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Which graphs are determined by their spectrum?

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Abstract

For almost all graphs the answer to the question in the title is still unknown. Here we survey the cases for which the answer is known. Not only the adjacency matrix, but also other types of matrices, such as the Laplacian matrix, are considered.

Keywords: Spectra of graphs; Eigenvalues; Cospectral graphs; Distance-regular graphs

1. Introduction

Consider the two graphs with their adjacency matrices, shown in Fig. 1. It is easily checked that both matrices have spectrum

$$\{[2]^1, [0]^3, [-2]^1\}$$

(exponents indicate multiplicities). This is the usual example of non-isomorphic cospectral graphs first given by Cvetković [19]. For convenience we call this couple the *Saltire pair* (since the two pictures superposed give the Scottish flag: Saltire). For graphs on less than five vertices, no pair with cospectral adjacency matrices exists, so each of these graphs is determined by its spectrum.

We abbreviate 'determined by the spectrum' to DS. The question 'which graphs are DS?' goes back for about half a century, and originates from chemistry. In 1956

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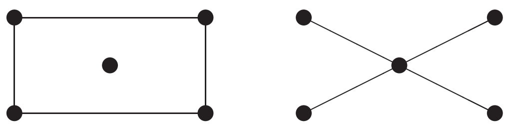


$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. 1. Two graphs with cospectral adjacency matrices.

Günthard and Primas [42] raised the question in a paper that relates the theory of graph spectra to Hückel's theory from chemistry (see also [23, Chapter 6]). At that time it was believed that every graph is DS until one year later Collatz and Sinogowitz [17] presented a pair of cospectral trees.

Another application comes from Fisher [35] in 1966, who considered a question of Kac [51]: 'Can one hear the shape of a drum?' He modeled the shape of the drum by a graph. Then the sound of that drum is characterised by the eigenvalues of the graph. Thus Kac's question is essentially ours.

After 1967 many examples of cospectral graphs were found. The most striking result of this kind is that of Schwenk [61] stating that almost all trees are non-DS (see Section 3.1). After this result there was no consensus for what would be true for general graphs (see, for example [38, p. 73]). Are almost all graphs DS, are almost no graphs DS, or is neither true? As far as we know the fraction of known non-DS graphs on n vertices is much larger than the fraction of known DS graphs (see Sections 3 and 5). But both fractions tend to zero as $n \to \infty$, and computer enumerations (Section 4) show that most graphs on 11 or fewer vertices are DS. If we were to bet, it would be for: 'almost all graphs are DS'.

Important motivation for our question comes from complexity theory. It is still undicided whether graph isomorphism is a hard or an easy problem. Since checking whether two graphs are cospectral can be done in polynomial time, the problem concentrates on checking isomorphism between cospectral graphs.

Our personal interest for the problem comes from the characterisation of distance-regular graphs. Many distance-regular graphs are known to be determined by their parameters, and some of these are also determined by their spectrum (see Section 6).

1.1. Some tools

We assume familiarity with basic results from linear algebra, graph theory, and combinatorial matrix theory. Some useful books are [11,23,38]. Nevertheless we start with some known but relevant matrix properties.

Lemma 1. *For $n \times n$ matrices A and B, the following are equivalent*:

(i) *A and B are cospectral.*
(ii) *A and B have the same chararacteristic polynomial.*
(iii) $\mathrm{tr}(A^i) = \mathrm{tr}(B^i)$ *for* $i = 1, \ldots, n$.

Proof. The equivalence of (i) and (ii) is obvious. By Newton's relations the roots $r_1 \geqslant \cdots \geqslant r_n$ of a polynomial of degree n are determined by the sums of the powers $\sum_{j=1}^{n} r_j^i$ for $i = 1, \ldots, n$. Now $\mathrm{tr}(A^i)$ is the sum of the eigenvalues of A^i which equals the sum of the ith powers of the roots of the chararacteristic polynomial. □

If A is the adjacency matrix of a graph, then $\mathrm{tr}(A^i)$ gives the total number of closed walks of length i (we assume that a closed walk has a distinguished vertex where the walk begins and ends). So cospectral graphs have the same number of closed walks of a given length i. In particular they have the same number of edges (take $i = 2$) and triangles (take $i = 3$).

Other useful tools are the following eigenvalue inequalities.

Lemma 2. *Suppose A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.*

(i) (*Interlacing*) *The eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_m$ of a principal submatrix of A of size m satisfy $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ for $i = 1, \ldots, m$.*
(ii) *Let s be the sum of the entries of A. Then $\lambda_1 \geqslant s/n \geqslant \lambda_n$, and equality on either side implies that every row sum of A equals s/n.*

1.2. The path

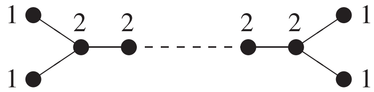
As a warm up we shall show how the results in the previous subsection can be used to prove that P_n, the path on n vertices, is DS.

Proposition 1. *The path with n vertices is determined by the spectrum of its adjacency matrix.*

Proof. The eigenvalues of P_n are

$$\lambda_i = 2\cos\frac{\pi i}{n+1}, \quad i = 1, \ldots, n$$

(see for example [23, p. 73]). So $\lambda_1 < 2$. Suppose Γ is cospectral with P_n. Then Γ has n vertices and $n-1$ edges. Furthermore, since the circuit has an eigenvalue 2, it cannot be an induced subgraph of Γ, because of eigenvalue interlacing (Lemma 2). Therefore Γ is a tree. Similarly, the star $K_{1,4}$ has an eigenvalue 2, so $K_{1,4}$ is not a subgraph of Γ. Also the following graph has an eigenvalue 2 (as can be seen from the given eigenvector):



So Γ is a tree with no vertex of degree at least 4 and at most one vertex of degree 3. Suppose x is a vertex of degree 3. Moving one branch at x to an endpoint of Γ, changes Γ into P_n. Since Γ and P_n are cospectral, this operation should not change the number of closed walks of length 4. But it clearly does (in a graph without 4 cycles, the number of closed walks of length 4 equals twice the number of edges plus four times the number of induced paths of length 2; and the operation decreases the latter number by one)! Hence Γ has no vertex of degree 3, so Γ is isomorphic to P_n. $\square$

2. The matrix

In the introduction we considered the usual adjacency matrix. But other matrices are customary too, and of course the answer to the main question depends on the choice of the matrix.

Suppose G is a graph on n vertices with adjacency matrix A. A linear combination of A, J (the all-ones matrix) and I (the identity matrix) with a non-zero coefficient for A, is called a *generalised adjacency matrix*. Let D be the diagonal matrix with the degrees of G on the diagonal (A and D have the same vertex ordering). In this paper we will mainly consider matrices that are a linear combination of a generalised adjacency matrix and D. The following matrices are distinguished:

1. The adjacency matrix A.
2. The adjacency matrix of the complement $\overline{A} = J - A - I$.
3. The Laplacian matrix $L = D - A$ (sometimes called Laplace matrix, or matrix of admittance).
4. The signless Laplacian matrix $|L| = D + A$.
5. The Seidel matrix $S = \overline{A} - A = J - 2A - I$.

Note that in this list A, $\overline{A}$ and S are generalised adjacency matrices. It is clear that for our problem it does not matter if we consider the matrix A or $\alpha A + \beta I$ (with $\alpha \neq 0$). Moreover the Laplacian matrix has the all-ones vector $\mathbf{1}$ as an eigenvector and therefore L and J have a common basis of eigenvectors. So two Laplacian matrices L_1 and L_2 are cospectral if and only if $\alpha L_1 + \beta I + \gamma J$ and $\alpha L_2 + \beta I + \gamma J$ (with $\alpha \neq 0$) are. In particular this holds for the Laplacian matrix of the complement $\overline{L} = nI - J - L$.

2.1. Regularity

If G is regular, the all-ones vector $\mathbf{1}$ is an eigenvector for every matrix considered above and so, as far as cospectrality is concerned, there is no difference between the matrices $A, \overline{A}, L, |L|$ and S. One must be careful here. The observation only holds within the class of regular graphs. In the next subsection we shall see that for the Seidel matrix a non-regular graph may be cospectral with a regular graph, whilst they are not cospectral with respect to one of the other matrices. In fact, we have the following result.

Proposition 2. *Let $\alpha \neq 0$. With respect to the matrix $Q = \alpha A + \beta J + \gamma D + \delta I$, a regular graph cannot be cospectral with a non-regular one, except possibly when $\gamma = 0$ and $-1 < \beta/\alpha < 0$.*

Proof. Without loss of generality we may assume that $\alpha = 1$ and $\delta = 0$. Let n be the number of vertices of the graph (which follows from the spectrum), and let d_i, $i = 1, \ldots, n$, be a putative sequence of vertex degrees.

First suppose that $\gamma \neq 0$. Then it follows from $\mathrm{tr}(Q)$ that $\sum_i d_i$ is determined by the spectrum of Q. Since $\mathrm{tr}(Q^2) = \beta^2 n^2 + (1 + 2\beta + 2\beta\gamma) \sum_i d_i + \gamma^2 \sum_i d_i^2$, it also follows that $\sum_i d_i^2$ is determined by the spectrum. Now Cauchy's inequality states that $(\sum_i d_i)^2 \leqslant v \sum_i d_i^2$ with equality if and only if $d_1 = d_2 = \cdots = d_n$. This shows that regularity of the graph can be seen from the spectrum of Q.

Next we consider the case $\gamma = 0$, and $\beta \leqslant -1$ or $\beta \geqslant 0$. Since $\mathrm{tr}(Q^2) = \beta^2 n^2 + (1 + 2\beta) \sum_i d_i$, also here it follows that $\sum_i d_i$ is determined by the spectrum of Q (we only use here that $\beta \neq -1/2$). Now Lemma 2 states that $\lambda_1(Q) \geqslant s/n \geqslant \lambda_n(Q)$, where $s = \beta n^2 + \sum_i d_i$ is the sum of the entries of Q, and equality on either side implies that every row sum of Q equals s/n. Thus equality (which can be seen from the spectrum of Q) implies that the graph is regular. On the other hand, if $\beta \geqslant 0$ ($\beta \leqslant -1$), then Q ($-Q$) is a non-negative matrix, hence if the graph is regular, then the all-ones vector is an eigenvector for eigenvalue $\lambda_1(Q) = s/n$ ($\lambda_n(Q) = s/n$). Thus also here regularity of the graph can be seen from the spectrum. □

If in this paper, we state that a regular graph is DS, without specifying the matrix, we mean that it is DS with respect to any generalised adjacency matrix for which regularity can be deduced from the spectrum. By the above proposition, this includes $A, \overline{A}, L$ and $|L|$ and thus we have:

Proposition 3. *A regular graph is DS if and only if it is DS with respect to A, $\overline{A}$, L or $|L|$.*

It is known (see [23, p. 398]) that all regular graphs on less than 10 vertices are DS, and that there are four pairs of cospectral regular graphs on 10 vertices. One such pair is given in Fig. 2 and the complements give another pair. In Section 3.3 we will present a method by which it can be seen that the two graphs of Fig. 2 are cospectral, without computing the spectra.

