Embedding partial geometries in Steiner designs

Andries E. Brouwer, Willem H. Haemers
Vladimir D. Tonchev

Abstract

We consider the following problem: given a partial geometry $\mathcal{P}$ with v points and k points on a line, can one add to the line set a set of k-subsets of points such that the extended family of k-subsets is a 2-$(v,k,1)$ design (or a Steiner system $S(2,k,v)$). We give some necessary conditions for such embeddings and several examples. One of these is an embedding of the partial geometry $PQ^+(7,2)$ into a 2-$(120,8,1)$ design.

1 Introduction

We consider the question whether, given a partial geometry $\mathcal{P} = (X, \mathcal{L})$, there is a Steiner 2-design $\mathcal{D} = (X, \mathcal{B})$ such that $\mathcal{L} \subseteq \mathcal{B}$. Clearly, the existence of such an embedding of $\mathcal{P}$ does not depend on the structure of $\mathcal{P}$, but only on its collinearity graph and line size.

Troughout $\mathcal{P} = (X, \mathcal{L})$ will denote a partial geometry with parameters s, t and α. The number v of points and the number l of lines of $\mathcal{P}$ are given by

$$v = (s+1)(st+\alpha)/\alpha\ ,\ l = (t+1)(st+\alpha)/\alpha\ .$$

The collinearity graph (or point graph) is strongly regular having eigenvalues $s(t+1)$, $s-\alpha$ and $-t-1$ with multiplicities

$$1,\ f = \frac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)},\ g = \frac{s(s+1-\alpha)(st+\alpha)}{\alpha(s+t+1-\alpha)},$$

respectively. For these and other results on partial geometries we refer to the survey paper by De Clerck and Van Maldeghem [3].

Suppose we have a collection $\mathcal{C}$ of $(s+1)$-cocliques in the point graph of $\mathcal{P}$ that cover all non-collinear pairs of points exactly once. Then $\mathcal{P}$ is embedded in the Steiner 2-design $\mathcal{D} = (X, \mathcal{C} \cup \mathcal{L})$ with parameters:

$$v,\ k = s+1,\ r = (st+t+\alpha)/\alpha,\ b = (st+\alpha)(st+t+\alpha)/\alpha^2.$$

Clearly a necessary condition for $\mathcal{P}$ to be embeddable is that r and b are integers. A quick inspection of the parameters of the known partial geometries shows that these divisibility conditions are satisfied very often. The following result excludes many more parameter sets.

Theorem 1.1 *Suppose $\mathcal{P}$ is embeddable in a Steiner 2-design $\mathcal{D}$ and suppose that $\mathcal{P}$ is not a Steiner 2-design or a net (i.e. $\alpha \neq s+1$ and $\alpha \neq t$). Then*

$$\alpha \leq \frac{t(s+1)}{t+s+1}.$$

If equality holds then $\alpha^f(t-\alpha)^g$ is the square of an integer.

This result is an immediate corollary of the following 'Fisher inequality'.

Theorem 1.2 *Let Γ be a strongly regular graph with parameters (v, k, λ, μ). If Γ is the collinearity graph of a partial linear space with l lines of size $s+1$, then either the given partial linear space is a partial geometry with parameters (s, t, α), or $l \geq v$. If $l = v$, then* $\det(A + (k/s)I)$ *is a square, where A is the adjacency matrix of Γ.*

Proof (of Theorem 1.2). Let N be the $v \times l$ point-line incidence matrix of the partial linear space, and A the adjacency matrix of Γ. Then $NN^{\top} = A + (t+1)I$, where $t+1 := k/s$ is the number of lines on each point. If $l = v$, then N is square, and $\det(A + (t+1)I) = (\det N)^2$. If $l < v$, then $NN^{\top}$ has rank less than v, so that A has eigenvalue $-t-1$, i.e., $(t+1)^2 - (\mu - \lambda)(t+1) + \mu - k = 0$. In this case, since $t+1$ divides k, it also divides μ, say $\mu = (t+1)\alpha$ for some nonnegative integer α, and we find $\lambda = s - 1 + \alpha t$, so that Γ has the parameters of the point graph of a pg(s, t, α), and since the lines are regular cliques, our partial linear space was in fact a partial geometry. □

