition value $v(\{1, 2\})$ is generated. Suppose now player 3 enters the scene, who would like to cooperate with player 1 and 2. But because the coalition $\{1, 2\}$ had been already formed before she entered the game, player 3 will actually cooperate with the existing coalition $\{1, 2\}$ instead of simply cooperating with 1 and 2 individually (Consensus is needed here). If $\{1, 2\}$ agrees as well, the coalition value $v(\{1,2,3\})$ will be generated. How to share it between $\{3\}$ and $\{1, 2\}$? Generally, no rule is better than splitting the joint surplus $v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\})$ equally and assigning half to each of the two parties in addition to their own values (Consensus is obtained once again). Then, what remains (the so-called *remainder*) for $\{1,2\}$ is $\frac{1}{2}(v(\{1,2,3\})+v(\{1,2\})-v(\{3\}))$. Apparently, 1 and 2 will share this remainder in a similar way: besides their individual values, each of them gets $\frac{1}{2}\left(\frac{1}{2}(v(\{1,2,3\})+v(\{1,2\})-v(\{3\}))-v(\{1\})-v(\{2\})\right)$. Extending this argument[3] to an n-player case, we then have a general method, which can be understood as a *standardized remainder rule* as we take the 2-player game standard solution as a base and apply it to solving games by taking the existing coalition as one player. Furthermore, since no order is pre-determined for a TU game, we take all possible ordering of players into account and average the corresponding outcomes, which serves as the final payoff for players.

Formally, this rule is defined as follows. For a given $\sigma \in \Pi(N)$ and $i \in \{1,2,...,n\}$ we define $\mathcal{S}_i^\sigma = \{\sigma(1),\sigma(2),...,\sigma(i)\}$ and $\mathcal{S}_0^\sigma = \emptyset$. We then define the recurrence relation

$$r(\mathcal{S}_i^\sigma) = \begin{cases} \frac{1}{2}\left(r(\mathcal{S}_{i+1}^\sigma)+v(\mathcal{S}_i^\sigma)-v(\{\sigma(i+1)\})\right) & \text{if } i \in \{1,2,...,n-1\} \\ v(N) & \text{if } i = n \end{cases}$$

where $r(\mathcal{S}_i^\sigma)$ is the *standardized remainder* for coalition $\mathcal{S}_i$ in a given order $\sigma \in \Pi(N)$: the value left for $\mathcal{S}_i^\sigma$ after allocating surplus to later entrants $\mathcal{S}_{-i}^\sigma$.

We construct the *individual standardized remainder vector* $s^\sigma(v)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1),\sigma(2),...,\sigma(n)$ and assign each player, besides her individual payoff $v(\{i\})$, half of the net surplus from the standardized remainder she[4] creates by joining the group of players already present. Formally, it is the vector in $\mathbb{R}^N$ defined by

$$s_{\sigma(i)}^\sigma = \begin{cases} \frac{1}{2}\left(r(\mathcal{S}_i^\sigma)-v(\mathcal{S}_{i-1}^\sigma)+v(\{\sigma(i)\})\right) & \text{if } i \in \{2,...,n\} \\ r(\mathcal{S}_1^\sigma) & \text{if } i = 1 \end{cases}$$

The consensus value is equal to the average of the individual standardized remainder vectors, i.e.

$$\gamma(v) = \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} s^\sigma(v).$$

A more descriptive name for the consensus value could be *the average serial standardized remainder value*. In the same spirit, an alternative name for the Shapley value could be *the average serial marginal contribution value*.

[3]Indeed, this argument is based on a backward process. Alternatively, we can construct a forward process to model the idea, which yields the same result. The consistency is provided in Ju, Borm and Ruys (2004).

[4]Exactly speaking, the value of the corresponding standardized remainder is created by her (the new entrant) and the existing coalition.

By Ju, Borm and Ruys (2004), surprisingly, the consensus value is in fact a linear combination of the Shapley value and the *equal surplus* solution:

$$\gamma_i(v) = \frac{1}{2}\Phi_i(v) + \frac{1}{2}\left(\frac{v(N) - \sum_{j\in N} v(\{j\})}{n} + v(\{i\})\right)$$

for all $i \in N$, where $\Phi_i(v)$ is the Shapley value of the game.

