

## Consistency of a time-stepping method for a class of piecewise linear networks

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*Abstract*—In this paper we will study the computation of transient solutions of a class of piecewise linear (PL) circuits. The network models will be so-called linear complementarity systems, which can be seen as dynamical extensions of the PL modeling structure as proposed by Van Bokhoven [1]. In particular, the numerical simulation will be based on a time-stepping method using the well-known backward Euler scheme. It will be demonstrated, by means of an example, that this widely applied time-stepping method does not necessarily produce useful output for arbitrary linear dynamical systems with ideal diode characteristics. Next the consistency of the method will be proven for PL networks that can be realized by linear passive circuit elements and ideal diodes by showing that the approximations generated by the method converge to the true solution of the system in a suitable sense. To give such a consistency proof, the fundamental framework developed in [2] is indispensable as it proposes a precise definition of a “solution” of a linear complementarity system and provides conditions under which solutions exist and are unique.

*Keywords*—circuit simulation, piecewise linear networks, switched circuits, linear complementarity problem, passivity.

### I. INTRODUCTION

Simulation of switched networks is a problem that has been studied extensively in circuit theory [1, 3–9]. Roughly speaking, there are two main approaches, namely event-tracking (see e.g. [4, 5]) and time-stepping methods (see [1, 7–9] for electrical networks and [10–14] for unilaterally constrained mechanical systems with friction phenomena). Representing a hybrid systems point of view (see for instance [15]), *event-tracking methods* are based on considering the simulation interval as a union of disjoint subintervals on which the circuit topology (called “mode” in the hybrid systems terminology) remains unchanged. On each of these subintervals we are dealing in general with *differential and algebraic equations* (DAE), which can be solved by standard integration routines (*DAE simulation*). As integration proceeds, one has to monitor certain indicators (mostly given by inequalities, e.g. related to currents through diodes being nonnegative) to determine when the subinterval ends (*event detection*). At this event time a mode transition occurs, which means that one has to determine what the new circuit topology will be on the next subinterval (*mode selection*). If the continuous state at the event time is not consistent with the selected mode, a state jump is necessary (*re-initialization*). The complete numerical simulation method is based on repetitive cycles consisting of DAE simulation, event detection, mode selection and re-initialization.

*Time-stepping methods* replace the describing equations directly by some “discretized” equivalent. Numerical integration routines are applied to approximate the system equations involving derivatives and all algebraic relations are enforced to hold at each time-step. In this way, one has to solve at each time-step an algebraic problem (sometimes called the “one-step problem”) involving information obtained from previous time-steps. In contrast with event-driven methods, time-stepping methods do not determine the event times accurately, but “overstep” them, which puts the convergence of the approximations in a suitable sense (called “consistency”) into question.

In this paper we will study the consistency of a time-stepping method that is based on the well-known backward Euler integration scheme for a class of piecewise linear (PL) electrical circuits. The used network models are so-called *linear complementarity systems* [15–19], which can be seen as dynamical extensions of the PL model structure that

has been introduced by Van Bokhoven [1, 8]. Van Bokhoven’s model is based on the linear complementarity problem of mathematical programming [20] and covers many well-known canonical PL descriptions [21] (see also [2]). As such complementarity modeling is very powerful and many dynamical PL circuits are captured by (dynamic) linear complementarity systems.

Time-stepping methods may be preferable to event-tracking methods in particular in situations where many mode changes take place. In fact there do exist examples of linear complementarity systems (see e.g. Example 2 below), for which the event times (the times at which the mode changes) accumulate, i.e., the system displays an infinite number of switches (mode transitions) in a finite time-interval. It is obvious that this behavior causes difficulties for an event-tracking method as simulation beyond the accumulation point is in principle not possible without using some heuristic tool. Time-stepping can be an effective alternative in such situations.

As mentioned before, the time-stepping method that we will study here is based on the well-known backward Euler scheme and has been described, for instance, in [1, 7, 8] for electrical networks. Similar methods have been used in a mechanical context in [10–14]. A nice feature of the method is that it is straightforward to implement and many algorithms (e.g. Lemke’s algorithm [20], Katzenelson’s algorithm [22] and others [8]) are available to solve the one-step problems consisting of linear complementarity problems.

Convergence problems of time-stepping methods for mechanical systems subject to unilateral constraints or friction have been studied by Stewart [11, 23]. He shows that for a broad class of nonlinear constrained mechanical systems there always exists a *subsequence* of approximating time functions that converge to a real solution of the mechanical model. In the context of mechanical systems subject to unilateral constraints or friction, the complementarity conditions appear between the *force* and *position* variables. A direct translation to electrical circuits would not yield networks with complementarity conditions between the voltage and current variables which is the case for ideal diodes. Therefore, the results that have been obtained in [11, 23] do not cover electrical networks containing ideal diodes, which are included in the class of PL networks studied in this paper. Therefore, the objective of the current paper is to show that for the class of PL circuits that can be realized by linear passive elements and ideal diodes (complementarity conditions) the backward Euler time-stepping method is consistent. Moreover, we will even prove that the whole sequence (and not only a subsequence) of the approximating time functions converges to the real transient solution of the network model, when the step size decreases to zero.

### II. NOTATION

Throughout the paper,  $\mathbb{R}$  ( $\mathbb{R}^n$ ) denotes the set of ( $n$ -tuples of) real numbers.  $\mathbb{R}_+$  denotes the set of nonnegative real numbers, i.e.,  $\mathbb{R}_+ = [0, \infty)$ . For the real part of the complex number  $\lambda$ , we write  $\text{Re}(\lambda)$ . For any  $x, y \in \mathbb{R}^n$ ,  $x \perp y$  means that  $x^\top y = 0$ . Inequalities for vectors are always meant to hold componentwise.

The Euclidean and maximum norm of a vector  $x \in \mathbb{R}^n$  will be denoted by  $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$  and  $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$ , respectively. For a real number  $r \in \mathbb{R}$ , we use the notation  $\lceil r \rceil$  to denote the smallest integer larger than or equal to  $r$ . We write  $O(x)$  for any function such that  $\limsup_{x \rightarrow \infty} |O(x)|/x < \infty$ . We say that a proposition  $\mathcal{P}(\delta)$  holds for all sufficiently small (large)  $x$  if there exists  $x_0 > 0$  such that it holds for all  $0 \leq x \leq x_0$  ( $x_0 \leq x$ ).

