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Abstract

We develop social learning environments as a unified framework for modeling and analyzing opinion dynamics. A social learning environment consists of a finite set of individuals and, for every individual, a set of possible opinions and an opinion updating rule. We establish conditions that guarantee the existence of an equilibrium and the convergence of learning paths to an equilibrium. Our framework unifies a wide range of models from the literature, including the voter, DeGroot, and Sznajd models, as well as the majority rule and a variation we call individual majority rule. This unification reveals a common underlying structure that explains the stability of these seemingly distinct models. Building on this foundation, we extend the theory to adaptive networks, in which network topology and opinion dynamics change simultaneously through a feedback loop. In particular, we introduce a novel class of adaptive networks called selective diffusion networks, where opinions are about the benefits provided by other individuals, and link formation is costly. We show that equilibria are absorbing when stability of opinions coincides with consensus within network components. Lastly, we develop potential functions as an alternative method for analyzing convergence. We provide an order-theoretic characterization of potential functions, which leads to a systematic method for their construction, and establish formal connections between potentials and conditions on social learning environments.

Keywords: Social learning environment, opinion dynamics, potentials, equilibrium.

JEL Classification: C62, D83, D85.

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1 Introduction

The study of how ideas spread over a society, known as opinion dynamics, is a foundational topic that bridges disciplines such as economics, sociology, computer science, and political science. The literature is vast, and there are a variety of modeling techniques, each presenting its own challenges and methods. For example, Bayesian learning models assume individuals are Bayesian learners who observe others in society and form their opinions by conditioning on a prior distribution of possible opinions, potentially centered around the true state of the world (Banerjee, 1992; Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011). This strand of literature generally relies on probability theory. In contrast, non-Bayesian models assume only that individuals use heuristic rules to form their opinions (Jadbabaie, Molavi, Sandroni, and Tahbaz-Salehi, 2012; Grabisch and Rusinowska, 2020; Noorazar, 2020). For instance, in the canonical model of DeGroot (1974), all agents simultaneously update their opinions by averaging the opinions of their neighbors in a predetermined, fixed network topology, using weights that reflect trust or influence. The theory of Markov chains is used to analyze the long-term behavior of the system.

This paper develops a general framework for analyzing opinion dynamics. We introduce the concept of a social learning environment, which consists of a set of individuals, and for every individual a set of possible opinions and an opinion updating rule. Each individual's opinion updating rule depends on the opinions of others and is allowed to be stochastic. A learning path consists of a sequence of opinion profiles that could emerge in the social learning environment. Consequently, an equilibrium in a social learning environment is an opinion profile that is not subject to change. The framework of a social learning environment, learning paths, and equilibrium are related to the analysis of social environments in Demuynck, Herings, Saulle, and Seel (2019) by means of dominance correspondences and the myopic stable set.

The concepts of increase-proof and decrease-proof opinion profiles are fundamental to the analysis presented in this paper. An opinion profile is said to be increase-proof if the updating of the opinions of the individuals can only lower opinions. Similarly, an opinion profile is decrease-proof if opinion updating can only result in higher opinions. Therefore, an opinion profile is an equilibrium if it is both increase-proof and decrease-proof. This alternative formulation of equilibrium is the key to the introduction of structured social learning environments. A social learning environment is said to be structured if, for every opinion profile that is increase-proof but not in equilibrium, there exists an opinion update that leads to an increase-proof opinion profile. We show that structured social learning environments are absorbing, i.e., from every opinion profile there is a learning path ending in an equilibrium. We first identify a key class of environments, which we call positively monotonic, where opinion updating rules are deterministic and non-decreasing functions of opinions in the society. We prove that any positively monotonic social learning environment is structured. Then, we demonstrate that many well-known models in the literature as discussed in Castellano, Fortunato, and Loreto (2009) lead to structured social learning environments. Examples are the voter model of Holley and Liggett (1975), the Degroot model (De Groot, 1974), the Keynesian beauty contest (Ledoux, 1981), and the median rule. Furthermore, we introduce the individual majority rule—a novel variation of the standard majority rule (Galam 2002)—as another key instance of a structured social learning environment.

For a social environment, we assume opinions to be represented by real numbers and individuals to update their opinion one at a time. In a generalized social learning environment, the set of possible opinions is an arbitrary partially ordered set and the individual updating rules are replaced by a transition correspondence. The transition correspondence indicates all the possible transitions that can occur from a given opinion profile. To extend the analysis of social learning environment to generalized social learning environments, we need another property,

called cohesiveness. Cohesiveness essentially requires that if there is a transition at an opinion profile where at least one opinion increases, then there is also a transition at that opinion profile where for every individual the opinion does not change or increases. We show that when a cohesive generalized social learning environment is structured, then it is absorbing. We demonstrate that the Sznajd model (Sznajd-Weron, 2005) and the majority rule (Galam, 2002) are cohesive generalized social learning environments which are structured and therefore absorbing.

Additionally, we contribute to the literature on adaptive networks, in which the network topology evolves alongside opinions through feedback mechanisms. Individuals update their opinions using the opinions of their friends, but at the same time adjust their friendships on the basis of their opinions. The adaptive networks literature has found applications in diverse fields such as neuroscience, physics, epidemiology, machine learning, and sociology (Berner, Gross, Kuehn, Kurths, and Yanchuk, 2023; Ehrhardt, Marsili, and Vega-Redondo, 2006; Bolletta and Pin, 2025; Gross and Blasius, 2008; Herrera, Cosenza, Tucci, and González-Avella, 2011; Rogers and Gross, 2013).

We model an adaptive network as a generalized social learning environment, where the set of possible opinions of an individual is no longer just about opinions, but also takes into account the set of friends of that individual. A transition from one state to another state either amounts to an individual opinion update or the addition of a friendship relation or the removal of a friend. A special case of an adaptive network is a selective diffusion network, where every individual has an opinion about the benefits provided by any other individual as well as costs to maintain a relationship with another individual. In this setting, a link between two individuals can be created if both think the benefits exceed the costs. A link between two individuals can be severed if for at least one of them the costs exceed the opinion about the benefits. While updating their opinions, individuals can only observe the opinions of their neighbors. Therefore, changes in the underlying network topology affects the opinion dynamics, which affect the topological dynamics. A selective diffusion network is shown to be absorbing if the opinions regarding each individual are structured and opinions are stable if and only if there is consensus within components in the network. These properties are satisfied if opinion updating takes place in accordance with the DeGroot model and the voter model.

Finally, we propose potentials as an alternative method to analyze the convergence of opinion dynamics. Potentials are commonly employed in game theory (Monderer and Shapley, 1996) to demonstrate the convergence of better-reply dynamics to Nash equilibria. The theory of potential games has grown rapidly and has recently been extended to social environments, including networks and matching markets (Rosenthal, 1973; Chwe, 1994; Chakrabarti and Gilles, 2007; Demuynck, Herings, Saulle, and Seel, 2019; Demuynck, Herings, and Seel, 2023). We develop potentials as a tool to study opinion dynamics and offer a novel order-theoretic characterization of potentials. Previously, potential functions in the game-theoretic context were characterized using order theory (Kukushkin, 1999; Voorneveld and Norde, 1997). Our characterization builds on and extends this approach, providing a systematic method for constructing such functions and analyzing the structure of equilibria in any opinion dynamics model that admits them. We show that both the voter model and the Sznajd model admit the average opinion in society as a potential, which in particular implies that the equilibria coincide with increase-proof opinion profiles.

Section 2 introduces social learning environments, gives conditions for them to be absorbing, and treats important models in the literature as special cases. Section 3 carries out the same program for generalized social learning environments and Section 4 incorporates adaptive networks and in particular selective diffusion networks as a special case of a generalized social learning environment. Section 5 introduces potentials as a tool to study convergence of opinion dynamics. Finally, a brief conclusion is presented in Section 6.

2 Social Learning Environments

In this section we develop the framework of a social learning environment in which different models of opinion dynamics can be analyzed. A social learning environment has a finite set of individuals N. The set of all coalitions, i.e., non-empty subsets of N, is denoted by $\mathcal{N}$. Each individual $i \in N$ is assumed to have a set of possible opinions X_i. The set X_i is assumed to be a finite subset of the real numbers in this section. Under this assumption, it is straightforward to define the concepts of *opinion increases* and *opinion decreases*. We relax this assumption in Section 3.

The specification of an opinion for every individual leads to an opinion profile. Opinion profiles therefore belong to $X = \prod_{i \in N} X_i$. For an opinion profile $x \in X$ and an individual $i \in N$, x_i denotes the opinion of i and $x_{-i} \in X_{-i} = \prod_{j \in N \setminus \{i\}} X_i$ the opinion profile of individuals other than i. The opinion of individual i is periodically updated according to an opinion updating rule that assigns a probability distribution over the set of opinions, $f_i : X \to \Delta(X_i)$, where $\Delta(X_i)$ denotes the set of probability distributions on X_i. In each period, the opinion of one of the individuals $i \in N$ at the current opinion profile $x \in X$ is updated according to the distribution $f_i(x)$. If the opinion of i changes to y_i, then the new opinion profile is $y = (y_i, x_{-i})$. Next, the process repeats itself from the new opinion profile y. The process of opinion dynamics corresponds to a Markov process.

A social learning environment is formally defined as follows.

Definition 2.1. A *social learning environment* $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ consists of

(i) a finite set of individuals $N = \{1, 2, \ldots, n\}$,

(ii) for every individual $i \in N$, a non-empty, finite set of possible opinions $X_i \subset \mathbb{R}$,

(iii) for every individual $i \in N$, an opinion updating rule $f_i : X \to \Delta(X_i)$.

To study the process of opinion dynamics, we introduce the concept of a learning path.

Definition 2.2. Let $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ be a social learning environment and let $m \in \mathbb{N} \cup \{\infty\}$. A sequence of opinion profiles $(x^k)_{k=1}^m$ in X is a *learning path* if for every $k \in \{1, \ldots, m-1\}$ there exist $i \in N$ such that $x_i \in \operatorname{supp}(f_i(x^k))$ and $x^{k+1} = (x_i, x^k_{-i})$.

A learning path is a sequence of opinion profiles that can be realized in a social learning environment. A learning path is finite if $m \in \mathbb{N}$ and infinite if $m = \infty$. Next, we introduce the notion of iterated external stability, a set-wise reachability concept.

Definition 2.3. Let $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ be a social learning environment and $Y \subset X$. The set of opinion profiles Y satisfies *iterated external stability* if for any $x \in X \setminus Y$ there exists $m \in \mathbb{N}$ and a finite learning path $(x^k)_{k=1}^m$ such that $x^1 = x$ and $x^m \in Y$.

To analyze stability and convergence of the process of opinion dynamics in a social learning environment, we define a transition correspondence that maps each opinion profile to its possible successors.

Definition 2.4. Let $\Gamma = (N, (X_i)_{i \in N}, (f_i)_{i \in N})$ be a social learning environment. The *transition correspondence* $h : X \to X$ for Γ is defined by

$$h(x) = \cup_{i \in N} \{(y_i, x_{-i}) \in X \mid y_i \in \operatorname{supp}(f_i(x))\}, \quad x \in X.$$

A pair $(x, y) \in X \times X$ such that $y \in h(x) \setminus \{x\}$ is a *transition.*

An opinion profile that admits no transitions is called an equilibrium.

Definition 2.5. Let $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ be a social learning environment. The set of all *equilibria* of Γ is given by

$$E = \{x \in X \mid h(x) = \{x\}\}.$$

The social learning environment is said to be absorbing if the set of equilibria is reachable from any given opinion profile.

Definition 2.6. The social learning environment $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is *absorbing* if E satisfies iterated external stability.

We now argue that many opinion dynamics models known in the literature converge due to a common underlying structure. To do so, it is useful to introduce the notions of opinion increase and opinion decrease. An opinion increase is a transition that leads to an increase in the opinion of the individual who updates and, similarly, an opinion decrease is a transition that leads to a decrease in the opinion of the updating individual. An equilibrium is therefore an opinion profile where neither opinion increases nor opinion decreases occur. For every $i \in N$, we define

(i) $F_i^+(x) = \{y_i \in X_i \mid y_i \in \text{supp}(f_i(x)),\ y_i > x_i\}, \quad x \in X,$

(ii) $F_i^-(x) = \{y_i \in X_i \mid y_i \in \text{supp}(f_i(x)),\ y_i < x_i\}, \quad x \in X.$

In essence, the sets $F_i^+(x)$ and $F_i^-(x)$ partition the set of possible opinion changes that individual i might exhibit at opinion profile x. Here, $F_i^+(x)$ consists of the possible updated opinions that are greater than the current opinion of individual i. Similarly, $F_i^-(x)$ contains the possible updated opinions which are less than the current opinion of individual i. One or both sets $F_i^+(x)$ and $F_i^-(x)$ can be empty.

An opinion profile is called increase-proof if no agent has a possible opinion increase, and decrease-proof if no agent has a possible opinion decrease.

Definition 2.7. Let $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ be a social learning environment. An opinion profile $x \in X$ is

(i) *increase-proof* if, for every $i \in N$, $F_i^+(x) = \emptyset$,

(ii) *decrease-proof* if, for every $i \in N$, $F_i^-(x) = \emptyset$.

The opinion profile in which everyone holds the highest possible opinion is increase-proof. Thus, an increase-proof opinion profile always exists. The same is true for decrease-proof opinion profiles. We denote the set of all increase-proof opinion profiles by X^+ and the set of all decrease-proof opinion profiles by X^-. It then holds that $E = X^+ \cap X^-$.

Proposition 2.8. *Let $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ be a social learning environment. Then, X^+ and X^- satisfy iterated external stability.*

Proof. This directly follows from the fact that X is a finite set. If one keeps sequentially increasing the opinions of individuals, the resulting learning path is finite, and the last opinion profile of the learning path will be in X^+. A similar argument applies to X^-. □

Any learning path consisting solely of opinion increases or opinion decreases must be non-cyclical. This implies that any learning path made entirely up of opinion increases or opinion decreases is finite, since the finiteness of the set X ensures that such transitions will eventually be exhausted. Consequently, both the set of all increase-proof

opinion profiles and the set of all decrease-proof opinion profiles satisfies iterated external stability.