If in Proposition 2, $\gamma = 0$ and $\beta/\alpha = -1/2$, Q is essentially the Seidel matrix, which is the subject of the next section. In case $\gamma = 0$, $-1 < \beta/\alpha < 0$ and

Fig. 2. Two cospectral regular graphs.

$\beta/\alpha \neq -1/2$ we do not know if a regular graph can be cospectral with a non-regular one.

2.2. Seidel switching

For a given partition of the vertex set of G, consider the following operation on the Seidel matrix S of G:

$$S = \begin{bmatrix} S_1 & S_{12} \\ S_{12}^\top & S_2 \end{bmatrix} \sim \begin{bmatrix} S_1 & -S_{12} \\ -S_{12}^\top & S_2 \end{bmatrix} = \widetilde{S}$$

Observe that $\widetilde{S} = \widetilde{I}S\widetilde{I}^{-1}$, where $\widetilde{I} = \widetilde{I}^{-1} = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$, which means that S and $\widetilde{S}$ are similar, and therefore S and $\widetilde{S}$ are cospectral. Let $\widetilde{G}$ be the graph with Seidel matrix $\widetilde{S}$. The operation that changes G into $\widetilde{G}$ is called Seidel switching. It has been introduced by Van Lint and Seidel [53] and further explored by Seidel (see for example [64]). Note that only in the case that S_{12} has equally many times a -1 as a $+1$, $\widetilde{G}$ has the same number of edges as G. So $\widetilde{G}$ is hardly ever isomorphic to G. And it is easy to check that S_{12} cannot have the mentioned property for all possible partitions. Thus we have:

Proposition 4. *With respect to the Seidel matrix, no graph with more than one vertex is DS.*

It is also clear that if G is regular, $\widetilde{G}$ is in general not regular.

2.3. The signless Laplacian matrix

There is a straighforward relation between the eigenvalues of the signless Laplacian matrix of a graph and the adjacency eigenvalues of its line graph.

Suppose G is a connected graph with n vertices and m edges. Let N be the $n \times m$ vertex-edge incidence matrix of G. It easily follows (see [60]) that $\operatorname{rank}(N) = n - 1$ if G is bipartite, and $\operatorname{rank}(N) = n$ otherwise. Moreover $NN^\top = |L|$, and $N^\top N = 2I + B$, where $|L| = A + D$ is the signless Laplacian matrix of G and B is the adjacency matrix of the line graph $L(G)$ of G. Since $NN^\top$ and $N^\top N$ have the same non-zero eigenvalues (including multiplicities), the spectrum of B follows from the spectrum of $|L|$ and vice versa. More precisely, suppose $\lambda \neq 0$, then λ is an

Fig. 3. Two graphs cospectral w.r.t. $|L|$, but not w.r.t. L.

eigenvalue of $|L|$ with multiplicity μ (say) if and only if $\lambda - 2$ is an eigenvalue of B with multiplicity μ. The matrix $N^\top N$ is positive semidefinite, hence the eigenvalues of B are at least -2 and the multiplicity of the eigenvalue -2 equals $m - n + 1$ if G is bipartite and $m - n$ otherwise.

For example, if G is the path P_n, then $L(G) = P_{n-1}$. In Section 1.2 we mentioned that the adjacency eigenvalues of P_{n-1} are $2\cos\frac{\pi i}{n}$ $(i = 1, \ldots, n-1)$. So -2 has multiplicity 0. Since P_n is bipartite, the signless Laplacian matrix $|L|$ of P_n has one eigenvalue 0 and the other eigenvalues are $2 + 2\cos\frac{\pi i}{n}$ for $i = 1, \ldots, n-1$.

Suppose G is bipartite. Then it is easily seen that the matrices L and $|L|$ are similar by a diagonal matrix with diagonal entries ± 1 (like we saw with Seidel switching), so they have the same spectrum. In particular the above eigenvalues are also the Laplacian eigenvalues of P_n. Also here some caution is needed. A non-bipartite graph may be cospectral with a bipartite graph with respect to L or $|L|$. So, for a bipartite graph, being DS with respect to one matrix does not have to imply being DS with respect to the other. For example the two graphs of Fig. 3 are cospectral with respect to $|L|$ (because they have the same line graph), but both graphs are DS with respect to L (see Section 4). Note that the second graph is bipartite, so for this graph L and $|L|$ have the same spectrum.

2.4. Generalised adjacency matrices

For matrices that are just a combination of A, I and J, the following theorem of Johnson and Newman [50] roughly states that cospectrality for two generalised adjacency matrices implies cospectrality for all.

Theorem 1. *For the adjacency matrix A of a graph, define $\mathscr{A} = \{A + \alpha J \mid \alpha \in \mathbb{R}\}$. If G and $\widetilde{G}$ are cospectral with respect to two matrices in $\mathscr{A}$, then G and $\widetilde{G}$ are cospectral with respect to all matrices in $\mathscr{A}$.*

Proof. Suppose that the two graphs are cospectral with respect to $A + \alpha J$ and $A + \beta J$, $\alpha \neq \beta$. Let A and $\widetilde{A}$ be the adjacency matrices of G and $\widetilde{G}$, respectively. Then

$$\mathrm{tr}((A+\alpha J)^i) = \mathrm{tr}((\widetilde{A}+\alpha J)^i) \quad \text{and}$$
$$\mathrm{tr}((A+\beta J)^i) = \mathrm{tr}((\widetilde{A}+\beta J)^i), \quad i = 1, \ldots, n.$$

From properties of the trace function like $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, $\mathrm{tr}(XJYJ) = \mathrm{tr}(XJ)\,\mathrm{tr}(YJ)$, and since $J^2 = nJ$, it follows that

$$\begin{aligned}\mathrm{tr}((A+\alpha J)^i) &= \mathrm{tr}(A^i) + i\alpha\mathrm{tr}(A^{i-1}J)\\ &\quad + f_i\big(\alpha, \mathrm{tr}(AJ), \mathrm{tr}(A^2J), \ldots, \mathrm{tr}(A^{i-2}J)\big)\end{aligned}$$

for some function f_i for $i = 1, \ldots, n$. For $\mathrm{tr}((\widetilde{A} + \alpha J)^i)$, $\mathrm{tr}((A + \beta J)^i)$, and $\mathrm{tr}((\widetilde{A} + \beta J)^i)$ we find similar expressions with the same function f_i for $i = 1, \ldots, n$. From the above equations, and by using induction on i, it can be deduced that $\mathrm{tr}(A^i) = \mathrm{tr}(\widetilde{A}^i)$ and $\mathrm{tr}(A^{i-1}J) = \mathrm{tr}(\widetilde{A}^{i-1}J)$ for $i = 1, \ldots, n$. Indeed, if $\mathrm{tr}(A^jJ) = \mathrm{tr}(\widetilde{A}^jJ)$ for $j = 1, \ldots, i-2$, then

$$\begin{aligned}&\mathrm{tr}(A^i) + i\alpha\mathrm{tr}(A^{i-1}J) = \mathrm{tr}(\widetilde{A}^i) + i\alpha\mathrm{tr}(\widetilde{A}^{i-1}J) \quad \text{and}\\ &\mathrm{tr}(A^i) + i\beta\mathrm{tr}(A^{i-1}J) = \mathrm{tr}(\widetilde{A}^i) + i\beta\mathrm{tr}(\widetilde{A}^{i-1}J)\end{aligned}$$

and therefore $\mathrm{tr}(A^i) = \mathrm{tr}(\widetilde{A}^i)$ and $\mathrm{tr}(A^{i-1}J) = \mathrm{tr}(\widetilde{A}^{i-1}J)$. Hence $\mathrm{tr}((A+\gamma J)^i) = \mathrm{tr}((\widetilde{A}+\gamma J)^i)$ for $i = 1, \ldots, n$ for any γ. Thus, according to Lemma 1, G and $\widetilde{G}$ are cospectral with respect to all matrices in $\mathscr{A}$. □

The above argument is due to Godsil and McKay [39, Theorem 3.6]. They used it for a related characterisation of graphs that are cospectral with respect to both A and $\overline{A}$. Note that $-\overline{A} - I \in \mathscr{A}$. Thus we have:

Corollary 1. *If two graphs are cospectral with respect to A and $\overline{A}$, then they are cospectral with respect to any generalised adjacency matrix.*

Note that also $-\frac{1}{2}(S + I) \in \mathscr{A}$, so we may replace A and $\overline{A}$ in Corollary 1 by A and S, or by $\overline{A}$ and S.

One might wonder if a similar result holds for linear combinations of A and D. This is not the case, as the example in Fig. 4 found by Spence (private communication) shows. The two graphs have the same spectrum with respect to the adjacency matrix A and the Laplacian matrix L, but not with respect to the signless Laplacian matrix $|L|$. Hence (see Section 2.3) also the line graphs of the graphs from Fig. 4 have different adjacency spectra.

Godsil and McKay [39, Table 4, third pair] already gave a pair of graphs which are cospectral with respect to the adjacency matrix A and with respect to the signless Laplacian matrix $|L|$ (so their line graphs are cospectral with respect to the adjacency matrix), but not with respect to D (i.e. they have different degree

Fig. 4. Two graphs cospectral w.r.t. A and L, but not w.r.t. $|L|$.

sequences). It turns out that the two graphs are also not cospectral with respect to the Laplacian matrix.

2.5. *Other matrices*

The distance matrix Δ is the matrix for which $(\Delta)_{i,j}$ gives the distance in the graph between vertex i and j. Note that $\Delta = 2J - 2I - A$ for graphs with diameter two. Since almost all graphs have diameter two, the spectrum of a distance matrix only gives additional information if the graphs have relatively few edges, such as trees (see Section 3.1). Other matrices that have been considered are polynomials in A or L, and Chung [16] prefers a scaled version of the Laplacian matrix: $D^{-1/2}LD^{-1/2}$. But with respect to all these matrices there exist cospectral non-isomorphic graphs. The examples come from finite geometry, more precisely from the classical generalised quadrangle $Q(4, q)$, where q is an odd prime power (see for example [58]). The point graph and the line graph of this geometry are cospectral (see Section 3.2) and non-isomorphic (in fact they are strongly regular, see Section 6.1). The automorphism group acts transitivily on vertices, edges and non-edges. This means that there is no combinatorial way to distinguish between vertices, between edges and between non-edges. Therefore the graphs will be cospectral with respect to every matrix mentioned so far (and to every other sensible matrix).