The set of real matrices with  $n$  rows and  $m$  columns is denoted by  $\mathbb{R}^{n \times m}$ . For any  $A \in \mathbb{R}^{n \times m}$ ,  $J \subseteq \{1, 2, \dots, n\}$ , and  $K \subseteq \{1, 2, \dots, m\}$ ,  $A_{JK}$  denotes the submatrix  $\{A_{jk}\}_{j \in J, k \in K}$ . If  $J = \{1, 2, \dots, n\}$  ( $K = \{1, 2, \dots, m\}$ ), we also write  $A_{\bullet K}$  ( $A_{J\bullet}$ ). For any  $A \in \mathbb{R}^{n \times m}$   $\|A\| := \sup_{\|x\|=1} \|Ax\|$  denotes the matrix norm induced by the Euclidean vector norm. A square matrix  $A \in \mathbb{R}^{n \times n}$  is

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said to be nonnegative (positive) definite if  $x^\top Ax \geq 0$  ( $x^\top Ax > 0$ ) for all  $0 \neq x \in \mathbb{R}^n$ . We write  $\sigma(A)$  for the set of eigenvalues of  $A$  and  $\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$  for the spectral radius of  $A$ . By the symmetric part of  $A$ , we mean the matrix  $\frac{1}{2}(A + A^\top)$ . The identity matrix is denoted by  $I$ . Given two matrices  $A \in \mathbb{R}^{n_a \times m}$  and  $B \in \mathbb{R}^{n_b \times m}$ , the matrix obtained by stacking  $A$  over  $B$  is denoted by  $\text{col}(A, B)$ .

The set of  $n$ -tuples of square integrable functions on  $(t_0, t_1)$  is denoted by  $\mathcal{L}_2^n(t_0, t_1)$ . The notation  $\langle x, y \rangle$  denotes the inner product of  $x, y \in \mathcal{L}_2^n(t_0, t_1)$ , i.e.,  $\langle x, y \rangle = \int_{t_0}^{t_1} x^\top(t)y(t)dt$ . The norm on  $\mathcal{L}_2^n(t_0, t_1)$  is defined by  $\|x\| = \langle x, x \rangle^{1/2}$ . Moreover, the time function  $x|_\Omega$  denotes the restriction of the time function  $x$  to the interval  $\Omega$ . We say that the sequence  $\{x_k\} \subset \mathcal{L}_2^n(t_0, t_1)$  converges (converges weakly) to  $x$  if  $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$  ( $\lim_{k \rightarrow \infty} \langle x_k - x, y \rangle = 0$  for all  $y \in \mathcal{L}_2^n(t_0, t_1)$ ).

The typewriter font will be used for distributions to distinguish them from functions. The space  $\mathcal{L}_\delta(0, \tau)$  consists of the distributions of the form  $u = u_{\text{imp}} + u_{\text{reg}}$  where  $u_{\text{imp}} = u_0 \delta$  is called the *impulsive part* with  $u_0 \in \mathbb{R}$  and  $u_{\text{reg}} \in \mathcal{L}_2(0, \tau)$  is called the *regular part*. We say that the sequence of distributions  $\{u_0^k \delta + u_{\text{reg}}^k\} \subset \mathcal{L}_\delta(0, \tau)$  converges (weakly) to  $u_0 \delta + u_{\text{reg}}$ , if  $\{u_0^k\}$  converges to  $u_0$  and  $\{u_{\text{reg}}^k\}$  converges (weakly) to  $u_{\text{reg}}$  in  $\mathcal{L}_2$ -sense.

The matrix triple  $(A, B, C)$  with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{m \times n}$  is said to be *minimal* if  $(A, B)$  is controllable and  $(C, A)$  is observable (see for instance [24]).

### III. LINEAR COMPLEMENTARITY SYSTEMS

We begin by briefly recalling the linear complementarity problem (LCP) of mathematical programming. For an extensive survey on the problem, the reader is referred to [20].

*Problem 1:* (LCP( $q, M$ )) Given  $q \in \mathbb{R}^m$  and  $M \in \mathbb{R}^{m \times m}$ , find  $z \in \mathbb{R}^m$  such that

$$0 \leq z \perp q + Mz \geq 0. \quad (1)$$

We say that  $z$  solves LCP( $q, M$ ) if  $z$  satisfies (1). The set of all solutions of LCP( $q, M$ ) will be denoted by  $\text{SOL}(q, M)$ . Note that the so-called complementarity (1) conditions also appear in the ideal diode characteristic  $v \leq 0$ ,  $i \geq 0$ , and  $iv = 0$ . Not surprisingly, the linear complementarity problem plays a major role in the analysis of networks with ideal diodes. As discussed in [2, Section 3], many dynamical piecewise linear electrical networks can be cast as linear complementarity systems by following the ideas developed in [1]. A linear complementarity system is given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2a)$$

$$y(t) = Cx(t) + Du(t) \quad (2b)$$

$$0 \leq u(t) \perp y(t) \geq 0 \quad (2c)$$

where  $u(t) \in \mathbb{R}^m$ ,  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^m$  and  $A, B, C$ , and  $D$  are matrices of appropriate dimensions. We denote (2) by LCS( $A, B, C, D$ ). For more details on LCS, we refer to [16–19].

### IV. SIMULATION OF LCS

The aim of this section is to discuss two approaches for the simulation of LCS.

#### A. Event-tracking Methods

From a hybrid system point of view, LCS( $A, B, C, D$ ) has  $2^m$  modes depending on the complementarity conditions (2c) that indicate which diodes are blocking and which ones are conducting. The system

is governed in mode  $K$  by the differential algebraic equation (DAE)

$$\dot{x} = Ax + Bu \quad (3a)$$

$$y = Cx + Du \quad (3b)$$

$$u_i = 0 \text{ if } i \notin K \quad (3c)$$

$$y_i = 0 \text{ if } i \in K \quad (3d)$$

as long as the inequality constraints

$$y_i \geq 0 \text{ if } i \notin K \quad (4a)$$

$$u_i \geq 0 \text{ if } i \in K \quad (4b)$$

hold. By starting in mode  $K \subseteq \{1, 2, \dots, m\}$ , an event-tracking method integrates the DAE (3) by standard routines and monitors the inequalities (4). In case of a violation of (4), the event time (the time just before the violation) has to be determined in order to find out the mode which will be active after the event. Once the new mode is determined, the above procedure repeats itself again. One of the main disadvantages of this type of approach arises if there is an accumulation of events. In principle, event-tracking methods cannot go beyond such an accumulation point without using some kind of heuristic tool. In what follows, we give an example of an LCS having accumulation of events.

*Example 2:* Consider the following example (its time-reversed version is due to Filippov [25, p. 116])

$$\dot{x}_1 = -\text{sgn } x_1 + 2\text{sgn } x_2$$

$$\dot{x}_2 = -2\text{sgn } x_1 - \text{sgn } x_2$$

where  $\text{sgn } x$  is the set-valued (relay) function given by

$$\text{sgn } x = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

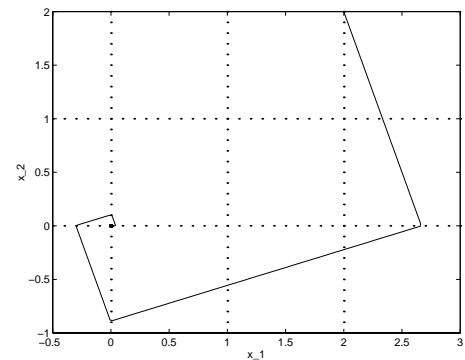


Fig. 1. Trajectory with initial state  $(2, 2)^\top$ .