Given that both opinion increases and opinion decreases lead to acyclic learning paths when only one type of transition is used, our next step is to exhaust both types of transitions, potentially in a specific order, and provide a condition that guarantees iterated external stability of the set of equilibria.

Definition 2.9. A social learning environment $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is *structured* if at least one of the following two conditions hold:

(i) for every $x \in X^+ \setminus X^-$ there exists $y \in h(x) \setminus \{x\}$ such that $y \in X^+$,

(ii) for every $x \in X^- \setminus X^+$ there exists $y \in h(x) \setminus \{x\}$ such that $y \in X^-$.

The idea behind a structured environment is to ensure that even if we are not yet in equilibrium, the system retains a pathway forward, either by continuing to decrease or to increase opinion profiles. Under this condition, we can establish that the social learning environment is absorbing.

Theorem 2.10. *Let $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ be a structured social learning environment. Then, Γ is absorbing.*

Proof. We consider only case (i) in Definition 2.9 as the proof for the other case is similar. From any initial opinion profile there is a learning path that ends in X^+ due to Proposition 2.8. Consider any opinion profile $x^1 \in X^+$. If $x^1 \in X^+ \cap X^- = E$, then the proof is done. Otherwise, we have $x^1 \in X^+ \setminus X^-$. This implies that an opinion update by an individual is an opinion decrease. Moreover, since Γ is structured, there exists some $x^2 \in h(x^1) \setminus \{x^1\}$ that belongs to X^+. By iteratively following such opinion profiles as long as possible, we obtain a sequence of opinion profiles $(x^k)_{k=1}^m$ in X^+, where m is finite since the sequence of opinion profiles is decreasing and X is finite. It follows that $x^m \in X^+ \cap X^- = E$, thereby completing our proof. □

The proof of Theorem 2.10 consists of first choosing opinion increases until an increase-proof opinion profile is reached. Once this is achieved, one proceeds with opinion decreases to opinion profiles that are increase-proof until an equilibrium is reached. This is feasible since the social learning environment is structured. Notice that the resulting learning path does not contain any cycles. This is because returning to a previous opinion profile would require at least one opinion increase and one opinion decrease along the associated learning path, bringing us back to the opinion profile where the initial increase was implemented. However, in our case, opinion increases are implemented outside the set X^+, whereas opinion decreases keep the system inside the set X^+. This makes returning to a previous opinion profile impossible. Figure 1 presents a graphical illustration of the proof.

We continue by showing that many of the well-known models as presented in the literature on opinion dynamics result in a structured social learning environment. In fact, we show the stronger result that these models lead to a strongly structured social learning environment as defined next.

Definition 2.11. A social learning environment $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is *strongly structured* if both of the following two conditions hold:

(i) for every $x \in X^+$, for every $y \in h(x)$, we have $y \in X^+$,

(ii) for every $x \in X^-$, for every $y \in h(x)$, we have $y \in X^-$.

A structured social learning environment requires that either X^+ or X^- has the property that, for any of its opinion profiles that are not an equilibrium, there exists an opinion update that leads to an opinion profile within the same set. A strongly structured environment, on the other hand, requires that both X^+ and X^- satisfy this property for all possible opinion updates. A strongly structured social learning environment ensures that society reaches

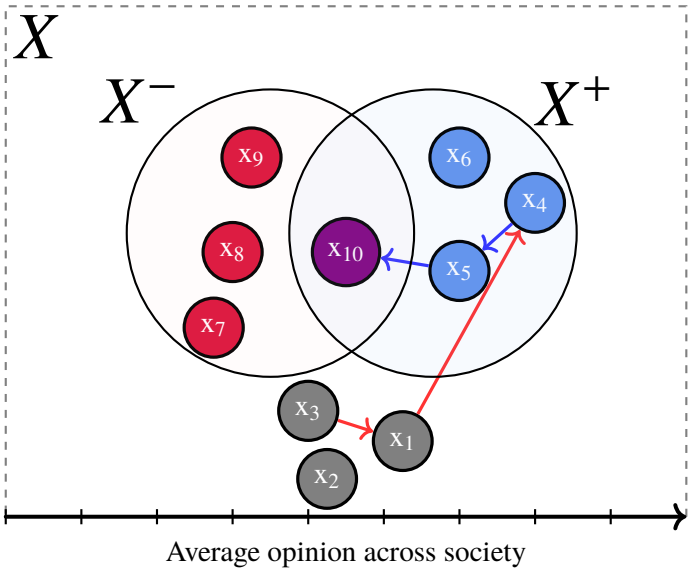


Figure 1: A graphical representation of the proof of Theorem 2.10 for the case of Definition 2.9.(i). Red arrows indicate opinion increases, whereas blue arrows reflect opinion decreases.

an equilibrium in a finite number of steps once an opinion profile in X^+ or X^- is reached. See Figure 2 for an illustration.

Example 2.12 presents a first example of a strongly structured social learning environment, corresponding to the voter model of Holley and Liggett (1975). We need some notation first. A network g on a society N is a set of links $g = \{ij \mid i, j \in N,\ i \neq j\}$. We say that individuals i and j are linked to each other if $ij \in g$. The set of all networks on society N is denoted by G^N. The set of neighbors of individual $i \in N$ in a network $g \in G^N$ is denoted by $N_i(g) = \{j \in N | ij \in g\}$. For $r \in \cup_{j \in N} X_j$, we define

$$N_i^r(g, x) = \{j \in N_i(g) \mid x_j = r\}$$

as the set of neighbors of individual i with opinion r in opinion profile x.

Example 2.12. *(Voter model)*
The voter model was introduced by Holley and Liggett (1975) as an interacting particle system. In this model, individuals hold binary opinions, either 'yes' or 'no', about a specific issue. Moreover, the individuals are located on a network $g \in G^N$ and observe the opinions of their neighbors. In each period, a randomly selected individual $i \in N$ adopts the opinion of one individual in $N_i(g)$. The opinion of an individual without neighbors never changes. The model can also be interpreted as a model of conflict, where each node represents a piece of land occupied by either army 1 or army 0. Armies are more likely to spread to a given area if they already have units of soldiers nearby. Formally, for every $i \in N$, the set of possible opinions is $X_i = \{0, 1\}$. For every $i \in N$, the opinion updating rule is defined by

$$f_i(x)(y_i) = \begin{cases} \frac{|N_i^1(g,x)|}{|N_i(g,x)|}, & \text{if } y_i = 1,\ N_i(g,x) \neq \emptyset, \\ \frac{|N_i^0(g,x)|}{|N_i(g,x)|}, & \text{if } y_i = 0,\ N_i(g,x) \neq \emptyset, \\ 1, & \text{if } y_i = x_i,\ N_i(g,x) = \emptyset, \\ 0, & \text{if } y_i \neq x_i,\ N_i(g,x) = \emptyset, \end{cases}$$

where $f_i(x)(y_i)$ denotes the probability that the updated opinion of individual i equals y_i if the current opinion profile is x. The social learning environment $(N, (X_i)_{i \in N}, (f_i)_{i \in N})$ corresponding to the voter model is denoted by Γ^{VM}. △

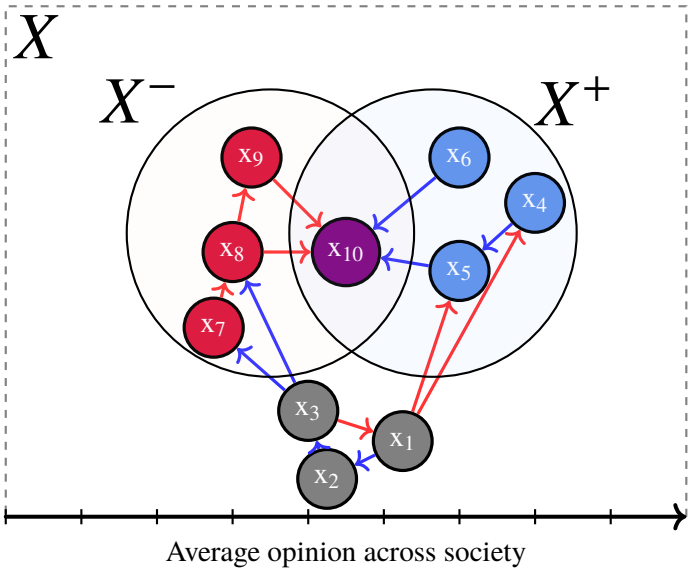


Figure 2: In a strongly structured social learning environment, cycles can only happen outside $X^+ \cup X^-$. Once an element in $X^+ \cup X^-$ is reached, opinion dynamics lead to an equilibrium in a finite number of steps.

Our next result establishes that the voter model of Holley and Liggett (1975) is indeed strongly structured.

Proposition 2.13. *The social learning environment* Γ^{VM} *corresponding to the voter model is strongly structured.*

Proof. We prove that $X^+ = X^+ \cap X^- = E$ so that the claim follows vacuously by (i) in Definition 2.11. Let $x \in X^+$. For every $i \in N$, we have by definition that $F_i^+(x) = \emptyset$, which implies that every $i \in N$ has the same opinion as the individuals in $N_i(g)$. To see why, suppose by contradiction that this is not the case. Then, there exist individuals $i, j \in N$ such that $ij \in g$ and $x_i < x_j$, so $x_i = 0$ and $x_j = 1$. It follows that $F_i^+(x) = \{1\}$, which contradicts the assumption that $x \in X^+$. Therefore, we conclude that each individual has the same opinion as the individuals in $N_i(g)$. This clearly implies that $x \in X^-$ by similar reasoning. Therefore, we conclude that $x \in X^+ \cap X^- = E$. A similar argument applies to $x \in X^-$, thereby completing the proof. □

If we generalize the voter model by allowing the opinion set of each individual to be the same arbitrary finite subset of the real numbers, then the generalized model can be shown to be still strongly structured.

Theorem 2.10 and Proposition 2.13 lead to the following corollary.

Corollary 2.14. *The social learning environment* Γ^{VM} *corresponding to the voter model is absorbing.*

Another example of a strongly structured social learning environment is the one corresponding to the individual majority rule. It is a variation of the classical majority rule (Galam, 2002), in which a randomly selected group of individuals adopts the majority opinion of that group. By contrast, our individual version selects just one individual who then adopts the majority opinion of their neighbors.

Example 2.15. *(Individual majority rule)*
Every individual $i \in N$ has a binary set of possible opinions $X_i = \{0, 1\}$. Individuals are located on a network $g \in G^N$. Individuals observe only the opinions of their neighbors. Each period, the opinion of a randomly selected individual i is updated to match the majority opinion among the individual in $N_i(g)$. If no opinion has a strict majority, then the opinion of i is not updated. To formally define the opinion updating rule, we define

$$M_i(g, x) = \arg\max_{r \in \{0,1\}} |N_i^r(g, x)|, \quad x \in X,$$

Then, for any $i \in N$ and $x \in X$, the probability that the opinion of i is updated to $y_i \in X_i$ is given by

$$f_i(x)(y_i) = \begin{cases} 1, & \text{if } y_i = x_i \in M_i(g,x) \\ 1, & \text{if } y_i \in M_i(g,x),\ x_i \notin M_i(g,x), \\ 0, & \text{otherwise.} \end{cases}$$

The social learning environment $(N, (X_i)_{i\in N}, (f_i)_{i\in N})$ corresponding to the individual majority rule is denoted by Γ^{IMR}. △

Proposition 2.16. *The social learning environment Γ^{IMR} corresponding to the individual majority rule is strongly structured.*

Proof. We first prove that condition (i) in Definition 2.11 is satisfied. Let $x \in X^+ \setminus X^-$ and $y \in h(x) \setminus \{x\}$. Since $x \in X^+$ and $y \neq x$, there exists $i \in N$ such that $x_i = 1$ and $y_i = 0$. Moreover, for every $j \in N \setminus \{i\}$, we have $x_j = y_j$. Also, for every $j \in N \setminus \{i\}$, if $M_j(g,x) \neq M_j(g,y)$, then $0 \in M_j(g,y) \setminus M_j(g,x)$ since only the opinion of individual i is updated. Hence, for every $j \in N \setminus \{i\}$, we have that $y_j = x_j \geq \max \operatorname{supp}(f_j(x)) \geq \max \operatorname{supp}(f_j(y))$, where the first inequality follows from $x \in X^+$. It also holds that $y_i = \max \operatorname{supp}(f_i(y)) = 0$, since the opinion update of individual i from 1 to 0 is only possible if $M_i(g,x) = \{0\}$, so also $M_i(g,y) = \{0\}$. We conclude that $y \in X^+$.

The proof that condition (ii) in Definition 2.11 is satisfied is similar. □

Corollary 2.17. *The social learning environment Γ^{IMR} corresponding to the individual majority rule is absorbing.*

Conditional on the current opinion profile, the opinion updating rules in Γ^{IMR} always assign probability 1 to one of the opinions. Such a social environment is said to be deterministic.

Definition 2.18. The social learning environment $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is *deterministic* if, for every $i \in N$, for every $x \in X$, $|\operatorname{supp}(f_i(x))| = 1$.

For deterministic social learning environments, we typically represent the opinion updating rule of individual $i \in N$ by a function $f_i : X \to X_i$. The profile of opinion updating rules is denoted by $f : X \to X$, so, for every $x \in X$, we have $f(x) = (f_i(x))_{i\in N}$. We show next that deterministic social learning environments with monotonic opinion updating rules are strongly structured. For any $x, y \in \mathbb{R}^m$, we write $x \leq y$ if, for every $i \in \{1, 2, \ldots, m\}$, $x_i \leq y_i$. Similarly, we write $x < y$ if $x \leq y$ and there exists $j \in \{1, 2, \ldots, m\}$ such that $x_j < y_j$.

