BUSINESS CYCLES IN A TWO-SECTOR MODEL OF ENDOGENOUS GROWTH

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Abstract

This paper analyzes the impact of cyclical volatility on endogenous growth: does growth increase or decrease with increased cyclical volatility? We construct a stochastic two-sector model of endogenous growth to analyze this question in detail. We will show that economic growth is higher in the presence of business cycles, since people devote more time to learning activities in an uncertain economic environment. Human capital is a hedge against future income uncertainty. Hence, the rate of economic growth will be higher in a stochastic environment. Based on a calibration of the model, we find that economic growth increases by 0.16%-point as a result of observed business cycle variability. When account is taken of the interaction between the model's general equilibrium and the cycle, welfare gains (measured in units of a permanent percentage increase in consumption) from eliminating business cycle volatility are about 0.12%.

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1. Introduction

Low frequency movements in per capita income are denoted as economic growth, while high frequency components are called business cycles. While average growth in U.S. per capita income over the 1950-1992 period amounted to 1.75% per year, the business cycle component deviates on average by 2.45%.[2] Apparently, the road to prosperity is not a smooth one. In the public discussion at least, these business cycle fluctuations are considered as undesirable events, being a major cause of unemployment, bankruptcies, demand slack and other miseries. In turn, all these business cycle reactions might have effects on long-term economic growth. The important issue of the relationship between business cycles and long-term growth thus arises. The aim of this paper is precisely to study this issue in detail.

What kind of relationship between long-term economic growth and cyclical variability is observed in the data? Does an increase in the intensity of business cycle fluctuations generally goes along with higher or lower economic growth? There are some empirical studies dealing with this issue. Based on data from the *International Financial Statistics* of the International Monetary Fund, Kormendi and Meguire (1985) use a cross-section of forty-seven countries over the 1950-77 period. Testing simultaneously a set of hypotheses, they find a significant positive effect of cyclical variability (measured as the standard deviation of real output growth) and the mean annual growth rate: their estimation results suggest that an increase of 2%-point in the standard deviation of the rate of economic growth yields an increase in the rate of economic growth of approximately 1%-point. In Figure 1 we plot Kormendi and Meguire's data on economic growth against business cycle intensity.

<Insert Figure 1>

Grier and Tullock (1989) construct pooled cross-section/time-series data on 113 countries, using data from Summers and Heston (1984). In line with Kormendi and Meguire, they

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[2]   Per capita income figures are taken from Summers and Heston, PWT 5.6. Series are logged and detrended by the Hodrick-Prescott filter (setting the smoothing parameter to 400).

find a positive and significant effect of the standard deviation of real GDP growth on mean economic growth.[3]

The purpose of this paper is to explore this interdependency between growth and cyclical variability within the context of a Real Business Cycle (RBC) model extended with endogenous growth. In particular, we will show that (i) economic growth is higher in the presence of uncertainty about future overall productivity compared to the deterministic model, (ii) economic growth is a negative function of the persistence of the imposed sequence of productivity shocks, and (iii) economic growth is a positive function of the variance of these shocks. Quantitatively, the model can replicate the findings by Kormendi and Meguire (1985) and Grier and Tullock (1989) under certain restrictions on the persistence parameters of the exogenous stochastic technology disturbances. A second objective of this paper is to re-estimate the welfare gains of eliminating cyclical variability. While these welfare gains are typically found to be very small in models of exogenous growth (see for instance Lucas 1987, Atkeson and Phelan 1994, and İmrohoroğlu 1989), our findings suggest that welfare gains (*i.e.* the increase in lifetime utility) from eliminating business cycle volatility are about 0.12% when account is taken of the interaction between the model's general equilibrium and the cycle.

The deterministic counterpart of the model is a discrete time variant of the Lucas-Uzawa two-sector "learning-or-doing" model of endogenous growth (Lucas 1988, Uzawa 1965). In the basic model, business cycle fluctuations are driven by two mechanisms for intertemporal substitution, *viz.* (i) the consumption-or-savings choice, and (ii) the learning-or-doing choice.[4] Savings are procyclical, while learning time typically moves

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[3]   Admittedly, these findings are not undisputed. Other studies reach different and sometimes opposite conclusions (*cf.* Levine and Renelt 1992, Ramey and Ramey 1995, Martin and Rogers 1996).

[4]   The motivation to use the Lucas-Uzawa learning-or-doing model and not (for example) a simpler "Y=AK" model is that we want to construct a model that is capable of mimicing observed time series patterns of important economic variables, both at high and low frequencies. Since there are no transitional dynamics in the Y=AK model, this type of model is not a suitable business cycle model: shocks cannot be propagated forwardly in time since agents do not substitute consumption and leisure intertemporally in response to a temporary productivity disturbance. A second reason why we think that this model is not appropriate for our purpose is that it abstracts from employment as a productive factor input. Since employment fluctuations are a key feature of observed business cycle fluctuations, models that try to mimic actual business cycle patterns should attempt to

countercyclically: the opportunity costs (in terms of foregone production) of productivity-enhancing learning activities are relatively low (high) during recessions (booms) so that more (less) employees will be allocated to the learning sector during an economic downturn (expansion) relative to the production sector (*cf.* Aghion and Saint-Paul 1991, Hall 1991). These intertemporal reallocations of workers across learning and production activities along the business cycle are supported by the empirical observation that human capital creation tends to be countercyclical (*cf.* Bean 1990, Davis and Haltiwanger 1989, Saint-Paul 1993).

The key ingredient in our analysis is that business cycle fluctuations induce precautionary savings. Agents want to insure against future income risk by increasing the accumulation of human capital. In the context of our two-sector model, increasing the rate of human capital accumulation implies an enlargement of the learning sector relative to the production sector; more labour needs to be allocated to the learning sector to accomplish faster accumulation of human capital. Since human capital accumulation determines growth, the rate of economic growth will be higher in the presence of cyclical variability.

The topic of precautionary savings has received much attention in the recent literature. For example, Deaton (1991) shows that agents behave "prudently" and accumulate assets as a buffer stock to protect consumption against bad states of the economy, assuming convex marginal utility and borrowing constraints. However, the analysis in Deaton is embedded into a partial equilibrium framework; all uncertainty is focused on labour income and the real interest rate is fixed. Similar to the methodology used in this paper, Skinner (1988) analyzes the role of precautionary savings against uncertain interest rates and earnings by taking a second-order expansion of the Euler equation. Skinner finds that "it is only to the extent that annual variations in earnings signal a permanent change in future earnings that precautionary savings become important" (p.238). The intuition of this result being that a given year's income fluctuation is only a small proportion of the present value of future income. Hence, precautionary savings are very small when a year's earnings fluctuation only has transitory effects on future income, but are potentially important when output variations signal permanent effects on future income.

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explain these employment movements.

Examples of contiguous studies on the relation between economic growth and the cycle include King and Rebelo (1988), Aghion and Saint-Paul (1991), Aghion and Howitt (1992), Dellas (1991), Caballero and Hammour (1994), Stiglitz (1993), and Benavie, Grinols, and Turnovsky (1996). King and Rebelo integrate Real Business Cycle theory in an endogenous growth model and study the effect of economic fluctuations on the path of economic activity. Contrary to the basic implication of the neoclassical model that temporary shocks only exert temporary effects on the level of economic activity, King and Rebelo show that temporary disturbances have permanent effects in a two-sector endogenous growth model. The propagation mechanism at work along the cycle in Aghion and Saint-Paul is the opportunity cost effect. To generate interaction between economic growth and business cycles, Aghion and Saint-Paul assume that the cost of R&D is convex. Optimal firm policies with respect to business cycle fluctuations are asymmetric in that case: the reallocation of employees to the research sector during recessions is larger than the reverse reallocation during economic expansions. Thus, average growth in the economy is higher in the presence of business cycle volatility. Aghion and Howitt construct an endogenous growth model in which vertical innovations lead to the replacement of incumbent firms through a Schumpeterian process of creative destruction. They find that economic growth and the variability of the growth rate increase with the size of the vertical innovations. Dellas investigates the relation between economic growth and the cycle by considering the effect of stabilization policy on the allocation of production factors. Since low skill employment is disproportionally affected by cyclical variability, agents may want to accumulate human capital more rapidly so as to increase job security. Hence, stabilization policies can retard economic growth by discouraging such behaviour. Caballero and Hammour focus on the cleansing and insulation effect of recessions: old production units can more easily be scrapped during economic downturns (cleansing), but units in place may partly be sheltered from recessions when creation of new vintages is reduced (insulation). Stiglitz develops a model in which firms facing capital market imperfections reduce R&D efforts during economic downturns, so that recessions have negative effects on long-term growth. Benavie, Grinols, and Turnovsky introduce costly investment in a stochastic "Y=AK" endogenous growth model and study the effects of fiscal policy on economic growth and its variability. Perhaps closest related to the ideas in this paper is Einarsson and Marquis (1994), constructing a stochastic

growth model drawn from the family of "intermediate" models for which economic growth is partly endogenous and partly exogenous. They do not allow for complete endogenous growth since "that would have posed significant problems for the numerical solution procedure that was employed" (p.2). In the present paper we construct a stochastic model without having to fall back on exogenous growth.

To assess the relation between growth and cycles quantitatively, we use the following procedure. In section 2 we introduce the model, derive the first order conditions, normalize the variables suitably, calculate the balanced growth path, and look at comparative statics. Section three analyzes the model's dynamic properties. Technically, we loglinearize the model around the balanced growth path and solve it with the method of undetermined coefficients (McCallum 1983, Campbell 1994, Uhlig 1995). We pay particular attention to an often ignored variance term arising in the loglinearized Euler equations, since that term represents the influence of precautionary savings. A quantitative example is presented in section four. Based on a calibration of the model, we find that economic growth will increase by about 0.16%-point due to the presence of uncertainty. An evaluation of the welfare gains of eliminating business cycle variability is presented in section five. Endogenous labour supply is introduced in section six. In section seven we investigate the role of human capital externalities. Finally, section eight concludes.

2.   The Lucas-Uzawa two-sector model of endogenous growth

2.1 The model

In this section we construct a discrete time stochastic version of the Lucas-Uzawa two-sector endogenous growth model (Lucas 1988, Uzawa 1965). In the production sector, physical capital K, human capital H, and labour L are combined in order to produce one single homogeneous commodity. The production function is given by $Y=Y(K, uHL)$. Y denotes aggregate output and is concave with respect to physical capital K and effective labour input uHL separately, and exhibits constant returns to scale when factor inputs are accumulated at a uniform rate. Effective labour input is determined by the total labour force L, human capital H, and the fraction of time that an employee is allocated to the production sector u. For simplicity, we abstract from population growth and normalize L to unity. We assume the functional form of the production function to be Cobb-Douglas,

i.e.

$$Y(t) = A(t)K(t-1)^{\alpha}[u(t)H(t-1)]^{1-\alpha}, \tag{1}$$

where t is a time index, α $(1-\alpha)$ is the production elasticity of physical capital (labour), and $A(t)$ is an exogenous productivity index. Uncertainty in the efficiency to transform inputs into output in the production sector is modelled by assuming that the logarithm of $A(t)$ follows a stochastic AR(1) process:

$$\ln(A(t+1)) = (1-\phi_A)\ln(A) + \phi_A\ln(A(t)) + \epsilon_A(t+1), \qquad \epsilon_A \sim N(0,\sigma_A^2) \textbf{ i.i.d.} \tag{2}$$

where $\ln(A)$ is the unconditional mean of $\ln(A(t))$, and $|\phi_A| \leq 1$ measures the persistence of the productivity shocks (with $|\phi_A|=1$ representing a random walk).

In standard RBC models, changes in $A(t)$ are assumed to represent temporary shocks in the production technology. Our present two-sector model allows us to explain changes in the technology available to production firms in a separate block. Hence, in the context of our two-sector model we can think of $A(t)$ as representing something more general, *viz. anything* that affects total factor productivity in the production sector (think of, for instance, climate, labour union behaviour, government policies, shifts in consumer preferences, maintenance of machinery, managerial conduct).