Proof (of Theorem 1.1). Apply Theorem 1.2 to the noncollinearity graph Γ of $\mathcal{P}$. We find either $b - l \geq v$, which reduces to $\alpha(s+t+1) \leq t(s+1)$, or Γ is the collinearity graph of a partial geometry $\mathcal{P}'$ with parameters (s', t', α'), where $s' = s$, $t' = s - \alpha$, $\alpha' = s - t$ and $(t-\alpha)(s+1-\alpha) = 0$. The adjacency matrix A of Γ has eigenvalues $st(s+1-\alpha)/\alpha$, $\alpha - 1 - s$, t with respective multiplicities 1, f, g, so if $b - l = v$ then $\alpha^f(s+t+1)^g$ is a square, and since $t^2 = (t-\alpha)(s+t+1)$ also $\alpha^f(t-\alpha)^g$ is a square. □

A strongly regular graph is called *imprimitive* when it or its complement is a vertex disjoint union of cliques (i.e., when $\mu = 0$ or $\mu = k$ or there are no (non)edges at all). A union of m-cliques is the collinearity graph of a partial linear space with lines of size c if and only if a 2-$(m, c, 1)$ design exists. The complement of a union of n m-cliques is the collinearity graph of a partial linear space with lines of size c if and only if a group divisible design $\mathrm{GD}(c, 1, m; nm)$ exists. Nothing nontrivial can be said here.

A partial geometry is called *improper* if $\alpha = 1$, s, $s+1$, t or $t+1$. For $\alpha = s$ or $\alpha = s+1$ the point graph is imprimitive. Otherwise, if $\alpha = t+1$, $\mathcal{P}$ is a dual Steiner system, which is not embeddable by Theorem 1.1. If $\alpha = t$, $\mathcal{P}$ is a net that we want to embed in a 2-$(k^2, k, 1)$ design, i.e. an affine plane. Therefore $\mathcal{P}$ is embeddable if and only if it is the union of some parallel classes of an affine plane.

Finally we consider the case $\alpha = 1$. Then $\mathcal{P}$ is a generalized quadrangle $GQ(s,t)$. In this case the divisibility conditions and the conditions of Theorem 1.1 are always fulfilled. Several of the known generalized quadrangles are constructed as a set of lines in a projective or affine space and hence are embeddable by construction, see Payne and Thas [6]. The smallest (with respect to v) open case is a possible embedding of $GQ(5,3)$ in a 2-$(96,6,1)$ design.

One might ask whether an embedding will be unique. But when there is an embedding into some Steiner system with lots of subsystems, like a projective or affine space, then by twisting one or more subsystems one will in general get lots of embeddings. For example, in the smallest non-trivial case, that of the generalized quadrangle $GQ(2,2)$, naturally embedded into $PG(3,2)$ as the $Sp(4,2)$ quadrangle, each plane contains three lines of the quadrangle and there are two ways of extending that set of three to a 2-(7,3,1) on that plane. Thus, a plane can be 'flipped'. Flipping one plane destroys all other planes except those that meet it in a line from the quadrangle. An arbitrary 2-(15,3,1) containing the lines of the quadrangle is obtained from $PG(3,2)$ by flipping 0, 1, 2 or 3 planes on a given line of the quadrangle, and we find precisely four nonisomorphic 2-$(15,3,1)$ designs that contain $GQ(2,2)$.

2 Proper partial geometries

Let us examine the known families of proper partial geometries for possible imbeddings.

First, we consider the class $\mathcal{S}(\mathcal{K})$. These geometries exist whenever there is a maximal arc $\mathcal{K}$ of degree d in a projective plane of order $q = dc$. The parameters are $s = d(c-1)$, $t = c(d-1)$, $\alpha = (c-1)(d-1)$. Substitution in Theorem 1.1 gives $d \leq 1 + c/(c-1)^2$, which is satisfied only if $d = c = 2$. Then $\mathcal{P}$ is $GQ(2,2)$, which has four embeddings, as we saw before. Note that the same conclusion holds for the dual geometries, since they belong to the same parameter family.

Next we consider the class $\mathcal{T}_2^*(\mathcal{K})$ with parameters $s = 2^h - 1$, $t = (2^h + 1)(2^m - 1)$, $\alpha = 2^m - 1$. They exist whenever m divides h and consist of all the points and a subset of the lines of $AG(3, 2^h)$, being all translates of the lines through the origin that correspond to a maximal arc $\mathcal{K}$ of degree 2^m in $PG(2, 2^h)$. So they all are embeddable. For the duals, all our necessary conditions are fulfilled, but we don't know whether any embeddings exists (except, of course, for the trivial case $m = h = 1$). For $m = 1$, $h = 2$ we get the open case $GQ(5, 3)$, that we mentioned earlier.