Definition 2.19. A deterministic social learning environment $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ is *positively monotonic* if, for every $x, y \in X$,

$$x \leq y \implies f(x) \leq f(y).$$

The condition of positive monotonicity is equivalent to the requirement that an increase in the opinion of a specific individual does not lead to a lower opinion update than before by any individual.

Theorem 2.20. *Let $\Gamma = (N, (X_i)_{i\in N}, (f_i)_{i\in N})$ be a deterministic positively monotonic social learning environment. Then, Γ is strongly structured.*

Proof. We first show that condition (i) in Definition 2.11 is satisfied. Let $x \in X^+ \setminus X^-$ and $y \in h(x) \setminus \{x\}$. Since $x \in X^+$ and $y \neq x$, there exists $i \in N$ such that $y_i = f_i(x) < x_i$, whereas for every $j \in N \setminus \{i\}$ we have $x_j = y_j$. We find that $y < x$, which implies that $f(y) \leq f(x)$ by positive monotonicity of Γ. Since $x \in X^+$, it holds that $f(x) \leq x$. Together with the properties we derived for y, it follows that $f(x) \leq y$. Combining the inequalities, we

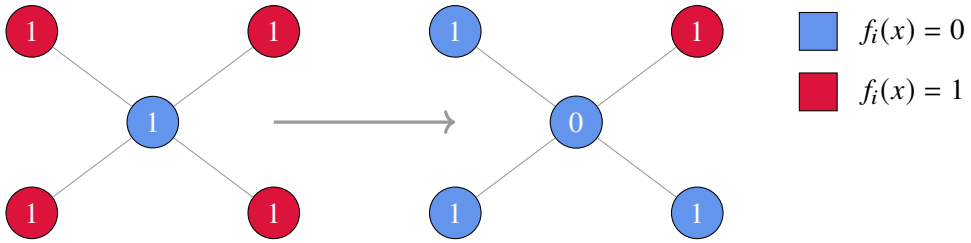


Figure 3: The social environment Γ is assumed to be positively monotonic. The opinion profile on the left is in X^+. A belief update can only change the belief updates of others in the same direction. The opinion profile on the right is therefore in X^+ as well.

obtain $f(y) \leq y$, hence $y \in X^+$.

The proof that condition (ii) in Definition 2.11 is satisfied is similar. □

Theorem 2.20 is illustrated in Figure 3.

We present next some examples in the literature that correspond to positively monotonic social learning environments. In Example 2.21 we develop a discrete version of the DeGroot model. As special cases of this model, we also obtain the Keynesian beauty contest and the median rule.

Example 2.21. *(DeGroot model, Keynesian beauty contest, and median rule)*
The individuals in the society N are located on a network $g \in G^N$. For every $i \in N$, let $\underline{x}_i$ be the lowest opinion in X_i, let $m_i : \mathbb{R}^n \to \mathbb{R}$ be any non-decreasing function, for instance a weighted average of the opinions of the neighbors of i, and let $r_i : \mathbb{R} \to X_i$ be the floor function that rounds its input downwards to the nearest element in X_i or to $\underline{x}_i$ if rounding downwards is not possible. More formally, for every $z \in \mathbb{R}$ we have $r_i(z) = \max(\{\underline{x}_i\} \cup \{x_i \in X_i | x_i \leq z\})$. For every $i \in N$ such that $N_i(g) \neq \emptyset$, the deterministic opinion updating rule is defined by

$$f_i(x) = r_i(m_i(x)), \quad x \in X,$$

whereas for every $i \in N$ such that $N_i(g) = \emptyset$, we have

$$f_i(x) = x_i, \quad x \in X.$$

The social learning environment $(N, (X_i)_{i \in N}, (f_i)_{i \in N}))$ corresponding to the DeGroot model is denoted by Γ^{DG}.

Simply taking the average opinion of i's neighbors and rounding it down to the nearest element in X_i, i.e., if $N_i(g) \neq \emptyset$, then

$$f_i(x) \quad = \quad r_i\left(\tfrac{1}{|N_i(g)|} \textstyle\sum_{j \in N_i(g)} x_j\right)$$

is a special case of this model. Another special case of this model is the game "Guess 2/3 of the Average," also known as the Keynesian beauty contest (Ledoux, 1981; Nagel, 1995). In this game, individuals are allowed to hold an opinion between zero and one hundred, so $X_i = \{0, 1, \ldots, 100\}$. Participants are rewarded only if their opinion is less than or equal to 2/3 of the others average opinion and closest to 2/3 of the others average opinion among all individuals. We take $g \in G^N$ to be the complete network and

$$m_i(x) \quad = \quad \left(\tfrac{2}{3} \cdot \tfrac{1}{|N_i(g)|} \textstyle\sum_{j \in N_i(g)} x_j\right), \quad x \in X.$$

Since m_i is equal to the best response function of individual i in the Keynesian beauty contest game, a proof that Γ^{DG} is absorbing provides a proof for the existence of a Nash equilibrium in this game. Moreover, since in the

Keynesian beauty contest, the set X^- consists only of the zero vector, it then follows that the zero vector is the only Nash equilibrium of the Keynesian Beauty Game.

A final special case of the DeGroot model concerns the median rule. We assume that, for every $i, j \in N$, $X_i = X_j$. Let $i \in N$ be such that $N_i(g) \neq \emptyset$. Then we let m_i correspond to the median of the opinions x_j of the neighbors $j \in N_i(g)$ of i. In case the median is not unique, the individual adopts the highest of the two median opinions. The function m_i is non-decreasing. Since, for every $x \in X$, $m_i(x) \in X_i$, we have that $r_i(m_i(x)) = x_i$. △

Proposition 2.22. *The social learning environment Γ^{DG} corresponding to the DeGroot model is strongly structured.*

Proof. Let $g \in G^N$ be the network on which the individuals are located and, for every $i \in N$, let $m_i : \mathbb{R} \to \mathbb{R}$ be the non-decreasing function that individuals use for updating their opinions. Let $x, y \in X$ be such that $x \leq y$. Let $i \in N$ be such that $N_i(g) = \emptyset$. Then it holds that

$$f_i(x) = x_i \leq y_i = f_i(y).$$

Let $i \in N$ be such that $N_i(g) \neq \emptyset$. Then it holds that

$$f_i(x) = r_i(m_i(x)) \leq r_i(m_i(y)) = f_i(y),$$

where the weak inequality uses that m_i is non-decreasing. It follows that Γ^{DG} is a deterministic positively monotonic social learning environment. Therefore, by Theorem 2.20, it is strongly structured. □

Corollary 2.23. *The social learning environment Γ^{DG} corresponding to the DeGroot model is absorbing.*

3 Generalized Social Learning Environments

In this section, we generalize social learning environments using order theory. Opinions no longer need to be real numbers, and any strict partial order on opinions replaces the strictly greater than relation that we have over the reals. In many real-world settings, opinions are categorical or multi-dimensional and transitions may involve coordinated changes by multiple individuals. To address this, we generalize the notion of a social learning environment by allowing opinions to lie in arbitrary partially ordered sets and by replacing the transition rule by a transition correspondence. Moreover, we relax the assumption $X = \prod_{i \in N} X_i$ to $X \subset \prod_{i \in N} X_i$. This generalization also paves the way for studying coalitional learning and adaptive networks within a unified framework. Finally, the set X_i may contain more information than only an opinion, like for instance the set of neighbors of i in Section 4. We often refer to elements of X_i as private states and to elements of X as states.

Definition 3.1. A *generalized social learning environment* $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ consists of

(i) a finite set of individuals $N = \{1, 2, \ldots, n\}$,

(ii) for every individual $i \in N$, a finite partially ordered set of possible private states $(X_i, \succeq_i)$,

(iii) a non-empty state space $X \subset \prod_{i \in N} X_i$,

(iv) a transition correspondence $h : X \to X$.

For every $i \in N$, we denote the strict partial order associated with $\succeq_i$ by $\succ_i$. Since private states are only assumed to be partially ordered, we allow for the possibility that private states cannot be compared to one another. The

associated *product order* $\geq$ is such that, for every $x, y \in X$, $x \geq y$ if and only if for every $i \in N$, $x_i \geq_i y_i$. The associated strict product order is denoted by $\succ$.

By definition of a correspondence, it holds that, for every $x \in X$, $h(x)$ is a non-empty subset of X.

Since, for every $i \in N$, X_i is a partially ordered set, we can formulate the notions of increase proofness and decrease proofness in this generalized setting. Similarly, the transition correspondence $h : X \to X$ serves as the generalization of the transition correspondence h introduced in Definition 2.4, where it was based on transition rules $(f_i)_{i \in N}$. In this section, we take h as a primitive.

We define the relations $h^+ : X \to X$ and $h^- : X \to X$ by

$$\begin{aligned} h^+(x) &= \{y \in h(x) \mid \text{there exists } i \in N \text{ such that } y_i \succ_i x_i\}, \quad x \in X, \\ h^-(x) &= \{y \in h(x) \mid \text{there exists } i \in N \text{ such that } x_i \succ_i y_i\}, \quad x \in X. \end{aligned}$$

Since h^+ and h^- may be empty-valued, we refer to them as relations rather than correspondences.

The generalized social learning environment comes with a few caveats, as we essentially treat the correspondence h as a black box without imposing any additional structure on it. Before addressing these complexities, however, we define learning paths, increase-proofness, decrease-proofness, and equilibrium in one comprehensive definition.

Definition 3.2. Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment and let $m \in \mathbb{N} \cup \{\infty\}$.

(i) A sequence of states $(x^k)_{k=1}^m$ in X is a *learning path* if, for every $k \in \{1, \ldots, m-1\}$, we have $x^{k+1} \in h(x^k)$.

(ii) Let $Y \subset X$. The set of states Y satisfies *iterated external stability* if for every $x \in X \setminus Y$ there exists a finite learning path $(x^k)_{k=1}^m$ such that $x^1 = x$ and $x^m \in Y$.

(iii) A state $y \in X$ is said to be *increase-proof* if $y \in X^+ = \{x \in X \mid h^+(x) = \emptyset\}$, *decrease-proof* if $y \in X^- = \{x \in X \mid h^-(x) = \emptyset\}$, and an *equilibrium* if $y \in E = \{x \in X \mid h(x) = \{x\}\}$.

(iv) The social environment Γ is *absorbing* if E satisfies iterated external stability.

In a generalized social learning environment, private states lie in partially ordered sets and updates are governed by a transition correspondence rather than individual updating rules. All key notions, like learning paths, increase proofness, decrease proofness, and equilibrium, still carry over to this setting. With this background, we are now ready to discuss the caveats associated with generalized social learning environments.

The first problem is the breakdown of the relation $X^+ \cap X^- = E$ in Section 2. To see why, note that in the context of social learning environments as discussed in Section 2, the order over individual opinions follows from the order over the set of real numbers. Since this order is complete, the lack of increases and decreases implies that there are no transitions. However, once we relax the assumption of completeness by assuming partial orders, even if there are no state increases or decreases in the system with respect to this order, there may be a transition to a state that is not comparable. Therefore, we only know that $E \subset X^+ \cap X^-$. This is a problem because our proof strategy relies on the reachability of the set $X^+ \cap X^-$, which is not sufficient to infer the reachability of the set E in case $E \subsetneq X^+ \cap X^-$.

The second problem is that transitions are no longer necessarily monotone. Recall that in Section 2 the iterated external stability of X^+ and X^- played an important role in identifying sufficient conditions to establish that social learning environments are absorbing. Specifically, our notion of structured social learning environments enabled us to reach X^+ first and then descend to X^- in a particular way, ensuring that we remained within X^+ as

well. Unfortunately, assuming a general transition correspondence h is not sufficient to guarantee the iterated external stability of the sets X^+ and X^-, making it impossible to carry out the same strategy. In fact, since X is not necessarily a product set, it is even possible that X^+ and X^- are empty. In social learning environments, individuals change their opinions independently, without affecting others. However, transitions in a generalized social learning environment might involve an increase and a decrease in the private states of two individuals simultaneously. As a result, non-escapable cycles can emerge, even if we attempt to exhaust state increases by repeatedly following transitions that lead to an increase in a single individual's private state.

To overcome these challenges, we impose some additional structure on the generalized social learning environment, thereby recovering the property that $X^+ \cap X^- = E$ and ensuring that cyclical behavior does not arise. We show that that many models are compatible with the additional structure, thereby making our generalized social learning environments both meaningful and applicable.

The product order allows us to compare states. If all individuals weakly improve, and at least one strictly improves, we say that the state has increased. This structure enables us to provide conditions that guarantee a monotonic development of the state. In particular, if we assume that for every $x \notin E$ there exists $y \in h(x)$ such that the states x and y are comparable with respect to the product order $\geq$, then, even though we may not be able to compare any two arbitrary states, we can at least compare certain states that are adjacent in terms of the transitions and we would have $X^+ \cap X^- = E$.

Similarly, we have discussed that under a general transition correspondence h the sets X^+ and X^- may fail to satisfy iterated external stability or even be empty. Therefore, we need to eliminate such transition behaviors. If we assume that for every $x \notin X^+$ there exists $y \in h(x)$ such that $y \succ x$, then cycles may still occur, but it is always possible to get out of them.

Motivated by these two properties of the product order, we now define the general concept of *cohesiveness*.

Definition 3.3. The generalized social learning environment $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ is *cohesive* if

(i) for every $x \in X \setminus E$ there exists $y \in h(x)$ such that $y \succ x$ or $x \succ y$.

(ii) for every $x \in X \setminus X^+$ there exists $y \in h(x)$ such that $y \succ x$.

In principle, we can define a weaker notion of cohesiveness by requiring that conditions (i) and (ii) are satisfied or conditions (i) and (iii), where condition (iii) states that for every $x \in X \setminus X^-$ there exists $y \in h(x)$ such that $x \succ y$. The results in this section, with straightforward modifications, would still be valid.