Both types of capital are predetermined by their last period's stocks. Physical capital accumulation follows from the economy's resource constraint

$$K(t) = Y(t) - C(t) + (1-\delta_K)K(t-1), \tag{3}$$

where C denotes consumption, and $\delta_K \geq 0$ is the rate at which physical capital depreciates. In the learning sector, accumulation of human capital is based on the linear Lucas-Uzawa function[5]

$$H(t) = H(t-1)(\chi(t)[1-u(t)] + 1 - \delta_H). \tag{4}$$

We assume human capital to depreciate at rate $\delta_H \geq 0$. According to eq. 4, the stock of human capital shrinks at rate δ_H if no labour time is devoted to the learning sector

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[5] In his 1990 paper, Lucas uses a more general learning technology in which learning time enters non-linearly. An evaluation of the consequences of this non-linear learning function for the relationship between economic growth and the cycle is left for future research.

($u(t){=}1$). If all workers are learning, human capital grows at its maximum rate $\chi(t){-}\delta_H$. Within this range, there are no diminishing returns to the stock of human capital: we will thus expect the model to deliver endogenous growth for the usual reasons.

Since we conceive of learning activity as an essentially uncertain process, we assume that the productivity of research workers is not constant over time. To capture the uncertainty inherent to the process of learning, and similar to the process for $A(t)$, we assume that the efficiency to transform learning time into human capital follows a stochastic AR(1) process:

$$\ln(\chi(t{+}1)) \; = \; (1{-}\varphi_\chi)\ln(\chi){+}\varphi_\chi\ln(\chi(t)){+}\epsilon_\chi(t{+}1), \qquad \epsilon_\chi \; \sim \; N(0,\sigma_\chi^2) \; \text{i.i.d.} \qquad (5)$$

So changes in $\chi(t)$ represent temporary shocks in the research technology. As before, $\ln(\chi)$ is the unconditional mean of $\ln(\chi(t))$, and $|\varphi_\chi|{\leq}1$ measures the persistence of productivity shocks (with $|\varphi_\chi|{=}1$ representing a random walk).

Turning to the preference side of our model, we assume that the decision maker wants to maximize the expected discounted stream of future utilities

$$U \; = \; E\left[\sum_{t=0}^{\infty} \beta^t \frac{C(t)^{1-\theta}-1}{1-\theta}\right] \qquad \theta{\neq}1, \qquad (6)$$

where E is the expectation operator, θ is the relative rate of risk aversion, and β is the discount factor. The social planner maximizes intertemporal utility defined by eq. 6 subject to 1, 3, 4, and the initial conditions $K(0){>}0$ and $H(0){>}0$. For the moment, we don't include leisure time in the utility function to keep the model simple. In other words, labour supply is inelastic. Labour movements along the cycle are thus entirely between sectors. This unrealistic assumption will be relaxed in section six.

2.2 Solving the model

Solving the model proceeds along the usual lines. Let $\lambda(t)$ and $\mu(t)$ be the Lagrange multipliers for the constraints (3) and (4), where (1) has been substituted into (3). The first order conditions for an optimal path are given by

$$C(t)^{-\theta} \; = \; \lambda(t) \qquad (7)$$

Equation 7 says that, on the margin, goods must be equally valuable in their two uses,

$$\lambda(t)(1-\alpha)\frac{Y(t)}{u(t)} \;=\; \mu(t)H(t-1)\chi(t). \tag{8}$$

consumption and physical capital accumulation. Similarly, equation 8 says that, on the margin, time must be equally valuable in its two uses, production and human capital accumulation. The dynamic path of the Lagrange multipliers is given by (E_t is the expectation operator, conditional on information up to time t)

$$\lambda(t) \;=\; \beta E_t\left[\lambda(t+1)\left(\alpha\frac{Y(t+1)}{K(t)}+1-\delta_K\right)\right] \tag{9}$$

$$\mu(t) \;=\; \beta E_t\left[\lambda(t+1)(1-\alpha)\frac{Y(t+1)}{H(t)}+\mu(t+1)\big(\chi(t+1)[1-u(t+1)]+1-\delta_H\big)\right] \tag{10}$$

In order to characterize the balanced growth path, we follow Mulligan and Sala-i-Martin (1993) and transform some variables to make them stationary. Define the human-to-physical capital ratio, the consumption-to-physical capital ratio, and the output-to-physical capital ratio as $h(t)\equiv H(t)/K(t)$, $c(t)\equiv C(t)/K(t-1)$, and $y(t)\equiv Y(t)/K(t-1)$ respectively. We will now characterize the balanced growth path in terms of h, c, y, s (the savings rate), and u, where the savings rate $s\equiv(Y-C)/Y\equiv(y-c)/y$. An *equilibrium* is defined as a set of paths $\{c(t), s(t), u(t), h(t), y(t)\}$ maximizing eq. 6 and satisfying eqs. 7-10.

We want to derive reduced form expressions for the balanced growth solution of our model in terms of the structural parameters. Substituting eq. 7 into 9, transforming C and Y to c and y, and substituting the resource constraint into $K(t)/K(t-1)$ gives

$$1 \;=\; \beta E_t\left[\left(\frac{c(t+1)}{c(t)}[s(t)y(t)+1-\delta_K]\right)^{-\theta}[\alpha y(t+1)+1-\delta_K]\right]. \tag{11}$$

This is the Euler equation for the optimal consumption path over time.

We turn to the key argument in our paper. To understand the mechanics of the model, it is useful to loglinearize equation 11 along the lines of Campbell (1993), resulting in

$$0 \;\approx\; \ln\beta + E_t\{-\theta[\ln(c(t+1))-\ln(c(t))]-\theta[s(t)y(t)-\delta_K]+\alpha y(t+1)-\delta_K\}+\frac{V_1}{2}, \tag{12}$$

where

$$V_1 \equiv Var_t\{-\theta[\ln(c(t+1)) - \ln(c(t))] - \theta s(t)y(t) + \alpha y(t+1)\}.$$

That is, we approximate the original Euler equation by taking a second-order Taylor expansion; the variance of the stochastic term is included as a kind of "uncertainty premium". Under convex marginal utility, a lottery decreases the agent's utility compared to a situation where the agent gets the average outcome with certainty. To be indifferent between playing a lottery and playing a deterministic game, the agent wants to be compensated in terms of an uncertainty premium in the former case.

More formally, we have assumed here that the stochastic terms in the Euler equation are jointly conditionally homoscedastic and lognormally distributed: this should hold at least approximately. The key term here is the variance term $\frac{1}{2}V_1$. It arises since, by Jensen's inequality, the logarithm of the expectation of a random variable is not equal to the expectation of the logarithm of that variable. Put differently, if x is a lognormally distributed random variable, then $\ln(E_t[x(t+1)]) = E_t[\ln(x(t+1))] + \frac{1}{2}Var_t[\ln(x(t+1))]$.

Often this variance term is ignored, but in this paper it plays a key role in the analysis. It is important to realize that this variance term represents the effect of business cycle uncertainty on intertemporal decisions by economic agents. In particular, the variance term in equation 12 will increase the optimal growth rate of consumption: economic agents insure against future consumption losses by means of precautionary savings (*cf.* Sandmo 1970, Mirman 1971, Skinner 1988, Deaton 1991).

Since c, y, and s are constant along a balanced growth path, eq. 12 implies

$$0 \approx \ln\beta - \theta(s^*y^* - \delta_K) + \alpha y^* - \delta_K + \frac{V_1}{2}, \tag{13}$$

where balanced growth outcomes are denoted by an asterisk (*). Thus, the presence of a variance term in this Euler equation affects the balanced growth values for s^* and y^*.

The second Euler equation follows from substituting the Lagrange multiplier $\mu(t)$ from eq. 8 into 10

$$1 = \beta E_t\left[\left(\frac{c(t+1)}{c(t)}[s(t)y(t) + 1 - \delta_K]\right)^{-\theta} \frac{Y(t+1)}{Y(t)} \frac{H(t-1)}{H(t)} \frac{u(t)}{u(t+1)} \frac{\chi(t)}{\chi(t+1)} [\chi(t+1) + 1 - \delta_H]\right] \tag{14}$$

For similar reasons, the log version of this equation takes the form
where $g_Y(t+1) \equiv [Y(t+1) - Y(t)]/Y(t)$, and so forth. V_2 in this equation is defined by

$$0 \approx \ln\beta + E_t\{ -\theta[\ln(c(t+1)) - \ln(c(t))] - \theta[s(t)y(t) - \delta_K] + g_Y(t+1) - g_H(t) - g_u(t+1) - g_\chi(t+1) + [\chi(t+1) - \delta_H]\} + \frac{V_2}{2}, \tag{15}$$

$$V_2 \equiv Var_t\{ -\theta[\ln(c(t+1)) - \ln(c(t))] - \theta s(t)y(t) + g_Y(t+1) - g_H(t) - g_u(t+1) - g_\chi(t+1) + \chi(t+1)\}.$$

The variance term in this Euler equation affects in particular the agent's decision on human capital formation. As a way to insure against future income losses, agents will accumulate human capital more rapidly in an uncertain economic environment. Since human capital formation is the engine of growth in this model, we thus found the transmission channel between economic growth and business cycles.

It is important to notice that the presence of uncertainty also increases the *unconditional* average of χ. Since

$$E[\chi(t+1)|\chi(t)] = \chi^{1-\phi_\chi}\chi(t)^{\phi_\chi}E\left[e^{\epsilon_\chi(t+1)}\right] = \chi^{1-\phi_\chi}\chi(t)^{\phi_\chi}E\left[e^{\sigma_\chi^2/2}\right],$$

the unconditional average of χ is increased by a factor $\sigma_\chi^2/2$. *Ceteris paribus*, this would already lead to an increase in the rate of economic growth. However, since σ_χ is small, this term can safely be ignored.[6]

Along a balanced growth path, the growth rate of aggregate output is equal to the growth rate of the stock of human capital ($g_Y = g_H$), and u, χ are constant so that

$$0 \approx \ln\beta - \theta[s^*y^* - \delta_K] + \chi - \delta_H + \frac{V_2}{2}. \tag{16}$$

From eq. 13 and 16 we can derive the balanced output-to-physical capital ratio and the savings rate. Then, from the accumulation functions of human and physical capital, one can find the optimal fraction of production time along a balanced growth path. From the definition of the savings rate we find the balanced consumption-to-physical capital ratio, and, from the production function we obtain an expression for the ratio of human to physical capital. Finally, from $g_Y = g_K$, we find an expression for the growth rate of aggregate production. The results are summarized below.

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[6]　In the simulations we have run, we also experimented with an unconditional mean of χ increased by half of the variance of the innovation term of the AR(1) process for χ. This did not affect our results quantitatively.

$$s^* = \frac{\alpha\zeta}{\theta\psi}; \quad u^* = 1 - \frac{\zeta}{\theta\chi} + \frac{\delta_K - \delta_H}{\chi}; \quad c^* = \frac{\psi}{\alpha} - \frac{\zeta}{\theta}$$

$$y^* = \frac{\psi}{\alpha}; \quad h^* = \left(\frac{\psi}{\alpha A}\right)^{\frac{1}{1-\alpha}}\left(1 - \frac{\zeta}{\theta\chi} + \frac{\delta_K - \delta_H}{\chi}\right)^{-1}; \quad g_Y^* = \frac{\zeta}{\theta} - \delta_K,$$

where $\psi \equiv \chi + \delta_K - \delta_H + \frac{1}{2}V_2 - \frac{1}{2}V_1$, $\zeta \equiv \ln\beta + \chi + \theta\delta_K - \delta_H + \frac{1}{2}V_2$ are auxiliary terms. The model's balanced growth path is stochastic around its stationary expressions for s^*, u^*, c^* and so on. Moreover, there is a general equilibrium effect of uncertainty in this economy: the stationary balanced growth path expressions of the model are affected by the presence of the variance terms V_1 and V_2.

Proposition: The presence of an "uncertainty premium" V_1 leaves the size of the learning sector relative to the production sector unchanged and does not affect the rate of economic growth. The presence of an "uncertainty premium" V_2 enlarges the learning sector relative to the production sector and increases the rate of economic growth.

Proof: Partial differentiation shows that $\partial u^*/\partial V_1 = 0$, $\partial g_Y^*/\partial V_1 = 0$, $\partial u^*/\partial V_2 < 0$, $\partial g_Y^*/\partial V_2 > 0$.