Example 3.2 *Consider a P-A situation* $\mathcal{P} = (N, M, R, \rho^0, \{f^\rho\}_{\rho\in R}) \in PAS^{N,M}$ *where* $N := \{1,2,3\}$, $M := \{A, B\}$, *and*

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
f^\rho(\{1\}) &= 10\rho_{1,A} + 1\rho_{1,B} + \rho_{1,A}\rho_{1,B} \\
f^\rho(\{2\}) &= 8\rho_{2,A} + 3\rho_{2,B} + \rho_{2,A}\rho_{2,B} \\
f^\rho(\{3\}) &= 6\rho_{3,A} + 5\rho_{3,B} + \rho_{3,A}\rho_{3,B} \\
f^\rho(\{1,2\}) &= \sum_{i\in\{1,2\}} f^\rho(\{i\}) + 3\left(\sum_{i\in\{1,2\}}\sum_{k\in M} \rho_{i,k}\right) \\
f^\rho(\{1,3\}) &= \sum_{i\in\{1,3\}} f^\rho(\{i\}) + 4\left(\sum_{i\in\{1,3\}}\sum_{k\in M} \rho_{i,k}\right) \\
f^\rho(\{2,3\}) &= \sum_{i\in\{2,3\}} f^\rho(\{i\}) + 2\left(\sum_{i\in\{2,3\}}\sum_{k\in M} \rho_{i,k}\right) \\
f^\rho(\{1,2,3\}) &= \sum_{i\in\{1,2,3\}} f^\rho(\{i\}) + 6\left(\sum_{i\in\{1,2,3\}}\sum_{k\in M} \rho_{i,k}\right)
\end{aligned}$$

It is easy to see that these payoff functions satisfy Assumption 1-4. The corresponding P-A game is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$v(S)$	0	12	0	19	0	17	27

with $\gamma(v) = \left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$ *and* $\Phi(v) = \left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$.

In our opinion, the consensus value well fits the admission structure of the P-A situations. Consider an existing coalition S and a new entrant i. Ex ante, S is a well formed coalition: they had reallocated projects with each other and now cooperate well; they also share the joint surplus in some way. Now, player i would join this coalition. What happens? Obviously, i could not work with any sub-coalition of S but only with S as a whole since S has already been formed over there, comparable to the case that two players cooperate. The immediate (standard) solution would then be to share the extra revenues from the cooperation equally between S and i.

However, the approach to model the whole P-A situation as one cooperative game is not completely satisfying: P-A games do not take all practical features of P-A situations into account. The underlying process of realizing and allocating the maximal gain of the grand coalition $f^{\rho^*(N)}(N)$ starting from the individual payoffs is still a blackbox.

4 A stage approach to compensation and surplus sharing problems in project-allocation situations

We now focus on an underlying process of obtaining and redistributing the maximal payoff of the grand coalition $f^{\rho^*(N)}(N)$ in a project-allocation situation $\mathcal{P} = (N, M, R, \rho^0, \{f^\rho\}_{\rho \in R})$.

- Stage 1: The *initial state*, ρ^0, is a static and natural state, in which no interactive behavior happens and players keep their shares as given. This state yields an *initial value distribution*

  $$\pi^0 := \left(f^{\rho^0}(\{i\})\right)_{i \in N}$$

- Stage 2: This stage consists of *project reallocation* towards the *optimal share profile* $\rho^*(N)$ for the grand coalition N. Since in the new profile $\rho^*(N)$, any player $i \in N$ has a new assignment of projects, $\rho_i^*(N)$, it immediately yields a new "direct" payoff: $f^{\rho^*(N)}(\{i\})$, which corresponds to the *direct reallocation value distribution*:

  $$\pi^* := \left(f^{\rho^*(N)}(\{i\})\right)_{i \in N}$$

  Taking coalitional behavior into account, we construct a *reallocating* game (N, w) by defining $w(S) = \sum_{i \in S} f^{\rho^*(S)}(\{i\})$, where $\rho^*(S) = \arg\max_{\rho \in \mathcal{F}(S, \rho^0)} f^\rho(S)$. So, in particular, $w(N) = \sum_{i \in N} \pi_i^*$.

Proposition 4.1 *Reallocating games are superadditive.*

Proof. Let $\mathcal{P} = (N, M, R, \rho^0, \{f^\rho\}_{\rho\in R})$ be a project-allocation situation and let the corresponding reallocating game be given by (N, w). We need to show $w(S\cup T) \geq w(S) + w(T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.

Let $\rho^*(S\cup T)$, $\rho^*(S)$ and $\rho^*(T)$ be the optimal share profiles for coalitions $S\cup T$, S, T, respectively. Let $\tilde{\rho} \in \mathcal{F}(S\cup T)$ be such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, we have

$$\begin{aligned}
& w(S\cup T) \\
= & \sum_{i\in S\cup T} f^{\rho^*(S\cup T)}(\{i\}) \\
\geq & \sum_{i\in S\cup T} f^{\tilde{\rho}}(\{i\}) \\
= & \sum_{i\in S} f^{\tilde{\rho}}(\{i\}) + \sum_{i\in T} f^{\tilde{\rho}}(\{i\}) \\
= & \sum_{i\in S} f^{\rho^*(S)}(\{i\}) + \sum_{i\in T} f^{\rho^*(T)}(\{i\}) \\
= & w(S) + w(T)
\end{aligned}$$

Here, the inequality follows from the fact $f^{\rho^*(S\cup T)}(S\cup T) \geq f^{\tilde{\rho}}(S\cup T)$ and Assumption 4. ■

In general, $\pi^* \neq \pi^0$; and apparently, the players incurred losses due to project reallocation need to be compensated. For this we will consider solutions of the associated *compensation* game $(N, \bar{w})$ by defining $\bar{w}(S) = w(S) - \sum_{i\in S}\pi^*_i$. Note that $\bar{w}(N) = 0$.