The question arises whether it is possible to define the matrix of G in a (not so sensible) way such that every graph becomes DS. This is indeed the case, as follows from the following example. Fix a graph F and define the corresponding matrix A_F of G by $(A_F)_{i,j} = 1$ if F is isomorphic to an induced subgraph of G that contains i and j $(i \neq j)$, and put $(A_F)_{i,j} = 0$ otherwise. If $F = K_2$, then $A_F = A$, the adjacency matrix. However, $A_F = J - I$ for $G = F$, and $A_F = O$ for every other graph on the same number of vertices, and so F is DS with respect to A_F. If it is required that the graph G can be reconstructed from its matrix, one can take $A + 2A_F$. And moreover, let g_n denote the number of non-isomorphic graphs on n vertices and let $F_1, F_2, \ldots, F_{g_n}$ be these graphs in some order, then every graph on n vertices is DS with respect to the matrix

$$A + 2\sum_{i=1}^{g_n} iA_{F_i}.$$

In [47], Halbeisen and Hungerbühler give a result of this nature in terms of a scaled Laplacian. They define $W = \operatorname{diag}(n^{-1}, n^{-2}, n^{-4}, \ldots, n^{-2^{n-1}})$ and show that two graphs G_1 and G_2 on n vertices are isomorphic if and only if there exist orderings of the vertices such that the scaled Laplacian matrices WL_1W and WL_2W are cospectral.

In both of the above cases, it is more work to check cospectrality of the matrices than testing isomorphism. If there would be an easily computable matrix for which every graph becomes DS, the graph isomorphism problem would be solved.

3. Constructing cospectral graphs

Nowadays, many constructions of cospectral graphs are known. Most constructions from before 1988 can be found in [23, Section 6.1] and [22, Section 1.3]; see also [38, Section 4.6]. More recent constructions of cospectral graphs are presented by Seress [65], who gives an infinite family of cospectral 8-regular graphs. Graphs cospectral to distance-regular graphs can be found in [8,28,44], and Section 3.2. Notice that the mentioned graphs are regular, so they are cospectral with respect to any generalised adjacency matrix, which in this case includes the Laplacian matrix.

There exist many more papers on cospectral graphs. On regular, as well as non-regular graphs, and with respect to the Laplacian matrix as well as the adjacency matrix. We mention [5,36,46,54,57,59], but do not claim to be complete.

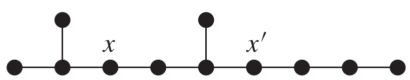
In the present paper we discuss three construction methods for cospectral graphs. One used by Schwenk to construct cospectral trees, one from incidence geometry to construct graphs cospectral with distance-regular graphs, and one presented by Godsil and McKay, which seems to be the most productive one.

3.1. Trees

Consider the adjacency spectrum. Suppose we have two cospectral pairs of graphs. Then the disjoint unions one gets by uniting graphs from different pairs, are clearly also cospectral. Schwenk [61] examined the case of uniting disjoint graphs by identifying a fixed vertex from one graph with a fixed vertex from the other graph. Such a union is called a *coalescence* of the graphs with respect to the fixed vertices. He proved the following (see also [23, p. 159] and [38, p. 65]).

Lemma 3. *Consider the adjacency spectrum. Let G and G' be cospectral graphs and let x and x' be vertices of G and G' respectively. Suppose that $G - x$ (that is the subgraph of G obtained by deleting x) and $G' - x'$ are cospectral too. Let Γ be an arbitrary graph with a fixed vertex y. Then the coalescence of G and Γ with respect to x and y is cospectral with the coalescence of G' and Γ with respect to x' and y.*

For example, let $G = G'$ be as given below, then $G - x$ and $G - x'$ are cospectral, because they are isomorphic.



Suppose $\Gamma = P_3$ and let y be the vertex of degree 2. Then Lemma 3 gives that the graphs in Fig. 5 are cospectral.

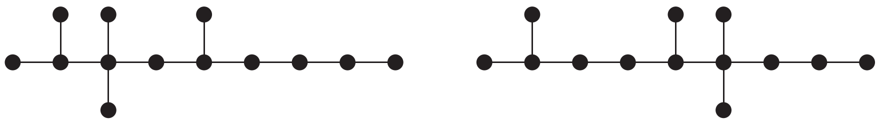


Fig. 5. Cospectral trees.

It is clear that Schwenk's method is very suitable for constructing cospectral trees. In fact, the lemma above enabled him to prove his famous theorem:

Theorem 2. *With respect to the adjacency matrix, almost all trees are non-DS.*

After Schwenk's result, trees were proved to be non-DS with respect to all kinds of matrices. Godsil and McKay [39] proved that almost all trees are non-DS with respect to the adjacency matrix of the complement $\overline{A}$, while McKay [55] proved it for the Laplacian matrix L (and hence also for $|L|$; see Section 2.3) and for the distance matrix Δ.

Others have also looked at stronger characteristics than the spectrum and showed that they are still not strong enough to determine trees. Cvetković [20] defined the angles of a graph and showed that almost all trees share eigenvalues and angles with another tree. Botti and Merris [4] showed that almost all trees share a complete set of immanental polynomials with another tree.

3.2. Partial linear spaces

A *partial linear space* consists of a (finite) set of points $\mathscr{P}$, and a collection $\mathscr{L}$ of subsets of $\mathscr{P}$ called lines, such that two lines intersect in at most one point (and consequently, two points are on at most one line). Let $(\mathscr{P}, \mathscr{L})$ be such a partial linear space and assume that each line has exactly q points, and each point is on q lines. Then clearly $|\mathscr{P}| = |\mathscr{L}|$. Let N be the point–line incidence matrix of $(\mathscr{P}, \mathscr{L})$. Then $NN^\top - qI$ and $N^\top N - qI$ both are the adjacency matrix of a graph, called the *point graph* (also known as *collinearity graph*) and *line graph* of $(\mathscr{P}, \mathscr{L})$, respectively. These graphs are cospectral, since $NN^\top$ and $N^\top N$ are. But in many examples they are non-isomorphic. In fact, the pairs of cospectral graphs coming from generalised quadrangles mentioned in Section 2.5 are of this type.

Here we present more explicitly an example from [44]. The points are all ordered q-tuples from the set $\{1, \ldots, q\}$. So $|\mathscr{P}| = q^q$. Lines are the sets consisting of q such q-tuples that are identical in all but one coordinate. The point graph of this geometry is the well-known Hamming graph $H(q, q)$. It is a famous distance-regular graph (see Section 6) of diameter q, with the property that any two vertices at distance two have exactly 2 common neighbours. If $q \geqslant 3$, the line graph does not have this property: two vertices at distance two have 1 or q common neighbours. In fact, this implies that the line graph is not even distance-regular. For $q = 3$ the geometry is

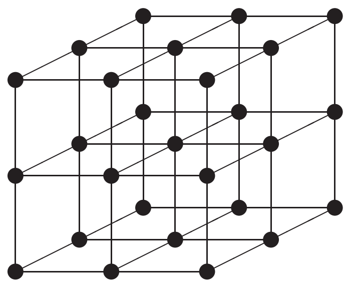


Fig. 6. The geometry of the Hamming graph $H(3,3)$.

displayed in Fig. 6. The point graph is defined on the points, with adjacency being collinearity. The vertices of the line graph are the lines, where adjacency is defined as intersection.

3.3. GM switching

In some cases Seidel switching (see Section 2.2) also leads to cospectral graphs for the adjacency spectrum (for example if the graphs G and $\widetilde{G}$ are regular of the same degree). Godsil and McKay [40] consider a more general version of Seidel switching and give conditions under which the adjacency spectrum is unchanged by this operation. We will refer to their method as GM switching. Though GM switching has been invented to make cospectral graphs with respect to the adjacency matrix, the idea also works for the Laplacian and the signless Laplacian matrix, as will be clear from the following formulation.

Theorem 3. *Let N be a $(0,1)$-matrix of size $b \times c$ (say) whose column sums are 0, b or $b/2$. Define $\widetilde{N}$ to be the matrix obtained from N by replacing each column $\mathbf{v}$ with $b/2$ ones by its complement $\mathbf{1} - \mathbf{v}$. Let B be a symmetric $b \times b$ matrix with constant row (and column) sums, and let C be a symmetric $c \times c$ matrix. Put*

$$M = \begin{bmatrix} B & N \\ N^\top & C \end{bmatrix} \quad \text{and} \quad \widetilde{M} = \begin{bmatrix} B & \widetilde{N} \\ \widetilde{N}^\top & C \end{bmatrix}.$$

Then M and $\widetilde{M}$ are cospectral.

Proof. Define

$$Q = \begin{bmatrix} \frac{2}{b}J - I_b & O \\ O & I_c \end{bmatrix}.$$

Then $Q^{-1} = Q$ and $QMQ^{-1} = \widetilde{M}$. $\square$

The matrix partition used in [40] is more general than the one presented here. But this simplified version suffices for our purposes: to show that GM switching produces many cospectral graphs.

If M and $\widetilde{M}$ are adjacency matrices of graphs then GM switching also gives cospectral complements and hence, by Theorem 1, it produces cospectral graphs with respect to any generalised adjacency matrix.

If one wants to apply GM switching to the Laplacian matrix L of a graph G, define $M = -L$. Then the requirement that B has constant row sums means that N must have constant row sums, that is, the vertices of B all have the same number of neighbours in C. In case $M = |L|$, the signless Laplacian matrix, all vertices corresponding to B must again have the same number of neighbours in C, but in addition, the subgraph of G induced by the vertices of B must be regular.

In the special situation that all columns of N have $b/2$ ones, GM switching is the same as Seidel switching. So the above theorem also gives sufficient conditions for Seidel switching to produce cospectral graphs with respect to the adjacency matrix A and the Laplacian matrix L.

If $b = 2$, GM switching just interchanges the two corresponding vertices, and we call it trivial. But if $b \geqslant 4$, GM switching almost always produces non-isomorphic graphs. In Figs. 7 and 8 we have two examples of pairs of cospectral graphs produced by GM switching. In both cases $b = c = 4$ and the upper vertices correspond to the matrix B and the lower vertices to C. In the example of Fig. 7, B corresponds to a regular subgraph and so the graphs are cospectral with respect to the adjacency matrix A, but also with respect to the adjacency matrix of the complement $\overline{A}$ and the Seidel matrix S.

In the example of Fig. 8 all vertices of B have the same number of neighbours in C, so the graphs are cospectral with respect to the Laplacian matrix L.

Also the two graphs of Fig. 2 are cospectral by GM switching (w.r.t. A, $\overline{A}$, L and $|L|$). Indeed, let B correspond to the four vertices on the corners of the rectangle.