Cohesiveness of a generalized social learning environment implies that the intersection of the sets of increase-proof and decrease-proof states coincides with the set of equilibria.

Proposition 3.4. *Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a cohesive generalized social learning environment. Then it holds that $X^+ \cap X^- = E$.*

Proof. Clearly, we have $E \subset X^+ \cap X^-$. Thus, it suffices to prove that $X^+ \cap X^- \subset E$. Let $x \in X^+ \cap X^-$. To obtain a contradiction, suppose $x \in X \setminus E$. By condition (i) in Definition 3.3 there exists $y \in h(x)$ such that $y \succ x$ or $x \succ y$. The former case violates the assumption that $x \in X^+$ and the latter case violates the assumption that $x \in X^-$. Therefore, we have $x \in E$. □

The following proposition establishes that cohesiveness of a generalized social learning environment is sufficient to guarantee iterated external stability of the set of increase-proof states.

Proposition 3.5. *Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a cohesive generalized social learning environment. Then X^+ satisfies iterated external stability.*

Proof. Let $x^1 \in X \setminus X^+$. By condition (ii) in Definition 3.3, there exists $x^2 \in h(x^1)$ such that $x^2 \succ x^1$. If $x^2 \notin X^+$, then, by the same condition, there exists $x^3 \in h(x^2)$ such that $x^3 \succ x^2$. Continuing this process as long as the state is not in X^+, we construct a sequence $(x^k)_{k=1}^m$ for some $m \in \mathbb{N} \cup \{\infty\}$, such that, for every $k = 1, \ldots, m-1$, $x^{k+1} \in h(x^k)$ and $x^{k+1} \succ x^k$. Since X is finite, this strictly increasing sequence must eventually terminate, so m is finite. We have that $x^m \in X^+$. □

Following this result, we can now introduce the concept of structured generalized social learning environments.

Definition 3.6. Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a generalized social learning environment. We say that Γ is *structured* if for every $x \in X^+ \setminus X^-$ there exists $y \in h(x) \setminus \{x\}$ such that $y \in X^+$.

Following Definition 3.6, the following generalization of Theorem 2.10 naturally follows.

Proposition 3.7. *Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a cohesive generalized social learning environment. If Γ is structured, then it is absorbing.*

Proof. From any initial state there is a learning path that ends in X^+ due to Proposition 3.5. Consider any state $x^1 \in X^+$. If $x^1 \in X^+ \cap X^-$, then the proof is done. Otherwise, we have $x^1 \in X^+ \setminus X^-$. Since Γ is structured, there exists some $x^2 \in h(x^1) \setminus \{x^1\}$ that belongs to X^+. By iteratively following such states, we obtain a sequence of states $(x^k)_{k=1}^m$ in X^+. Since the sequence of states is decreasing and X is finite, m is finite and $x^m \in X^-$. Hence, we obtain $x^m \in X^+ \cap X^- = E$, where the equality follows from Proposition 3.4. □

We now proceed to some examples of generalized social learning environments, where we examine several variations of the social learning environments introduced in Section 2. Specifically, we relax the assumption that opinion updates are done by only one individual at a time. Instead, several individuals may change their opinions simultaneously, which corresponds to simultaneous learning. Although individuals may not update their opinions at the exact same moment, one can interpret this scenario as a case where individuals do not continuously observe the opinions of others but rather check them periodically. As long as multiple opinion updates occur while no one is observing the opinions of others, this situation can be regarded as simultaneous learning.

The transition correspondence h is also straightforward to construct. In the case of social learning environments, each individual is endowed with an opinion updating rule $f_i : X \to \Delta(X_i)$. We extend this concept by allowing any non-empty subset of individuals to have an opinion updating rule. Thus, rather than selecting individual agents in each period, we consider a collection of individuals who update their opinions simultaneously.

Given an opinion profile x and a coalition $S \in \mathcal{N}$, let x_S denote the vector containing the opinions of the individuals in S and x_{-S} denote the vector of opinions of the individuals in $N \setminus S$. We denoted $\prod_{i\in S} X_i$ by X_S. We also define coalitional opinion updating rules $f_S : X \to \Delta(X_S)$, where, for every $x \in X$, $f_S(x)$ yields a probability distribution over the opinions of the coalition S. Coalitional opinion updating rules are deterministic if, for any given opinion profile, they assign probability 1 to a single, specific opinion vector. A deterministic updating rule of coalition $S \in \mathcal{N}$ is typically represented by a function $f_S : X \to X_S$.

We now present our first example, the Sznajd model.

Example 3.8. *(Sznajd model)*
The Sznajd model, also known as “united we stand, divided we fall," is a socio-physics model designed to capture social validation and conformity within societies. The literature on this model is vast (Behera and Schweitzer,

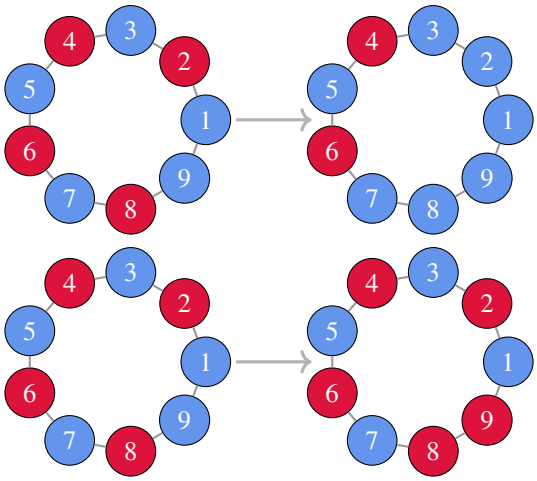


Figure 4: The Sznajd model of Example 3.8. The figure at the top demonstrates the consensus rule for the coalition $\{8, 9, 1, 2\}$, whereas the figure at the bottom demonstrates the disagreement rule for the coalition $\{9, 1, 2, 3\}$.

2003; Stauffer, 2002; Stauffer, 2003; Sznajd-Weron, Sznajd, and Weron, 2021). The model operates on the principle that if a group of individuals shares the same opinion, they can influence their neighbors to adopt the same view. Conversely, if there are differing opinions within a group, the neighbors of that group begin to argue with them.

There are different variations on the Sznajd model, each defining what constitutes a group of individuals in a distinct way. In the one-dimensional version of the model, the individuals in a society with at least five individuals are arranged in a circular structure, where each individual has an opinion of 0 or 1. At each point in time, four neighboring individuals, k, $k+1$, $k+2$, and $k+3$, are randomly selected, where all these numbers are taken modulo n. The dynamics of the model follow from applying the consensus rule and the disagreement rule.

Consensus rule: If $k+1$ and $k+2$ share the same opinion, then individuals k and $k+3$ also adopt this opinion.

Disagreement rule: If $k+1$ and $k+2$ hold opposing opinions, then k adopts the opposite of the opinion of $k+1$, and $k+3$ adopts the opposite of the opinion of $k+2$.

To formulate the Sznajd model as a generalized social learning environment, let $N = \{1, 2, \ldots, n\}$ with $n \geq 5$ be the set of individuals in the society, for every $i \in N$, the set of possible opinions is $X_i = \{0, 1\}$ and the order $\succeq_i$ is the usual order on the real numbers. We set $X = \prod_{i \in N} X_i$. Let $S \in \mathcal{N}$ be a coalition of four consecutive individuals, so S is of the form $\{k, k+1, k+2, k+3\}$. Then the opinion updating rule $f_S : X \to X_S$ is defined by

$$f_S(x) = \begin{cases} (x_{k+1}, x_{k+1}, x_{k+1}, x_{k+1}), & x \in X \text{ such that } x_{k+1} = x_{k+2}, \\ (x_{k+2}, x_{k+1}, x_{k+2}, x_{k+1}), & x \in X \text{ such that } x_{k+1} \neq x_{k+2}. \end{cases}$$

For any $S \in \mathcal{N}$ that is not of the form $\{k, k+1, k+2, k+3\}$ for some $k \in \mathbb{N}$, we simply define $f_S(x) = x_S$. Accordingly, the transition correspondence $h : X \to X$ is given by

$$h(x) = \cup_{S \in \mathcal{N}} \{(f_S(x), x_{-S})\}, \quad x \in X.$$

The generalized social learning environment $(N, (X_i, \succeq_i)_{i \in N}, X, h)$ corresponding to the Sznajd model is denoted by Γ^{SZ}. △

Proposition 3.9. *The generalized social learning environment Γ^{SZ} corresponding to the Sznajd model is cohesive.*

Proof. We first show that condition (i) of Definition 3.3 is satisfied. Let $x \in X \setminus E$. There are at least two consecutive individuals who share the same opinion as otherwise the opinions alternate, which implies $x \in E$ by the disagreement rule. We can choose $k \in N$ such that $x_k \neq x_{k+1} = x_{k+2}$ since otherwise all opinions of individuals in N would be the same and $x \in E$ by the consensus rule. If $x_{k+1} = x_{k+2} = 1$, then an opinion update by coalition $S = \{k, k+1, k+2, k+3\}$ follows the consensus rule and leads from a state x such that $x_S = (0, 1, 1, x_{k+3})$ to a state $y \in h(x)$ such that $y_S = (1, 1, 1, 1)$, so $y \succ x$. If $x_{k+1} = x_{k+2} = 0$ and $x_{k-1} = 0$, then an opinion update by coalition $S = \{k-1, k, k+1, k+2\}$ follows the disagreement rule and leads from a state x such that $x_S = (0, 1, 0, 0)$ to a state $y \in h(x)$ such that $y_S = (0, 1, 0, 1)$, so $y \succ x$. If $x_{k+1} = x_{k+2} = 0$ and $x_{k-1} = 1$, then an opinion update by coalition $S = \{k-2, k-1, k, k+1\}$ follows the consensus rule and leads from a state x such that $x_S = (x_{k-2}, 1, 1, 0)$ to a state $y \in h(x)$ such that $y_S = (1, 1, 1, 1)$, so $y \succ x$. We have shown that condition (i) of Definition 3.3 holds.

We show next that condition (ii) of Definition 3.3 is satisfied. Let $x \in X \setminus X^+$. It follows that $x \notin E$. Following the same reasoning to show that condition (i) of Definition 3.3 is satisfied, we find that there exists $y \in h(x)$ such that $y \succ x$. This proves that condition (ii) of Definition 3.3 holds. □

Proposition 3.10. *The generalized social learning environment* Γ^{SZ} *corresponding to the Sznajd model is structured.*

Proof. By Proposition 3.9, Γ^{SZ} is cohesive, so by Proposition 3.4, $X^+ \cap X^- = E$. We prove that $X^+ \subset E$, from which the claim follows. To this end, assume the contrary, and let $x \in X^+ \setminus X^-$. Since $x \notin X^-$, we have $x \in X \setminus E$. There are at least two consecutive individuals who share the same opinion as otherwise the opinions alternate, which implies $x \in E$ by the disagreement rule. We can choose $k \in N$ such that $x_k \neq x_{k+1} = x_{k+2}$ since otherwise all opinions of individuals in N would be the same and $x \in E$ by the consensus rule. Suppose $x_{k+1} = x_{k+2} = 1$. Then an opinion update by coalition $S = \{k, k+1, k+2, k+3\}$ follows the consensus rule and leads from a state x such that $x_S = (0, 1, 1, x_{k+3})$ to a state $y \in h(x)$ such that $y_S = (1, 1, 1, 1)$, which contradicts that $x \in X^+$. Consequently, $x_k \neq x_{k+1} = x_{k+2} = 0$. An opinion update by coalition $S = \{k-1, k, k+1, k+2\}$ follows the disagreement rule and leads from a state x such that $x_{k+2} = 0$ to a state $y \in h(x)$ such that $y_{k+2} = 1$. This contradicts $x \in X^+$. Consequently, it holds that $X^+ \subset E$. □

Corollary 3.11. *The generalized social learning environment* Γ^{SZ} *corresponding to the Sznajd model is absorbing.*

Our second example is the majority rule model. This model is similar to the individual majority rule model discussed earlier. However, the key difference is that in the majority rule model individuals can update their beliefs simultaneously, similar to the Sznajd model.

Example 3.12. *(Majority rule)*
The majority rule model is due to Galam (2002). Each individual in the society is either in favor of or against a proposal to change the status quo. For every $i \in N$, the set of possible opinions is $X_i = \{0, 1\}$ and is ordered in the usual way. The state space X is equal to $\prod_{i \in N} X_i$. In each time period, a coalition $S \in \mathcal{N}$ of randomly selected individuals of size k is selected with probability $p_k \geq 0$, where $\sum_{k=1}^{n} p_k = 1$, and discusses the issue. At the end of the discussion, all individuals in the group adopt the opinion that holds the majority among the selected individuals. In the case of a tie, the individuals in the group prefer the status quo and decide to be against the reform. For every $S \in \mathcal{N}$ with $p_{|S|} > 0$, the deterministic opinion updating rule is given by

$$f_S(x) = \begin{cases} (1, 1, \ldots, 1), & x \in X \text{ such that } \sum_{i \in S} x_i > \frac{|S|}{2}, \\ (0, 0, \ldots, 0), & x \in X \text{ such that } \sum_{i \in S} x_i \leq \frac{|S|}{2}. \end{cases}$$

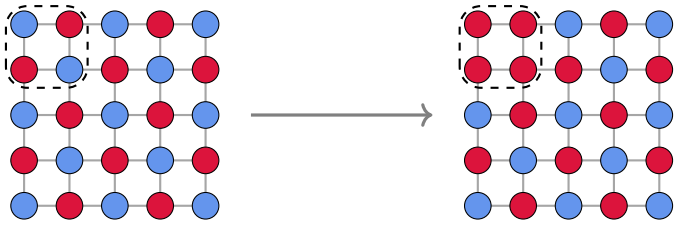


Figure 5: The majority rule model of Example 3.12. In the case of a tie, individuals prefer the status quo and decide to be against the reform.