The partial geometries $PQ^+(4n-1, 2)$ were constructed by De Clerck, Dye and Thas [2] using a non-singular hyperbolic quadric Q in $PG(4n-1, 2)$ with a spread (i.e a partition of the point set of Q into maximal totally singular subspaces). The points of the partial geometry are the points of $PG(4n-1, 2)$ that are not on Q and the lines are the hyperplanes in the subspaces of the spread. A point x is on a line L if x lies in the polar space of L with respect to Q. The parameters of $PQ^+(4n-1, 2)$ are $s = 2^{2n-1} - 1$, $t = 2^{2n-1}$, $\alpha = 2^{2n-2}$.

We found, first by computer and later by hand, that for $n = 2$ this geometry indeed has an embedding. For an extensive discussion, see the next section.

Theorem 2.1 *The partial geometry $PQ^+(4n-1, 2)$ is embeddable in a Steiner 2-design if and only if $n \leq 2$.*

Proof First suppose the geometry is embeddable. Then the blocks of the embedding are cocliques of size $k = 2^{2n-1}$ in the point graph of $PQ^+(4n-1, 2)$, which is the orthogonality graph on the nonsingular points. The Gram matrix of the vectors spanning the points of any such coclique is $J - I$, which is nonsingular, and hence these vectors are linearly independent and their number cannot exceed the dimension of the space. That is, $2^{2n-1} \leq 4n$, so $n \leq 2$. The case $n = 1$ is trivial: $K_{3,3}$ can be extended to K_6. It remains to show embeddibility in case $n = 2$. That is, we have to construct a system of 8-cliques, one on each edge, for the nonorthogonality graph on the nonsingular points for $O_8^+(2)$. As follows: Pick a good system $\mathcal{O}$ of ovoids, one on each pair of nonorthogonal singular points, and pick a good totally singular 4-space V. For each ovoid O in $\mathcal{O}$ we find a unique point $O \cap V = \langle p \rangle$, and a *base* (basis consisting of 8 mutually nonorthogonal vectors) $B = \{a + p | \langle a \rangle \in O,\ a \neq p\}$. The set of 120 bases thus obtained is the required system of 8-cliques. For details on the choice of $\mathcal{O}$ and V (not any V will do), see the next section. □

Recently, Mathon and Street [4] and De Clerck [1] have derived new partial geometries from $PQ^+(4n - 1, 2)$, but with the same parameters. We don't know if any of these admits an embedding. The non-existence argument for $n > 2$ doesn't work anymore, because these new geometries have other point graphs.

The parameter sets under consideration all meet the bound of Theorem 1.1, but the condition there is always fulfilled. For the related parameter sets $s = 2^{2m}-1$, $t = 2^{2m}$ and $\alpha = 2^{2m-1}$, no geometry is known, but for $m > 1$ they may exist. The embedding however, can not exist by Theorem 1.1 (indeed, $f + g = v - 1$ is odd, so $2^{(2m-1)(f+g)}$ is not a square). For $n \neq 1$, the dual parameters never satisfy the conditions of Theorem 1.1.

The partial geometries $PQ^+(4n-1, 3)$ (the construction method is due to Thas and is only known to work for $n = 1$) have parameters $s = 3^{2n-1} - 1$, $t = 3^{2n-1}$, $\alpha = 2 \cdot 3^{2n-1}$. These geometries and their duals have no embedding by Theorem 1.1.

Finally we consider the two known sporadic proper partial geometries. The one with parameters $s = 4$, $t = 17$ and $\alpha = 2$, constructed by the second author, does not satisfy the divisibility conditions, so has no embedding, but the dual may have one. The other one due to Van Lint and Schrijver with $s = t = 5$ and $\alpha = 2$ looks more interesting. An embedding would lead to a 2-$(81, 6, 1)$ design and such a design is not known to exist. By (incomplete) computer search we were able to extend quite far, but not far enough. Probably the embedding does not exist.

Remark The 120 points and the 120 blocks of the embedding of $PQ^+(7, 2)$ given in Theorem 2.1, form a (flag-transitive) partial linear space, with an incidence graph that is not a bipartite distance-regular graph of diameter 4 or 5, and yet, both the point and the block graph are primitive strongly regular graphs (in fact they are isomorphic). This seems to be a remarkable property. Examples with imprimitive strongly regular graphs are given by the elliptic semiplanes.