Fig. 7. Two graphs cospectral w.r.t. any generalised adjacency matrix.

Fig. 8. Two graphs cospectral w.r.t. the Laplacian matrix.

3.4. Lower bounds

GM switching gives lower bounds for cospectral graphs with respect to several types of matrices.

Let G be a graph on $n-1$ vertices and fix a set X of three vertices. There is a unique way to extend G by one vertex x to a graph G', such that $X \cup \{x\}$ induces a regular graph in G' and that every other vertex in G' has an even number of neighbours in $X \cup \{x\}$. Thus the adjacency matrix of G' admits the structure of Theorem 3, where B corresponds to $X \cup \{x\}$. This implies that from a graph G on $n-1$ vertices one can make $\binom{n-1}{3}$ graphs with a cospectral mate on n vertices (with respect to any generalised adjacency matrix) and every such n-vertex graph can be obtained in four ways from a graph on $n-1$ vertices. Of course some of these graphs may be isomorphic, but the probability of such a coincidence tends to zero as $n \to \infty$ (see [45] for details). So, if g_n denotes the number of non-isomorphic graphs on n vertices, then:

Theorem 4. *The number of graphs on n vertices which are non-DS with respect to any generalised adjacency matrix is at least*

$$n^3 g_{n-1}\left(\tfrac{1}{24} - \mathrm{o}(1)\right).$$

The fraction of graphs with the required condition with $b = 4$ for the Laplacian matrix is roughly $2^{-n} n\sqrt{n}$. This leads to the following lower bound (again see [45] for details):

Theorem 5. *The number of non-DS graphs on n vertices with respect to the Laplacian matrix is at least*

$$rn\sqrt{n}g_{n-1}$$

for some constant $r > 0$.

In fact, a lower bound like the one in Theorem 5 can be obtained for any matrix of the form $A + \alpha D$, including the signless Laplacian matrix $|L|$.

4. Computer results

The mentioned papers [39,40] of Godsil and McKay also give interesting computer results for cospectral graphs. In [40] all graphs up to nine vertices are generated and checked on cospectrality. Recently, this enumeration has been extended to 11 vertices, and cospectrality was tested with respect to the adjacency matrix A, the set of generalised adjacency matrices (A & $\overline{A}$), the Laplacian matrix L, and the signless Laplacian matrix $|L|$, by Haemers and Spence [45]. The results are in Table 1, where we give the fractions of non-DS graphs for each of the four cases. The last

Table 1
Fractions of non-DS graphs

n	# graphs	A	A & $\overline{A}$	L	$\|L\|$	GM-A	GM-L	GM-$\|L\|$
2	2	0	0	0	0	0	0	0
3	4	0	0	0	0	0	0	0
4	11	0	0	0	0.182	0	0	0
5	34	0.059	0	0	0.118	0	0	0
6	156	0.064	0	0.026	0.103	0	0	0
7	1044	0.105	0.038	0.125	0.098	0.038	0.069	0
8	12346	0.139	0.094	0.143	0.097	0.085	0.088	0
9	274668	0.186	0.160	0.155	0.069	0.139	0.110	0
10	12005168	0.213	0.201	0.118	0.053	0.171	0.080	0.001
11	1018997864	0.211	0.208	0.090	0.038	0.174	0.060	0.001

columns give the fractions of graphs for which the GM switching gives cospectral non-isomorphic graphs with respect to the adjacency matrix (GM-A), the Laplacian matrix (GM-L) and the signless Laplacian matrix (GM-$|L|$). So column GM-A gives a lower bound for column A & $\overline{A}$ (and, of course, for column A), column GM-L is a lower bound for column L, and column GM-$|L|$ is a lower bound for column $|L|$.

Notice that for $n \leqslant 4$ there are no cospectral graphs with respect to A or to L, but there is one such pair with respect to $|L|$. This is the pair given in Fig. 3. For $n = 5$ there is just one pair with respect to A. This is of course the Saltire pair.

An interesting result from the table is that the fraction of non-DS graphs is non-decreasing for small n, but starts to decrease at $n = 10$ for A, at $n = 9$ for L, and at $n = 6$ for $|L|$. Especially for the Laplacian and the signless Laplacian matrix, these data arouse the expectation that the fraction of DS graphs tends to 1 as $n \to \infty$. In addition, the table shows that the majority of non-DS graphs with respect to A & $\overline{A}$ and L comes from GM switching (at least for $n \geqslant 7$). If this tendency continues, the lower bounds given in Theorems 4 and 5 will be asymptotically tight (with maybe another constant) and almost all graphs will be DS for all three cases. Indeed, the fraction of graphs that admit a non-trivial GM switching tends to zero as n tends to infinity, and the partitions with $b = 4$ account for most of these switchings (see also [40]).

5. DS graphs

In Section 3 we saw that many constructions for non-DS graphs are known, and in the previous section we remarked that it is more likely that almost all graphs are DS, than that almost all graphs are non-DS. Yet much less is known about DS graphs than about non-DS graphs. For example, we do not know of a satisfying counterpart to the lower bounds for non-DS graphs given in Section 3.4. The reason is that it is not easy to prove that a given graph is DS. We saw an example in the introduction, and in the coming sections we will give some more graphs which can be shown to be

DS. Like in Proposition 1, the approach goes via structural properties of the graph that follow from the spectrum. So let us start with a short survey of such properties.

5.1. Spectrum and structure

Lemma 4. *For the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix of a graph G, the following can be deduced from the spectrum*:

(i) *The number of vertices.*
(ii) *The number of edges.*
(iii) *Whether G is regular.*
(iv) *Whether G is regular with any fixed girth.*

For the adjacency matrix the following follows from the spectrum:

(v) *The number of closed walks of any fixed length.*
(vi) *Whether G is bipartite.*

For the Laplacian matrix the following follows from the spectrum:

(vii) *The number of components.*
(viii) *The number of spanning trees.*

Proof. Item (i) is clear, while (ii) and (v) have been proved in Section 1.1. Item (vi) follows from (v), since G is bipartite if and only if G has no closed walks of odd length. Item (iii) follows from Proposition 2, and (iv) follows from (iii) and the fact that in a regular graph the number of closed walks of length less than the girth depends on the degree only. The last two statements follow from well-known results on the Laplacian matrix, see for example [11]. Indeed, the corank of L equals the number of components and if G is connected, the product of the non-zero eigenvalues equals n times the number of spanning trees (the matrix-tree theorem). □

Remark that the Saltire pair shows that (vii) and (viii) do not hold for the adjacency matrix. The two graphs of Fig. 9 have cospectral Laplacian matrices. They illustrate that (v) and (vi) do not follow from the Laplacian spectrum. The two graphs given in Fig. 3 show that (v)–(viii) are false for the signless Laplacian matrix.

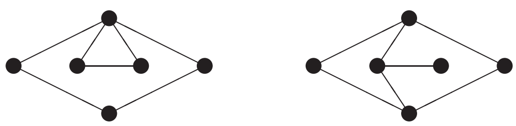


Fig. 9. Two graphs cospectral w.r.t. the Laplacian matrix.

5.2. *Some DS graphs*

Lemma 4 immediately leads to some DS graphs.

Proposition 5. *The complete graph K_n, the regular complete bipartite graph $K_{m,m}$, the cycle C_n and their complements are DS.*

Recall that a regular graph is said to be DS, if it is DS with respect to any generalised adjacency matrix for which regularity can be deduced from the spectrum; see Section 2.1.

Proof (*of Proposition 5*). By Proposition 3, we only need to show that these graphs are DS with respect to the adjacency matrix. A graph cospectral with K_n has n vertices and $n(n-1)/2$ edges and therefore equals K_n. A graph cospectral with $K_{m,m}$ is regular and bipartite with $2m$ vertices and m^2 edges, so it is isomorphic to $K_{m,m}$. A graph cospectral with C_n is 2-regular with girth n, so it equals C_n. $\square$

Proposition 6. *The disjoint union of k complete graphs, $K_{m_1} + \cdots + K_{m_k}$, is DS with respect to the adjacency matrix.*

Proof. The spectrum of the adjacency matrix A of any graph cospectral with $K_{m_1} + \cdots + K_{m_k}$ equals $\{[m_1-1]^1, \ldots, [m_k-1]^1, [-1]^{n-k}\}$, where $n = m_1 + \cdots + m_k$. This implies that $A + I$ is positive semidefinite of rank k, and hence $A + I$ is the matrix of inner products of n vectors in $\mathbb{R}^k$. All these vectors are unit vectors, and the inner products are 1 or 0. So two such vectors coincide or are orthogonal. This clearly implies that the vertices can be ordered in such a way that $A + I$ is a block diagonal matrix with all-ones diagonal blocks. The sizes of these blocks are non-zero eigenvalues of $A + I$. $\square$

In general, the disjoint union of complete graphs is not DS with respect to $\overline{A}$ and L. The Saltire pair shows that $K_1 + K_4$ is not DS for $\overline{A}$, and $K_5 + 5K_2$ is not DS for L, because it is cospectral with the Petersen graph extended by five isolated vertices (both graphs have Laplacian spectrum $\{[5]^4, [2]^5, [0]^6\}$). Note that the above proposition also shows that a complete multipartite graph is DS with respect to $\overline{A}$.

In Section 1.2 we saw that P_n, the path with n vertices, is DS with respect to A. In fact, P_n is also DS with respect to $\overline{A}$, L, and $|L|$. The result for $\overline{A}$, however, is non-trivial and the subject of [33]. For the Laplacian and the signless Laplacian matrix, there is a short proof for a more general result.

Proposition 7. *The disjoint union of k disjoint paths, $P_{n_1} + \cdots + P_{n_k}$, is DS with respect to the Laplacian matrix L and the signless Laplacian matrix $|L|$.*

Proof. The Laplacian and the signless Laplacian eigenvalues of P_n are $2+2\cos\frac{\pi i}{n}$, $i=1,\dots,n$; see Section 2.3. Suppose G is a graph cospectral with $P_{n_1}+\cdots+P_{n_k}$ with respect to L. Then all eigenvalues of L are less than 4. Lemma 4 implies that G has k components and $n_1+\cdots+n_k-k$ edges, so G is a forest. By eigenvalue interlacing (Lemma 2) every diagonal entry of L is less than 4. So every degree of G is at most 3. Let L' be the Laplacian matrix of $K_{1,3}$. The spectrum of L' equals $\{[4]^1,[1]^2,[0]^1\}$. If degree 3 occurs then $L'+D$ is a principal submatrix of L for some diagonal matrix D with non-negative entries. But then $L'+D$ has largest eigenvalue at least 4, a contradiction. So the degrees in G are at most two and hence G is the disjoint union of paths. The length m (say) of the longest path follows from the largest eigenvalue. Then the other lengths follow recursively by deleting P_m from the graph and the eigenvalues of P_m from the spectrum.