As shown in [26, 27], this type of systems can be cast as LCS. Solutions of the system are spiraling towards the origin, which is an equilibrium. Since  $\frac{d}{dt}(|x_1(t)| + |x_2(t)|) = -2$  when  $x(t) \neq 0$  along trajectories  $x$  of the system, solutions reach the origin in finite time (see Figure 1 for a trajectory). Every crossing from one quadrant to another corresponds to an event (relay switch). Therefore, on a finite time interval there are infinitely many events, i.e., events do accumulate.

This example shows that the event-tracking methods might not be the most efficient methods for the simulation of LCS.

#### B. The Backward Euler Time-stepping Method

An alternative for event-tracking is the time-stepping method. Typically, such a method tries to replace approximately the overall system

description by a discretized equivalent instead of considering several linear DAE as an event-tracking method does. A frequently used time-stepping scheme (see [1, 7–9]) is based on the well known backward Euler method. For LCS the method consists of discretizing the system description by applying the backward Euler integration routine and imposing the complementarity conditions at every time step. This comes down to the computation of  $u_{k+1}^h$ ,  $y_{k+1}^h$ , and  $x_{k+1}^h$  given  $x_k^h$  through the linear complementarity problem given by

$$\frac{x_{k+1}^h - x_k^h}{h} = Ax_{k+1}^h + Bu_{k+1}^h \quad (5a)$$

$$y_{k+1}^h = Cx_{k+1}^h + Du_{k+1}^h \quad (5b)$$

$$0 \leq y_{k+1}^h \perp u_{k+1}^h \geq 0. \quad (5c)$$

Note that we use roman font for the numerical approximations. In the above relations,  $\bullet_k^h$  denotes the value at the  $k$ th step of the corresponding variable for the step size  $h > 0$ . Based on this scheme, one can construct approximations of the transient response of an LCS on an simulation interval  $[0, \tau]$  by applying the algorithm below.

*Algorithm 3:*  $(\{u_k^h\}, \{x_k^h\}, \{y_k^h\}) = \text{App.}(A, B, C, D, \tau, h, x_0)$

1.  $N_h = \lceil \frac{\tau}{h} \rceil$
2.  $x_{-1}^h := x_0$
3.  $k := -1$
4. solve the *one-step problem*

$$y_{k+1}^h = C(I - hA)^{-1}x_k^h + [D + hC(I - hA)^{-1}B]u_{k+1}^h$$

$$0 \leq u_{k+1}^h \perp y_{k+1}^h \geq 0$$

5.  $x_{k+1}^h := (I - hA)^{-1}x_k^h + h(I - hA)^{-1}Bu_{k+1}^h$
6.  $k := k + 1$
7. if  $k < N_h$  goto 4
8. stop.

The one-step problem in step 4 is given by a linear complementarity problem. In general a linear complementarity problem may have multiple solutions or have no solutions at all. We shall proceed by assuming unique solvability of the problem. The assumption is introduced here for reasons of generality; later on we will prove that the assumption is implied by passivity.

*Assumption 4:* For all sufficiently small  $h > 0$ ,  $\text{LCP}(C(I - hA)^{-1}\bar{x}, G(h^{-1}))$  has a unique solution for all  $\bar{x}$ , where  $G(h^{-1})$  is given by  $D + hC(I - hA)^{-1}B$ .

This assumption implies that for all sufficiently small  $h > 0$ , Algorithm 3 generates an output, which is unique. Hence, for a given step size  $h > 0$  (sufficiently small), we can define the approximations  $(u^h, x^h, y^h) \in \mathcal{L}_\delta(0, \tau)$  given by

$$u_{\text{imp}}^h = hu_0^h \delta \quad (6a)$$

$$x_{\text{imp}}^h = hx_0^h \delta \quad (6b)$$

$$y_{\text{imp}}^h = hy_0^h \delta \quad (6c)$$

$$\left. \begin{aligned} u_{\text{reg}}^h(t) &= u_l^h \\ x_{\text{reg}}^h(t) &= x_l^h \\ y_{\text{reg}}^h(t) &= y_l^h \end{aligned} \right\} \text{ whenever } (l-1)h \leq t < lh, \quad (6d)$$

where  $u_k^h$ ,  $x_k^h$  and  $y_k^h$ ,  $k = 0, 1, \dots, N_h$  have been obtained from Algorithm 3. One of the main goals of the paper is to prove that for a passive system these approximations converge in a suitable sense. This property is called *consistency* of the numerical method. In the following example, we illustrate that Algorithm 3 is not always consistent even if Assumption 4 holds.

*Example 5:* Consider the linear complementarity system (consisting of a triple integrator with complementarity conditions)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad y = x_1$$

$$0 \leq u \perp y \geq 0$$

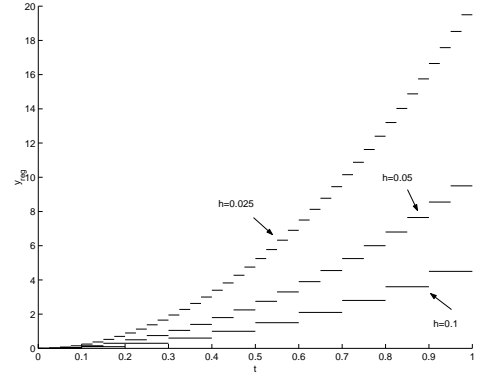


Fig. 2. Nonconvergence of backward Euler approximations for the triple integrator with ideal diode characteristic.

with the initial state  $x_0 = (0 \quad -1 \quad 0)^\top$ . Algorithm 3 gives

$$(u_k^h, y_k^h) = \begin{cases} (h^{-2}, 0) & \text{if } k = 0 \\ (0, \frac{k(k+1)}{2}h) & \text{if } k \neq 0. \end{cases}$$

It follows from (6d) that

$$\|y_{\text{reg}}^h\| \geq \left( \int_{(N_h-2)h}^{(N_h-1)h} \|y_{(N_h-1)}^h\|^2 dt \right)^{1/2} = O(h^{-1/2})$$

whenever  $N_h \geq 2$ . Therefore,  $y_{\text{reg}}^h$  is far from being convergent as it is not bounded as  $h$  converges to zero. For three different values of  $h$ , the trajectories of  $y_{\text{reg}}^h$  on  $[0, 1]$  are depicted in Fig. 2.

This example indicates that one should be cautious in applying a time-stepping method to a general LCS. As a consequence, verification of the numerical scheme in the sense of showing consistency is needed.

## V. PRELIMINARIES

### A. Passivity of a Linear System

In the sequel, we are mainly concerned with linear passive complementarity systems.

*Definition 6:* [28] The system  $(A, B, C, D)$  given by (2a)-(2b) is said to be *passive (dissipative with respect to the supply rate  $u^\top y$ )* if there exists a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  (a *storage function*), such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geq V(x(t_1))$$

holds for all  $t_0$  and  $t_1$  with  $t_1 \geq t_0$ , and all  $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$  satisfying (2a)-(2b).