3 Triality, ovoids, spreads and bases

Let X be an 8-dimensional vector space over a field K, provided with a nondegenerate quadratic form Q of (maximal) Witt index 4. Let L be the collection of totally singular (t.s.) lines, and let Z_0, Z_1, Z_2 be the sets of singular points and of t.s. 4-spaces of the first and second kind, respectively. Put $Z = Z_0 \cup Z_1 \cup Z_2$. Natural incidence (symmetrized containment) defines a bipartite graph Γ on $Z \cup L$ with bipartition $\{Z, L\}$. This graph has automorphism group $G \simeq O_8^+(K).\mathrm{Sym}(3)$. The group G is transitive on Z and L and preserves $\{Z_0, Z_1, Z_2\}$. The subgroup $G_0 \simeq O_8^+(K)$ preserves the sets Z_0, Z_1, Z_2. The phenomenon that the three sets Z_0, Z_1, Z_2 can be permuted arbitrarily is called *triality*.

Let N_0 be the set of nonsingular points. We need to interpret these in terms of the graph Γ so that we can apply triality and also get sets N_1, N_2. One way of doing that is by representing a nonsingular point $\langle n \rangle$ by the reflection

$$r_{\langle n\rangle} \; : \; x \mapsto x - \frac{(x,n)}{Q(n)} n \; .$$

Let R_0 be this set of reflections. There is a 1-1 correspondence between N_0 and R_0. We have $R_0 \subset G$ and R_0 is closed under conjugation by $G_0.2 \simeq PGO_8^+(K)$. It follows that we can find three sets R_0, R_1, R_2 of reflections under conjugation by G, where R_i consists of the reflections that preserve Z_i and interchange Z_j and Z_k for $\{i,j,k\} = \{0,1,2\}$.

Lemma 3.1 (cf. Tits [7]). *Let $r \in R_i$ and $s \in R_j$ with $i \neq j$. Then $(rs)^3 = 1$.*

Proof We may suppose $r \in R_0$, $s \in R_1$. Since $srsrs \in R_0$ it suffices to show that r and $srsrs$ fix the same singular points. Let $p \in Z_0$ be fixed by r, and put $W = sp$ and $V = rW$ so that $V \in Z_1$ and $W \in Z_2$. Now $\pi := V \cap W$ is a plane containing p and s fixes each line on p in π so that V and sV have at least a plane in common. But $sV \in Z_1$, so $V = sV$, i.e., $rsp = srsp$. □

So far the field was arbitrary, but from now on we take $K = \mathbf{F}_2$. The property that only holds in this case is: *If m, n are nonsingular vectors orthogonal to the t.s. plane π, then $(m,n) = 0$.* Indeed, $\pi^\perp$ is the union of three totally isotropic (t.i.) 4-spaces on π, of which two are t.s., so m and n are both contained in the third.

Now that $K = \mathbf{F}_2$, let us use $+$ between projective points instead of the spanning vectors, and write $\langle a\rangle + \langle b\rangle := \langle a+b\rangle$.

Let a *base* be a set of 8 mutually nonorthogonal nonsingular points. Let an *ovoid* be a set of 9 mutually nonorthogonal singular points. Let a *spread* be a set of 9 pairwise disjoint t.s. 4-spaces. All elements of a spread are of the same kind, and we talk about a *j-spread* when the spread is a subset of Z_j ($j = 1,2$). If we call ovoids *0-spreads*, then i-spreads ($i = 0,1,2$) correspond under triality.

If B is a base, then $b_0 := \sum_{b\in B} b$ is singular, and $O_B := \{b_0\} \cup \{b_0 + b \mid b \in B\}$ is an ovoid. Conversely, if O is an ovoid, and $p \in O$, then $B_{O,p} := \{a + p \mid a \in O, p \neq a\}$ is a base. Thus, we find a 9-1 correspondence between bases and ovoids.

Proposition 3.2 *Let S be a 1-spread. Then $\mathcal{O} := \{rS \mid r \in R_2\}$ is a system of ovoids, one on each pair of nonorthogonal singular points.*

These are the sets $\mathcal{O}$ called 'good' in the previous section.

Proof The numbers fit, so we have to show that no pair of noncollinear points is covered twice. Interchanging types 0 and 2, we have to show that no two disjoint 4-spaces W, W' are contained in both rS and $r'S$ for $r, r' \in R_0$. Let $r = r_m$ and $r' = r_n$. Then $m^\perp$ and $n^\perp$ meet W and W' in the same plane

π and π', respectively. Both m and n lie in the t.i. but not t.s. 4-spaces Y and Y' on π and π'. But then Y and Y' have the line l spanned by m and n in common, and l hits the disjoint planes π, π' in distinct points, so has at least 4 points, contradiction. □

Proposition 3.3 *Take S and $\mathcal{O}$ as above, and fix an element $V \in S$. The set $\mathcal{B}$ of 120 bases $B_{O,p}$ with $O \in \mathcal{O}$ and $p = O \cap V$ is a system of 8-cliques, one on each edge, for the nonorthogonality graph on the 120 nonsingular points.*

Of course the existence of $\mathcal{B}$ is the whole point of this section (in fact, of this paper).