□

Results from comparative statics are reported in Table 1. Since comparative statics for the deterministic version of the model are well-understood (*cf.* Mulligan and Sala-i-Martin 1993, Faig 1995), we concentrate on the impact of both variance terms on the balanced growth path.[7] An increase in V_1 stimulates savings, and depresses the consumption-to-physical capital ratio, the production-to-physical capital ratio, and the ratio of human to physical capital. Secondly, an increase in V_2 stimulates savings, lowers the optimal production time, and raises the consumption-to-physical capital ratio, the production-to-physical capital ratio, the ratio of human to physical capital, and the rate of economic growth. It is the presence of a variance term in the Euler equation determining the optimal

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[7]   Formally, this is not completely correct. In the comparative statics analysis, we treat V_1 and V_2 *as if* they are exogenous while they are actually endogenously determined in the model. These variance terms are complicated functions of the ultimate exogenous stochastic processes. As long as V_1 and V_2 are increasing functions of the standard deviations σ_A and σ_χ (which seems plausible), the comparative statics results also hold for the exogenous standard deviations.

accumulation of human capital that stands at the heart of our key result that growth and cycles are interrelated: business cycles create business cycle risk, and agents want to insure against this income risk by increasing human capital accumulation, and thereby the rate of economic growth.

<Insert Table 1>

These results are driven by the fact that people want to guard against future income declines in such an uncertain environment. Their means to do so is "prudent" behaviour in the sense that human capital is accumulated more rapidly. Thus, the effect of uncertainty on capital formation is not symmetric across both types of capital; there is a bias towards human capital formation. To develop some intuition behind this result, we proceed by introducing two new variables. The expected return to physical capital (conditional on information up to time t) is defined as

$$E_t[R_K(t+1)] = E_t[\alpha y(t+1) + 1 - \delta_K]. \tag{17}$$

In a similar fashion we define the expected return to human capital as

$$E_t[R_H(t+1)] = E_t\left[\frac{Y(t+1)}{Y(t)}\frac{H(t-1)}{H(t)}\frac{u(t)}{u(t+1)}\frac{\chi(t)}{\chi(t+1)}[\chi(t+1)+1-\delta_H]\right]. \tag{18}$$

Recall that both returns are implicitly present in the original Euler equations 11 and 14. The reason why the definition of the return to human capital is so complicated is that units of time must be transformed into units of the good.

Along a balanced growth path, it holds that $R_K^* = \alpha y^* + 1 - \delta_K$ and $R_H^* = \chi + 1 - \delta_H$. For the deterministic economy we thus find the arbitrage condition $R_K^* = R_H^*$. The required return to physical capital in a stochastic economy will increase (*cf.* the balanced growth expression for y^*) to the extent that uncertainty increases V_2 *relative to* V_1, whereas the rate of return to human capital will be unaffected. (Below it will be shown that in a calibrated version of the model this indeed holds true: uncertainty increases V_2 relative to V_1). In order to realize the higher balanced growth return on physical capital, physical capital must become relatively scarce. That is, h^* must be higher in the stochastic equilibrium compared to the deterministic economy. Also from the balanced growth expression for h^*

it can be seen that an increase in the ratio of human to physical capital can only be accomplished by an increase in V_2 relative to V_1. From the expression for the balanced growth rate it then directly follows that the asymmetry towards human capital accumulation goes along with an increase in research activity and in the rate of economic growth.

3.   The method of undetermined coefficients

To determine V_1 and V_2, we solve a loglinearized version of our model with the method of undetermined coefficients (McCallum 1983, Campbell 1994, Uhlig 1995).[8] Further details can be found in Appendix 1. Let $\tilde{y}(t) \equiv \ln(y(t)) - \ln(y^*)$ denote the log-deviation of $y(t) = Y(t)/K(t-1)$ from its balanced growth value y^*, and define $\tilde{c}(t)$, $\tilde{u}(t)$, $\tilde{s}(t)$, $\tilde{h}(t)$, $\tilde{A}(t)$, and $\tilde{\chi}(t)$ in a similar way. Using a linear approximation to the equations characterizing the equilibrium, the method in Uhlig (1995) yields the recursive equilibrium laws of motion in the form

$$\tilde{y}(t) \;=\; \eta_{yh}\tilde{h}(t-1) + \eta_{yA}\tilde{A}(t) + \eta_{y\chi}\tilde{\chi}(t),$$

where η_{yh} is the partial elasticity of $\tilde{y}(t)$ with respect to $\tilde{h}(t\text{-}1)$, η_{yA} is the partial elasticity of $\tilde{y}(t)$ with respect to $\tilde{A}(t)$, and $\eta_{y\chi}$ is the partial elasticity of $\tilde{y}(t)$ with respect to $\tilde{\chi}(t)$.

Given balanced growth values y^*, c^*, u^*, s^*, and h^*, the method delivers the coefficients η_{yh}, η_{yA}, $\eta_{y\chi}$, ..., η_{hh}, η_{hA}, $\eta_{h\chi}$, *i.e.* the recursive equilibrium laws of motion

$$\tilde{h}(t) \;=\; \Omega_P\,\tilde{h}(t-1) + \Omega_Q \begin{bmatrix} \tilde{A}(t) \\ \tilde{\chi}(t) \end{bmatrix} \tag{19}$$

and

$$\begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \\ \tilde{c}(t) \\ \tilde{s}(t) \end{bmatrix} = \Omega_R \tilde{h}(t-1) + \Omega_S \begin{bmatrix} \tilde{A}(t) \\ \tilde{\chi}(t) \end{bmatrix}, \tag{20}$$

where $\Omega_P,...,\Omega_S$ are matrices containing the partial elasticities we are looking for. Given these laws of motion, we can in turn back out

$$V_1 \equiv Var_t\{\theta[\tilde{c}(t) - \tilde{c}(t+1)] - \theta s^* y^*[\tilde{s}(t) + \tilde{y}(t)] + \alpha y^* \tilde{y}(t+1)\}$$

and

$$V_2 \equiv Var_t\{\theta[\tilde{c}(t) - \tilde{c}(t+1)] + (1-\theta)s^* y^*[\tilde{s}(t) + \tilde{y}(t)] + \tilde{y}(t+1) - \tilde{y}(t) \\ + (1+\chi u^*)\tilde{u}(t) - \tilde{u}(t+1) + (1+\chi u^* - \chi)\tilde{\chi}(t) + (\chi - 1)\tilde{\chi}(t+1)\},$$

and then determine the balanced growth values y^*, c^*, u^*, s^*, and h^*. In sum, we get a fixed point problem which we solve by iteration: usually few iterations suffice to achieve convergence.

4. Interaction between economic growth and business cycles: A quantitative assessment.

To obtain quantitative results, the model needs to be calibrated. The assumption of a Cobb-Douglas technology in the production sector implies that the production elasticity of physical capital, α, equals the capital share in national income. Following other RBC studies (for example Hansen 1985), we set α=0.36. We set the discount factor β equal to 0.96 (see Kydland and Prescott, 1982), interpreting one period to correspond to a year. Physical capital depreciation is set at 6% anually (*cf.* Stokey and Rebelo, 1995). We set the rate of human capital depreciation at 1.5%, which roughly corresponds to values reported in Mincer (1974). Empirical estimates for the rate of risk aversion vary widely, but are typically larger than unity. Here we take θ=1.5. The scaling factor *A* is set at 1. In a model with elastic labour supply, Jones, Manuelli, and Rossi (1993) estimate the quantities of work in the market sector and in human capital formation to be 0.17 and 0.12, respectively. To keep the same relative sector sizes in the context of our basic model

with inelastic labour supply, we choose $u^*=0.59$ in the deterministic economy. A typical value for the rate of economic growth is 2%. These two numbers are replicated by the model when the transformation rate of research time into learning, χ, is set at 0.0865. Finally we want to select reasonable parameters for the exogenous stochastic processes. Following the RBC literature, we set the persistence parameter for the AR(1) process generating shocks to the production sector at 0.81, which corresponds to a commonly used value of 0.95 for quarterly series; the standard deviation of the corresponding innovation term is set at 0.011. To obtain a reasonable overall fit to the actual U.S. business cycle experience, we set the persistence parameter for the AR(1) process generating shocks to the learning sector at 0.2; the standard deviation of the corresponding innovation term is set at 0.015.[9] Hence, shocks to the learning sector are more volatile and less persistent than shocks to the production sector.

The solution for the stochastic growth model is presented in Table 2. As part a of the table shows, V_1 is approximately zero and V_2 is in the order of magnitude of 0.005. This result is fairly robust to changes in the parameter values. In the baseline case we find the following solution: $s^*=22.1\%$; $u^*=57.1\%$; $c^*=0.29$; $y^*=0.37$; $h^*=0.37$; $g_Y^*=2.21\%$. Compared to the deterministic case, economic growth is 0.16%-point higher in the presence of uncertainty. Part b of the table reports the recursive laws of motion for the baseline parameter constellation, as well as for some other parameter choices (the parameter ranges loosely cover the empirically relevant possibilities). The solution is invariant to changes in A (not reported), and is also not very sensitive with respect to the choice of χ, δ_H, β and θ. Variations in α and δ_K have larger effects on the equilibrium solution. In response to a shock, transition dynamics to the new balanced growth path is faster when α is lower and δ_K is higher. For instance, an imbalance in the ratio of human to physical capital disappears at an annual rate of 28% when $\alpha=0.26$, while it disappears at an annual rate of only 14% when $\alpha=0.46$.

<Insert Table 2>

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[9] Einarsson and Marquis (1994) calibrate a similar type of model. They use a standard deviation of 0.055 for an AR(1) process that generates stochastic depreciation of human capital, in combination with a standard deviation of 0.0107 for an AR(1) process that generates stochastic technology shocks to the production sector. The persistence parameters are set at 0.5 and 0.81, respectively.

The interpretation of these recursive laws of motion is the following. A 1% deviation of last period's human capital to physical capital ratio will cause the current human capital to physical capital ratio to deviate from its equilibrium value by approximately 0.81% (*ceteris paribus*). That is, an initial imbalance between the stocks of human and physical capital gradually disappears and the economy converges to h^*. Similarly, a 1%-productivity shock to the production sector lowers the human to physical capital ratio by 0.48% in the current period. The intuition behind this result is that the decision maker finds it profitable to enlarge the production sector relative to the learning sector during times when productivity in the production sector is high, since the opportunity costs of learning are relatively high (*cf.* Aghion and Saint-Paul 1991, Bean 1990, Davis and Halti-wanger 1989, Hall 1991).[10] A positive 1%-productivity shock to the learning sector instantaneously raises the ratio of human to physical capital by 0.52%. The opportunity cost effect now works in reverse direction: a temporary increase of productivity in the learning sector encourages a temporary reallocation of employees towards the learning sector. Similar interpretations can be given for the other recursive laws of motion.

Impulse-response functions in case of a productivity shock of 1% in the production sector and the learning sector are illustrated in Figure 2 and 3. Normalized variables tend to return to their initial value (panel A in both figures). The untransformed variables Y, K, H, and C are permanently affected by a temporary productivity shocks, as illustrated in panel B in the figures. That is, the model exhibits hysteresis. Production, the stocks of physical and human capital, and consumption increase permanently by about 1.11% when the economy is hit by a temporary one standard deviation shock to the production sector. In case of a one standard deviation shock to the learning sector, the long-term effect is something like 0.82%.

Interesting result: Temporary productivity shocks to the production sector and the learning sector have permanent effects on Y, K, H, and C.

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[10]   In Canton (1996) it is shown that in a two-sector model of endogenous growth similar to the one analyzed in this paper but *without physical capital* (like in Aghion and Saint-Paul 1991), consumption smoothing arguments tend to outweigh opportunity cost effects for plausible values of the intertemporal elasticity of substitution (*i.e.* $\theta > 1$).

<Insert Figure 1>

Let us confront our model to the post-war U.S. experience to see how well it replicates some important business cycle characteristics. In Table 3 we summarize some statistics of the U.S. economy and our artificial economy. By-and-large, business cycle movements in the artificial economy replicate the actual experience reasonably well. Consumption is less variable than output, but consumption is too smooth in the artificial economy compared to the U.S. experience. Investments are more volatile than production and the standard deviation generated in the model economy is in the same order of magnitude as the one observed in the U.S. economy. The standard deviation of physical capital is slightly below the actually observed number. Variability in the stock of human capital strongly differs in both economies. Whereas the correlation of human capital with output is negative in the artificial economy, Einarsson and Marquis (1994) report a positive correlation. However, the findings in the artificial economy are consistent with other studies concluding that human capital accumulation is countercyclical (*cf.* Bean 1990, Davis and Haltiwanger 1989, and Saint-Paul 1993).