We state a well-known theorem on passive systems which is sometimes called the positive real lemma.

*Lemma 7:* [28] Assume that  $(A, B, C)$  is minimal. Then the following statements are equivalent:

1.  $(A, B, C, D)$  is passive.
2. The matrix inequalities

$$K = K^\top \geq 0 \text{ and } \begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leq 0$$

have a solution.

3.  $G(s)$  is positive real, i.e.,  $G(\lambda) + G^\top(\bar{\lambda}) \geq 0$  for all  $\lambda \in \mathbb{C}$  with  $\lambda \notin \sigma(A)$  and  $\text{Re}(\lambda) > 0$ .

Moreover, if  $(A, B, C, D)$  is passive all solutions of the matrix inequalities in item 2 are positive definite.

Throughout the paper, we will frequently use the following assumption.

*Assumption 8:*  $(A, B, C)$  is a minimal representation and  $B$  is of full column rank.

### B. Solution Concept for LCS

Before precisely defining the solution concept of  $\text{LCS}(A, B, C, D)$ , we need to mention several spaces of functions and distributions. The space  $\mathcal{B}$  denotes the space of Bohl functions, i.e., functions having rational Laplace transforms. The space  $\mathcal{B}_\delta$  consists of the distributions of the form  $\mathbf{u} = \mathbf{u}_{\text{imp}} + \mathbf{u}_{\text{reg}}$ , where  $\mathbf{u}_{\text{imp}} = u_0\delta$  is called the *impulsive part* with  $u_0 \in \mathbb{R}$  and  $\mathbf{u}_{\text{reg}} \in \mathcal{B}$  is called the *regular part*. A distribution  $\mathbf{u} \in \mathcal{B}_\delta^n$  is said to be *initially nonnegative*, if its Laplace transform  $\hat{\mathbf{u}}(s)$  satisfies  $\hat{\mathbf{u}}(\sigma) \geq 0$  for all sufficiently large  $\sigma \in \mathbb{R}$ .

Next, we recall the notion of an *initial solution* which is of considerable importance in the analysis of linear complementarity systems.

**Definition 9:** The triple  $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in \mathcal{B}_\delta^{m+n+m}$  is an *initial solution* of  $\text{LCS}(A, B, C, D)$  with initial state  $x_0$  if there exists an index set  $I \subseteq \{1, 2, \dots, m\}$  such that

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + x_0\delta, \quad \mathbf{y} = C\mathbf{x} + D\mathbf{u} \\ \mathbf{y}_i &= 0 \text{ if } i \in I, \quad \mathbf{u}_i = 0 \text{ if } i \notin I \end{aligned}$$

hold in the distributional sense (for more details see [2]), and  $\mathbf{u}$  and  $\mathbf{y}$  are initially nonnegative.

Now, we can give a precise definition of what is meant by a solution of  $\text{LCS}(A, B, C, D)$ . Actually, the (global) solution concept for general linear complementarity systems (see [16]) is more complicated than the one we will present. In the case of linear passive complementarity systems, it can be trimmed as shown in [2].

**Definition 10:** The triple  $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in \mathcal{L}_\delta^{m+n+m}(0, \tau)$  is a (*global*) *solution* of  $\text{LCS}(A, B, C, D)$  on  $[0, \tau]$  with initial state  $x_0$  if the following conditions hold.

1. There exists an initial solution  $(\bar{\mathbf{u}}, \bar{\mathbf{x}}, \bar{\mathbf{y}})$  such that

$$(\mathbf{u}_{\text{imp}}, \mathbf{x}_{\text{imp}}, \mathbf{y}_{\text{imp}}) = (\bar{\mathbf{u}}_{\text{imp}}, \bar{\mathbf{x}}_{\text{imp}}, \bar{\mathbf{y}}_{\text{imp}}).$$

2. The equations

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + x_0\delta \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} \end{aligned}$$

hold in the distributional sense.

3. For almost all  $t \in [0, \tau]$ ,  $0 \leq \mathbf{u}_{\text{reg}}(t) \perp \mathbf{y}_{\text{reg}}(t) \geq 0$ .

Notice that the above definition is equivalent to the integral form given in [2, Definition VII.1]. The proof of the following theorem can be found in [2] and deals with the existence and uniqueness of solutions to linear passive complementarity systems.

**Theorem 11:** Suppose that  $(A, B, C, D)$  is such that Assumption 8 holds and  $(A, B, C, D)$  is passive. Let  $\tau > 0$  be given. For each  $x_0$ , there exists a unique solution  $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in \mathcal{L}_\delta^{m+n+m}(0, \tau)$  of  $\text{LCS}(A, B, C, D)$  on  $[0, \tau]$  with initial state  $x_0$ .

## VI. MAIN RESULTS

The following theorem is the basis of our consistency proof as it states sufficient conditions that imply consistency. The theorem is based on Assumption 4 rather than directly on the passivity property. Due to space constraints, we cannot include the proof here; see [29] or [30, Chapter 6] for full details.

**Theorem 12:** [29] Consider  $\text{LCS}(A, B, C, D)$  such that Assumption 4 holds and  $D$  is nonnegative definite. Let  $\tau > 0$  and  $x_0 \in \mathbb{R}^n$  be given. Also let  $(\mathbf{u}^h, \mathbf{x}^h, \mathbf{y}^h)$  be given by (6) via Algorithm 3. Suppose that there exists an  $\alpha > 0$  such that for all sufficiently small  $h$   $\|h\mathbf{u}_0^h\| \leq \alpha$  and  $\|\mathbf{u}_{\text{reg}}^h\| \leq \alpha$ . For any sequence  $\{h_k\}$  that converges to zero, we have the following statements:

1. There exists a subsequence  $\{h_{k_l}\} \subseteq \{h_k\}$  such that  $(\{\mathbf{u}^{h_{k_l}}\}, \{\mathbf{y}^{h_{k_l}}\})$  converges weakly to some  $(\mathbf{u}, \mathbf{y})$  and  $\{\mathbf{x}^{h_{k_l}}\}$  converges to some  $\mathbf{x}$ .
2. The triple  $(\mathbf{u}, \mathbf{x}, \mathbf{y})$  is a solution of  $\text{LCS}(A, B, C, D)$  on  $[0, \tau]$  with the initial state  $x_0$ .

3. If the solution  $(\mathbf{u}, \mathbf{x}, \mathbf{y})$  is unique for the initial state  $x_0$  in the sense of Definition 10, then the complete sequence  $(\{\mathbf{u}^{h_k}\}, \{\mathbf{y}^{h_k}\})$  converges weakly to  $(\mathbf{u}, \mathbf{y})$  and  $\{\mathbf{x}^{h_k}\}$  converges to  $\mathbf{x}$ .

Note that these conditions do not hold for the system that has been considered in Example 5. We shall show in the appendix that the conditions of Theorem 12 are satisfied in the case of passive complementarity systems so that the following result holds.