For every $S \in \mathcal{N}$ with $p_S = 0$, we simply have, for every $x \in X$, $f_S(x) = x_S$. The transition correspondence $h : X \to X$ is defined in the usual way,

$$h(x) = \cup_{S \in \mathcal{N}} \{(f_S(x), x_{-S})\}, \quad x \in X.$$

The generalized social learning environment $(N, (X_i, \succeq_i)_{i \in N}, X, h)$ corresponding to the majority rule model is denoted by Γ^{MR}.

△

Proposition 3.13. *The generalized social learning environment Γ^{MR} corresponding to the majority rule model is cohesive.*

Proof. We first show condition (i) in Definition 3.3 is satisfied. Let $x \in X \setminus E$. It holds that $h(x) \setminus \{x\} \neq \emptyset$. Let $y \in h(x) \setminus \{x\}$ and let $S \in \mathcal{N}$ be such that $y = (f_S(x), x_{-S})$. Since $y \neq x$, it holds that $p_{|S|} > 0$, so $f_S(x) \in \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}$ and either $y \succ x$ or $x \succ y$.

Next, we verify that condition (ii) in Definition 3.3 is satisfied. Let $x \in X \setminus X^+$. We have that $h^+(x) \neq \emptyset$, so there exists $i \in N$ and $y \in h(x)$ such that $y_i > x_i$. Let $S \in \mathcal{N}$ be such that $y = (f_S(x), x_{-S})$. Since $y \neq x$, it holds that $p_{|S|} > 0$ and $i \in S$. Moreover, since $y_i > x_i$, we have that $f_S(x) = (1, \ldots, 1)$. It follows that $y \succ x$. □

Proposition 3.14. *The generalized social learning environment Γ^{MR} corresponding to the majority rule model is structured.*

Proof. Let $x \in X^+ \setminus X^-$. Since $x \notin E$, there is $y \in h(x) \setminus \{x\}$. Moreover, since $x \in X^+$, we have $x \succ y$. We distinguish two cases.

Case 1. For every $k \geq 3$, $p_k = 0$.
Let $S \in \mathcal{N}$ be such that $|S| = 2$. An opinion increase of an individual is impossible, since there cannot be a strict majority of opinion 1 in coalition S, while at least one individual in S holds opinion 0. Therefore, we have that $y \in X^+$.

Case 2. There is $k \geq 3$ with $p_k > 0$.
Let k be the smallest number with these properties. Consider $T = \{i \in N \mid x_i = 1\}$. Since $x \in X \setminus E$, we have that $T \neq N$, so there is $j \in N$ with $x_j = 0$. It holds that $|T| \leq \lfloor k/2 \rfloor$, as otherwise we can construct a coalition S of cardinality k, containing individual j and at least $\lfloor k/2 \rfloor + 1$ individuals from T. Coalition S is chosen with positive probability and $f_S(x) = (1, 1, \ldots, 1)$, which contradicts that $x \in X^+$. Since $x \succ y$, it holds that the cardinality of $T' = \{i \in N \mid y_i = 1\}$ is less than or equal to $\lfloor k/2 \rfloor - 1$.

Let $S \in \mathcal{N}$ be a coalition for which $p_{|S|} > 0$. If $|S| = 1$ or $|S| = 2$, then $f_S(y) \leq y_S$. If $|S| \geq k$, then at most $\lfloor k/2 \rfloor - 1$ individuals in S hold opinion 1, so $f_S(y) = (0, 0, \ldots, 0) \leq y_S$. We have shown that $y \in X^+$. □

Corollary 3.15. *The generalized social learning environment* Γ^{MR} *corresponding to the majority rule model is absorbing.*

4 Adaptive Networks

So far in our analysis, the neighbors of an individual in a social learning environment remained fixed throughout the learning process. This assumption essentially means that individuals do not change their learning sources and keep their contacts the same over time. However, such an assumption is likely to be violated since individuals also revise their links with others, potentially changing the way information flows in the society. In the networks literature, there are two main strands that try to capture this feature.

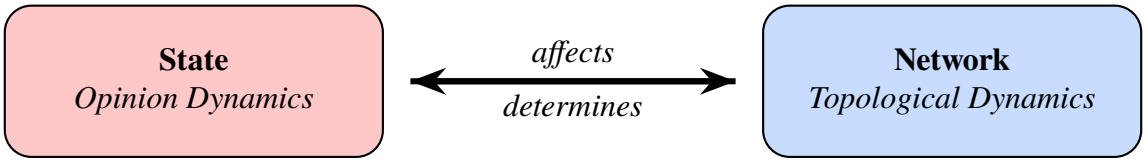


Figure 6: Adaptive networks involve a feedback loop between opinion dynamics and topological dynamics.

If the change in the network is exogenous, then such networks are studied under the name of evolving networks. On the other hand, if the changes in the network are caused by the way information spreads, then such networks are called adaptive networks. In this sense, adaptive networks involve a feedback process between two layers of interaction: opinion dynamics affect topological dynamics, which in turn affect the opinion dynamics. Although adaptive networks have a wide range of applications, from epidemiology to machine learning, there is still much to discover in this area. The twofold aim of this section is to analyze the convergence of adaptive networks using the tools provided so far and to devise a new class of adaptive network models called selective diffusion networks.

We now formulate the model of adaptive networks as a generalized social learning environment. The set N corresponds to the individuals in the society. The set X_i consists of the product of a finite set of possible opinions Y_i and a collection $2^{N\setminus\{i\}}$ of possible sets of neighbors of i. For every $i \in N$, the partial order $\succeq_i$ over X_i can be arbitrary. The state space X is equal to the subset of $\prod_{i\in N} X_i$ whose elements are compatible with an underlying undirected network. More precisely, for a state $(y_i, B_i)_{i\in N}$ to be in X, the sets of neighbors of all individuals need to be consistent with each other, so

$$X = \{(y_i, B_i)_{i\in N} \in \textstyle\prod_{i\in N} X_i | \text{for every } j, k \in N,\ k \in B_j \iff j \in B_k\}.$$

A state $(y_i, B_i)_{i\in N} \in X$ induces the unique network g such that $jk \in g$ if and only if $k \in B_j$. We therefore write the state space X as $Y \times G^N$ in this section, where $Y = \prod_{i\in N} Y_i$.

Lastly, we define the transition correspondence $h : Y \times G^N \to Y \times G^N$. In adaptive networks, there are two different types of changes: either the opinions change or the network topology changes. In each period, either the opinion of an individual changes or a single link of the network is added or broken resulting in an adjacent network. As in Section 2, we consider the case where at each time period there is an opinion update by only one individual.

Definition 4.1. An *adaptive network* is a generalized social learning environment $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ such that

(i) for every $i \in N$, $X_i = Y_i \times 2^{N\setminus\{i\}}$,

(ii) $X = Y \times G^N$,

(iii) the transition correspondence $h : Y \times G^N \to Y \times G^N$ is such that for every $x' \in h(y,g) \setminus \{(y,g)\}$ either there is $i \in N$ and $z_i \in Y_i$ such that $x' = ((z_i, y_{-i}), g)$ or there is a network g' adjacent to g such that $x' = (y, g')$.

Since the set of possible opinions in a generalized social learning environment is finite according to Definition 3.2, for every $i \in N$, the set Y_i is finite.

We now identify a class of adaptive networks called *selective diffusion networks*. The individuals in the society would like to engage in lucrative bilateral relationships with each other. Individuals connect to each other if and only if the associated benefits exceed the cost for both parties. The opinion of individual $i \in N$ on the benefits of the bilateral relationship with individual $j \in N \setminus \{i\}$ are denoted by $y_{ij} \in \mathbb{R}$. Therefore, the opinion of individual i is a vector y_i of length $n-1$,

$$y_i = (y_{i1}, y_{i2}, \ldots, y_{i(i-1)}, y_{i(i+1)}, \ldots, y_{in}).$$

The maintenance of links is costly. The costs for an individual $i \in N$ to have a link with individual $j \in N \setminus \{i\}$ are equal to c_{ij}, and the associated cost vector is denoted by

$$c_i = (c_{i1}, c_{i2}, \ldots, c_{i(i-1)}, c_{i(i+1)}, \ldots, c_{in}).$$

Every time period, there is either an opinion update or a network update. If there is a network update, then two distinct individuals i and j are selected. If they have a link between them, then it gets broken if one of the individuals opinion about the benefits of the link is below the individual's cost to maintain the link. On the contrary, if there is no link between them, they decide to form the link if the opinion on its benefit is at least equal to its cost for both individuals and strictly greater for at least one. The changes in the network there correspond to the ideas behind pairwise stability (Jackson and Wolinsky, 1996; Jackson and Watts, 2001; Watts, 2001; Jackson and Watts, 2002; Jackson, 2005).

Definition 4.2. An adaptive network $\Gamma = (N, (Y_i \times 2^{N\setminus\{i\}}, \succeq_i)_{i\in N}, X, h)$ is a *selective diffusion network* if, for every $i \in N$, $Y_i \subset \mathbb{R}^{n-1}$ and there exist a cost vector $c_i \in \mathbb{R}^{n-1}$ such that, for every $(y,g) \in X$,

(a) for every $ij \in g$, $(y, g - ij) \in h(y,g)$ if and only if $y_{ij} < c_{ij}$ or $y_{ji} < c_{ji}$,

(b) for every $ij \notin g$, $(y, g + ij) \in h(y,g)$ if and only if $(y_{ij}, y_{ji}) > (c_{ij}, c_{ji})$.

Notice that whenever the network topology changes, this potentially affects the updating of opinions in subsequent periods, thus changing the spread of information across society. The question is: under which conditions does the selective diffusion model converge to an equilibrium, that is, to a state such that there are no changes in opinions given the topology and no changes in topology given the opinions?

To answer this question, we first consider the possible changes in opinions given some fixed network. For every $i \in N$, we define the correspondences $h_i^+ : X \to X$ and $h_i^- : X \to X$ by

$$\begin{aligned} h_i^+(y,g) &= \{(z,g) \in h(y,g) \mid \text{there exists } j \in N \setminus \{i\} \text{ such that } z_{ji} > y_{ji}\}, \quad (y,g) \in X, \\ h_i^-(y,g) &= \{(z,g) \in h(y,g) \mid \text{there exists } j \in N \setminus \{i\} \text{ such that } z_{ji} < y_{ji}\}, \quad (y,g) \in X. \end{aligned}$$

Note that in the definition of h_i^+ and h_i^- attention is restricted to changes in opinions, whereas changes in the network are not taken into account. It holds that $(z,g) \in h_i^+(g)$ if and only if z can result as an opinion update from

y given the network g and there is at least one individual $j \in N \setminus \{i\}$ whose opinion on the benefits of the bilateral relationship with individual i has increased. The definition of $h_i^-(g)$ is similar, except that now there is at least one individual $j \in N \setminus \{i\}$ whose opinion on the benefits of the bilateral relationship with individual i has decreased.

The sets of states for which the opinion on i can only decrease, respectively increase, are defined by by

$$\begin{aligned} X_i^+ &= \{x \in X \mid h_i^+(x) = \emptyset\}, \\ X_i^- &= \{x \in X \mid h_i^-(x) = \emptyset\}. \end{aligned}$$

Next, we define the concepts of opinion-wise strongly structured social learning environments and opinion-wise stability.

Definition 4.3. The selective diffusion network $\Gamma = (N, (Y_i \times 2^{N\setminus\{i\}}, \succeq_i)_{i\in N}, X, h)$ is *strongly opinion-wise structured* if

(i) for every $i \in N$, for every $(y, g) \in X_i^+$, for every $(z, g) \in h(y, g)$, we have $(z, g) \in X_i^+$,

(ii) for every $i \in N$, for every $(y, g) \in X_i^-$, for every $(z, g) \in h(y, g)$, we have $(z, g) \in X_i^-$.

The notion of strongly opinion-wise structured in Definition 4.3 corresponds to the concept of strongly structured of Definition 2.11, when applied to one individual at a time and when restricting to the case where the network does not change.

The next definition focuses on the states where opinions do not change, a necessary requirement for a state to be an equilibrium.

Definition 4.4. Let $\Gamma = (N, (Y_i \times 2^{N\setminus\{i\}}, \succeq_i)_{i\in N}, X, h)$ be a selective diffusion network. A state $(y, g) \in X$ is *opinion-wise stable* if for every $(y', g) \in h(y, g)$ it holds that $y' = y$.

It is immediately verified that a state $(y, g) \in X$ is opinion-wise stable if and only if, for every $i \in N$, $h_i^+(y, g) = h_i^-(y, g) = \emptyset$. We denote the set of all opinion-wise stable states by X^{o}.

It is also convenient to define the notion of component-wise consensus, which means that all individuals in a component hold the same opinion regarding the bilateral benefits of other individuals.

Definition 4.5. Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a selective diffusion network. A state $(y, g) \in X$ exhibits *component-wise consensus* if, for any component C of g, for every $i, j \in C$ with $i \neq j$, for every $k \in N \setminus \{i, j\}$, we have $y_{ik} = y_{jk}$.

The set of all states that exhibit component-wise consensus is denoted by X^{c}. For many opinion updating models, it holds that a state is opinion-wise stable if and only if it exhibits component-wise consensus. The next result states that for such opinion updating models, strongly opinion-wise structured selective diffusion networks are absorbing.