<Insert Table 3>

Next we want to study how changes in the characteristics of the exogenous stochastic processes affect the balanced growth rate. A number of simulations are undertaken. Results can be summarized as follows:

1. *Shocks in the productivity of production workers have no effect on human capital formation and long-term economic growth.*
2. *Shocks in the productivity of research workers have important effects on human capital formation and long-term economic growth.*
3. *The interaction between economic growth and the cycle becomes more pronounced when shocks to the learning sector are less persistent.*

The intuition of these results being that precautionary investments in human capital are more effective as a guard against future income uncertainty. Increased investments in

physical capital accumulation as a means of hedging against bad times only affects the income level, whereas increased investments in human capital formation increases the rate of economic growth.

In Figure 4 we illustrate the growth premium as a function of the standard deviation of the innovation term ε_χ, for alternative choices of the persistence parameter ϕ_χ. The calibrated value for the standard deviation of the innovation term is 1.5%. The increase in economic growth due to the presence of these exogenous stochastic disturbances becomes larger when the standard deviation of the innovation term is increased. However, the quantitative effects crucially depend on the persistence of the imposed technology shocks. Business cycles have weak effects on economic growth when shocks are relativily persistent but the effects are strong when the technology shocks are close to an i.i.d. process with no persistence at all. In order to gain some intuition behind this result, one should realize that the *change* in $\tilde{\chi}$ enters as an argument in the definition of V_2. Using the definition of the AR(1) process for technology shocks to the learning sector, one finds that

$$Var_t\{E_t[\tilde{\chi}(t+1)] - \tilde{\chi}(t)\} \ = \ (\phi_\chi - 1)^2 Var_t\{\tilde{\chi}(t)\}$$

So the conditional variance of the difference in $\tilde{\chi}$ between two points in time is decreasing in ϕ_χ: changes in the productivity of research workers become more predictable when shocks to their productivity are more persistent. Therefore, the variance term V_2 is a decreasing function of the persistence parameter ϕ_χ.

<Insert Figure 4>

5. On welfare gains of eliminating business cycle variability

An important topic in business cycle research is how detrimental income uncertainty is for a nation's welfare, since such an evaluation of the welfare costs of business cycles would reflect (an upperbound of) the social benefits to be expected from government policies aimed at smoothing cyclical income movements. In his seminal work on business cycles, Lucas finds that the welfare costs of business cycles are typically small: "eliminating

aggregate consumption variability of this magnitude entirely, would [...] be the equivalent in utility terms of an increase in *average* consumption of something less than one tenth of a percentage point" (p.27, Lucas 1987). In related work, Atkeson and Phelan (1994) calculate the costs of business cycles in an economy with incomplete markets for insuring individual income risk to be something like 0.02% of aggregate consumption.[11] This small number brings them to the conclusion that "the potential welfare gains from countercyclical policy are essentially zero" (p. 189).

Welfare costs of business cycles might be underestimated because of the assumption of stationary consumption streams. Allowing for a unit root in the stochastic process for log consumption, Obstfeld (1994) recalculates the welfare cost of business cycles. His results suggest that welfare losses to society are substantially higher, typically in the order of magnitude of 0.2%. Non-stationarity of the consumption stream also prevails in the dynamic stochastic model of endogenous growth constructed in this paper. Doing a similar type of welfare study in the context of the model from this paper is more complicated, since the stochastic process for log consumption is not a simple martingale as in Obstfeld. Furthermore, after eliminating business cycle risk the two-sector economy will go through a transition period before settling down on the new balanced growth path. This complicates matters even further.

A first experimental design to evaluate the welfare cost of business cycle fluctuations is to compare expected lifetime utility of a representative agent living in the stochastic economy to lifetime utility of an agent in a deterministic economy without exogenous productivity disturbances. However, comparability requires initial endowments $K(0)$ and $H(0)$ to be equal in both economies. As we have seen, the human-to-physical capital ratio along a balanced growth path will increase in a stochastic economy compared to a deterministic setting since agents speed up the accumulation of human capital as a hedge against future income uncertainty. An appropriate experimental set-up should take account of this imbalance between the human-to-physical capital ratio in both economies. Let us therefore consider an experiment in which we compare expected lifetime utility of a representative agent living in the stochastic economy to lifetime utility of an agent living

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[11] However, İmrohoroğlu (1989) finds much larger costs of business cycles in a model with labour indivisibilities and liquidity constraints.

in an economy in which the exogenous stochastic shocks are set to zero from $t=0$ onwards. This deterministic economy will start with an excessive human-to-physical capital stock, inherited from its history as a stochastic economy. Therefore, the deterministic economy will initially go through a transitional dynamic period to run down the precautionary excess human capital stock before arriving at its deterministic balanced growth path.

To finish the description of our experimental design, one additional comment needs to be made. In section 2 we saw that the presence of uncertainty increases the *unconditional* average of χ by a factor $\sigma_\chi^2/2$, since

$$\mathrm{E}[\chi(t+1)|\chi(t)] = \chi^{1-\phi_\chi}\chi(t)^{\phi_\chi}\mathrm{E}\!\left[e^{\epsilon_\chi(t+1)}\right] = \chi^{1-\phi_\chi}\chi(t)^{\phi_\chi}\mathrm{E}\!\left[e^{\sigma_\chi^2/2}\right],$$

To assure that the unconditional mean value of χ in the stochastic economy is identical to χ in the deterministic setting, we have to use χ-½σ_χ^2 instead of χ. The expected average value of χ is thereby exactly equal in both economies. Similarly, to assure that the unconditional mean value of A in the stochastic economy is identical to A in the deterministic setting, we use A-½σ_A^2 instead of A.

We resort to numerical simulations since our experimental design does not allow for an analytical solution. Running 100 experiments of 400 periods yields the following results. When account is taken of the interaction between the model's general equilibrium and the cycle, welfare gains (*i.e.* the increase in lifetime utility) from eliminating business cycle volatility are about 0.12%. At the beginning of the experiment, the state variable $\tilde{h}(0)=0.06262$, capturing the deviation in h^* in the stochastic equilibrium compared to the deterministic outcome. In other words, the economy is passing through a transition period before settling down in the new equilibrium. These transition dynamics are merely responsible for the beneficial effects of a policy that eliminates business cycle fluctuations. The general equilibium effects associated with business cycle fluctuations will no longer exist when the stochastic exogenous disturbances are eliminated. After elimination of the technology shocks, agents face an excessive stock of human capital which they will decrease by increasing their consumption possibilities until the economy settles down on a deterministic balanced growth path. Thus, to wind up this section, welfare gains from eliminating cyclical variability in the present model are relatively high, since people run down the "precautionary" stock of human capital by increasing consumption in the short term.

6.   Endogenous labour supply

In the prior analysis it has been assumed that labour supply is exogenous, implying that labour movements along the cycle are entirely between the production sector and the research sector. This is an unrealistic assumption: cyclical fluctuations in employment are quite large in reality. In this section we account for employment variations along business cycles by introducing endogenous labour supply. Let us assume that people are indifferent between working in the production sector and the research sector, but they dislike to supply labour time. Denoting leisure time by $l(t)$ and normalizing the total endowment of time to unity, labour time $1-l(t)$ can be divided into production time $u(t)$ and learning time $1-u(t)-l(t)$. We assume a Cobb-Douglas form of the intratemporal utility function, changing the social planner's objective function to

$$U = E\left[\sum_{t=0}^{\infty} \beta^t [\xi \ln C(t) + (1-\xi)\ln l(t)]\right] \tag{6'}$$

where, in order to find an interior solution, the relative rate of risk aversion is set at unity. Parameter $0<\xi<1$ $(1-\xi)$ measures the relative weight that is attributed to consumption (leisure). The only other equation from the basic model that needs modification in case of endogenous labour supply is the accumulation function of human capital. Eq. 4 changes into

$$H(t) = H(t-1)(\chi(t)[1-u(t)-l(t)]+1-\delta_H). \tag{4'}$$

Solving the model proceeds in a similar fashion as in section 2. The first order conditions for consumption and leisure take the form

$$\frac{\xi}{C(t)} = \lambda(t) \tag{7'}$$

$$\frac{1-\xi}{l(t)} = \mu(t)H(t-1)\chi(t). \tag{21}$$

In words, condition 21 equalizes the marginal benefit of an additional unit of leisure to the marginal cost in terms of decreased accumulation of human capital. Combining conditions 7', 21, and 8 yields the following relation between leisure time, production time, and

savings

$$\frac{1-\xi}{l(t)} = \frac{\xi}{u(t)}\frac{1-\alpha}{1-s(t)}. \tag{22}$$

The evolution of Lagrange multiplier μ is described by

$$\mu(t) = \beta E_t\left[\lambda(t+1)(1-\alpha)\frac{Y(t+1)}{H(t)}+\mu(t+1)\big(\chi(t+1)[1-u(t+1)-l(t+1)]+1-\delta_H\big)\right]. \tag{10'}$$

It can easily be shown that along the balanced growth path the model's variables satisfy

$$s^* = \frac{\alpha(\zeta-\chi l^*)}{\psi-\chi l^*}; \quad u^* = 1-\frac{\zeta}{\chi}+\frac{\delta_K-\delta_H}{\chi}; \quad c^* = \frac{\psi}{\alpha}-\zeta+\left(1-\frac{1}{\alpha}\right)\chi l^*;$$

$$y^* = \frac{\psi-\chi l^*}{\alpha}; \quad h^* = \left(\frac{\psi-\chi l^*}{\alpha A}\right)^{\frac{1}{1-\alpha}}\left(1-\frac{\zeta}{\chi}+\frac{\delta_K-\delta_H}{\chi}\right)^{-1}; \quad g_Y^* = \zeta-\chi l^*-\delta_K$$

This solution is intuitive: introducing leisure time in the agent's decision problem lowers output per unit of physical capital and thereby the incentive to save; it leaves the expression for the optimal allocation of workers to the production sector unaffected; it lowers the ratio of human to physical capital, consumption to physical capital, and the rate of economic growth. The optimal fraction of leisure time follows from

$$l_{1,2}^* = \frac{-a_2\pm\sqrt{a_2^2-4a_1a_3}}{2a_1}$$

$$a_1 \equiv \frac{(\alpha-1)\xi\chi^2}{(1-\xi)(\chi-\zeta+\delta_K-\delta_H)}; \quad a_2 \equiv \frac{(1-\alpha)\xi\chi\psi}{(1-\xi)(\chi-\zeta+\delta_K-\delta_H)}+\chi(1-\alpha); \quad a_3 \equiv \alpha\zeta-\psi.$$

To rule out complex solutions, we assume the discriminant $a_2^2-4a_1a_3$ to be nonnegative.[12] To assure nonnegative savings, we impose the constraint $l^*\leq\zeta/\chi$. Also, because of the time constraint, it must be the case that $1-u-l\geq0$.

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[12] When the discriminant is negative, the solution for the optimal fraction of leisure time can be rewritten to

$$l_{1,2}^* = \frac{-a_2\pm i\sqrt{4a_1a_3-a_2^2}}{2a_1},$$

where $i\equiv\sqrt{(-1)}$. In that case, the optimal fraction of leisure time might show oscillating patterns over time. This issue is left for future research.

We solve a similar fixed point problem as in section 4. The relative weight of consumption (ξ) is set at 0.2242 in order to replicate leisure demand to be equal to 71% in the deterministic economy. The transformation rate of research time into human capital accumulation (χ) is set at 0.2389 in order to reproduce the observation that about 17% of the total time endowment is allocated to market activity. The observed numbers for production time and leisure time are taken from Jones, Manuelli and Rossi (1993). In Table 4 we present the numerical solution for this model. As before we find that the variance term showing up in the Euler equation determining the accumulation of human capital is about 0.005. But the model with elastic labour supply predicts a much stronger interaction between economic growth and the cycle: the presence of business cycle fluctuations now increases growth from 1.35% in the deterministic economy to 2.97% in a stochastic environment.[13] Interestingly, these quantitative implications for the model with elastic labour supply roughly correspond to the findings by Kormendi and Meguire (1985) and Grier and Tullock (1989). The standard deviation for the rate of economic growth in the artificial economy is 3.23%; the empirical estimates of Kormendi and Meguire suggest that this would increase mean economic growth by something like 1.6%-point. The model with elastic labour supply roughly replicates this finding. This fit of the model to the findings by Kormendi and Meguire also motivates our choice to set the persistence parameter of shocks to the learning sector to 0.2.