For a graph G' cospectral with $P_{n_1}+\cdots+P_{n_k}$ with respect to $|L|$, the first step is to see that G' is bipartite. This follows by eigenvalue interlacing (Lemma 2): a circuit in G' gives a submatrix L' in $|L|$ with all row sums at least 4. So L' has an eigenvalue at least 4, a contradiction, and hence G' is bipartite. Since for bipartite graphs, L and $|L|$ have the same spectrum, G' is also cospectral with $P_{n_1}+\cdots+P_{n_k}$ with respect to L. Hence $G'=P_{n_1}+\cdots+P_{n_k}$. □

In fact, $P_{n_1}+\cdots+P_{n_k}$ is also DS with respect to A. It is straightforward to adapt the proof of Proposition 1 for this more general case. But with respect to $\overline{A}$ we do not know the answer. The proof that P_n is DS with respect to $\overline{A}$ is already involved, and there is no obvious way to generalise it.

The above two propositions show that for A, $\overline{A}$, L, and $|L|$ the number of DS graphs on n vertices is bounded below by the number of partitions of n, which is asymptotically equal to $2^{\alpha\sqrt{n}}$ for some constant α. This is clearly a very poor lower bound, but we know of no better one.

In the above we saw that the disjoint union of some DS graphs is not necessarily DS. One might wonder whether the disjoint union of regular DS graphs with the same degree is always DS. The disjoint union of cycles is DS, as can be shown by a similar argument as in the proof of Proposition 7. Also the disjoint union of some copies of a strongly regular DS graph is DS; see also Proposition 10. In general we expect a negative answer, however.

5.3. Line graphs

It is well-known (see Section 2.3) that the smallest adjacency eigenvalue of a line graph is at least -2. Other graphs with least adjacency eigenvalue -2 are the cocktailparty graphs ($\overline{mK_2}$, the complement of m disjoint edges) and the so-called generalised line graphs, which are common generalisations of line graphs and cocktailparty graphs (see [22, Chapter 1]). We will not need the definition of a generalised line graph, but only use the fact that if a generalised line graph is regular, it is a line graph or a cocktailparty graph. Graphs with least eigenvalue -2 have been characterised by Cameron, Goethals, Seidel and Shult [14]. They prove that such a graph is a generalised line graph or is in a finite list of exceptions that comes from root systems. Graphs in this list are called *exceptional graphs*. A consequence of the above characterisation is the following result of Cvetković and Doob [21, Theorem 5.1] (see also [22, Theorem 1.8]).

Theorem 6. *Suppose a regular graph Γ has the adjacency spectrum of the line graph $L(G)$ of a connected graph G. Suppose G is not one of the 15 regular 3-connected graphs on eight vertices, or $K_{3,6}$, or the semiregular bipartite graph with nine vertices and 12 edges. Then Γ is the line graph $L(G')$ of a graph G'.*

We would like to deduce from this theorem that the line graph of a connected regular DS graph, which is not one of the mentioned exceptions, is DS itself. This, however, is not possible. The reason is that the line graph $L(G)$ of a regular DS graph G can be cospectral with the line graph $L(G')$ of a graph G', which is not cospectral with G. Take for example $G = L(K_6)$ and $G' = K_{6,10}$, or $G = L(\text{Petersen})$ and $G' = \text{IG}(6, 3, 2)$, the incidence graph of the 2-(6, 3, 2) design. The following lemma gives neccessary conditions for this phenomenon (cf. [12, Theorem 1.7]).

Lemma 5. *Let G be a k-regular connected graph on n vertices and let G' be a connected graph such that $L(G)$ is cospectral with $L(G')$. Then G' is cospectral with G, or G' is a semiregular bipartite graph with $n+1$ vertices and $nk/2$ edges, where $(n, k) = (\beta^2 - 1, \alpha\beta)$ for integers α and β, with $\alpha \leqslant \frac{1}{2}\beta$.*

Proof. Suppose that G has m edges. Since $L(G')$ is cospectral with $L(G)$, $L(G')$ is regular and hence G' is regular or semiregular bipartite. Next we apply the results of Section 2.3. If G' is not bipartite, G' is regular with n vertices and hence G and G' are cospectral. Otherwise G' is semiregular bipartite with $n+1$ vertices and m edges with parameters (n_1, n_2, k_1, k_2) (say). Then $m = nk/2 = n_1k_1 = n_2k_2$ and $n = n_1 + n_2 - 1$. Also the signless Laplacian matrices $|L|$ and $|L'|$ of G and G' have the same non-zero eigenvalues. In particular the largest eigenvalues are equal. So $2k = k_1 + k_2$. Write $k_1 = k - \alpha$ and $k_2 = k + \alpha$, then

$$(n_1 + n_2 - 1)k = nk = n_1k_1 + n_2k_2 = n_1(k - \alpha) + n_2(k + \alpha).$$

Hence $k = (n_1 - n_2)\alpha$, which among others implies that $\alpha \neq 0$. Now $n_1(k - \alpha) = n_2(k + \alpha)$ gives

$$\alpha n_1(n_1 - n_2 - 1) = \alpha n_2(n_1 - n_2 + 1),$$

which leads to $(n_1 - n_2)^2 = n_1 + n_2$. Put $\beta = n_1 - n_2$, then $(n, k) = (\beta^2 - 1, \alpha\beta)$. Since $k_1 \leqslant n_2$ and $k_2 \leqslant n_1$, it follows that $\alpha \leqslant \frac{1}{2}\beta$. $\square$

Now the following can be concluded from Theorem 6 and Lemma 5.

Theorem 7. *Suppose G is a connected regular DS graph, which is not a 3-connected graph with eight vertices, or a regular graph with $\beta^2 - 1$ vertices and degree $\alpha\beta$ for some integers α and β, with $\alpha \leqslant \frac{1}{2}\beta$. Then also the line graph $L(G)$ of G is DS.*

This theorem enables us to construct recursively infinitely many DS graphs by repeatedly taking line graphs or complements. We have to avoid the 3-connected graphs on eight vertices, which is not a problem, and if a graph with parameters $(\beta^2 - 1, \alpha\beta)$ arises we take the complement. As a starting graph we can take any connected regular graph with $k \geqslant 3$, $n \leqslant 9$ and $n \neq 8$, or one of the graphs from Proposition 5 with $n \neq 8$. Though this construction gives many DS graphs, they all are line graphs and complements of line graphs of regular graphs. In particular they do not give examples for every n and hence do not improve the lower bound mentioned in the previous subsection.

Bussemaker, Cvetković, and Seidel [12] determined all connected regular exceptional graphs (see also [24]). There are exactly 187 such graphs, of which 32 are DS. This leads to the following characterisation.

Theorem 8. *Suppose Γ is a connected regular DS graph with all its adjacency eigenvalues at least -2, then one of the following occurs:*

(i) *Γ is the line graph of a connected regular DS graph.*
(ii) *Γ is the line graph of a connected semiregular bipartite graph, which is DS with respect to the signless Laplacian matrix.*
(iii) *Γ is a cocktailparty graph.*
(iv) *Γ is one of the 32 connected regular exceptional DS graphs.*

Proof. Suppose Γ is not an exceptional graph or a cocktailparty graph. Then Γ is the line graph of a connected graph G, say. Whitney [67] has proved that G is uniquely determined from Γ, unless $\Gamma = K_3$. If this is the case then $\Gamma = L(K_3) = L(K_{1,3})$, so i holds. Suppose G' is cospectral with G with respect to the signless Laplacian $|L|$. Then Γ and $L(G')$ are cospectral with respect to the adjacency matrix, so $\Gamma = L(G')$ (since Γ is DS). Hence $G = G'$. Because Γ is regular, G must be regular, or semiregular bipartite. If G is regular, DS with respect to $|L|$ is the same as DS. $\square$

All four cases from Theorem 8 do occur. For (i) and (iv) this is obvious, and (iii) occurs because the cocktailparty graphs are DS (by Propositions 3 and 6). Examples for Case (ii) are the complete graphs $K_n = L(K_{1,n})$ with $n \neq 3$. Thus the fact that K_n is DS implies that $K_{1,n}$ is DS with respect to $|L|$ if $n \neq 3$.

6. Distance-regular graphs

All regular DS graphs constructed so far have the property that either the adjacency matrix A or the adjacency matrix $\overline{A}$ of the complement has smallest eigenvalue at least -2. In this section we present other examples.

Consider a connected graph G on n vertices with diameter d. For vertices x and y of G at distance $d(x, y)$, let $b_{x,y}$ denote the number of neighbours of y at distance $d(x, y) + 1$ from x, and let $c_{x,y}$ denote the number of neighbours of y at distance $d(x, y) - 1$ from x. Suppose that for all x and y, the value of $b_{x,y}$ and $c_{x,y}$ only depends on $d(x, y)$. Then G is called *distance-regular* and we write $b_{d(x,y)}$ and $c_{d(x,y)}$ instead of $b_{x,y}$ and $c_{x,y}$. Let x be a vertex of G, then it follows that the number k_i of vertices at distance i from x is independent of x. In particular G is regular of degree $k_1 = b_0$. The array

$$\Upsilon = \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$$

is called the *intersection array* of G. The numbers n, d, b_i, c_i, k_i and $a_i = k_1 - b_i - c_i$ (take $b_d = c_0 = 0$) are the *parameters* of G. They satisfy the relations $k_0 = c_1 = 1$, $k_i c_i = k_{i-1} b_{i-1}$ for $i = 1, \ldots, d$, and $\sum_{i=0}^{d} k_i = n$. Thus all parameters are determined by Υ. Distance-regular graphs were introduced by Biggs [2]. The best reference for the subject is the monograph by Brouwer, Cohen, and Neumaier [8]; see also [38, Chapter 11]. If $d = 2$, G is called *strongly regular*, a concept first defined by Bose [3]. A basic result is that a distance-regular graph with diameter d has $d + 1$ distinct eigenvalues and that its (adjacency) spectrum

$$\Sigma = \left\{[\lambda_0]^1, [\lambda_1]^{m_1}, \ldots, [\lambda_d]^{m_d}\right\}$$

can be obtained from the intersection array and vice versa (see for example [28]). However, in general the spectrum of a graph does not tell you wether it is distance-regular or not. A famous distance-regular graph is the Hamming graph $H(d, q)$, and for $q = d \geqslant 3$ we have constructed graphs cospectral with, but non-isomorphic to $H(d, q)$ in Section 3.2. Many more examples are given in [44].