Theorem 4.6. *Let $\Gamma = (N, (Y_i \times 2^{N\setminus\{i\}}, \succeq_i)_{i\in N}, X, h)$ be a selective diffusion network. If Γ is strongly opinion-wise structured and $X^{\mathrm{o}} = X^{\mathrm{c}}$, then Γ is absorbing.*

Proof. Consider a state $(y^1, g^1) \in X$. We first keep the network g^1 fixed and start a process of sequential opinion increases to reach a state in $\cap_{i\in N} X_i^+$. If $(y^1, g^1) \in \cap_{i\in N} X_i^+$, then we are done. Otherwise, select the individual i' with the lowest index such that $(y^1, g^1) \notin X_{i'}^+$. Then there is $j \in N \setminus \{i'\}$ and $(y^2, g^1) \in h(y^1, g^1)$ such that $y^2_{-j} = y^1_{-j}$ and $y^2_{ji'} > y^1_{ji'}$. By Condition (i) of Definition 4.3 we know that, for every $i < i'$, $(y^2, g^1) \in X_i^+$. We repeat the opinion increases on i' until none remains, so after a finite number of, say k, iterations we obtain

$(y^k, g^1) \in \cap_{i \leq i'} X_i^+$. If $(y^k, g^1) \in \cap_{i \in N} X_i^+$, then we are done. Otherwise, we select the individual i' with the lowest index such that $(y^k, g^1) \notin X_{i'}^+$. After a finite number of steps, say ℓ^1, we find $(y^{\ell^1}, g^1) \in \cap_{i \in N} X_i^+$.

Our next goal is to find a state that belongs to $\cap_{i \in N} X_i^+$ and is such that link additions are not possible. If (y^{ℓ^1}, g^1) satisfies these properties, then we are done. Otherwise, there is $(y^{\ell^1}, g^2) \in h(y^{\ell^1}, g^1)$ where, for some $ij \notin g^1$, $g^2 = g^1 + ij$. We can now proceed as in the first paragraph to find a state $(y^{\ell^2}, g^2) \in \cap_{i \in N} X_i^+$. Since the possible number of link additions is finite, we find a state $(y'^1, g'^1) \in X$ with all the desirable properties after a finite number of link additions and subsequent opinion increases.

We now keep the network g'^1 fixed and start a process of sequential opinion decreases to reach a state in $\cap_{i \in N} X_i^-$. If $(y'^1, g'^1) \in \cap_{i \in N} X_i^-$, then we are done. Otherwise, select the individual i' with the lowest index such that $(y'^1, g'^1) \notin X_{i'}^-$. Then there is $j \in N \setminus \{i'\}$ and $(y'^2, g'^1) \in h(y'^1, g'^1)$ such that $y'^2_{-j} = y'^1_{-j}$ and $y'^2_{ji'} < y^1_{'ji'}$. By Condition (ii) of Definition 4.3 we know that, for every $i < i'$, $(y'^2, g'^1) \in X_i^-$. We repeat the opinion decreases on i' until none remains, so after a finite number of, say k, iterations we obtain $(y'^k, g'^1) \in \cap_{i \leq i'} X_i^-$. If $(y'^k, g'^1) \in \cap_{i \in N} X_i^+$, then we are done. Otherwise, we select the individual i' with the lowest index such that $(y'^k, g'^1) \notin X_{i'}^-$. After a finite number of steps, say ℓ, we find $(y'^\ell, g'^1) \in \cap_{i \in N} X_i^-$. Since Γ is strongly opinion-wise structured, we have by Condition (i) of Definition 4.3 that, for every $k \in \{1, \ldots, \ell\}$, $(y'^k, g'^1) \in \cap_{i \in N} X_i^+$. We conclude that for every $(y, g) \in h(y'^\ell, g'^1)$ it holds that $y = y'^\ell$, so $(y'^\ell, g'^1) \in X^{\mathrm{o}}$.

For every $(y, g) \in h(y'^1, g'^1)$ it holds that $g \subset g'^1$, so by definition of a selective diffusion network, for every $ij \notin g'^1$,

$$y'^1_{ij} < c_{ij} \text{ or } y'^1_{ji} < c_{ji} \text{ or } (y'^1_{ij}, y'^1_{ji}) = (c_{ij}, c_{ji}).$$

Since $y'^\ell \leq y'^1$, we have that for every $ij \notin g'^1$,

$$y'^\ell_{ij} < c_{ij} \text{ or } y'^\ell_{ji} < c_{ji} \text{ or } (y'^\ell_{ij}, y'^\ell_{ji}) = (c_{ij}, c_{ji}).$$

It follows that, for every $(y, g) \in h(y'^\ell, g'^1)$, $g \subset g'^1$.

We now start a process of sequential link deletions until none remains. If for every $(y, g) \in h(y'^\ell, g'^1)$ it holds that $g = g'^1$, then $(y'^\ell, g'^1) \in E$ and we are done. Otherwise, for some $k \in \mathbb{N}$, let $(y'^\ell, g'^k) \in X$ be such that there exists $(y, g) \in h(y'^\ell, g'^k)$ such that $g \subsetneq g'^k$. Let (y'^ℓ, g'^{k+1}) be such that $g'^{k+1} \subsetneq g'^k$. Since the number of links is finite, after a finite number of steps, we find a state $(y'^\ell, \underline{g}) \in X$ such that, for every $(y, g) \in h(y'^\ell, \underline{g})$ it holds that $\underline{g} \subset g$. For every $ij \notin \underline{g}$, it holds that either $ij \notin g'^1$, so

$$y'^\ell_{ij} < c_{ij} \text{ or } y'^\ell_{ji} < c_{ji} \text{ or } (y'^\ell_{ij}, y'^\ell_{ji}) = (c_{ij}, c_{ji})$$

or ij was deleted during the sequential link deletion process, so

$$y'^\ell_{ij} < c_{ij} \text{ or } y'^\ell_{ji} < c_{ji}.$$

We conclude that for every $(y, g) \in h(y'^\ell, \underline{g})$ it holds that $g = \underline{g}$.

The fact that $(y'^\ell, g'^1) \in X^{\mathrm{o}}$ together with the assumption $X^{\mathrm{o}} = X^{\mathrm{c}}$, implies $(y'^\ell, g'^1) \in X^{\mathrm{c}}$. Since $\underline{g} \subset g'^1$, it follows that $(y'^\ell, \underline{g}) \in X^{\mathrm{c}}$ and by the assumption $X^{\mathrm{o}} = X^{\mathrm{c}}$, $(y'^\ell, \underline{g}) \in X^{\mathrm{o}}$. Therefore, we obtain $(y'^\ell, \underline{g}) \in E$. □

Next, we introduce and study the adaptive version of the DeGroot model.

Example 4.7. *(DeGroot selective diffusion model)* We develop the adaptive version of the DeGroot model of Example 2.21. Each individual holds a vector of opinions regarding every other individual. Opinion updates

therefore amount to averaging over opinion vectors, thereby changing the opinion of one individual with respect to all other individuals in the society. We assume that individuals assign equal weight to the opinions of each of their neighbors. There is a finite set $O \subset \mathbb{R}$ of opinions with smallest element $\underline{o}$, and, for every $i \in N$, the set of possible opinions vectors is the product set $Y_i = \prod_{j \in N \setminus \{i\}} O$. Opinions regarding a particular individual are updated by taking the average opinion regarding that individual among neighbors. Let $r : \mathbb{R} \to O$ be the floor function that rounds its input downwards to the nearest element in O or to $\underline{o}$ if rounding downwards is not possible, so for every $z \in \mathbb{R}$ we have $r_i(z) = \max(\{\underline{o}\} \cup \{o \in O | o \leq z\})$.

Let $i \in N$ and $(y, g) \in X$. For every $j \in N \setminus \{i\}$ such that $N_i(g) \setminus \{j\} \neq \emptyset$, the deterministic opinion updating rule f_i updates the opinion of i regarding j to

$$f_{ij}(y, g) \quad = \quad r(\tfrac{1}{|N_i(g) \setminus \{j\}|} \textstyle\sum_{k \in N_i(g) \setminus \{j\}} y_{kj}),$$

whereas for every $j \in N \setminus \{i\}$ such that $N_i(g) \setminus \{j\} = \emptyset$, we have

$$f_{ij}(y, g) \quad = \quad y_{ij}.$$

At any time period, at a state $(y, g) \in X$, one of two things happens: either the opinions of an individual $i \in N$ are updated according to f_i or the topology of the network is modified according by adding a link $ij \notin g$ to g or deleting a ling $ij \in g$ from g in accordance with the selective diffusion model. In this way, we derive the transition correspondence $h : X \to X$. The selective diffusion network $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ corresponding to the DeGroot selective diffusion model is denoted by Γ^{DGS}. △

We show that the DeGroot selective diffusion model is absorbing by verifying that it satisfies the properties of Theorem 4.6. The first property is that Γ^{DGS} is strongly opinion-wise structured.

Proposition 4.8. *The selective diffusion network Γ^{DGS} corresponding to the DeGroot selective diffusion model is strongly opinion-wise structured.*

Proof. Let $i \in N$ and $(y, g) \in X_i^+$. Since, by definition of X_i^+, $h_i^+(y, g) = \emptyset$, it holds that, for every $(z, g) \in h(y, g)$, for every $j \in N \setminus \{i\}$, $z_{ji} \leq y_{ji}$.

Let $(z, g) \in h(y, g)$. Suppose $(z, g) \notin X_i^+$. Then there is $(z', g) \in h(z, g)$ and $j \in N \setminus \{i\}$ such that $z'_{ji} > z_{ji}$. By definition of the updating rule f_j, it holds that $N_j(g) \setminus \{i\} \neq \emptyset$ and

$$z'_{ji} \quad = \quad r(\tfrac{1}{|N_j(g) \setminus \{i\}|} \textstyle\sum_{k \in N_j(g) \setminus \{i\}} z_{ki}).$$

There are two possibilities. Either $z_{ji} = y_{ji}$, so we have

$$r(\tfrac{1}{|N_j(g) \setminus \{i\}|} \textstyle\sum_{k \in N_j(g) \setminus \{i\}} y_{ki}) \quad \geq \quad r(\tfrac{1}{|N_j(g) \setminus \{i\}|} \textstyle\sum_{k \in N_j(g) \setminus \{i\}} z_{ki}) = z'_{ji} > z_{ji} = y_{ji},$$

which contradicts $(y, g) \in X_i^+$. Or $z_{ji} < y_{ji}$, which implies

$$z_{ji} \quad = \quad r(\tfrac{1}{|N_j(g) \setminus \{i\}|} \textstyle\sum_{k \in N_j(g) \setminus \{i\}} y_{ki}) \geq r(\tfrac{1}{|N_j(g) \setminus \{i\}|} \textstyle\sum_{k \in N_j(g) \setminus \{i\}} z_{ki}) = z'_{ji} > z_{ji},$$

a contradiction. Consequently, it holds that $(z, g) \in X_i^+$.

In a similar way, we can prove that, for every $i \in N$, for every $(y, g) \in X_i^-$, for every $(z, g) \in h(y, g)$, we have $(z, g) \in X_i^-$. □

The second property of Γ^{DGS} to be verified is that opinion-wise stability coincides with opinion-wise consensus.

Proposition 4.9. *Let $\Gamma^{\text{DGS}} = (N, (Y_i \times 2^{N\setminus\{i\}}, \succeq_i)_{i\in N}, X, h)$ be a selective diffusion network corresponding to the DeGroot selective diffusion model. Then it holds that $X^{\text{c}} = X^{\text{o}}$.*

Proof. Let $(y, g) \in X^{\text{c}}$. Consider some component C of g. If C is a singleton, say $C = \{i\}$, then $f_i(y, g) = y_i$. Otherwise, the cardinality of C is greater than or equal to 2. Let $i \in C$ and $j \in N \setminus \{i\}$. If $N_i(g) \setminus \{j\} = \emptyset$, then we have

$$f_{ij}(y, g) \quad = \quad y_{ij},$$

whereas if $N_i(g) \setminus \{j\} \neq \emptyset$, then

$$\begin{aligned} f_{ij}(y, g) &= r(\tfrac{1}{|N_i(g)\setminus\{j\}|} \textstyle\sum_{k\in N_i(g)\setminus\{j\}} y_{kj}) \\ &= r(\tfrac{1}{|N_i(g)\setminus\{j\}|}|N_i(g) \setminus \{j\}|y_{ij}) \\ &= r(y_{ij}) \\ &= y_{ij}. \end{aligned}$$

It follows that $(y, g) \in X^{\text{o}}$.

Let $(y, g) \in X^{\text{o}}$. Consider some component C of g. If C is a singleton, then consensus within C holds trivially, so assume the cardinality of C is greater than or equal to 2. Let $i', j' \in C$ with $i' \neq j'$ and let $k \in N \setminus \{i', j'\}$.

Suppose $y_{i'k} \neq y_{j'k}$. Let $j \in C$ be such that, for every $i \in C$, $y_{ik} \leq y_{jk}$, so y_{jk} is the highest opinion on k among individuals in C. We have that

$$f_{jk}(y, g) \quad = \quad r(\tfrac{1}{|N_j(g)\setminus\{k\}|} \textstyle\sum_{i\in N_j(g)\setminus\{k\}} y_{ik}) < r(y_{jk}) = y_{jk},$$

where the strict inequality follows from the fact that j has the highest opinion on k among individuals in C, $i', j' \in N_j(g) \setminus \{k\}$ and $y_{i'k} < y_{jk}$ or $y_{j'k} < y_{jk}$. It follows that $(y, g) \notin X^{\text{o}}$, a contradiction. Consequently, $y_{i'k} = y_{j'k}$.

It now follows that $(y, g) \in X^{\text{c}}$. □

Corollary 4.10. *The selective diffusion network Γ^{DGS} corresponding to the DeGroot selective diffusion model is absorbing.*

We can use the same methodology to study the selective diffusion version of other models in Sections 2 and 3. For example, one can verify that the selective diffusion version of the voter model in Example 2.12 is absorbing.