To assess whether this extended two-sector model can explain employment variability along the cycle, we again compare artificially generated data with the post-war U.S. experience. Running 100 experiments over 100 periods, we find that the standard deviation of hours devoted to market activity is equal to 1.12%[14], and its correlation with output is 0.61. Employment fluctuations in the artificial economy fail to accurately replicate the actual U.S. experience (*cf.* Table 3). Both the standard deviation and the correlation of output for the artificially generated series are below their actually observed

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[13] This change in the magnitude of the interaction is due to the change in the parameterization of the model: the transformation rate of research time into human capital accumulation is set at a higher value since (because of leisure time) there is less time available to learn.

[14] In Appendix 2 we develop a procedure to calculate this unconditional variance as a subroutine in the MATLAB programs.

values of 0.0160 and 0.856, respectively. This failure in explaining labour market fluctuations seems common to the RBC literature (*cf.* Christiano and Eichenbaum 1992).

<Insert Table 4>

7.   Externalities

In the basic model from section 2 it has been assumed that human capital is a purely private commodity in the sense that an individual's human capital only affects his own productivity. However, it is often emphasized that learning is a social activity, involving groups of people. To formalize these human interactions, we follow Lucas (1988) and assume that human capital also contributes to factor productivity through an external effect. In particular, the average stock of human capital, defined by

$$H_a \equiv \frac{\int_0^\infty HL(H)dH}{\int_0^\infty L(H)dH}$$

where $L(H)$ is the mass of workers with skill level H, affects factor productivity favourably. The production function is now given by

$$Y(t) = A(t)K(t-1)^\alpha [u(t)H(t-1)]^{1-\alpha}H_a(t-1)^\Gamma \tag{1'}$$

The term H_a^Γ captures the external effect of human capital, $\Gamma \geq 0$. Since we assume that all workers are identical, the average skill level in the economy is equal to the individual stock of human capital, *i.e.* $H_a(t)=H(t)$. In the presence of externalities, the market outcome is in general not efficient. In our case of an external effect of human capital, firms do not take account of the complete effect of human capital on total factor productivity when deciding on how much labour to allocate to the research sector. Let us first analyze the social planner's problem. Such a central decision maker takes account of both the internal and the external effect of human capital, so that the evolution of μ is now described by

$$\mu(t) = \beta E_t \left[\lambda(t+1)(1-\alpha+\Gamma)\frac{Y(t+1)}{H(t)} + \mu(t+1)(\chi(t+1)[1-u(t+1)] + 1 - \delta_H) \right]. \qquad (10'')$$

An *optimal* path is defined as a contingency plan for $c(t)$, $s(t)$, $u(t)$, $h(t)$, $H_a(t)$, and $y(t)$ that maximizes utility (eq. 6), subject to 1', 7, 8, 9, 10", and the constraint $H_a(t)=H(t)$. In that case, the balanced growth path is described by

$$s^* = \frac{\alpha\left(\zeta + \frac{\Gamma}{1-\alpha}(\chi-\delta_H)\right)}{\theta\left(\psi + \frac{\Gamma}{1-\alpha}(\chi-\delta_H)\right)}; \quad u^* = 1 - \frac{\Gamma}{\theta(1-\alpha+\Gamma)} - \frac{1-\alpha}{1-\alpha+\Gamma}\frac{\zeta}{\theta\chi} + \frac{\frac{1-\alpha}{1-\alpha+\Gamma}\delta_K - \delta_H}{\chi};$$

$$y^* = \frac{\psi + \frac{\Gamma}{1-\alpha}(\chi-\delta_H)}{\alpha}; \quad g_Y^* = \frac{\zeta + \frac{\Gamma}{1-\alpha}(\chi-\delta_H)}{\theta} - \delta_K; \quad g_Y^* = g_C^* = g_K^* = \frac{1-\alpha+\Gamma}{1-\alpha}g_H^*.$$

In a market economy this external effect of human capital is not internalized by the agents deciding about the allocation of time across both sectors, *i.e.* the path of μ is given by equation 10. An *equilibrium* path is defined as a contingency plan for $c(t)$, $s(t)$, $u(t)$, $h(t)$, $H_a(t)$, and $y(t)$ that maximizes utility (eq. 6), subject to 1', 7, 8, 9, and 10. The balanced growth path is described by (denoting the market economy by a hat)

$$\hat{s}^* = \frac{\alpha\zeta}{\theta\psi}; \quad \hat{u}^* = 1 - \frac{1-\alpha}{1-\alpha+\Gamma}\frac{\zeta}{\theta\chi} + \frac{\frac{1-\alpha}{1-\alpha+\Gamma}\delta_K - \delta_H}{\chi};$$

$$\hat{y}^* = \frac{\psi}{\alpha}; \quad \hat{g}_Y^* = \frac{\zeta}{\theta} - \delta_K; \quad \hat{g}_Y^* = \hat{g}_C^* = \hat{g}_K^* = \frac{1-\alpha+\Gamma}{1-\alpha}\hat{g}_H^*.$$

Notice that, since the Euler equations in the market economy with externalities are identical to those from the basic model, the balanced growth outcome for the savings rate $\hat{s}^*$ and the output to physical capital ratio $\hat{y}^*$ do not change.

The presence of an external effect of human capital creates a wedge between the balanced growth rate of physical and human capital in both the social planner's economy and the market economy. However, human capital formation is larger in the former type of economy since the externality is internalized by the decision maker. Consequently, economic growth is higher along an optimal path. The wedge between the equilibrium growth rate of both types of capital causes the ratio of human to physical capital to decline along a balanced growth path.

Solving the recursive equilibrium laws of motion and the fixed point problem simultaneously delivers the solution as presented in Table 5. Also in the presence of external effects of human capital, the previous findings about the interaction between economic growth and the cycle still hold.

<Insert Table 5>

8.   Evaluation and conclusion

This paper analyzed the impact of cyclical volatility on endogenous growth. By constructing a stochastic version of the two-sector learning-or-doing model of endogenous growth, we have shown that economic growth is higher in the presence of business cycles since people devote more time to learning activities in an uncertain economic environment. Thus, the transmission channel of the interaction between economic growth and business cycles is here not an acceleration in the accumulation of physical capital, but instead an increase in the accumulation of human capital. The model predicts a strong interdependency between growth and cycles. We found that economic growth increases by about 0.16%-point as a result of observed business cycle variability.

The analysis presented in this paper can be extended in several directions. In this paper we assumed that formation of human capital takes place in the learning sector. More employees need to be allocated to the research sector in order to increase the accumulation of human capital. There are several other ways to model the process of knowledge creation. Human capital formation through learning-on-the-job or learning-by-doing, for instance, has received considerable attention in the literature (*cf.* Arrow 1962, Lucas 1993, Young 1991). The idea is that human capital is a by-product of production activity; more (less) human capital is created in expansions (recessions). One way to analyze the interaction between economic growth and business cycles is to endogenize labour supply. In this context the relevant question would be: do people supply more labour time in an uncertain economic environment in order to hedge against bad draws of income via increased accumulation of human capital and higher economic growth? Another interesting extension of the analysis in this paper is to assume that both physical and human capital are involved in the technology to build human capital, along the lines of Rebelo (1991),

Mulligan and Sala-i-Martin (1993), and Bond, Wang and Yip (1996). A difficulty with this extension is that no closed-form solutions for the balanced growth path can be obtained in this case. Since physical capital now enters as a productive input in the learning technology, business cycle variability might now also increase precautionary investments in physical capital. These issues are left for future research.

References

Aghion P., P. Howitt (1992): "A Model of Growth through Creative Destruction", *Econometrica*, vol. 60, 2, pp.323-351.

Aghion P., G. Saint-Paul (1991): "On the Virtue of Bad Times", *CEPR Working Paper*, no. 578.

Arrow K.J. (1962): "The Economic Implications of Learning by Doing", *Review of Economic Studies*, 29, pp.155-173.

Atkeson A., C. Phelan (1994): "Reconsidering the Costs of Business Cycles with Incomplete Markets", *NBER Macroeconomics Annual 1994*, eds. S. Fischer and J.J. Rotemberg, the MIT Press, Cambridge (Massachusetts) and London (England).

Bean C. (1990): "Endogenous Growth and the Pro-Cyclical Behaviour of Productivity", *European Economic Review*, 34, pp.355-363.

Benavie A., E. Grinols, S.J. Turnovsky (1996): "Adjustment Costs and Investment in a Stochastic Endogenous Growth Model", *Journal of Monetary Economics*, 38, pp.77-100.

Bond E.W., P. Wang, C. Yip (1996): "A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics", *Journal of Economic Theory*, 68, pp.149-173.

Caballero R.J., M.L. Hammour (1994): "The Cleansing Effect of Recessions", *American Economic Review*, vol. 84, 5, pp.1350-1368.

Campbell J. (1993): "Intertemporal Asset Pricing without Consumption Data", *American Economic Review*, 83, 3, pp.487-512.

Campbell J. (1994): "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model", *Journal of Monetary Economics*, 33, pp.453-506.

Canton E. (1996): "Learning During Recessions", *Mimeo*, Tilburg University.

Christiano L.J., M. Eichenbaum (1992): "Current Real-Business-Cycle Theories and Aggregate Labor-Market Fluctuations", *American Economic Review*, vol. 82, 4, pp.430-450.

Cooley T.F., E.C. Prescott (1995): "Economic Growth and Business Cycles", in Cooley T.F. (editor): *Frontiers of Business Cycle Research*, Princeton University Press, Princeton, New Jersey.

Davis S.J., J. Haltiwanger (1992): "Gross Job Creation, Gross Job Destruction, and Employment Reallocation", *Quarterly Journal of Economics*, pp.819-863.

Deaton A. (1991): "Saving and Liquidity Constraints", *Econometrica*, 59, 5, pp.1221-1248.

Dellas H. (1991): "Stabilization Policy and Long Term Growth: Are They Related?", *Mimeo*, University of Maryland.

Einarsson T., M.H. Marquis (1994): "An RBC Model with Growth: The Role of Human Capital", *Mimeo*, University of Iceland and Florida State University.

Faig M. (1995): "A Simple Economy with Human Capital: Transitional Dynamics, Technology Shocks, and Fiscal Policies", *Journal of Macroeconomics*, 17, pp.421-446.

Grier K.B., G. Tullock (1989): "An Empirical Analysis of Cross-National Economic Growth, 1951-80", *Journal of Monetary Economics*, 24, pp.259-276.

Hall R.E. (1991): "Recessions as Reorganizations", *NBER Macroeconomics Annual*, NBER, Cambridge, MA.

Hansen G.D. (1985): "Indivisible Labor and the Business Cycle", *Journal of Monetary Economics*, 16, pp.309-327.

İmrohoroğlu A. (1989): "Cost of Business Cycles with Indivisibilities and Liquidity Constraints", *Journal of Political Economy*, 97, 6, pp.1364-1383.

Jones L., R. Manuelli, P. Rossi (1993): "Optimal Taxation in Models of Endogenous Growth", *Journal of Political Economy*, 101, No. 3, pp.485-517.

King R.G., S.T. Rebelo (1988): "Business Cycles with Endogenous Growth", *Unpublished Manuscript*, University of Rochester.

King R.G., C.I. Plosser, S.T. Rebelo (1988): "Production, Growth, and Business Cycles: II. New

Directions", *Journal of Monetary Economics*, 21, pp.309-341.

Kormendi R.C., P.G. Meguire (1985): "Macroeconomic Determinants of Growth: Cross-Country Evidence", *Journal of Monetary Economics*, 16, pp.141-163.

Kydland F.E., E.C. Prescott (1982): "Time to Build and Aggregate Fluctuations", *Econometrica*, 50, pp.1345-1370.

Ladrón-de-Guevara A., S. Ortigueira, M. Santos (1994): "Equilibrium Dynamics in Two-Sector Models of Endogenous Growth", *Discussion Paper Centro de Investigación Económica*, No. 9403.