In the theory of distance-regular graphs an important question is: ‘Which graphs are determined by their intersection array Υ?’ For many distance-regular graphs this is known to be the case. The question ‘Which distance-regular graphs are determined by Σ?’ is a natural restriction. Candidates are distance-regular graphs determined by Υ. For these candidates, we have to investigate whether it can be deduced from the spectrum that the graph is distance-regular. An important class of graphs for which this is the case is the class of strongly regular graphs.

6.1. Strongly regular DS graphs

A connected regular graph with three distinct (adjacency) eigenvalues is strongly regular. Indeed, the Hoffman polynomial gives that A^2 is a linear combination of A, I and J (for a graph with distinct adjacency eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_t$, the Hoffman polynomial h is defined by $h(x) = \prod_{i \neq 0}(x - \lambda_i)$; if the graph is regular and connected with adjacency matrix A, then $h(A) = \frac{h(\lambda_0)}{n}J$, cf. [48]). Therefore $(A^2)_{i,j}$, the number of walks of length 2 between i and j, only depends on whether i and j are adjacent, non-adjacent, or coincide. Hence the graph is distance-regular with diameter 2. The disjoint union of k ($k \geqslant 2$) complete graphs of size ℓ, denoted by kK_ℓ, is also defined to be a strongly regular graph (this makes the set of strongly regular graphs closed under taking complements). Other examples of strongly regular graphs are the line graphs of K_n and $K_{m,m}$ (also known as the *triangular graphs* and the *lattice graphs*, respectively). By Propositions 5 and 6 and Theorem 7, we find the following infinite families of strongly regular DS graphs.

Proposition 8. *If $n \neq 8$ and $m \neq 4$, the graphs kK_ℓ, $L(K_n)$ and $L(K_{m,m})$ and their complements are strongly regular DS graphs.*

For $n = 8$ and $m = 4$ cospectral graphs exist. There is exactly one graph cospectral with $L(K_{4,4})$, the Shrikhande graph (see [66]), and there are three graphs cospectral with $L(K_8)$, the so called Chang graphs (see [15]). Besides the graphs of Proposition 8, only a few strongly regular DS graphs are known; these are surveyed in Table 2 ($GQ(3, 9)$ is the point graph of the unique generalised quadrangle of order $(3, 9)$, and a *local* graph of a graph G is the subgraph induced by the neighbours of a vertex of G). Being DS seems to be a very strong property for strongly regular graphs. Most strongly regular graphs have (many) cospectral mates. For example, there are exactly 32,548 non-isomorphic strongly regular graphs with spectrum $\{[15]^1, [3]^{15}, [-3]^{20}\}$ (cf. [56]). Other examples can be found in Brouwer's survey [7]. The list of strongly regular DS graphs is not growing rapidly. The latest result

Table 2
The known sporadic strongly regular DS graphs (up to complements)

n	Spectrum	Name	Reference
5	$\{[2]^1, [-\frac{1}{2}+\frac{1}{2}\sqrt{5}]^2, [-\frac{1}{2}-\frac{1}{2}\sqrt{5}]^2\}$	Pentagon	
13	$\{[6]^1, [-\frac{1}{2}+\frac{1}{2}\sqrt{13}]^6, [-\frac{1}{2}-\frac{1}{2}\sqrt{13}]^6\}$	Paley	[63]
16	$\{[5]^1, [1]^{10}, [-3]^5\}$	Clebsch	[62]
17	$\{[8]^1, [-\frac{1}{2}+\frac{1}{2}\sqrt{17}]^8, [-\frac{1}{2}-\frac{1}{2}\sqrt{17}]^8\}$	Paley	[63]
27	$\{[10]^1, [1]^{20}, [-5]^6\}$	Schläfli	[62]
50	$\{[7]^1, [2]^{28}, [-3]^{21}\}$	Hoffman-Singleton	[44]
56	$\{[10]^1, [2]^{35}, [-4]^{20}\}$	Gewirtz	[9,37]
77	$\{[16]^1, [2]^{55}, [-6]^{21}\}$	Local Higman-Sims	[6]
81	$\{[20]^1, [2]^{60}, [-7]^{20}\}$	Local GQ(3,9)	[10]
100	$\{[22]^1, [2]^{77}, [-8]^{22}\}$	Higman-Sims	[37]
112	$\{[30]^1, [2]^{90}, [-10]^{21}\}$	GQ(3,9)	[13]
162	$\{[56]^1, [2]^{140}, [-16]^{21}\}$	Local McLaughlin	[13]
275	$\{[112]^1, [2]^{252}, [-28]^{22}\}$	McLaughlin	[41]

concerns a graph on 81 vertices, and dates from 1992 (cf. [10]). Although we do not have enough evidence to conjecture that there are only finitely many strongly regular DS graphs besides the ones from Proposition 8, we do expect that only very few more strongly regular DS graphs will ever be found.

6.2. Distance-regularity from the spectrum

If $d \geqslant 3$ only in some special cases does it follow from the spectrum of a graph that it is distance-regular. The following result surveys the cases known to us.

Theorem 9. *If G is a distance-regular graph with diameter d and girth g satisfying one of the following properties, then every graph cospectral with G is also distance-regular, with the same parameters as G:*

(i) $g \geqslant 2d - 1$,
(ii) $g \geqslant 2d - 2$ *and G is bipartite*,
(iii) $g \geqslant 2d - 2$ *and* $c_{d-1}c_d < -(c_{d-1} + 1)(\lambda_1 + \cdots + \lambda_d)$,
(iv) *G is a generalised odd graph, that is,* $a_1 = \cdots = a_{d-1} = 0,\ a_d \neq 0$,
(v) $c_1 = \cdots = c_{d-1} = 1$,
(vi) *G is the dodecahedron, or the icosahedron*,
(vii) *G is the coset graph of the extended ternary Golay code*,
(viii) *G is the Ivanov–Ivanov–Faradjev graph.*

For result (i), (iv), and (vi) we refer to [9] (see also [43]), [49], and [44], respectively. Results (ii), (iii), (v), and (vii) are proved in [28] (in fact, (ii) is a special case of (iii)) and (viii) is proved [29]. Notice that the polygons C_n and the strongly regular graphs are special cases of (i), while bipartite distance-regular graphs with $d = 3$ (these are the incidence graphs of symmetric block designs, see also [23, Theorem 6.9]) are a special case of (ii).

An important result on spectral characterisations of distance-regular graphs is the following theorem of Fiol and Garriga [34].

Theorem 10. *Suppose $\widetilde{G}$ is cospectral with a distance-regular graph G with diameter d. If for every vertex x of $\widetilde{G}$, the number of vertices at distance d from x has the right value: k_d, then $\widetilde{G}$ is distance-regular.*

(In fact, Fiol and Garriga have proved a stronger result. They do not require $\widetilde{G}$ to be cospectral with a distance-regular graph, but assume $\widetilde{G}$ to be regular and connected and define d to be the number of distinct eigenvalues. Then $\widetilde{G}$ is distance-regular if and only if for each vertex x the number of vertices at distance d from x satisfies a certain expression in terms of the spectrum of $\widetilde{G}$. This expression equals k_d if $\widetilde{G}$ has the spectrum of a distance-regular graph.) Let us illustrate the use of Theorem 10 by proving case (i) of Theorem 9. Since the girth and the degree follow from the spectrum, any graph $\widetilde{G}$ cospectral with G also has girth g and degree k_1. Fix a vertex x in $\widetilde{G}$. Clearly $c_{x,y} = 1$ for every vertex y at distance less than $(g-1)/2$ from x, and $a_{x,y} = 0$ (where $a_{x,y}$ is the number of neighbours of y at distance $d(x, y)$ from x) if the distance between x and y is less then $(g-2)/2$. This implies that the number $\widetilde{k}_i$ of vertices at distance i from x equals $k_1(k_1-1)^{i-1}$ for $i = 1, \ldots, d-1$. Hence $\widetilde{k}_i = k_i$ for $i = 1, \ldots, d-1$. But then also $\widetilde{k}_d = k_d$ and $\widetilde{G}$ is distance-regular by Theorem 10.

6.3. Distance-regular DS graphs

The book by Brouwer, Cohen, and Neumaier [8] gives many distance-regular graphs determined by their intersection array. We only need to check which ones satisfy one of the properties of Theorem 9. First we give the known infinite families:

Proposition 9. *The following distance-regular graphs are DS*:

(i) *The polygons* C_n.
(ii) *The complete bipartite graphs minus a complete matching*.
(iii) *The odd graphs* O_{d+1}.
(iv) *The folded* $(2d+1)$*-cubes*.

As mentioned earlier, item (i) follows from property (i) of Theorem 9 (and from Proposition 5). Item (ii) follows from property (ii) of Theorem 9, and the graphs of items (iii) and (iv) are all generalised odd graphs, so the result follows from property (iv), due to Huang and Liu [49].

The remaining known distance-regular DS graphs are presented in Tables 3 and 4. In these tables we denote by $\mathrm{IG}(v, k, \lambda)$ the point–block incidence graph of a 2-(v, k, λ) design, and by GH, GO, and GD the point graph of a generalised hexagon, generalised octagon, and generalised dodecagon, respectively. By $\mathrm{IG}(\mathrm{AG}(2, q)\backslash\mathrm{pc})$ we denote the point–line incidence graph of the affine plane of order q minus a parallel class of lines (sometimes called a *bi-affine plane*). For all but one graph the fact that they are unique (that is, determined by their parameters) can be found in [8]. Uniqueness of the Perkel graph has been proved only recently [18]. The last columns in the tables refer to the relevant theorems by which distance-regularity follows from the spectrum.

Note that Proposition 9 and Tables 3 and 4 include some famous distance-regular graphs, such as the Heawood graph, the Pappus graph, the line graph of the Petersen graph and Tutte's 8-cage. We remark finally that also the complements of distance-regular DS graphs are DS (but not distance-regular, unless $d = 2$).