**Theorem 13:** Consider  $\text{LCS}(A, B, C, D)$  such that Assumption 8 holds and  $(A, B, C, D)$  is passive. Let  $\tau > 0$  and  $x_0 \in \mathbb{R}^n$  be given. Let  $(\mathbf{u}, \mathbf{x}, \mathbf{y})$  be the unique solution of  $\text{LCS}(A, B, C, D)$  on  $[0, \tau]$  with the initial state  $x_0$ . Also let  $(\mathbf{u}^h, \mathbf{x}^h, \mathbf{y}^h)$  be given by (6) via Algorithm 3. Then,  $(\{\mathbf{u}^h\}, \{\mathbf{y}^h\})$  converges weakly to  $(\mathbf{u}, \mathbf{y})$  and  $\{\mathbf{x}^h\}$  converges to  $\mathbf{x}$  as the step size  $h$  tends to zero.

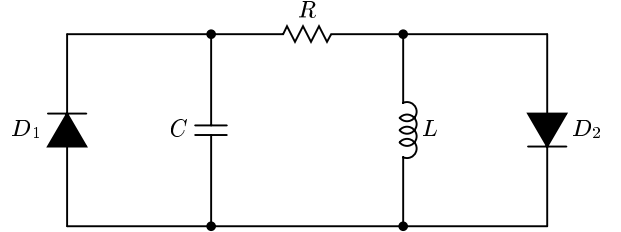


Fig. 3. RLC circuit with ideal diodes

We illustrate Theorem 13 in a simple example.

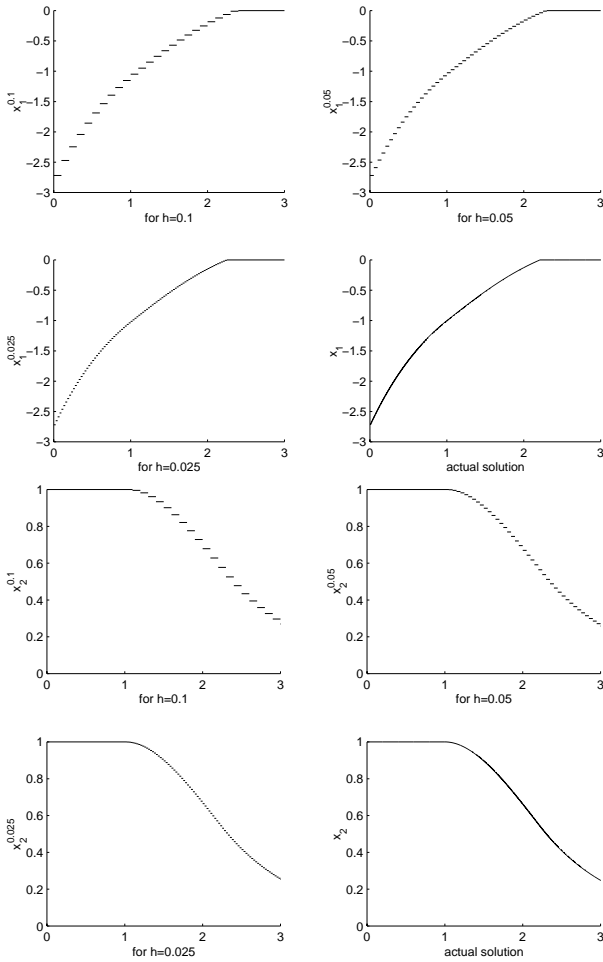
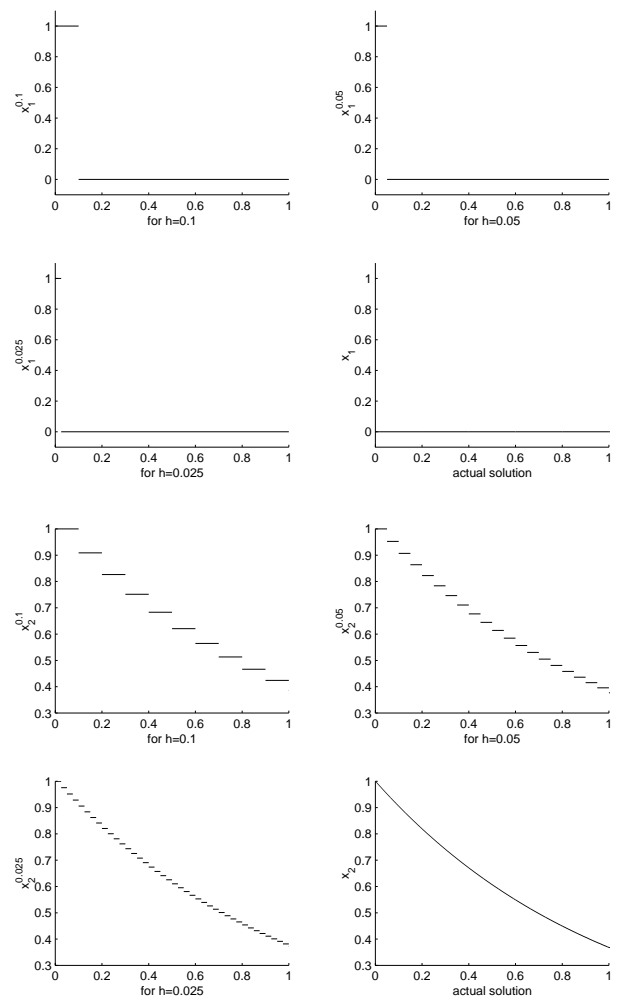
**Example 14:** Consider the linear RLC circuit (with  $R = 1 \Omega$ ,  $L = 1$  H and  $C = 1$  F) coupled to two ideal diodes as shown in Figure 3. The network is described by

$$\begin{aligned} \dot{x}_1 &= x_2 - u_1 + u_2, \quad \dot{x}_2 = -x_1 - x_2 - u_2 \\ y_1 &= -x_1, \quad y_2 = x_1 + x_2 + u_2 \\ 0 &\leq u \perp y \geq 0 \end{aligned}$$

where  $x_1$  is the voltage across the capacitor,  $x_2$  is the current through the inductor,  $u_1$  and  $u_2$  are the current through, and  $y_1$  and  $y_2$  are (minus) the voltage across diode 1 and 2, respectively. For two different initial states, we apply the backward Euler time-stepping method. The first initial state is  $x_0 = \text{col}(-e, 1)$ . In Figure 4, the approximating state trajectories for the step sizes 0.1, 0.5 and 0.025 are depicted. Note that there are two events (topology changes) of the circuit. The second initial state we consider is  $x_0 = \text{col}(1, 1)$ . As shown in [2, Example 6.3], this initial state is *inconsistent* in the sense that the corresponding solution contains a Dirac impulse in the  $u$ -trajectory and hence a discontinuity in the state. As expected from Theorem 13, the approximating state trajectories converge to the actual ones. In Figure 5, the approximating trajectories are depicted for the step sizes 0.1, 0.5 and 0.025. For reasons of clarity we draw the successive computed values of the approximations as horizontal lines; in practice of course one would use for instance piecewise linear interpolation. Note that we also picture the solution with inequalities in (6d) replaced by  $lh \leq t < (l+1)h$ .