Levine R., D. Renelt (1992): "A Sensitivity Analysis of Cross-Country Growth Regressions", *American Economic Review*, 82, pp.942-963.

Lucas R.E. (1987): *Models of Business Cycles*, Basil Blackwell Inc., New York.

Lucas R.E. (1988): "On the Mechanics of Economic Development", *Journal of Monetary Economics*, 22, pp.3-42.

Lucas R.E. (1990): "Supply-Side Economics: An Analytical Review", *Oxford Economic Papers*, 42, pp.293-316.

Lucas R.E. (1993): "Making a Miracle", *Econometrica*, 61, 2, pp.251.

Martin P., C. Rogers (1996): "Long-Term Growth and Short-Term Economic Instability", *CEPR Discussion Paper*, No.1281.

McCallum B.T. (1983): "On Non-Uniqueness in Rational Expectations Models", *Journal of Monetary Economics*, 11, pp.139-168.

Mincer J. (1974): *Schooling, Experience, and Earnings*, New York, Columbia University Press (for NBER).

Mulligan C.B., X. Sala-i-Martin (1993): "Transitional Dynamics in Two-Sector Models of Endogenous Growth", *Quarterly Journal of Economics*, 108, pp.739-775.

Obstfeld M. (1994): "Evaluating Risky Consumption Paths: The Role of Intertemporal Substitutability", *European Economic Review*, 38, pp.1471-1486.

Ramey G., V. Ramey (1995): "Cross-Country Evidence on the Link between Volatility and Growth", *American Economic Review*, 85, No. 5, pp.1138-1151.

Rebelo S. (1991): "Long-Run Policy Analysis and Long-Run Growth", *Journal of Political Economy*, 99, 3, pp.500-521.

Romer P.M. (1986): "Increasing Returns and Long-Run Growth", *Journal of Political Economy*, 94, pp.1002-1037.

Saint-Paul G. (1993): "Productivity Growth and the Structure of the Business Cycle", *European Economic Review*, 37, pp.861-890.

Skinner J. (1988): "Risky Income, Life Cycle Consumption, and Precautionary Savings", *Journal of Monetary Economics*, 22, pp.237-255.

Stiglitz J.E. (1993): "Endogenous Growth and Cycles", *NBER Working Paper*, No. 4286.

Stokey N., S. Rebelo (1995): "Growth Effects of Flat-Tax Rates", *Journal of Political Economy*, vol. 103, no. 3, pp.519-550.

Strang G. (1988): *Linear Algebra and its Applications*, third edition, Harcourt Brace Jovanovich, Publishers, San Diego.

Taylor J., H. Uhlig (1990): "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods", *Journal of Business and Economic Statistics*, 8, pp.1-19.

Uhlig H. (1995): "A Toolkit for Analyzing Nonlinear Dynamic Models Easily", *Discussion Paper Institute for Empirical Macroeconomics*, No. 101 (also available as *CentER Discussion Paper*, No. 9597).

Uzawa H. (1965): "Optimal Technical Change in an Aggregative Model of Economic Growth", *International Economic Review*, 6, pp.18-31.

Young A. (1991): "Learning by Doing and the Dynamic Effects of International Trade", *Quarterly Journal of Economics*, 106, 2, pp.369-405.

Appendix 1: Computational details

In this Appendix we describe the solution procedure. First we derive a loglinear approximation of the model around the balanced growth path, thereby transforming the model into a system of linear (expectational) difference equations. This linear system is then solved by using the method of undetermined coefficients.

Exploiting *e.g.* $y(t)=y^*e^{\tilde{y}(t)}\approx y^*(1+\tilde{y}(t))$, transforms the production function (eq. 1), the human capital accumulation function (eq. 4), the savings rate, and the two Euler equations (eqs. 11, 14) into

$$\tilde{y}(t) = \tilde{A}(t)+(1-\alpha)[\tilde{u}(t)+\tilde{h}(t-1)] \qquad \text{(A1)}$$

$$\tilde{h}(t) \approx \tilde{h}(t-1)-s^*y^*[\tilde{s}(t)+\tilde{y}(t)]-\chi u^*\tilde{u}(t)+(\chi-\chi u^*)\tilde{\chi}(t) \qquad \text{(A2)}$$

$$\tilde{s}(t) \approx \frac{1-s^*}{s^*}[\tilde{y}(t)-\tilde{c}(t)] \qquad \text{(A3)}$$

$$0 \approx \mathrm{E}_t\{\theta[\tilde{c}(t)-\tilde{c}(t+1)]-\theta s^*y^*[\tilde{s}(t)+\tilde{y}(t)]+\alpha y^*\tilde{y}(t+1)\} \qquad \text{(A4)}$$

$$0 \approx \mathrm{E}_t\{\theta[\tilde{c}(t)-\tilde{c}(t+1)]+(1-\theta)s^*y^*[\tilde{s}(t)+\tilde{y}(t)]+\tilde{y}(t+1)-\tilde{y}(t)+ \\ (1+\chi u^*)\tilde{u}(t)-\tilde{u}(t+1)+(1+\chi u^*-\chi)\tilde{\chi}(t)+(\chi-1)\tilde{\chi}(t+1)\}. \qquad \text{(A5)}$$

For further details on how to obtain such approximations, see Uhlig (1995).

We proceed by writing eqs. A1-A5 and the two exogenous stochastic processes eqs. 2 and 5 (in loglinear form) in matrix notation to:

$$0 = \Omega_A\kappa(t)+\Omega_B\kappa(t-1)+\Omega_C\pi(t)+\Omega_D\tau(t) \qquad \text{(A6)}$$

$$0 = \mathrm{E}_t[\Omega_F\kappa(t+1)+\Omega_G\kappa(t)+\Omega_H\kappa(t-1)+\Omega_J\pi(t+1)+ \\ \Omega_K\pi(t)+\Omega_L\tau(t+1)+\Omega_M\tau(t)] \qquad \text{(A7)}$$

$$\tau(t+1) = \Omega_N\tau(t)+\epsilon(t+1) \qquad \text{(A8)}$$

$\kappa(t)$ is the endogenous state vector, $\pi(t)$ is a vector of other endogenous variables, $\tau(t)$ is a vector of exogenous stochastic processes, $\epsilon(t+1)=[\epsilon_A(t+1)\ \epsilon_\chi(t+1)]'$, and $\Omega_A,...,\Omega_N$ are matrices.

We want to find the recursive equilibrium laws of motion

$$\kappa(t) = \Omega_P \kappa(t-1) + \Omega_Q \tau(t) \tag{A9}$$

$$\pi(t) = \Omega_R \kappa(t-1) + \Omega_S \tau(t) \tag{A10}$$

where the elements in $\Omega_P, ..., \Omega_S$ are the partial elasticities we are looking for. In this notation, the method in Uhlig (1995) can directly be applied to get an approximate *analytical* solution. The solution procedure to solve this system is based on the following theorem.

Theorem (Uhlig, 1995)
If there is a recursive equilibrium law of motion solving equations A6, A7, and A8, then the coefficient matrices can be found as follows. Let Ω_C^+ be the pseudo-inverse of Ω_C. Let Ω_C^0 be an $(\ell\text{-n}) \times \ell$ matrix, whose rows form a basis of the null space of Ω_C'.
    1. Ω_P satisfies the (matrix) quadratic equations

$$0 = \Omega_C^0 \Omega_A \Omega_P + \Omega_C^0 \Omega_B$$

$$0 = (\Omega_F - \Omega_J \Omega_C^+ \Omega_A)\Omega_P^2 - (\Omega_J \Omega_C^+ \Omega_B - \Omega_G + \Omega_K \Omega_C^+ \Omega_A)\Omega_P - \Omega_K \Omega_C^+ \Omega_B + \Omega_H$$

           The equilibrium described by the recursive equilibrium law of motion A9, A10, and by A8 is stable iff all eigenvalues of Ω_P are smaller than unity in absolute value.
    2. Ω_R is given by

$$\Omega_R = -\Omega_C^+ (\Omega_A \Omega_P + \Omega_B)$$

    3.   Given Ω_P and Ω_R, let Ω_V be the matrix

$$\Omega_V = \begin{bmatrix} \Omega_I \otimes \Omega_A & \Omega_I \otimes \Omega_C \\ \Omega_N' \otimes \Omega_F + \Omega_I \otimes (\Omega_F \Omega_P + \Omega_J \Omega_R + \Omega_G) & \Omega_N' \otimes \Omega_J + \Omega_I \otimes \Omega_K \end{bmatrix}$$

           where Ω_I is the identity matrix of size k×k (k is the number of exogenous stochastic processes). If the matrix Ω_V is invertible, then

$$\begin{bmatrix} \text{vec}(\Omega_Q) \\ \text{vec}(\Omega_S) \end{bmatrix} = -\Omega_V^{-1} \begin{bmatrix} \text{vec}(\Omega_D) \\ \text{vec}(\Omega_L \Omega_N + \Omega_M) \end{bmatrix}$$

           where vec(.) denotes columnwise vectorization.

For technical details and a proof of the theorem, see Uhlig (1995). Details on matrix algebra can be found in Strang (1988).
        To perform the computations in this paper, we have used the MATLAB programs available for Uhlig (1995) at the Federal Reserve Bank of Minneapolis web site, http://res.mpls.frb.fed.us/research/res.html. The programs can be found under the "Archive of

Paper" and "Discussion Papers from the Institute for Empirical Macroeconomics", Discussion Paper 101.

Next we provide details about the input format that was used to run this MATLAB program. Since this program calculates variance-covariance matrices - and we need to know the variances that show up in the Euler equations - we have chosen the following input structure. $\kappa(t)'=[\tilde{h}(t)\ \tilde{u}(t)\ \tilde{y}(t)\ \tilde{c}(t)\ \tilde{s}(t)]$ is the endogenous state vector. Since $h(t)$ is the only state variable, the Ω_P matrix should be a null matrix except for the first column (which indeed turns out to be the case). $\pi(t)'=[\tilde{p}(t)\ \tilde{q}(t)]$ is a vector of other endogenous variables, $\tau(t)'=[\tilde{A}(t)\ \tilde{\chi}(t)]$ is a vector of exogenous stochastic processes. We introduce the following auxiliary variables:

$$\tilde{p}(t) \equiv \theta[\tilde{c}(t-1)-\tilde{c}(t)]-\theta s^*y^*[\tilde{s}(t-1)+\tilde{y}(t-1)]+\alpha y^*\tilde{y}(t)$$

$$\tilde{q}(t) \equiv \theta[\tilde{c}(t-1)-\tilde{c}(t)]+(1-\theta)s^*y^*[\tilde{s}(t-1)+\tilde{y}(t-1)]+\tilde{y}(t)-\tilde{y}(t-1)+$$
$$(1+\chi u^*)\tilde{u}(t-1)-\tilde{u}(t)+(1+\chi u^*-\chi)\tilde{\chi}(t-1)+(\chi-1)\tilde{\chi}(t).$$

So $\tilde{p}(t)$ and $\tilde{q}(t)$ are the loglinear Euler equations, lagged by one period. From the loglinearized model and the vector definitions we find the following matrix-structure:

$$\Omega_A = \begin{bmatrix} 0 & 1-\alpha & -1 & 0 & 0 \\ -1 & -\chi u^* & -s^*y^* & 0 & -s^*y^* \\ 0 & 0 & \dfrac{1-s^*}{s^*} & -\dfrac{1-s^*}{s^*} & -1 \\ 0 & 0 & \alpha y^* & -\theta & 0 \\ 0 & -1 & 1 & -\theta & 0 \end{bmatrix}$$

$$\Omega_B = \begin{bmatrix} 1-\alpha & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta s^*y^* & \theta & -\theta s^*y^* \\ 0 & 1+\chi u^* & (1-\theta)s^*y^*-1 & \theta & (1-\theta)s^*y^* \end{bmatrix}$$

$$\Omega_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Omega_D = \begin{bmatrix} 1 & 0 \\ 0 & \chi(1-u^*) \\ 0 & 0 \\ 0 & 0 \\ 0 & \dfrac{1+\chi u^*-\chi}{\omega_\chi}+\chi-1 \end{bmatrix}$$

$$\Omega_J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Omega_N = \begin{bmatrix} \phi_A & 0 \\ 0 & \phi_\chi \end{bmatrix}.$$

Furthermore, Ω_F is a 2×5 null matrix, Ω_G is a 2×5 null matrix, Ω_H is a 2×5 null matrix, Ω_K is a 2×2 null matrix, Ω_L is a 2×2 null matrix, and Ω_M is a 2×2 null matrix. A complication in the solution procedure is that the MATLAB program needs to know the variance terms V_1, V_2 in advance. An iteration procedure is used. The program performs the calculations for any arbitrary values for V_1 and V_2. The solution, *i.e.* a set of matrices that contain the partial elasticities of the recursive equilibrium laws of motion, is used to calculate variance-covariance matrices. It thereby can be checked whether the imposed value of V_1 and V_2 is consistent with the actual value. If not, the program changes the imposed values until imposed and actual values are (approximately) identical. Usually, only few iterations suffice to achieve convergence.