5 Potentials in Social Learning Environments

In this section, we introduce potential functions as an alternative method for studying the convergence properties of generalized social learning environments. Essentially, a potential function is a function that monotonically changes as a dynamical system evolves, and the existence of a potential function is equivalent to convergence of the dynamical system in finite settings. Several notions of potential functions were defined and discussed in Monderer and Shapley (1996) to study normal-form games. In the context of games, the dynamics correspond to better-reply dynamics. Beginning from an arbitrary strategy profile, a better reply from a randomly selected agent changes the strategy profile into another one. Implicitly, this defines a Markov process. The existence of the potential function implies that this Markov process is absorbing. Therefore, if a potential function exists for a given game, then

better-reply dynamics eventually converges to a Nash equilibrium. The aim of this section is to extend this theory to social learning environments in order to identify convergence from this alternative perspective.

In our analysis, we define two different types of potential functions. The first one is called a strong learning potential, which is a function that strictly increases in value whenever agents update their opinion to any possible opinion in accordance with their opinion updating rule. The existence of strong learning potentials is equivalent to the absence of cycles in any learning path. This property essentially indicates that any learning path is finite, and thus learning processes in a social learning environment with a strong learning potential monotonically converge to an equilibrium. The second type of potential function is called a weak potential function, which is a function that strictly increases in value for some opinion update by some agent. Weak potentials, in contrast to strong potentials, allow for the existence of cycles in learning paths. However, the probability of getting trapped in such a cycle converges to zero. Effectively, such a social learning environment is absorbing.

We begin by introducing the concept of potentials in generalized social learning environments.

Definition 5.1. Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment. The function $P : X \to \mathbb{R}$ is a

(i) *weak learning potential* if for every $x \in X \setminus E$ there exists $y \in h(x)$ such that $P(y) > P(x)$,

(ii) *strong learning potential* if, for every $x \in X \setminus E$, for every $y \in h(x) \setminus \{x\}$, $P(y) > P(x)$.

Now, we establish the connections between the convergence properties of a given social learning environment and the different types of potential functions introduced so far. To do so, we prove a result that is analogous to Proposition 3.3 in Demunyck et al. (2023), which establishes the links between the existence of different types of potentials and the types of convergence these functions imply in social environments. A social environment is a general object that includes games, matching problems, and networks as specific cases (Demunyck et al., 2019). The authors prove that the existence of strong potentials is equivalent to all better reply paths being finite. This property is known as the finite improvement property in the game theory literature (Milchtaich, 1996; Young, 2004). The existence of weak reply potentials is equivalent to the existence of finite better reply paths from any given state to the core. This assumption is known as weak acyclicity in the game theory literature. Our result states that a similar structure exists in the realm of generalized social learning environments.

Proposition 5.2. *Let* $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ *be a generalized social learning environment. Then,*

(i) Γ *admits a weak learning potential if and only if* Γ *is absorbing,*

(ii) Γ *admits a strong learning potential if and only if every learning path in* Γ *is finite.*

Proof. To prove (i), assume that a generalized social learning environment admits a weak learning potential, say $P : X \to \mathbb{R}$. Consider any $x^1 \in X \setminus E$. By assumption, there exists $x^2 \in h(x^1)$ such that $P(x^2) > P(x^1)$. If $x^2 \in E$, then we are done. Otherwise, we can apply the same procedure repeatedly to construct a finite learning path $(x^k)_{k=1}^m$ such that $x^m \in E$. This follows from the finiteness of X, which implies that the function value of P can only increase a finite number of times.

For the converse, assume that Γ is absorbing. Then, for any state in $X \setminus E$, there exists a finite learning path that ends in an equilibrium. Now, for any $x \in X$, define $P(x)$ as the negative of the length of the shortest learning path from x to an equilibrium. By construction, P is a weak learning potential, as moving along a shortest learning path strictly increases the value of P.

To prove (ii), assume that a generalized social learning environment admits a strong learning potential, say

$P : X \to \mathbb{R}$. Suppose the sequence $(x^k)_{k=1}^{\infty}$ in X is an infinite learning path in Γ. The infinite sequence $(P(x^k))_{k=1}^{\infty}$ is strictly increasing, which contradicts the finiteness of X. For the converse, assume that every learning path in Γ is finite. Then there cannot exist a learning path that contains a cycle. Hence, the diagram representing the underlying Markov process is a forest, and therefore equivalently a Hasse diagram. Defining P to be any order-preserving function over this Hasse diagram yields the desired result. □

In the following statements, we characterize potential functions using order theory. These characterizations will ultimately pave the way toward answering the question of whether an absorbing generalized social learning environment is always cohesive. We next consider the learning paths that can be generated by a given set of transitions. Briefly, learning paths associated with a set of transitions T are the set of all learning paths that can be constructed by concatenating the transitions contained in T. The properties that such learning paths admit are significant for studying weak learning potentials that increase in value whenever a transition from the set T is realized. As we show, if an arbitrary set of transitions T satisfies certain properties, then any function that increases in value whenever a transition from T is realized defines a weak learning potential. In that sense, regardless of the specific values that a function $P : X \to \mathbb{R}$ assigns to a certain state, the elements in the set T reveals a specific positioning of the elements, thus implicitly defining an order over the states that potential functions preserve. To this end, let us proceed by further developing our theory.

Definition 5.3. Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment and let T be a set of transitions. We define the set of *learning paths associated with* T by

$$L(T) = \cup_{m \in \mathbb{N} \setminus \{1\}} \{(x^1, \ldots, x^m) \in X^m \mid \text{for every } k \in \{1, 2, \ldots, m-1\},\ (x^k, x^{k+1}) \in T\}.$$

A set of transitions is *acyclic* if the learning paths associated with this set of transitions do not contain cycles.

Similar to the concept of increase-proof states, we can define the concept of T-proof states for a set of transitions T.

Definition 5.4. Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment and let T be a set of transitions. A state $x \in X$ is *T-proof* if $h(x) \cap T = \emptyset$.

We denote the set of T-proof states by X^T, so $X^T = \{x \in X \mid h(x) \cap T = \emptyset\}$.

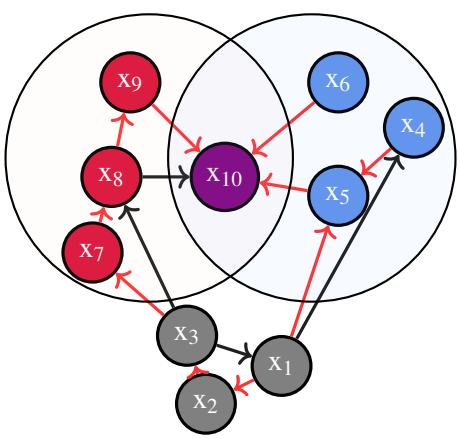
The following theorem shows that the acyclicity of a set of transitions T implies that the set of T-proof states satisfies iterated external stability.

Proposition 5.5. *Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment and let T be an acyclic set of transitions. Then X^T satisfies iterated external stability.*

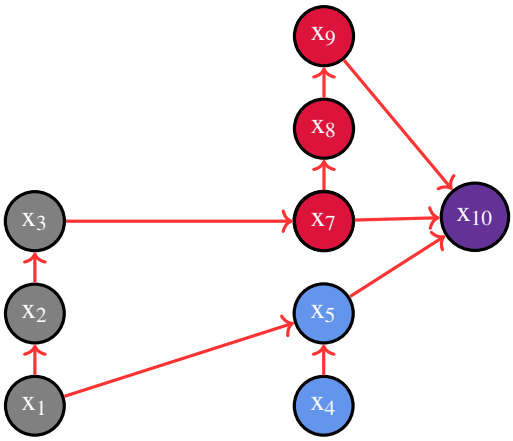
Proof. Let $x^1 \in X \setminus X^T$. For some $k \in \mathbb{N}$, assume $x^k \in X \setminus X^T$. Since $x^k \notin X^T$, it holds that $h(x^k) \cap T \neq \emptyset$. Choose $(x^k, x^{k+1}) \in T$. Since T is acyclic, the states $x^1, \ldots, x^{k+1}$ are all distinct. Since X is finite, for some $k \in \mathbb{N}$, we have $x^{k+1} \in X^T$. □

Proposition 5.5 generalizes Proposition 2.8, which states that the set of increase-proof states satisfies iterated external stability. As we shall see, the study of T-proof states is a stepping stone toward understanding potential functions, as the two concepts turn out to be closely related.

Analogous to the concept of strictly increasing functions, which preserve the ordering of real numbers with respect to the usual order, an order-preserving function is a monotone function that preserves the order of elements in a given partially ordered set. Below, we define these concepts formally.



(a) The set of transitions T, marked with red arrows, satisfies acyclicity and $X^T \subset E$.



(b) Hasse-diagram of the order induced by the set T of red transitions in Figure 7a.

Figure 7: Visualizations of the set of transitions T and its corresponding induced order.

Definition 5.6. Let $(X, \succeq_X)$ and $(Y, \succeq_Y)$ be two partially ordered sets. A function $f : X \to Y$ is *order-preserving* if $x^2 \succeq_X x^1$ implies $f(x^2) \succeq_Y f(x^1)$.

We next define the concept of transitive closure.

Definition 5.7. Let X be an arbitrary set and $T \subset X \times X$ be a relation over X. The *transitive closure* of T, denoted by T^+, is the smallest transitive relation such that $T \subset T^+$.

We next present the order-theoretic characterization of weak learning potentials.

Proposition 5.8. *Let $\Gamma = (N, (X_i, \succeq_i)_{i \in N}, X, h)$ be a generalized social learning environment. The function $P : X \to \mathbb{R}$ is a weak learning potential if and only if there exists an acyclic set of transitions T with $X^T \subset E$ such that the function $P : X \to \mathbb{R}$ is an order-preserving function of T^+.*

Proof.
Assume that $P : X \to \mathbb{R}$ is a weak learning potential. We define

$$T = \{(x, y) \in X \times X \mid y \in h(x),\ P(y) > P(x)\}.$$

As long as a state is not an equilibrium, there exists a transition that increases the value of the weak learning potential. Therefore, we have that $X^T \subset E$. Now consider the transitive closure T^+ of T, which defines a strict partial order over the set X, i.e., T^+ is an irreflexive, transitive, and asymmetric relation (Davey and Priestley, 2002). We show that P is an order-preserving function with respect to T^+. Consider any $(x, y) \in T^+$. Clearly, there must be a sequence of transitions in T beginning at state x and ending at state y. By construction, each such transition increases the value of the function P, so $P(y) > P(x)$.

Assume that there exists an acyclic set of transitions T with $X^T \subset E$ such that the function $P : X \to \mathbb{R}$ is an order-preserving function of T^+. We know that T^+ defines a strict partial order over the set of states X. Consider any $x \in X \setminus E$. Since $x \notin E$, we have $x \notin X^T$. Therefore, there exists $y \in h(x)$ such that $(x, y) \in T$. This implies that $(x, y) \in T^+$ and since P is an order-preserving function of T^+, $P(y) > P(x)$. □

Figures 7(a) and 7(b) illustrate Proposition 5.8 via the exemplary social learning environment given in Figure 2. Figure 7(b) rearranges Figure 7(a) to reveal the underlying Hasse diagram. Note also that the red arrows in Figure 7(a) form a proper subset of all transitions, as including the cycle that contains x^1, x^2, and x^3 would violate

transitivity.

The potentials constructed in the proof of Proposition 5.2 are rather abstract. For the case where possible opinions are real numbers, a natural potential to consider would be the average opinion in society P^{a}, defined by

$$P^{\mathrm{a}}(x) \quad = \quad \tfrac{1}{n} \textstyle\sum_{i \in N} x_i, \quad x \in X.$$

Proposition 5.9. *Let $\Gamma = (N, (X_i, \geq_i)_{i \in N}, X, h)$ be a cohesive generalized social learning environment such that, for every $i \in N$, $X_i \subset \mathbb{R}$. The average opinion in society P^{a} is a weak learning potential if and only if $X^+ = E$.*

Proof. Assume P^{a} is a weak learning potential. By Proposition 5.8, there is an acyclic set T of transitions with $X^T \subset E$ such that P^{a} is an order-preserving function of T^+. Let $x \in X^+$, so $h^+(x) = \emptyset$. For every transition $(x, y) \in T$ it holds that $P^{\mathrm{a}}(x) < P^{\mathrm{a}}(y)$. Now $h^+(x) = \emptyset$ implies $h(x) \cap T = \emptyset$, so $x \in X^T \subset E$.

Assume $X^+ = E$. Define the set of transitions T by

$$T = \{(x, y) \in X \times X \mid y \in h(x),\ P^{\mathrm{a}}(y) > P^{\mathrm{a}}(x)\}.$$

Then T is an acyclic set of transitions. Let $x \in X^T$. Then there is no $y \in h(x)$ such that $P^{\mathrm{a}}(y) > P^{\mathrm{a}}(x)$. By the first condition in Definition 3.3 of cohesiveness this implies hat $x \in X^+$, which by assumption is equal to E. Clearly, P^{a} is an order-preserving function of T^+. By Proposition 5.8, P^{a} is a weak learning potential. □

As we did in previous sections, we continue with providing examples to illustrate the applicability of our results. Let us begin with the study of the voter model of Example 2.12. In Proposition 2.13 we argued that the voter model forms a structured social learning environment, so it follows from Theorem 2.10 that it is absorbing. Here, we provide an alternative proof demonstrating that the average opinion in society P^{a} is a weak learning potential.

Proposition 5.10. *The social learning environment Γ^{VM} corresponding to the voter model admits the average opinion in society P^{a} as a weak learning potential.*

Proof. Let $g \in G^N$ be the network on which the individuals are located and let $x \in X \setminus E$. Since x is not an equilibrium, there exists $ij \in g$ such that $x_i = 0$ and $x_j = 1$. The transition in which the opinion of individual i is updated from 0 to 1 increases P^{a}. □

As a consequence of Proposition 5.9 and Proposition 5.10, we obtain the following corollary for the voter model.

Corollary 5.11. *In the social learning environment Γ^{VM} corresponding to the voter model it holds that $X^+ = E$.*

We can also use potentials to prove that the generalized social learning environment corresponding to the Sznajd model is absorbing. In particular, we can show that average opinion across society is a weak learning potential, just in the case of voter model.