## VII. CONCLUSIONS

In this paper, we studied the consistency of a time-stepping method based on the backward Euler integration routine. The method has proven itself already in practice for the transient simulation of piecewise linear electrical circuits and constrained mechanical systems. However, one cannot indiscriminately apply this method for general classes of discontinuous systems as shown by an example in this paper. The main result of the paper has presented a rigorous proof of the consistency of the backward Euler time-stepping method when applied to a class of networks that can be modeled as linear passive electrical networks with ideal diodes (or stated differently, can be modeled as

Fig. 4. State trajectories for the initial state  $x_0 = \text{col}(-e, 1)$ .Fig. 5. State trajectories for the initial state  $x_0 = \text{col}(1, 1)$ .

linear passive complementarity systems). In spite of the mixed continuous and discrete behaviour of the circuit and the possibility of Dirac impulses occurring at the initial time, we have shown the convergence of the approximations to the actual transient solution of the network model.

Of course, it would be interesting to generalize these results to other systems of a mixed continuous and discrete nature. In particular, we are currently studying the consistency of the backward Euler method for dynamical systems with relays (see [27] as a first step in this direction) and for other linear complementarity systems. For systems where the backward Euler time-stepping scheme does not generate proper output (like the triple integrator), it is useful to consider extensions of the time-stepping algorithm that are consistent.

### VIII. APPENDIX

In the following lemma, we state some results for the matrix inverse  $(I - hA)^{-1}$ .

*Lemma 15:* Let  $A \in \mathbb{R}^{n \times n}$ . The following statements hold:

1.  $\|(I - hA)^{-1}\| \leq \frac{1}{1 - \lambda h}$  for all  $h$  with  $\lambda h < 1$  where  $\lambda$  is the largest eigenvalue of  $\frac{1}{2}(A + A^T)$ .
2. There exists an  $\alpha > 0$  such that  $\|(I - hA)^{-1}\| \leq \alpha$  for all sufficiently small  $h$ .

*Proof:* 1: By the Wazewski inequality (see e.g. [31, Theorem 8.1]),  $\|e^{At}\| \leq e^{\lambda t}$  for all  $t$  where  $\lambda$  is the largest eigenvalue of  $\frac{1}{2}(A + A^T)$ . Theorem 1.5.3 in [32] gives now the desired inequality.

2: It can easily be verified by using item 1 that  $\|(I - hA)^{-1}\| \leq 1/(1 - \beta)$  whenever  $\lambda h \leq \beta < 1$ . ■

#### A. Rational Complementarity Problem

It can be shown that there is a one-to-one relation between the initial solutions to  $\text{LCS}(A, B, C, D)$  with initial state  $x_0$  and the *proper* solutions of the so-called *rational complementarity problem* (see for instance [17, 19]).

*Problem 16:* ( $\text{RCP}(x_0, A, B, C, D)$ ) Given  $x_0 \in \mathbb{R}^n$  and  $(A, B, C, D)$  with appropriate sizes, find  $\hat{u}(s) \in \mathbb{R}^m(s)$  and  $\hat{y}(s) \in \mathbb{R}^m(s)$  such that

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]\hat{u}(s)$$

and  $\hat{u}(s) \perp \hat{y}(s)$  for all  $s \in \mathbb{C}$ , and  $\hat{u}(\sigma) \geq 0$  and  $\hat{y}(\sigma) \geq 0$  for all sufficiently large  $\sigma \in \mathbb{R}$ .

The following proposition states the above mentioned one-to-one relation which is given by the Laplace transform and its inverse.

*Proposition 17:* [17] The triple  $(u, x, y)$  is an initial solution of  $\text{LCS}(A, B, C, D)$  with initial state  $x_0$  if and only if its Laplace transform  $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$  is such that  $(\hat{u}(s), \hat{y}(s))$  is a proper solution of  $\text{RCP}(x_0, A, B, C, D)$  and  $\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s)$ .

We shall use the following proposition which establishes the relation between the solutions of the one-step problem and the solutions of the rational complementarity problem.

*Proposition 18:* Consider a matrix quadruple  $(A, B, C, D)$  such that Assumption 4 holds. We have the following statements for all  $x_0 \in \mathbb{R}^n$ .

1. RCP( $x_0, A, B, C, D$ ) has a unique solution.
2. For all sufficiently small  $h$ ,

$$\hat{u}(h^{-1}) = hu_0^h, \quad \hat{x}(h^{-1}) = hx_0^h, \quad \hat{y}(h^{-1}) = hy_0^h$$

where  $(\hat{u}(s), \hat{y}(s))$  is the solution of RCP( $x_0, A, B, C, D$ ),  $\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s)$  and  $(u_0^h, x_0^h, y_0^h)$  is the solution of the one-step problem of Algorithm 3 for  $k = 0$ .

*Proof:* 1: Observe the basic fact that if LCP( $q, M$ ) is solvable then LCP( $\alpha q, M$ ) is also solvable provided that  $\alpha \geq 0$ . As a consequence, Assumption 4 implies together with the identity  $h(I - hA)^{-1} = (h^{-1}I - A)^{-1}$  that for all sufficiently small  $h$ , LCP( $C(h^{-1}I - A)^{-1}x_0, G(h^{-1})$ ) has a unique solution. From [17, Theorem 4.1 and Corollary 4.10], we can conclude that RCP( $x_0, A, B, C, D$ ) has a unique solution.

2: Let  $(\hat{u}(s), \hat{y}(s))$  be the solution of RCP( $x_0, A, B, C, D$ ). It can be easily seen that  $\hat{u}(h^{-1})$  solves LCP( $C(h^{-1}I - A)^{-1}x_0, G(h^{-1})$ ) for all sufficiently small  $h$ . Note that if  $z$  is a solution of LCP( $q, M$ ) then  $\alpha z$  is a solution of LCP( $\alpha q, M$ ) provided  $\alpha \geq 0$ . Therefore,  $h^{-1}\hat{u}(h^{-1})$  solves LCP( $C(I - hA)^{-1}x_0, G(h^{-1})$ ) for all sufficiently small  $h$ . In other words, for all sufficiently small  $h$

$$\hat{u}(h^{-1}) = hu_0^h, \quad \hat{x}(h^{-1}) = hx_0^h, \quad \hat{y}(h^{-1}) = hy_0^h \quad (7a)$$

where  $\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s)$ .  $\blacksquare$

### B. Some Results on LCPs

We will present in this subsection some results on LCPs, that will be needed to prove the main result (Theorem 13) for linear passive complementarity systems.

*Proposition 19:* Let  $M \in \mathbb{R}^{m \times m}$  be a positive definite matrix and  $z_i$  the unique solution of LCP( $q_i, M$ ) for  $i = 1, 2$ . Then,

$$\|z_1 - z_2\| \leq \frac{m^{3/2}}{\mu(M)} \|q_1 - q_2\|$$

where  $\mu(M)$  denotes the smallest eigenvalue of the symmetric part of  $M$ , i.e.,  $\frac{1}{2}(M + M^\top)$ .