Recall from eqs. 11, 14 that V_1 and V_2 are *conditional* variances. Therefore, by defintion, we have $V_1 = V_t[\tilde{p}(t+1)]$ and $V_2 = V_t[\tilde{q}(t+1)]$. To calculate V_1, we propose the following procedure. The recursive equilibrium law of motion for $\tilde{p}(t+1)$ is given by

$$\tilde{p}(t+1) = \Omega_R(1,:)\kappa(t)+\Omega_S(1,:)\tau(t+1),$$

where $\Omega_R(1,:)$ is the first row of matrix Ω_R (that is, $\Omega_R(1,:)$ are the elements from matrix Ω_R pertaining to $\tilde{p}(t+1)$). The variance of $\tilde{p}(t+1)$, conditional on information up to time t is defined by

$$V_t[\tilde{p}(t+1)] = E_t[\{\tilde{p}(t+1)-E_t[\tilde{p}(t+1)]\}^2].$$

Denote the term on the RHS between accolades, the surprise in $\tilde{p}(t+1)$ conditional on information up to time t, as $\epsilon_p(t+1)$. Substitution of the recursive equilibrium law of motion for $\tilde{p}(t+1)$ into the term on the RHS between accolades yields

$$\epsilon_p(t+1) = \Omega_R(1,:)\kappa(t)+\Omega_S(1,:)\tau(t+1)-E_t[\Omega_R(1,:)\kappa(t)+\Omega_S(1,:)\tau(t+1)].$$

Or,

$$\epsilon_p(t+1) \;=\; \Omega_S(1,:)[\tau(t+1) - E_t[\tau(t+1)]].$$

Since $\tau(t+1) = \Omega_N \tau(t) + \epsilon(t+1)$, we thus find $\epsilon_p(t+1) = \Omega_S(1,:)\epsilon(t+1)$.

From $E_t[\epsilon_p(t+1)^2] = E_t[\epsilon_p(t+1)\epsilon_p(t+1)'] = E_t[\Omega_S \epsilon(t+1)\epsilon(t+1)'\Omega_S']$ it follows that

$$V_1 \;=\; V_t[\tilde{p}(t+1)] \;=\; \Omega_S(1,:)\Omega_\epsilon \Omega_S(1,:)',$$

where $\Omega_\epsilon \equiv E_t[\epsilon(t+1)\epsilon(t+1)']$. Similarly, one can derive that

$$V_2 \;=\; V_t[\tilde{q}(t+1)] \;=\; \Omega_S(2,:)\Omega_\epsilon \Omega_S(2,:)'.$$

Now we can readily calculate V_1 and V_2 in MATLAB.

Appendix 2: Calculation of the unconditional variance of *l*

In this Appendix we derive an expression for the unconditional variance of leisure time. We add $\tilde{l}(t)$ as an additional endogenous variable to the computational procedure described in Appendix 1. That is, $\pi(t)' = [\tilde{p}(t) \; \tilde{q}(t) \; \tilde{l}(t)]$. (Some matrices need a slight modification.) The recursive equilibrium law of motion for $\tilde{l}(t)$ is given by

$$\tilde{l}(t) \;=\; \Omega_R(3,:)\kappa(t-1) + \Omega_S(3,:)\tau(t).$$

The unconditional variance of $\tilde{l}(t)$ is given by

$$\begin{aligned}
V[\tilde{l}(t)] \;=\; E[\tilde{l}(t)^2] \;=\;\; & E[\Omega_R(3,:)\kappa(t-1)\kappa(t-1)'\Omega_R(3,:)'] + E[\Omega_R(3,:)\kappa(t-1)\tau(t)'\Omega_S(3,:)'] + \\
& E[\Omega_S(3,:)\tau(t)\kappa(t-1)'\Omega_R'] + E[\Omega_S(3,:)\tau(t)\tau(t)'\Omega_S(3,:)'].
\end{aligned}$$

Define $\Omega_{\kappa\kappa} = E[\kappa(t)\kappa(t)']$, $\Omega_{\kappa\tau} = E[\kappa(t-1)\tau(t)']$, and $\Omega_{\tau\tau} = E[\tau(t)\tau(t)']$. The previous expression thus rewrites to

$$V[\tilde{l}] \;=\; \Omega_R(3,:)\Omega_{\kappa\kappa}\Omega_R(3,:)' + \Omega_R(3,:)\Omega_{\kappa\tau}\Omega_S(3,:)' + \Omega_S(3,:)\Omega_{\kappa\tau}'\Omega_R(3,:)' + \Omega_S(3,:)\Omega_{\tau\tau}\Omega_S(3,:)'.$$

Next we want to obtain expressions for $\Omega_{\kappa\kappa}$, $\Omega_{\kappa\tau}$, and $\Omega_{\tau\tau}$. Firstly, we write $\Omega_{\kappa\tau}$ to

$$\begin{aligned}
\Omega_{\kappa\tau} \;=\; & E[\kappa(t-1)\tau(t)'] \;=\; E[\kappa(t-1)\tau(t-1)\Omega_N'] \;=\; E[\kappa(t)\tau(t)']\Omega_N' \;=\; \\
& E[\Omega_P \kappa(t-1)\tau(t)']\Omega_N' + E[\Omega_Q \tau(t)\tau(t)']\Omega_N'.
\end{aligned}$$

Or,

$$\Omega_{\kappa\tau} \;=\; \Omega_P \Omega_{\kappa\tau}\Omega_N' + \Omega_Q \Omega_{\tau\tau}\Omega_N'.$$

Secondly, we write $\Omega_{\kappa\kappa}$ to

$$\Omega_{\kappa\kappa} = E[\kappa(t)\kappa(t)'] = \Omega_P E[\kappa(t-1)\kappa(t-1)']\Omega_P' + \Omega_P E[\kappa(t-1)\tau(t)']\Omega_Q' +$$
$$\Omega_Q E[\tau(t)\kappa(t-1)']\Omega_P' + \Omega_Q E[\tau(t)\tau(t)']\Omega_Q'.$$

Or,

$$\Omega_{\kappa\kappa} = \Omega_P \Omega_{\kappa\kappa} \Omega_P' + \Omega_P \Omega_{\kappa\tau} \Omega_Q' + \Omega_Q \Omega_{\kappa\tau}' \Omega_P' + \Omega_Q \Omega_{\tau\tau} \Omega_Q'.$$

Finally, from $\tau(t) = \Omega_N \tau(t-1) + \varepsilon(t)$ and $\Omega_{\tau\tau} = E[\tau(t)\tau(t)']$ we find

$$\Omega_{\tau\tau} = \Omega_N \Omega_{\tau\tau} \Omega_N' + \Omega_\epsilon.$$

To solve for $\Omega_{\tau\tau}$, we proceed by considering a columnwise vectorization of $\Omega_{\tau\tau} = \Omega_N \Omega_{\tau\tau}\Omega_N' + \Omega_\epsilon$, *viz.*

$$\text{vec}(\Omega_{\tau\tau}) = \text{vec}(\Omega_N \Omega_{\tau\tau} \Omega_N') + \text{vec}(\Omega_\epsilon).$$

From matrix algebra we use that $\text{vec}(\Omega_N \Omega_{\tau\tau} \Omega_N') \equiv (\Omega_N \otimes \Omega_N)\text{vec}(\Omega_{\tau\tau})$. Hence, it is easy to see that $(\Omega_I - \Omega_N \otimes \Omega_N)\text{vec}(\Omega_{\tau\tau}) = \text{vec}(\Omega_\epsilon)$, or

$$\text{vec}(\Omega_{\tau\tau}) = (\Omega_I - \Omega_N \otimes \Omega_N)^{-1} \text{vec}(\Omega_\epsilon).$$

By rewriting $\text{vec}(\Omega_{\tau\tau})$ to matrix notation we obtain an expression for $\Omega_{\tau\tau}$. In a similar fashion we obtain:

$$\text{vec}(\Omega_{\kappa\tau}) = (\Omega_I - \Omega_P \otimes \Omega_N)^{-1}(\Omega_Q \otimes \Omega_N)\Omega_{\tau\tau}$$

$$\text{vec}(\Omega_{\kappa\kappa}) = (\Omega_I - \Omega_P \otimes \Omega_P)^{-1}[(\Omega_P \otimes \Omega_Q)\text{vec}\Omega_{\kappa\tau} + (\Omega_Q \otimes \Omega_P)\text{vec}(\Omega_{\kappa\tau}') + (\Omega_Q \otimes \Omega_Q)\text{vec}(\Omega_{\tau\tau})]$$

After reshaping back to matrix format, the expressions for $\Omega_{\kappa\kappa}$, $\Omega_{\kappa\tau}$, and $\Omega_{\tau\tau}$ are substituted into the formula for $V[\tilde{l}]$. This is the expression for the unconditional variance of $\tilde{l}$ we were looking for.

The MATLAB command for $\otimes$ is kron, *e.g.* $\Omega_P \otimes \Omega_P$ becomes kron(Ω_P, Ω_P); the MATLAB command for vec is :, *e.g.* vec(Ω_P) becomes $\Omega_P(:)$.

	s^*	u^*	c^*	y^*	h^*	g_Y^*
A	0	0	0	+	-	0
χ	+	-	+	+	+	+
α	+	-	-	-	-	0
δ_K	+	0	+	+	+	0
$\delta_H{}^a$	+	-	-	-	-	-
β	+	-	-	0	+	+
θ	-	+	+	0	-	-
V_1	+	0	-	-	-	0
$V_2{}^b$	+	-	+	+	+	+

[a]   For $\theta>1$.

[b]   The comparative statics results for s^* and c^* hinge on the assumption that $\psi>\zeta$ and $\alpha<\theta$ respectively. The latter condition is met for realistic parameter values; for V_1 close to zero, we can rewrite the former condition to $(\theta-1)\delta_K<1-\beta$. This condition is also met for our baseline parameter choices, but might be violated for other choices: an increase in V_2 will lead to a decrease in savings when households become more risk averse (θ increases).

Note: The savings rate - for instance - raises when the maximum growth rate of human capital in the learning sector is higher, when physical capital is more productive, when capital depreciation is higher, when agents are more patient, when agents are less risk averse, and when the variance terms in the Euler equations are higher.

Table 1: Comparative statics in the Lucas-Uzawa model.

	Base-line	Variation in χ		Variation in α		Variation in δ_K		Variation in δ_H		Variation in β		Variation in θ	
		0.0665	0.1065	0.26	0.46	0.02	0.10	0	0.03	0.94	0.98	1	2
s^*(%)	22.1	21.7	22.3	15.9	28.2	16.1	25.3	22.3	21.8	18.3	25.8	25.0	20.6
u^*(%)	57.1	64.1	52.7	57.1	57.1	57.1	57.1	62.8	51.3	73.2	41.2	44.3	63.5
c^*	0.29	0.25	0.33	0.43	0.21	0.22	0.36	0.32	0.26	0.30	0.28	0.28	0.30
y^*	0.37	0.32	0.43	0.52	0.29	0.26	0.48	0.41	0.33	0.37	0.37	0.37	0.37
h^*	0.37	0.26	0.50	0.72	0.18	0.22	0.56	0.40	0.35	0.29	0.52	0.48	0.34
g_Y^*(%)	2.21	0.89	3.54	2.21	2.21	2.22	2.21	3.22	1.21	0.81	3.59	3.32	1.66
V_1	0	0	0	0	0	0	0	0	0	0	0	0	0
V_2 (*10^{-3})	50	52	49	50	50	51	50	51	50	52	50	51	49

Note: The baseline parameters are: A=1; α=0.36; β=0.96; δ_K=0.06; δ_H=0.015; θ=1.5; χ=0.0865.