Proposition 5.12. *The social learning environment Γ^{SZ} corresponding to the Sznajd model admits the average opinion in society P^{a} as a weak learning potential.*

Proof. Consider any state $x \in X \setminus E$. Since $x \notin E$, there are at least two individuals with differing opinions. Moreover, there must be two consecutive individuals who hold the same opinion as otherwise the opinions would alternate, which constitutes an equilibrium state by the disagreement rule. If there are two consecutive individuals with opinion 1, then one can select such a pair of individuals who have at least one neighbor holding the opinion 0. If we update the opinions using these individuals and their neighbors, those who believe in 0 update their opinions to 1, thereby increasing P^{a}. Now, consider the case where there are two consecutive individuals hold opinion

0. Among such pairs, one can find a pair that neighbors an individual with opinion 1. The other neighbor of that individual has opinion 0 or 1, so the possible opinion configurations of these individuals are $(0, 0, 1, 0)$ and $(0, 0, 1, 1)$. The latter case was already covered. The disagreement rule applied to the first configuration changes the opinions to $(1, 0, 1, 0)$, thereby increasing P^{a}. □

By Proposition 3.9 we know that Γ^{SZ} is cohesive. By Proposition 5.9, we therefore obtain the following corollary of Proposition 5.12.

Corollary 5.13. *In the social learning environment Γ^{SZ} corresponding to the Sznajd model it holds that $X^+ = E$.*

So far in our analysis, the concept of structured social learning environments has played a key role, raising the natural question of whether it is possible to establish meaningful links between such environments and potential functions. The answer is affirmative and is most easy understood using *lexicographic potentials*, introduced by Demuynck, Herings, and Seel (2023) in the context of social environments.

Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a generalized social learning environment and let $\mathcal{P} = (P_1, P_2, \ldots, P_\ell)$ be a profile of functions, where for every $k \in \{1, 2, \ldots, \ell\}$, we have $P_k : X \to \mathbb{R}$. We define the lexicographic order $\succeq_{\mathcal{P}}$ on X as follows:

$$x \succeq_{\mathcal{P}} y \iff \exists k^* \in \{1, 2, \ldots, \ell\}, \text{ for every } k < k^*,\ P_k(x) = P_k(y) \text{ and, if } k^* \neq \ell + 1, P_{k^*}(x) > P_{k^*}(y).$$

Accordingly, this order leads to the concept of a lexicographic potential system.

Definition 5.14. Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a generalized social learning environment, and let $\mathcal{P} = (P_1, P_2, \ldots, P_\ell)$ be such that, for every $k \in \{1, 2, \ldots, \ell\}$, $P_k : X \to \mathbb{R}$. The profile $\mathcal{P}$ is a *lexicographic weak potential system* if

$$\text{for every } x \in X \setminus E, \text{ there exists } y \in h(x) \text{ such that } y \succ_{\mathcal{P}} x.$$

It is clearly the case that a potential function corresponds to a lexicographic weak potential system consisting of a single function. The other way around is true as well. If $\mathcal{P}$ is a lexicographic weak potential system for Γ, then Γ also admits a weak potential function. In fact, any order-preserving function P of $\succeq_{\mathcal{P}}$ would serve as a weak potential and the finiteness of X guarantees that such an order-preserving function exists.

Proposition 5.15. *Let $\Gamma = (N, (X_i, \succeq_i)_{i\in N}, X, h)$ be a cohesive generalized social learning environment such that, for every $i \in N$, $X_i \subset \mathbb{R}$. Then Γ is structured if and only if $\mathcal{P} = (P_1, P_2)$ defined by*

$$\begin{aligned} P_1(x) &= 1, & x \in X^+ \\ P_1(x) &= -1, & x \notin X^+, \end{aligned}$$

$$P_2(x) = -P_1(x) \cdot P^{\mathrm{a}}(x), \quad x \in X.$$

is a lexicographic weak potential system.

Proof.
Assume Γ is structured. Let $x \in X \setminus E$. We first consider the case where $x \notin X^+$. Then there exists $y \in h^+(x)$. By cohesiveness, there exists $y \in h(x)$ such that $y > x$. If $y \in X^+$, then $y \succ_{\mathcal{P}} x$ since $P_1(y) = 1 > -1 = P_1(x)$. If $y \notin X^+$, then $P_1(y) = -1 = P_1(x)$ and

$$P_2(y) = P^{\mathrm{a}}(y) > P^{\mathrm{a}}(x) = P_2(x),$$

so $y \succ_{\mathcal{P}} x$.

We next consider the case where $x \in X^+ \setminus X^-$. Since Γ is structured, there exists $y \in h(x) \setminus \{x\}$ such that $y \in X^+$, so $P_1(x) = 1 = P_1(y)$. Moreover, since $x \in X^+$, it holds that $y < x$. Therefore, we have

$$P_2(y) = -P^{\mathrm{a}}(y) > -P^{\mathrm{a}}(x) = P_2(x),$$

proving that $y \succ_{\mathcal{P}} x$.

Assume $\mathcal{P}$ is a lexicographic weak potential system. Let $x \in X^+ \setminus X^-$. There there exists $y \in h(x)$ such that $y \succ_{\mathcal{P}} x$. Since $P_1(x) = 1$ and $y \succ_{\mathcal{P}} x$, it holds that $P_1(y) = 1$, so $y \in X^+$. □

Finally, let us comment on the intuition behind Proposition 5.15. Note that in structured generalized social learning environments, one learning path to equilibrium involves increasing opinions until reaching X^+ and then decreasing opinions until reaching E. This suggests to use P^{a} as a potential function over the region $X \setminus X^+$. Similarly, the negative of P^{a} serves as a potential for states in $X^+ \setminus X^-$. The construction of the lexicographic weak potential system achieves this decomposition.

6 Conclusion

This paper develops social learning environments as a unified framework for the analysis of opinion dynamics. We introduce the notion of structured and strongly structured social learning environments, classes of models that guarantee convergence to equilibrium through directional and acyclic learning paths. Our analysis reveals that many well-known models from the literature, including the voter model, individual majority rule, and the DeGroot model are obtained as special cases.

Beyond these classical settings, we extend our theory to generalized social learning environments, where sets of possible opinions are partially ordered and transitions may involve coordinated updates. This allows us to study phenomena such as simultaneous learning and coalitional updates. A key insight in this setting is the importance of cohesiveness, a structural property ensuring that agents move consistently, thereby avoiding cycles and ensuring convergence. We now obtain the Sznajd model and the majority rule as special cases.

We also explore the dynamics of adaptive networks, where individuals can modify their social links in response to changing opinions. In doing so, we introduce the concept of selective diffusion networks, where agents form or sever ties based on their evolving opinions about mutual benefit. Using our framework, we show that such networks are cohesive and converge to equilibria where both opinions and the network remain unchanged.

Lastly, we introduce a novel approach to convergence based on potential functions, drawing inspiration from potentials as defined in abstract games. We provide a order-theoretic characterization of such functions and demonstrate how they can be used to study opinion dynamics. Moreover, we link this approach to structured social learning environments through the use of lexicographic potential systems.

In sum, this paper presents a general and flexible framework for understanding how opinions evolve and eventually reach an equilibrium in decentralized settings. The tools we develop here have the potential to be applied across a wide range of domains, including economics, sociology, and network science. We hope that the accessibility of our methods will increase the attention this field receives, ultimately accelerating its development.

References

Acemoglu, D., M.A. Dahleh, I. Lobel, and A. Ozdaglar (2011), "Bayesian Learning in Social Networks," *Review of Economic Studies, 78,* 1201–1236.

Banerjee, A.V. (1992), "A Simple Model of Herd Behavior," *Quarterly Journal of Economics, 107,* 797–817.

Behera, L., and F. Schweitzer (2003), "On Spatial Consensus Formation: Is the Sznajd Model Different from a Voter Model?," *International Journal of Modern Physics C, 14,* 1331–1354.

Berner, R., T. Gross, C. Kuehn, J. Kurths, and S. Yanchuk (2023), "Adaptive Dynamical Networks," *Physics Reports, 1031,* 1–59.

Bolletta, U., and P. Pin (2025), "Dynamic Opinion Updating with Endogenous Networks," *European Economic Review, 176,* 105045, 1–30.

Castellano, C., S. Fortunato, and V. Loreto (2009), "Statistical Physics of Social Dynamics," *Reviews of Modern Physics, 81,* 591–646.

Chakrabarti, S., and R.P. Gilles (2007), "Network Potentials," *Review of Economic Design, 11,* 13–52.

Chwe, M.S.Y. (1994), "Farsighted Coalitional Stability," *Journal of Economic Theory, 63,* 299–325.

Davey, B.A., and H.A. Priestley (2002), *Introduction to Lattices and Order,* Cambridge University Press, Cambridge.

DeGroot, M.H. (1974), "Reaching a Consensus," *Journal of the American Statistical Association, 69,* 118–121.

Demuynck, T., P.J.J. Herings, R.D. Saulle, and C. Seel (2019), "The Myopic Stable Set for Social Environments," *Econometrica, 87,* 111–138.

Demuynck, T., P.J.J. Herings, and C. Seel (2023), "Potentials in Social Environments," CentER Discussion Paper 2023-022, Tilburg University, Tilburg, pp. 1–45.

Ehrhardt, G.C., M. Marsili, and F. Vega-Redondo (2006), "Phenomenological Models of Socioeconomic Network Dynamics," *Physical Review E – Statistical, Nonlinear, and Soft Matter Physics, 74,* 036106, 1–11.

Elboim, D., Y. Peres, and R. Peretz (2024), "The Asynchronous DeGroot Dynamics," *Random Structures and Algorithms, 65,* 857–895.

Galam, S. (2002), "Minority Opinion Spreading in Random Geometry," *European Physical Journal B, 25,* 403–406.

Golub, B., and E. Sadler (2016), "Learning in Social Networks," in Y. Bramoullé, A. Galeotti, and B.W. Rogers (eds.), *The Oxford Handbook of the Economics of Networks,* Oxford University Press, Oxford, UK, pp. 504–542.

Grabisch, M., and A. Rusinowska (2020), "A Survey on Nonstrategic Models of Opinion Dynamics," *Games, 11,* 65, 1–29.

Gross, T., and B. Blasius (2008), "Adaptive Coevolutionary Networks: A Review," *Journal of the Royal Society Interface, 5,* 259–271.

Herrera, J.L., M.G. Cosenza, K. Tucci, and J.C. González-Avella (2011), "General Coevolution of Topology and Dynamics in Networks," *Europhysics Letters, 95,* 58006, 1–6.

Holley, R.A., and T.M. Liggett (1975), "Ergodic Theorems for Weakly Interacting Infinite Systems and the

Voter Model," *Annals of Probability, 3,* 643–663.

Jackson, M.O. (2005), "A Survey of Network Formation Models: Stability and Efficiency," in G. Demand and M. Wooders (eds.), *Group formation in Economics: Networks, Clubs, and Coalitions,* 11–57.

Jackson, M.O. (2008), *Social and Economic Networks,* Princeton University Press, Princeton, New Jersey.

Jackson, M.O., and A. Watts (2001), "The Existence of Pairwise Stable Networks," *Seoul Journal of Economics, 14,* 299–321.

Jackson, M.O., and A. Watts (2002), "The Evolution of Social and Economic Networks," *Journal of Economic Theory, 106,* 265–295.

Jackson, M.O., and A. Wolinsky (1996), "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory, 71,* 44–74.

Jadbabaie, A., P. Molavi, A. Sandroni, A., and A. Tahbaz-Salehi (2012), "Non-Bayesian Social Learning," *Games and Economic Behavior, 76,* 210–225.

Kukushkin, N.S. (1999), "Potential Games: A Purely Ordinal Approach," *Economics Letters, 64,* 279–283.

Ledoux, A. (1981), "Concours résultats complets. Les victimes se sont plu á jouer le 14 d'atout. *Jeux et Stratégie, 2,* 10-–11.

Milchtaich, I. (1996), "Congestion Games with Player-specific Payoff Functions," *Games and Economic Behavior, 13,* 111–124.

Monderer, D., and L.S. Shapley (1996), "Potential Games," *Games and Economic Behavior, 14,* 124–143.

Nagel, R. (1995), "Unraveling in Guessing Games: An Experimental Study," *American Economic Review, 85,* 1313–1326.

Noorazar, H. (2020), "Recent Advances in Opinion Propagation Dynamics: A 2020 Survey," *European Physical Journal Plus, 135,* 521, 1–20.

Rogers, T., and T. Gross (2013), "Consensus Time and Conformity in the Adaptive Voter Model," *Physical Review E, 88,* 030102, 1–17.

Rosenthal, R.W. (1973), "A Class of Games Possessing Pure-strategy Nash Equilibria," *International Journal of Game Theory, 2,* 65–67.

Stauffer, D. (2002), "Sociophysics: The Sznajd Model and Its Applications," *Computer Physics Communications, 146,* 93–98.

Stauffer, D. (2003), "How to Convince Others? Monte Carlo Simulations of the Sznajd Model," *AIP Conference Proceedings, 690,* American Institute of Physics, pp. 147–155.

Sznajd-Weron, K. (2005), "Sznajd Model and Its Applications," arXiv Preprint Physics/0503239, 1–12.

Sznajd-Weron, K., J. Sznajd, and T. Weron (2021), "A Review on the Sznajd Model – 20 Years After," *Physica A: Statistical Mechanics and its Applications, 565,* 125537, 1–12.

Voorneveld, M., and H. Norde (1997), "A Characterization of Ordinal Potential Games," *Games and Economic Behavior, 19,* 235–242.

Watts, A. (2001), "A Dynamic Model of Network Formation," *Games and Economic Behavior, 34,* 331–341.

Young, H.P. (2004), *Strategic Learning and Its Limits,* Oxford University Press, Oxford.