7. Graphs with few eigenvalues

Like distance-regular graphs, graphs with few distinct eigenvalues have a lot of structure. The regular graphs with two (complete graphs) or three eigenvalues

Table 3
Sporadic distance-regular DS graphs with diameter 3

n	Spectrum	g	Name	Theorem
12	$\{[5]^1, [\sqrt{5}]^3, [-1]^5, [-\sqrt{5}]^3\}$	3	Icosahedron	9(vi)
14	$\{[3]^1, [\sqrt{2}]^6, [-\sqrt{2}]^6, [-3]^1\}$	6	Heawood; IG(7, 3, 1); GH(1, 2)	9(i)
14	$\{[4]^1, [\sqrt{2}]^6, [-\sqrt{2}]^6, [-4]^1\}$	4	IG(7, 4, 2)	9(ii)
15	$\{[4]^1, [2]^5, [-1]^4, [-2]^5\}$	3	L(Petersen)	7
21	$\{[4]^1, [1+\sqrt{2}]^6, [1-\sqrt{2}]^6, [-2]^8\}$	3	GH(2, 1); L(IG(7, 3, 1))	9(v), 7
22	$\{[5]^1, [\sqrt{3}]^{10}, [-\sqrt{3}]^{10}, [-5]^1\}$	4	IG(11, 5, 2)	9(ii)
22	$\{[6]^1, [\sqrt{3}]^{10}, [-\sqrt{3}]^{10}, [-6]^1\}$	4	IG(11, 6, 3)	9(ii)
26	$\{[4]^1, [\sqrt{3}]^{12}, [-\sqrt{3}]^{12}, [-4]^1\}$	6	IG(13, 4, 1); GH(1, 3)	9(i)
26	$\{[9]^1, [\sqrt{3}]^{12}, [-\sqrt{3}]^{12}, [-9]^1\}$	4	IG(13, 9, 6)	9(ii)
36	$\{[5]^1, [2]^{16}, [-1]^{10}, [-3]^9\}$	5	Sylvester	9(i)
42	$\{[6]^1, [2]^{21}, [-1]^6, [-3]^{14}\}$	5	Second subconstituent Hoffman-Singleton	9(i)
42	$\{[5]^1, [2]^{20}, [-2]^{20}, [-5]^1\}$	6	IG(21, 5, 1); GH(1, 4)	9(i)
42	$\{[16]^1, [2]^{20}, [-2]^{20}, [-16]^1\}$	4	IG(21, 16, 12)	9(ii)
52	$\{[6]^1, [2+\sqrt{3}]^{12}, [2-\sqrt{3}]^{12}, [-2]^{27}\}$	3	GH(3, 1); L(IG(13, 4, 1))	9(v), 7
57	$\{[6]^1, [\frac{3+\sqrt{5}}{2}]^{18}, [\frac{3-\sqrt{5}}{2}]^{18}, [-3]^{20}\}$	5	Perkel	9(i)
62	$\{[6]^1, [\sqrt{5}]^{30}, [-\sqrt{5}]^{30}, [-6]^1\}$	6	IG(31, 6, 1); GH(1, 5)	9(i)
62	$\{[25]^1, [\sqrt{5}]^{30}, [-\sqrt{5}]^{30}, [-25]^1\}$	4	IG(31, 25, 20)	9(ii)
105	$\{[8]^1, [5]^{20}, [1]^{20}, [-2]^{64}\}$	3	GH(4, 1); L(IG(21, 5, 1))	9(v), 7
114	$\{[8]^1, [\sqrt{7}]^{56}, [-\sqrt{7}]^{56}, [-8]^1\}$	6	IG(57, 8, 1); GH(1, 7)	9(i)
114	$\{[49]^1, [\sqrt{7}]^{56}, [-\sqrt{7}]^{56}, [-49]^1\}$	4	IG(57, 49, 42)	9(ii)
146	$\{[9]^1, [\sqrt{8}]^{72}, [-\sqrt{8}]^{72}, [-9]^1\}$	6	IG(73, 9, 1); GH(1, 8)	9(i)
146	$\{[64]^1, [\sqrt{8}]^{72}, [-\sqrt{8}]^{72}, [-64]^1\}$	4	IG(73, 64, 56)	9(ii)
175	$\{[21]^1, [7]^{28}, [2]^{21}, [-2]^{125}\}$	3	L(Hoffman-Singleton)	7
186	$\{[10]^1, [4+\sqrt{5}]^{30}, [4-\sqrt{5}]^{30}, [-2]^{125}\}$	3	GH(5, 1); L(IG(31, 6, 1))	9(v), 7
456	$\{[14]^1, [6+\sqrt{7}]^{56}, [6-\sqrt{7}]^{56}, [-2]^{343}\}$	3	GH(7, 1); L(IG(57, 8, 1))	9(v), 7
506	$\{[15]^1, [4]^{230}, [-3]^{253}, [-8]^{22}\}$	5	M_{23} graph	9(i)
512	$\{[21]^1, [5]^{210}, -3]^{280}, [-11]^{21}\}$	4	Coset graph doubly truncated binary Golay code	9(iii)
657	$\{[16]^1, [7+\sqrt{8}]^{72}, [7-\sqrt{8}]^{72}, [-2]^{512}\}$	3	GH(8, 1); L(IG(73, 9, 1))	9(v), 7
729	$\{[24]^1, [6]^{264}, [-3]^{440}, [-12]^{24}\}$	3	Coset graph extended ternary Golay code	9(vii)
819	$\{[18]^1, [5]^{324}, [-3]^{468}, [-9]^{26}\}$	3	GH(2, 8)	9(v)
2048	$\{[23]^1, [7]^{506}, [-1]^{1288}, [-9]^{253}\}$	4	Coset graph binary Golay code	9(iii), (iv)
2457	$\{[24]^1, [11]^{324}, [3]^{468}, [-3]^{1664}\}$	3	GH(8, 2)	9(v)

(strongly regular graphs) have been considered in the above. Here we consider the non-regular graphs with three adjacency, or three Laplacian eigenvalues, and the regular graphs with four eigenvalues. From the results on these graphs (see for example [25–27,30]) we find that there are among them some (families of) DS graphs. We remark that recently some bipartite DS graphs with four eigenvalues have been found; for these we refer to the forthcoming paper [31].

Table 4
Sporadic distance-regular DS graphs with diameter at least 4

n	Spectrum	d	g	Name	Theorem
18	$\{[3]^1, [\sqrt{3}]^6, [0]^4, [-\sqrt{3}]^6, [-3]^1\}$	4	6	Pappus; IG(AG(2, 3)\pc)	9(ii)
20	$\{[3]^1, [\sqrt{5}]^3, [1]^5, [0]^4, [-2]^4, [-\sqrt{5}]^3\}$	5	5	Dodecahedron	9(vi)
28	$\{[3]^1, [2]^8, [-1+\sqrt{2}]^6, [-1]^7, [-1-\sqrt{2}]^6\}$	4	7	Coxeter	9(i)
30	$\{[3]^1, [2]^9, [0]^{10}, [-2]^9, [-3]^1\}$	4	8	Tutte's 8-cage; GO(1, 2)	9(i)
32	$\{[4]^1, [2]^{12}, [0]^6, [-2]^{12}, [-4]^1\}$	4	6	IG(AG(2, 4)\pc)	9(ii)
45	$\{[4]^1, [3]^9, [1]^{10}, [-1]^9, [-2]^{16}\}$	4	3	GO(2, 1); L(GO(1, 2))	9(v), 7
50	$\{[5]^1, [\sqrt{5}]^{20}, [0]^8, [-\sqrt{5}]^{20}, [-5]^1\}$	4	6	IG(AG(2, 5)\pc)	9(ii)
80	$\{[4]^1, [\sqrt{6}]^{24}, [0]^{30}, [-\sqrt{6}]^{24}, [-4]^1\}$	4	8	GO(1, 3)	9(i)
98	$\{[7]^1, [\sqrt{7}]^{42}, [0]^{12}, [-\sqrt{7}]^{42}, [-7]^1\}$	4	6	IG(AG(2, 7)\pc)	9(ii)
102	$\{[3]^1, [\frac{1+\sqrt{17}}{2}]^9, [2]^{18}, [\theta_1]^{16}, [0]^{17}, [\theta_2]^{16}, [\frac{1-\sqrt{17}}{2}]^9, [\theta_3]^{16}\}$ ($\theta_1, \theta_2, \theta_3$ roots of $\theta^3+3\theta^2-3=0$)	7	9	Biggs–Smith	9(v)
126	$\{[3]^1, [\sqrt{6}]^{21}, [\sqrt{2}]^{27}, [0]^{28}, [-\sqrt{2}]^{27}, [-\sqrt{6}]^{21}, [-3]^1\}$	6	12	GD(1, 2)	9(i)
128	$\{[8]^1, [\sqrt{8}]^{56}, [0]^{14}, [-\sqrt{8}]^{56}, [-8]^1\}$	4	6	IG(AG(2, 8)\pc)	9(ii)
160	$\{[6]^1, [2+\sqrt{6}]^{24}, [2]^{30}, [2-\sqrt{6}]^{24}, [-2]^{81}\}$	4	3	GO(3, 1); L(GO(1, 3))	9(v), 7
170	$\{[5]^1, [\sqrt{8}]^{50}, [0]^{68}, [-\sqrt{8}]^{50}, [-5]^1\}$	4	8	GO(1, 4)	9(i)
189	$\{[4]^1, [1+\sqrt{6}]^{21}, [1+\sqrt{2}]^{27}, [1]^{28}, [1-\sqrt{2}]^{27}, [1-\sqrt{6}]^{21}, [-2]^{64}\}$	6	3	GD(2, 1); L(GD(1, 2))	9(v), 7
330	$\{[7]^1, [4]^{55}, [1]^{154}, [-3]^{99}, [-4]^{21}\}$	4	5	M_{22} graph	9(v)
425	$\{[8]^1, [3+\sqrt{8}]^{50}, [3]^{68}, [3-\sqrt{8}]^{50}, [-2]^{256}\}$	4	3	GO(4, 1); L(GO(1, 4))	9(v), 7
990	$\{[7]^1, [5]^{42}, [4]^{55}, [\frac{-1+\sqrt{33}}{2}]^{154}, [1]^{154}, [0]^{198}, [-3]^{99}, [\frac{-1-\sqrt{33}}{2}]^{154}, [-4]^{21}\}$	8	5	Ivanov–Ivanov–Faradjev	9(viii)

7.1. Regular DS graphs with four eigenvalues

Many regular graphs with four eigenvalues can be constructed from other regular graphs with at most four eigenvalues. For example, the complement of the disjoint union of some copies of a strongly regular graph has four adjacency eigenvalues. It is easy to show that if this strongly regular graph is DS, then the corresponding regular graph with four eigenvalues (and its complement, of course) is also DS. Hence, by considering the strongly regular DS graphs in Section 6.1, we find an infinite family of DS graphs.

Another way to produce regular graphs with four eigenvalues is the following product construction. For a graph G with adjacency matrix A, we define $G \otimes J_t$ as the graph with adjacency matrix $A \otimes J_t$, where $\otimes$ denotes the Kronecker product, and J_t the $t \times t$ all-ones matrix. It is shown in [25] that the graphs $C_5 \otimes J_t$ and $H \otimes J_t$ (and their complements), where H is the complement of the distance-regular graph obtained by removing a complete matching from the complete bipartite graph $K_{m,m}$ (the incidence graph IG$(m, m-1, m-2)$ of a $2-(m, m-1, m-2)$ design), are regular DS graphs with four eigenvalues. To summarize the above:

Proposition 10. *The following graphs and their complements, which have at most four eigenvalues, are regular DS graphs*:

(i) *The disjoint union of t copies of a strongly regular DS graph.*
(ii) $C_5 \otimes J_t$.
(iii) $\overline{\text{IG}(m, m-1, m-2)} \otimes J_t$.