*Proof:* By Lemma 7.3.10 and Proposition 5.10.10 in [20], we have  $\|z_1 - z_2\|_\infty \leq (m/\mu(M))\|q_1 - q_2\|_\infty$ . It yields  $\|z_1 - z_2\| \leq (m^{3/2}/\mu(M))\|q_1 - q_2\|$  since  $\|z\| \leq m^{1/2}\|z\|_\infty$  and  $\|z\|_\infty \leq \|z\|$  for all  $z \in \mathbb{R}^m$ .  $\blacksquare$

Using the passivity property, we can compute a lower bound on  $\mu(G(h^{-1}))$  with  $G(s) := D + C(sI - A)^{-1}B$ , that will be useful for the application of Proposition 19.

*Lemma 20:* Consider a matrix quadruple  $(A, B, C, D)$  such that Assumption 8 holds and  $(A, B, C, D)$  is passive. Let  $\mu(N)$  denote the smallest eigenvalue of the symmetric part of a matrix  $N$ . The following statements hold.

1.  $D \geq 0$ .
2.  $u \neq 0$  and  $u^\top Du = 0$  imply that  $u^\top CBu > 0$ .
3. There exists an  $\alpha > 0$  such that  $\mu(D + hCB) \geq \alpha h$  for all sufficiently small  $h$ .
4. There exists a  $\beta > 0$  such that  $\mu(G(h^{-1})) \geq \beta h$  for all sufficiently small  $h$  where  $G(s) = D + C(sI - A)^{-1}B$ .

*Proof:* 1-2: It follows from [30, Lemma 3.8.2].

3: It follows from [30, Lemma 5.7.6].

4: It is known from matrix theory (see e.g. [33, Property 9.13.4.9]) that  $\mu(N_1 + N_2) \geq \mu(N_1) + \mu(N_2)$  for all square matrices  $N_1$  and  $N_2$ . Hence, we get from item 3 that  $\mu(G(h^{-1})) \geq \mu(D + hCB) + O(h^2) \geq \beta h$  for some  $\beta > 0$  and all sufficiently small  $h$ .  $\blacksquare$

The following auxiliary lemma will be needed in the sequel.

*Lemma 21:* Let  $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \geq b\}$  be a given nonempty polyhedron with  $A \in \mathbb{R}^{n \times m}$  and  $b \in \mathbb{R}^m$  and let  $x^*$  be equal to  $\arg \min_{x \in \mathcal{P}} \|x\|$ . There exists an index set  $J \subseteq \{1, 2, \dots, n\}$  such that  $x^* = \arg \min_{A_J \bullet, x=b_J} \|x\|$ .

*Proof:* Consider the convex quadratic optimization problem  $\min_{Ax \geq b} \frac{1}{2}x^\top x$ . The well-known Kuhn-Tucker conditions are necessary and sufficient for this problem because of its convexity (see for instance [20, Chapter 1.2]), i.e.  $x^*$  is the solution of the optimization problem above if and only if there exists a  $u \in \mathbb{R}^m$  such that  $x^* = A^\top u$ ,  $Ax^* \geq b$ ,  $u \geq 0$ ,  $u^\top (Ax^* - b) = 0$ . Take such a vector  $u$ . Let  $J = \{i \mid u_i > 0\}$  and  $v = u_J$ . Then,  $x^*$  satisfies  $x^* = (A_J \bullet)^\top v$ ,  $A_J \bullet x^* = b_J$ . Note that these are necessary and sufficient (Kuhn-Tucker) conditions for the convex quadratic minimization problem  $\min_{A_J \bullet, x=b_J} \frac{1}{2}x^\top x$ .  $\blacksquare$

To formulate the next lemma, we need to recall the concept of a dual cone.

*Definition 22:* For any nonempty set  $\mathcal{Q} \subset \mathbb{R}^m$ , the set  $\{w \in \mathbb{R}^m \mid w^\top v \geq 0 \text{ for all } v \in \mathcal{Q}\}$  is called the *dual cone* of  $\mathcal{Q}$  and is denoted by  $\mathcal{Q}^*$ .

*Lemma 23:* Let  $M \in \mathbb{R}^{m \times m}$  be nonnegative definite and  $\mathcal{Q} = \text{SOL}(0, M)$ . We have the following statements.

1. LCP( $q, M$ ) is solvable if and only if  $q \in \mathcal{Q}^*$ .
2. For each  $q \in \mathcal{Q}^*$ , there exists a unique least-norm solution  $z^* \in \text{SOL}(q, M)$  such that  $\|z^*\| \leq \|z\|$  for all  $z \in \text{SOL}(q, M)$ .
3. There exists  $\alpha > 0$  such that for all  $q \in \mathcal{Q}^*$   $\|z^*(q)\| \leq \alpha \|q\|$ , where  $z^*(q)$  denotes the least-norm solution (see item 2) of LCP( $q, M$ ).

*Proof:* 1: It follows from [20, Exercise 3.12.1 and Corollary 3.8.10].

2: This follows from the fact that  $\text{SOL}(q, M)$  is a nonempty polyhedron whenever  $q \in \mathcal{Q}^*$  [20, Theorem 3.1.7(c)].

3: Define

$$\alpha(A) = \max_{Ax=y} \{ \|x\| \mid y \in \text{im } A \text{ and } \|y\| = 1 \}$$

if  $A \neq 0$  and zero otherwise. Take

$$\alpha = \sqrt{2} \max_{J \subseteq \mathcal{A}_m} \max_{K \subseteq \mathcal{A}_m} \alpha((\text{col}(I, -I_{J^c \bullet}, M, -M_{J \bullet}))_{K \bullet})$$

where  $\mathcal{A}_k = \{1, 2, \dots, k\}$ . For any  $q \in \mathcal{Q}^*$ , we know from the items 1 and 2 that LCP( $q, M$ ) is solvable and that there exists a unique least-norm solution  $z^*(q)$ . Let  $J = \{i \mid z_i^*(q) > 0\}$ . Clearly,  $\mathcal{P} = \{v \mid v_J \geq 0, v_{J^c} = 0, q_J + M_{JJ}v_J = 0, \text{ and } q_{J^c} + M_{J^c J}v_J \geq 0\} \subseteq \text{SOL}(q, M)$  and  $z^*(q) \in \mathcal{P}$ . Note that  $\mathcal{P}$  is a polyhedron, since  $\mathcal{P} = \{v \mid Av \geq b\}$  where  $A = \text{col}(I, -I_{J^c \bullet}, M, -M_{J \bullet})$  and  $b = \text{col}(0, 0, -q, q_J)$ . Moreover, it is obvious that  $z^*(q) = \arg \min_{Av \geq b} \|v\|$ . Then, according to Lemma 21 there exists  $K \subseteq \{1, 2, \dots, 3m\}$  such that  $z^*(q) = \arg \min_{A_K \bullet, v=b_K} \|v\|$ . Thus, we have  $\|z^*(q)\| \leq \alpha(A_{K \bullet}) \|b_K\|$ . Note that  $\|b_K\|^2 \leq \|b\|^2 \leq \|q\|^2 + \|q_J\|^2 \leq 2\|q\|^2$  and  $\sqrt{2}\alpha(A_{K \bullet}) \leq \alpha$ . Consequently,  $\|z^*(q)\| \leq \alpha \|q\|$ .  $\blacksquare$

### C. Proof of Theorem 13

After these results on LCPs, the proof of the main result on linear passive complementarity systems is in order. The proof will be based on showing that the requirements of Theorem 12 are fulfilled for this class of linear complementarity systems.