- Stage 3: This stage considers cooperation after the reallocation in the second stage, i.e. co-working on projects based on the optimal share profile $\rho^*(N)$. This type of cooperation yields a *co-working* game (N, ω) defined as the project-allocation game corresponding to a project-allocation situation $(N, M, R, \rho^*(N), \{f^\rho\}_{\rho\in R})$.

Proposition 4.2 *Co-working games are superadditive.*

Proof. Apparently, co-working games are superadditive as they are project-allocation games. ■

Indeed, this game still takes project reallocation into consideration so that players are allowed to reallocate shares before joint production. However, the initial share profile itself in this situation is the optimal share profile and $\omega(N) = v(N) = f^{\rho^*(N)}(N)$, players do not reallocate projects in the grand coalition any more (although it may happen in theory within sub-coalitions) but directly work with each other with their current shares. So no compensation is needed. What entails is only surplus sharing. For this aspect, we consider solutions of the associated *surplus sharing* game $(N, \bar{\omega})$ given by $\bar{\omega}(S) = \omega(S) - \sum_{i \in S} \pi_i^*$. It is obvious that $\sum_{i \in N} \pi_i^* + \bar{w}(N) + \bar{\omega}(N) = v(N)$.

The above description on the three stages in a project-allocation situation implies a natural and reasonable way to share the maximal payoff $f^{\rho^*(N)}(N)$. Firstly, a player i has her direct reallocation value due to optimal project reallocation. In addition, to determine the compensations, one solves the compensation game $\bar{w}$; and to solve the surplus sharing problem, one solves the surplus sharing game $\bar{\omega}$. A player's final payoff is the sum of these three parts.

Solving both games with the same one-point solution concept yields a (stage-based) consistent one-point solution concept for P-A situations:

$$\psi_i^*(\mathcal{P}) := \pi_i^* + \psi_i(\bar{w}) + \psi_i(\bar{\omega})$$

where $\psi : TU^N \longrightarrow \mathbb{R}^N$ is a one-point solution concept for TU games.

Generally, it will be the case that the immediate application of ψ to the P-A game will yield a different solution, i.e., $\psi_i^*(\mathcal{P}) \neq \psi_i(v)$. Then, it is natural to ask under what condition both approaches might give the same result. We require two weak conditions on ψ, i.e., $\psi(0) = 0$ and translation invariance $\psi(v + b) = \psi(v) + b$ for all $v \in TU^N$ and $b \in \mathbb{R}^N$ (b is an additive game), and strengthen Assumption 3 in the following way.

Assumption 3' *f is additive with respect to coalitions, i.e. $f^\rho(S \cup T) = f^\rho(S) + f^\rho(T)$ for all $\rho \in R$ and for all $S, T \in 2^N$ with $S \cap T = \emptyset$.*

Now, formally, we have

Proposition 4.3 *With Assumption 3', if ψ satisfies translation invariance and $\psi(0) = 0$, then $\psi_i^*(\mathcal{P}) = \psi_i(v)$, where ψ, $\mathcal{P}$ and v are defined as above.*

Proof. Clearly, assumption 3' implies that $f^{\rho^*(N)}(N) = \sum_{i \in N} f^{\rho^*(N)}(\{i\})$. Consequently, $w(S) = v(S)$ for all $S \in 2^N$ and $\bar{\omega}(S) = 0$ for all $S \in 2^N$. What remains is obvious: $\psi^*(\mathcal{P}) = \psi(v)$. ■

In particular, $\Phi^*(\mathcal{P})$ of a project-allocation situation $\mathcal{P} \in PAS^{N,M}$ is given by

$$\Phi_i^*(\mathcal{P}) = \pi_i^* + \Phi_i(\bar{w}) + \Phi_i(\bar{\omega}) \text{ for all } i \in N$$

Similarly, $\gamma^*(\mathcal{P})$ of a project-allocation situation $\mathcal{P} \in PAS^{N,M}$ is given by

$$\gamma_i^*(\mathcal{P}) = \pi_i^* + \gamma_i(\bar{w}) + \gamma_i(\bar{\omega}) \text{ for all } i \in N$$