Table 2a: Balanced growth values for various parameter constellations.

	Base-line	Variation in χ		Variation in α		Variation in δ_K		Variation in δ_H		Variation in β		Variation in θ	
		0.0665	0.1065	0.26	0.46	0.02	0.10	0	0.03	0.94	0.98	1	2
η_{hh}	0.81	0.83	0.79	0.72	0.86	0.86	0.76	0.79	0.83	0.81	0.81	0.81	0.81
η_{hA}	-0.48	-0.44	-0.51	-0.63	-0.38	-0.41	-0.53	-0.50	-0.45	-0.48	-0.47	-0.51	-0.46
$\eta_{h\chi}$	0.52	0.46	0.56	0.83	0.35	0.45	0.57	0.54	0.49	0.52	0.51	0.51	0.52
η_{uh}	0.40	0.42	0.39	0.40	0.36	0.41	0.39	0.39	0.42	0.39	0.42	0.31	0.46
η_{uA}	0.87	0.98	0.77	0.83	0.86	1.12	0.68	0.79	0.95	0.87	0.86	1.08	0.76
$\eta_{u\chi}$	-1.82	-1.93	-1.72	-2.07	-1.60	-2.00	-1.67	-1.76	-1.89	-1.82	-1.83	-1.91	-1.78
η_{yh}	0.90	0.91	0.89	1.06	0.73	0.90	0.89	0.89	0.91	0.89	0.91	0.84	0.93
η_{yA}	1.56	1.63	1.49	1.62	1.47	1.71	1.44	1.51	1.61	1.56	1.55	1.69	1.49
$\eta_{y\chi}$	-1.17	-1.24	-1.10	-1.53	-0.87	-1.28	-1.07	-1.13	-1.21	-1.16	-1.17	-1.22	-1.14
η_{ch}	0.56	0.55	0.56	0.67	0.45	0.52	0.59	0.56	0.55	0.54	0.58	0.47	0.61
η_{cA}	0.49	0.46	0.53	0.56	0.44	0.42	0.55	0.52	0.47	0.50	0.49	0.56	0.46
$\eta_{c\chi}$	-0.15	-0.14	-0.16	-0.23	-0.10	-0.11	-0.19	-0.17	-0.14	-0.16	-0.15	-0.22	-0.11
η_{sh}	1.20	1.28	1.13	2.04	0.71	2.01	0.89	1.15	1.26	1.57	0.94	1.10	1.25
η_{sA}	3.75	4.22	3.37	5.55	2.63	6.74	2.62	3.46	4.09	4.73	3.06	3.39	3.98
$\eta_{s\chi}$	-3.58	-3.95	-3.28	-6.88	-1.95	-6.08	-2.61	-3.35	-3.84	-4.50	-2.94	-2.30	-3.95

Note: The baseline parameters are: $A=1$; $\alpha=0.36$; $\beta=0.96$; $\delta_K=0.06$; $\delta_H=0.015$; $\theta=1.5$; $\chi=0.0865$.

Table 2b: The recursive equilibrium laws of motion.

Variable	U.S. Economy 1950-1989		Two-Sector Model of Endogenous Growth	
x	σ_x	ρ_{xY}	σ_x	ρ_{xY}
Y	2.45	1.00	2.52	1.00
C	1.80	0.87	0.79	0.76
s	8.32	0.78	7.04	0.97
$1-l$	1.60	0.86		
u			2.89	0.88
K	2.42	0.39	1.26	0.59
H	0.77	0.21	0.27	-0.47

Note: Variables are logged and detrended by the Hodrick-Prescott filtering technique, setting the smoothing parameter at 400 (a common choice for annual data). Summary statistics for the U.S. economy are taken from Einarsson and Marquis (1994, Table 1). σ_x denotes the standard deviation of variable x (in %); ρ_{xY} denotes the correlation of variable x with output Y; $1-l$ is the fraction of hours devoted to market activity (or employment). The artificially generated second moments and correlation figures are averages across 100 runs of 100 periods.

Table 3: Summary statistics.

Endogenous labour supply	
$l^*(\%)$	65.3
$s^*(\%)$	24.7
$u^*(\%)$	16.0
c^*	0.27
y^*	0.36
h^*	1.28
$g_Y^*(\%)$	2.96
V_1	0
V_2 ($*10^{-3}$)	50

Note: The relative weight of consumption (ξ) is set at 0.2242 in order to replicate leisure demand to be equal to 71% in the deterministic economy. The transformation rate of research time into human capital accumulation (χ) is set at 0.2389 in order to reproduce the observation that about 17% of the total time endowment is allocated to market activity.

Table 4: Endogenous labour supply.

External effects of human capital			
Γ	0.2	0.3	0.4
$s^*(\%)$	22.3 [22.1]	22.5 [22.1]	22.6 [22.1]
$u^*(\%)$	50.1 [63.2]	47.7 [65.2]	45.8 [66.9]
c^*	0.34 [0.29]	0.36 [0.29]	0.38 [0.29]
y^*	0.43 [0.37]	0.47 [0.37]	0.50 [0.37]
$g_h^*(\%)$	-0.88 [-0.53]	-1.42 [-0.71]	-1.99 [-0.85]
$g_H^*(\%)$	2.82 [1.69]	3.03 [1.51]	3.19 [1.37]
$g_Y^*(\%)$	3.70 [2.22]	4.45 [2.22]	5.18 [2.22]
V_1	0 [0]	0 [0]	0 [0]
V_2 $(*10^{-3})$	50 [51]	50 [51]	47 [52]

Note: Γ represents the external effect of human capital. The table reports the optimal solution (in a social planner's economy) above the equilibrium solution (in a market economy).

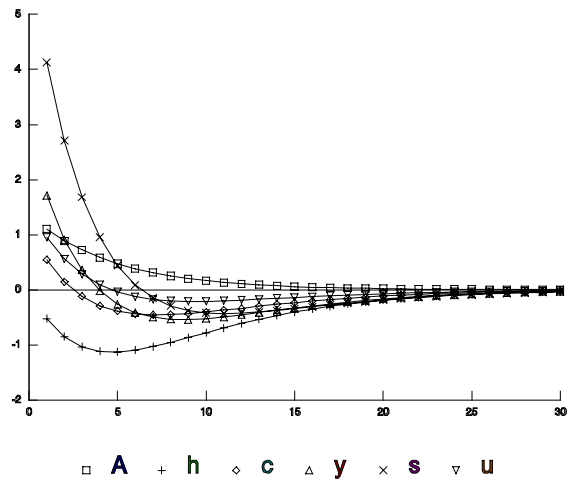Table 5: External effects of human capital.
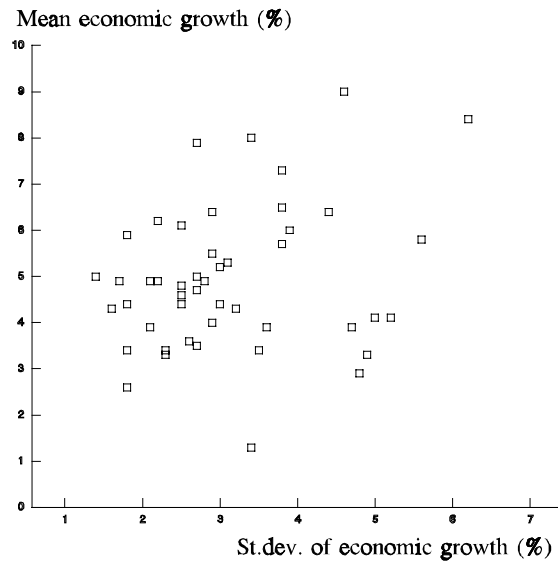
Figure 1: Kormendi and Meguire's data on economic growth and cyclical variability
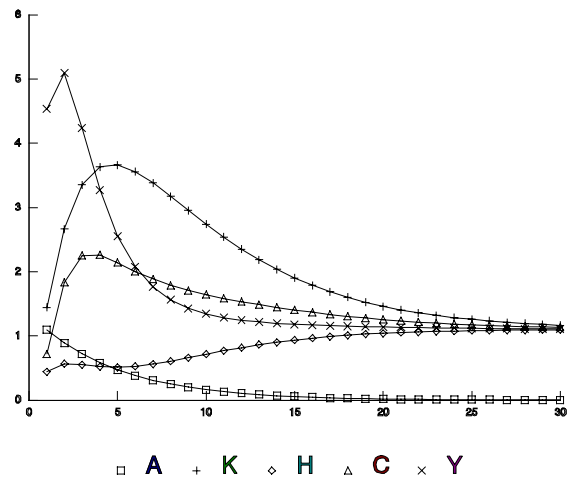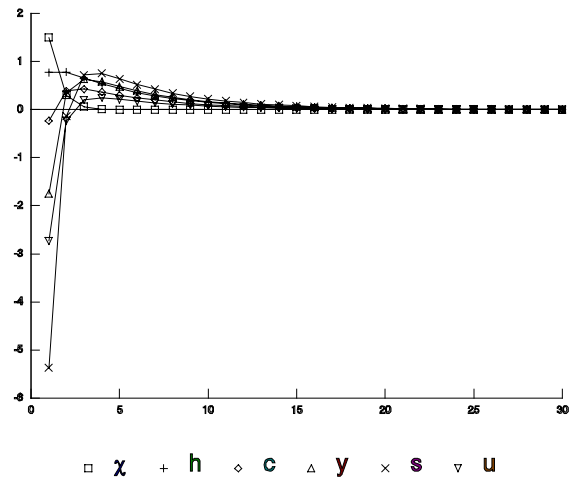
PANEL A

PANEL B

Figure 2: Impulse-responses to a one standard deviation shock to the production sector, stationary variables in Panel A and non-stationary variables in Panel B.
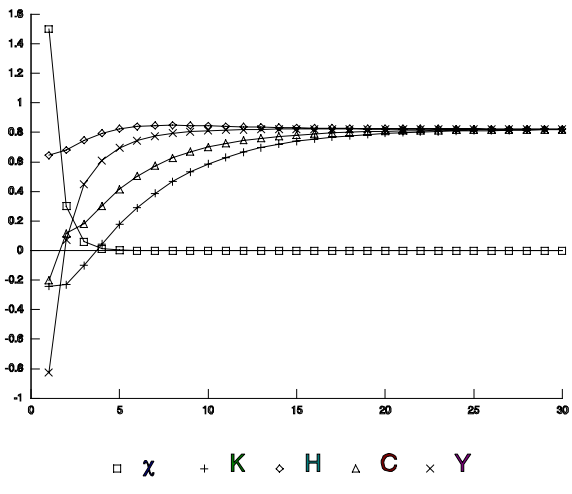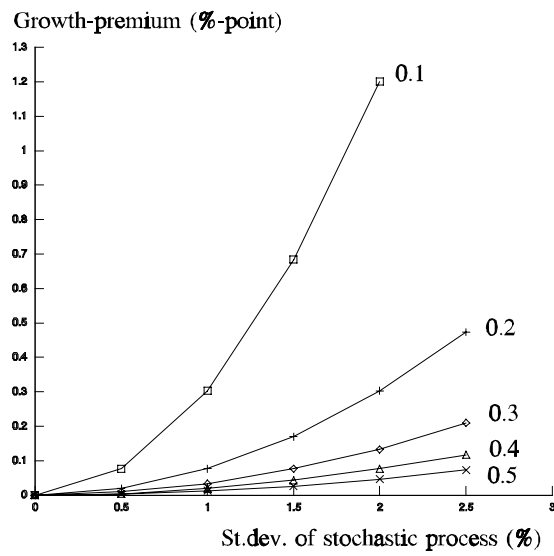
PANEL A

PANEL B

Figure 3: Impulse-responses to a one standard deviation shock to the learning sector, stationary variables in Panel A and non-stationary variables in Panel B.

Growth-premium (%-point)

St.dev. of stochastic process (%)

Note: The growth-premium (in %-point) is measured on the vertical axis; the standard deviation of the imposed exogenous stochastic process σ_χ (in %) is on the horizontal axis; the persistence parameter ω_χ runs from 0.1 to 0.5.

Figure 4: Interaction between economic growth and business cycles
when shocks hit the learning sector only.