*Lemma 24:* Consider LCS( $A, B, C, D$ ) such that Assumption 8 holds and  $(A, B, C, D)$  is passive. For all sufficiently small  $h$ , LCP( $hC(I - hA)^{-1}\bar{x}, G(h^{-1})$ ) has a unique solution for each  $\bar{x} \in \mathbb{R}^n$ .

*Proof:* The statement follows from the positive definiteness of  $G(h^{-1})$  for all sufficiently small  $h$  (Lemma 20 item 4 together with Theorem 3.1.6 of [20]).  $\blacksquare$

*Lemma 25:* Consider LCS( $A, B, C, D$ ) such that Assumption 8 holds and  $(A, B, C, D)$  is passive. Let  $\tau > 0$  and  $\mathcal{Q} = \text{SOL}(0, D)$ ,

i.e.  $\mathcal{Q} = \{z \in \mathbb{R}^m \mid z \geq 0, Dz \geq 0 \text{ and } z^\top Dz = 0\}$  be given. Also let  $(\{u_k^h\}, \{x_k^h\}, \{y_k^h\})$  be produced by Algorithm 3. The following statements hold for all sufficiently small  $h$ .

1.  $Cx_k^h \in \mathcal{Q}^*$  for all  $k \neq -1$ .
2. There exists an  $\alpha > 0$  independent of  $x_0$  such that  $\|u_k^h\| \leq \alpha \|x_0\|$  for all  $k \neq 0$ .

*Proof:* 1: It is evident from (5b) and (5c) that  $u_k^h$  solves LCP( $Cx_k^h, D$ ) when  $k \neq -1$ . Since  $D$  is nonnegative definite (Lemma 20 item 1),  $Cx_k^h \in \mathcal{Q}^*$  due to [20, Corollary 3.8.10].

2: All inequalities involving  $h$  are meant to hold for all sufficiently small  $h$ , and  $\alpha_1, \alpha_2, \dots, \alpha_6$  are suitably chosen positive constants in this proof. Note that LCP( $Cx_k^h, D$ ) is solvable for all  $k \neq -1$  due to item 1 and [20, Corollary 3.8.10]. Let  $u^*$  be the least-norm solution of LCP( $Cx_k^h, D$ ). Clearly,  $u^*$  solves also LCP( $Cx_k^h - hC(I - hA)^{-1}Bu^*, G(h^{-1})$ ). According to Proposition 19, we have

$$\|u_{k+1}^h - u^*\| \leq \frac{m^{3/2}}{\mu(G(h^{-1}))} \|C(I - hA)^{-1}x_k^h - Cx_k^h + hC(I - hA)^{-1}Bu^*\|,$$

since  $u_{k+1}^h$  solves LCP( $C(I - hA)^{-1}x_k^h, G(h^{-1})$ ) and  $G(h^{-1}) > 0$  for all sufficiently small  $h$ . By using the triangle inequality and Lemma 20 item 4, we obtain

$$\|u_{k+1}^h - u^*\| \leq \frac{\alpha_1}{h} \|C[(I - hA)^{-1} - I]x_k^h\| + \alpha_1 \|C(I - hA)^{-1}Bu^*\|.$$

Note that  $(I - hA)^{-1} - I = hA(I - hA)^{-1}$ . It can be easily verified that Lemma 15 item 2 and Lemma 23 item 3 result in

$$\|u_{k+1}^h - u^*\| \leq \alpha_2 \|x_k^h\|. \quad (8)$$

Consequently, we get

$$\|u_{k+1}^h\| \leq \|u^*\| + \|u_{k+1}^h - u^*\| \leq \alpha_3 \|x_k^h\| \quad (9)$$

by applying the triangle inequality and employing Lemma 23 item 3 and (8). It follows that

$$\begin{aligned} \|x_{k+1}^h\| &\leq \|x_k^h\| + \|x_{k+1}^h - x_k^h\| \\ &\leq \|x_k^h\| + \|[(I - hA)^{-1} - I]x_k^h + h(I - hA)^{-1}Bu_{k+1}^h\| \quad (\text{from (5a)}) \\ &\leq (1 + \alpha_4 h) \|x_k^h\|. \quad (\text{from Lemma 15 item 2}) \end{aligned}$$

This implies that

$$\|x_k^h\| \leq \alpha_5 \|x_0^h\| \quad (10)$$

for some  $\alpha_5 > 0$  since  $\lim_{h \rightarrow 0} (1 + \alpha_4 h)^{N_h} = e^{\alpha_4 \tau}$  (Lemma 15 item 3). Here  $N_h = \lceil \frac{\tau}{h} \rceil$ . Note that we have

$$\|x_0^h\| = \|x_0 + hBu_0^h\| \leq \alpha_6 \|x_0\| \quad (11)$$

from Lemma 6.3 item 2. Finally, (9), (10) and (11) establish the desired inequality. ■

After all these preliminaries, we can prove Theorem 13.

*Proof of Theorem 13:* According to Lemma 24, Assumption 4 holds. Then, Proposition 18 item 1 implies that RCP( $x_0, A, B, C, D$ ) has a unique solution, say  $(\hat{u}(s), \hat{y}(s))$ . It is known from [2, Theorem 3.6] that  $\hat{u}(s)$  is proper. Therefore, boundedness of  $\|h\hat{u}_0^h\|$  for all sufficiently small  $h$  follows from Proposition 18 item 2. On the other hand,  $D$  is nonnegative definite due to item 1 of Lemma 20 and

$$\|u_{\text{reg}}^h\| = \left( \int_0^\tau \|u_{\text{reg}}^h(t)\| dt \right)^{1/2} \leq \alpha \tau^{1/2} \|x_0\| \quad (12)$$

due to (6) and Lemma 25 item 2. Finally, it is known from Theorem 11 that  $(u, x, y)$  is the unique solution on  $[0, \tau]$  with the initial state  $x_0$ . As a consequence of Theorem 12 item 3,  $\{(u^{h_k}, y^{h_k})\}$  converges weakly to  $(u, y)$  and  $\{x^{h_k}\}$  converges to  $x$  for any sequence  $\{h_k\}$  that converges to zero. In other words,  $\{(u^h, y^h)\}$  converges weakly to  $(u, y)$  and  $\{x^h\}$  converges to  $x$  as  $h$  tends to zero. ■

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