5 An example: disintegration in the water sector

Let us look at an example considering the reform of disintegration and reallocation in the water sector. In this setting, we have three players $N := \{1, 2, 3\}$: player 1 is a provincial government, 2 is a local government, and 3 is a company; two projects: water business (A) and related business (B) such as a golf club or recreation park built on the water source land. So, $M := \{A, B\}$. Initially, both projects are owned by the local government. Consequently, the initial share profile is

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Unlike the for-profit project B, water business is usually seen as a public utility. So, the payoff of running the water project can be interpreted as the social welfare/value instead of individual profit. Moreover, we assume that the company has speciality in operating a commercial business while the provincial government may create higher social value if she controls the water project. However, they do have some relative weaknesses. For example, the private company is not good at running public utilities. This type of situation is modelled by the corresponding payoff functions, which are provided in Example 3.2.

Without cooperation, players' individual payoffs come from two parts: the stand-alone payoffs generated from project A or B and the payoff due to the cross-subsidy effect between two projects. With cooperation, in addition to players' individual payoffs, there are some extra gains from cooperation, which, for instance, is expressed by $2\left(\sum_{i\in\{2,3\}}\sum_{k\in M}\rho_{i,k}\right)$ in the payoff function of coalition $\{2, 3\}$.

The optimal reform plans for the various coalitions will be: $\rho^*(\{1\}) = \rho^0$;

$$\rho^*(\{1,2\}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \rho^*(\{1,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \rho^*(\{2,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for grand coalition $\{1, 2, 3\}$,

$$\rho^*(\{1,2,3\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One readily checks that $\pi^0 := (0, 12, 0)$, and $\pi^* = (10, 0, 5)$.

Moreover, beside the project-allocation game (N, v) for the whole situation in this example, we have a compensation game and a surplus sharing game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$v(S)$	0	12	0	19	0	17	27
$\bar{w}(S)$	-10	12	-5	3	-15	8	0
$\bar{\omega}(S)$	0	0	0	3	8	2	12

The solutions based on the Shapley value or the consensus value can be found in the following two tables.

$\Phi(v)$	$\left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$
π^*	$(10, 0, 5)$
$\Phi(\bar{w})$	$\left(-9\frac{1}{6}, 13\frac{1}{3}, -4\frac{1}{6}\right)$
$\Phi(\bar{\omega})$	$\left(5\frac{1}{6}, 2\frac{1}{6}, 4\frac{2}{3}\right)$
$\Phi^*(\mathcal{P})$	$\left(6, 15\frac{1}{2}, 5\frac{1}{2}\right)$

$\gamma(v)$	$\left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$
π^*	$(10, 0, 5)$
$\gamma(\bar{w})$	$\left(-9\frac{1}{12}, 13\frac{1}{6}, -4\frac{1}{12}\right)$
$\gamma(\bar{\omega})$	$\left(4\frac{7}{12}, 3\frac{1}{12}, 4\frac{1}{3}\right)$
$\gamma^*(\mathcal{P})$	$\left(5\frac{1}{2}, 16\frac{1}{4}, 5\frac{1}{4}\right)$

Hence, according to the consensus value, the local government is compensated by the provincial government and the company with a total amount $13\frac{1}{6}$ due to project reallocation, and obtains $3\frac{1}{12}$ from the joint surplus generated by joint production based on the new share profile.

References

[1] Curiel, I., G. Pederzoli, and S. Tijs (1989), Sequencing Games, *European Journal of Operational Research*, 40, 344-351.

[2] Hamers, H. (1995), *Sequencing and Delivery Situations: a Game Theoretic Approach*, CentER Dissertation Series, Tilburg, The Netherlands.

[3] Ju, Y., P. Borm, and P.H.M. Ruys (2004), The Consensus Value: a New Solution Concept for Cooperative Games, *mimeo*.

[4] Klijn, F. (2000), *A Game Theoretic Approach to Assignment Problems*, CentER Dissertation Series, Tilburg, The Netherlands.

[5] Moulin, H. (1987), Egalitarian-Equivalent Cost Sharing of a Public Good, *Econometrica*, 55, 4, 963-976.

[6] Owen, G. (1975), On the Core of Linear Production Games, *Mathematical Programming*, 9, 358-370.

[7] Shapley, L.S., (1953), A Value for n-Person Games, In: Kuhn, H., Tucker, A.W. (Eds.), *Contributions to the Theory of Games*, Vol. II. Princeton University Press, Princeton, pp. 307-317.

[8] Tijs, S. and R. Branzei (2002), Cost Sharing in a Joint Project, *ZiF Working Paper*, IMW WP No. 336, Institute of Mathematical Economics, Universität Bielefeld.