About line graphs we can be more specific than in Section 5.3 (cf. [32, Theorems 3 and 5]). A connected regular line graph with four eigenvalues must be the line graph of a strongly regular graph, or the line graph of the incidence graph of a symmetric 2-design, or the line graph of a complete bipartite graph. Moreover, $L(K_{m,n})$ is *not* DS if and only if $\{m, n\} = \{4, 4\}$, $\{m, n\} = \{6, 3\}$, or $\{m, n\} = \{2t^2 + t, 2t^2 - t\}$ and there exists a strongly regular graph with spectrum $\{[2t^2]^1, [t]^{2t^2-t-1}, [-t]^{2t^2+t-1}\}$ (such a strongly regular graph comes from a symmetric Hadamard matrix with constant diagonal of size $4t^2$).

The line graph of a strongly regular graph G is DS if and only if G is DS, G is not $K_{4,4}$ or CP(4), and G does not have spectrum $\{[2t^2]^1, [t]^{2t^2-t-1}, [-t]^{2t^2+t-1}\}$.

The line graph of the incidence graph of a symmetric design is DS if and only if the design is determined, up to duality, by its parameters, and its incidence graph is not the Cube or $K_{4,4}$.

Note that the non-DS graph $L(K_{4,4})$ has only three distinct eigenvalues, i.e., it is strongly regular (see Section 6.1).

Besides the line graphs and the graphs from Proposition 10, there are some sporadic regular DS graphs with four eigenvalues (cf. [30]). Except for the distance-regular ones, we list them in Table 5 (up to complements); for explanation of the names of the graphs we refer to [30].

For completeness we mention that the line graph of the incidence graph of a non-symmetric 2-$(v, k, 1)$ design with $[(v-1)/(k-1)] + k > 18$ (a regular graph with five eigenvalues) is DS if and only if the design is uniquely determined by its parameters (cf. [23, Theorem 6.22]).

7.2. Non-regular DS graphs with three eigenvalues

In [26], the connected non-regular graphs with three adjacency eigenvalues have been studied. Among others, all such graphs with least eigenvalue -2 have been determined. Among them are five graphs which are DS with respect to the adjacency matrix, one of them being the cone over the Petersen graph (obtained by adding one vertex adjacent to all vertices of the Petersen graph). The sixth graph in Table 6 of sporadic non-regular DS graphs with three adjacency eigenvalues (we exclude the

Table 5
Sporadic regular DS graphs with four (adjacency) eigenvalues

n	Spectrum	Name
13	$\{[4]^1, [\theta_1]^4, [\theta_2]^4, [\theta_3]^4\}$ ($\theta_1, \theta_2, \theta_3$ roots of $\theta^3 + \theta^2 - 4\theta + 1 = 0$)	Cycl(13)
18	$\{[5]^1, [3]^1, [\frac{-1+\sqrt{13}}{2}]^8, [\frac{-1-\sqrt{13}}{2}]^8\}$	Twisted double $L(K_{3,3})$
18	$\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$	$K_{3,3} \oplus K_3$
18	$\{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$	BCS_{179} [12]
19	$\{[6]^1, [\theta_1]^6, [\theta_2]^6, [\theta_3]^6\}$ ($\theta_1, \theta_2, \theta_3$ roots of $\theta^3 + \theta^2 - 6\theta - 7 = 0$)	Cycl(19)
20	$\{[5]^1, [\sqrt{5}]^7, [-1]^5, [-\sqrt{5}]^7\}$	2-cover of $\overline{C_5 \otimes J_2}$
24	$\{[5]^1, [3]^6, [-1]^{14}, [-3]^3\}$	2-cover of $\overline{\overline{C_6} \otimes J_2}$
24	$\{[12]^1, [-2+2\sqrt{5}]^3, [0]^{17}, [-2-2\sqrt{5}]^3\}$	$\overline{\text{Icosahedron}} \otimes J_2$
24	$\{[14]^1, [4]^4, [2]^2, [-2]^{17}\}$	BCS_{183} [12]
24	$\{[14]^1, [\sqrt{7}]^8, [-2]^7, [-\sqrt{7}]^8\}$	Distance 2 graph of Klein graph
26	$\{[7]^1, [5]^1, [\frac{-1+\sqrt{17}}{2}]^{12}, [\frac{-1-\sqrt{17}}{2}]^{12}\}$	Twisted double Paley(13)
27	$\{[8]^1, [5]^4, [-1]^{20}, [-4]^2\}$	
27	$\{[8]^1, [\frac{1+\sqrt{45}}{2}]^6, [-1]^{14}, [\frac{1-\sqrt{45}}{2}]^6\}$	3-cover of K_9
30	$\{[20]^1, [2]^5, [0]^{19}, [-6]^5\}$	$\overline{L(\text{Petersen})} \otimes J_2$
34	$\{[9]^1, [7]^1, [\frac{-1+\sqrt{21}}{2}]^{16}, [\frac{-1-\sqrt{21}}{2}]^{16}\}$	Twisted double Paley(17)

Table 6
Sporadic non-regular DS graphs with three eigenvalues w.r.t. A

n	Spectrum	Name
11	$\{[5]^1, [1]^5, [-2]^5\}$	Cone over Petersen
14	$\{[8]^1, [1]^6, [-2]^7\}$	IG(7, 4, 2) plus clique on blocks
22	$\{[14]^1, [2]^7, [-2]^{14}\}$	Graph on points and planes of AG(3, 2)
24	$\{[11]^1, [3]^7, [-2]^{16}\}$	"Switched" $L(K_{5,5})$
36	$\{[21]^1, [5]^7, [-2]^{28}\}$	Switched $L(K_9)$
57	$\{[14]^1, [2]^{35}, [-4]^{21}\}$	Cone over Gewirtz

complete bipartite graphs) is the cone over the Gewirtz graph. It would be interesting to see if also other cones over strongly regular DS graphs are DS.

In [27], the connected non-regular graphs with three Laplacian eigenvalues have been studied. Among them is one infinite family of DS graphs: the so-called bird cages. Such a graph is constructed by connecting a clique and a coclique (of the same size) by a complete matching, and adding one extra vertex, which is adjacent to all vertices of the coclique.

Proposition 11. *The bird cages and their complements are DS with respect to the Laplacian matrix.*

Table 7
Sporadic non-regular DS graphs with three eigenvalues w.r.t. L

n	Spectrum	Name
11	$\{[5+\sqrt{3}]^5, [5-\sqrt{3}]^5, [0]^1\}$	$P(11,5,2)$
13	$\{[4+\sqrt{3}]^6, [4-\sqrt{3}]^6, [0]^1\}$	$P(13,4,1)$
13	$\{[\frac{11+\sqrt{17}}{2}]^6, [\frac{11-\sqrt{17}}{2}]^6, [0]^1\}$	$G(7,3,1)$
21	$\{[7]^{10}, [3]^{10}, [0]^1\}$	$P(21,5,1)$ with 5 absolute points
21	$\{[7]^9, [3]^{11}, [0]^1\}$	$P(21,5,1)$ with 9 absolute points; construction 4b from AG(2, 3)
21	$\{[\frac{17+\sqrt{37}}{2}]^{10}, [\frac{17-\sqrt{37}}{2}]^{10}, [0]^1\}$	$G(11,5,2)$
21	$\{[12]^6, [7]^{14}, [0]^1\}$	switched $L(K_7)$
25	$\{[\frac{11+\sqrt{21}}{2}]^{12}, [\frac{11-\sqrt{21}}{2}]^{12}, [0]^1\}$	Construction 4c from AG(2, 3)
25	$\{[\frac{19+\sqrt{61}}{2}]^{12}, [\frac{19-\sqrt{61}}{2}]^{12}, [0]^1\}$	$G(13,4,1)$
31	$\{[6+\sqrt{5}]^{15}, [6-\sqrt{5}]^{15}, [0]^1\}$	$P(31,6,1)$
36	$\{[9]^{16}, [4]^{19}, [0]^1\}$	Construction 4b from AG(2, 4)
41	$\{[7+\sqrt{8}]^{20}, [7-\sqrt{8}]^{20}, [0]^1\}$	Construction 4c from AG(2, 4)
55	$\{[11]^{25}, [5]^{29}, [0]^1\}$	Construction 4b from AG(2, 5)
61	$\{[\frac{17+\sqrt{45}}{2}]^{30}, [\frac{17-\sqrt{45}}{2}]^{30}, [0]^1\}$	Construction 4c from AG(2, 5)
105	$\{[15]^{49}, [7]^{55}, [0]^1\}$	Construction 4b from AG(2, 7)
113	$\{[\frac{23+\sqrt{77}}{2}]^{56}, [\frac{23-\sqrt{77}}{2}]^{56}, [0]^1\}$	Construction 4c from AG(2, 7)
136	$\{[17]^{64}, [8]^{71}, [0]^1\}$	Construction 4b from AG(2, 8)
145	$\{[13+\sqrt{24}]^{72}, [13-\sqrt{24}]^{72}, [0]^1\}$	Construction 4c from AG(2, 8)

All other known connected non-regular DS graphs with three eigenvalues with respect to L are listed in Table 7 (up to complements). For explanations of the names of these graphs we refer to [27].

8. Concluding remarks

Answering the question in the title for adjacency or Laplacian matrices seems out of reach. Proving that graphs are non-DS is easier than proving that they are DS. If it is indeed the case that almost all graphs are DS, then it will be very difficult to make a substantial step forward in proving this, with the described methods. The tools that we use for proving DS only seem to work for graphs with special structure, such as distance-regular graphs, but for these kind of graphs, being DS seems to be a much more special property than for arbitrary graphs. However, one could try to solve some more modest problems such as:

- Which trees are DS? Or, more modestly: Which trees are not cospectral to another tree? Some non-trivial results can be found in [33,52].

- Which linear combination of D, A, and J gives the most DS graphs? There is some evidence that the signless Laplacian matrix $|L| = D + A$ is a good candidate, see Table 1.
- Improve the lower bounds $2^{\alpha\sqrt{n}}$ from Section 5.2 for the number of DS graphs with respect to A, $\overline{A}$, L, or $|L|$.
- Extend the list of distance-regular DS graphs. Especially another unique strongly regular graph would be very interesting. A good candidate is the graph of Berlekamp, Van Lint, and Seidel [1].
- Which graphs with least adjacency eigenvalue -2 are DS? For regular graphs we saw an almost complete answer in Section 5.3. But the non-regular case is open.

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