The above description does not show the symmetry between the 120 nonsingular points and the 120 bases. A more symmetric description of the same configuration: Take a base B and put $O := O_B$. Let $\mathcal{S}_1 = \{rO | r \in R_2\}$ and $\mathcal{S}_2 = \{rO | r \in R_1\}$, and join $S \in \mathcal{S}_1$ to the 8 spreads $r_b S$ ($b \in B$) in $\mathcal{S}_2$.

Or, in terms of reflections: Take a base B and join $r \in R_2$ to the 8 reflections $r_b r r_b$ in R_1.

Proof We have to show that both descriptions are equivalent, and that they actually work. As to the former, interchanging types 0 and 1 we see that the spread S with fixed element V, the collection $\mathcal{O} = \{rS | r \in R_2\}$, an ovoid $O = rS$ in $\mathcal{O}$, the point $p = O \cap V$ and the set of 8 reflections fixing 7 points of O and interchanging p with the ninth point correspond to, respectively, the ovoid O with fixed element b_0, the collection $\mathcal{S}_1 = \{rO | r \in R_2\}$, a spread $S = rO$ in $\mathcal{S}_1$, the 4-space $rb_0 \in S$ containing b_0 and the set of 8 reflections $\{r r_b r | b \in B\}$. Since $r r_b r = r_b r r_b$, and $r_b O = O$, this shows that both descriptions are equivalent.

Since the bases by definition are 8-cliques in the nonorthogonality graph on the nonsingular points, and the numbers fit, we only have to check that no two bases $B_{O,p}$ have a pair in common, or, equivalently, that no two sets $\{r r_b r | b \in B\}$ and $\{s r_b s | b \in B\}$ have a pair in common (for $r, s \in R_2$). But if $r r_a r = s r_b s$ and $r r_c r = s r_d s$ ($a, b, c, d \in B$, $a \neq c$, $b \neq d$) then $r_a r r_a = r_b s r_b$, $r_c r r_c = r_d s r_d$, i.e., $r = r_a r_b s r_b r_a = r_a r_b r_d r_c r r_c r_d r_b r_a$. With $S = rO$ this means that $r_a r_b r_d r_c S = S$. But if a, b, c, d are all distinct, then $r_a r_b r_d r_c$ has order 5 (as is seen by its action on O) hence must fix some element $V \in S$. On the other hand, both $\langle a, b, c, d\rangle$ and $\langle a, b, c, d\rangle^\perp$ are elliptic quadrics, and $r_a r_b r_d r_c x = x + (x, c)c + (x, c+d)d + (x, b+d)b + (x, a+b)a$ while a, b, c, d and sums of two of them are not in Z_0, so when two of the inner products (x, c), $(x, c+d)$, $(x, b+d)$, $(x, a+b)$ vanish for $x \in V$, all do. But $V \cap \langle a, b, c, d\rangle^\perp$ does not contain a line, contradiction. If a, b, c, d are not all distinct, say $a = d$, then $r_a r_b r_d r_c = r_a r_b r_a r_c = r_e r_c$ for $e = r_a b = a + b$. But if $r_e r_c S = S$, then, since $r_e r_c$ fixes at least a line on each element of S, $r_e r_c$ must fix all elements of S. But $c \neq e$ and $r_e r_c x = x + (x, c)c + (x, e)e$, so we find that $(x, c) = (x, e)$ on each $V \in S$, so on all of X, so $c = e$, contradiction. □

What about the automorphism group of these constructions? The 8 reflections r_b ($r \in B$) generate a Sym(9) and clearly this is the full stabilizer of O in G. The stabilizer of O in G_0 is Alt(9), and that is also the full group of the system $\mathcal{O}$. It follows that the full group of the system $\mathcal{B}$ is the stabilizer of b_0 in the previous, i.e., is Alt(8).

As mentioned earlier, De Clerck-Dye-Thas construct a partial geometry pg(7,8,4) on the nonsingular points by fixing a spread S and taking the sets of nonsingular points on the t.i. but not t.s. 4-spaces meeting some element of S in a plane. (This is the dual of the pg(8,7,4) obtained from $\mathcal{O}$. Indeed, the DDT system has point set R_0, and its lines are the planes in some element of S, where r is incident with π if $r\pi = \pi$. If we let S be a 1-spread, then these planes can be identified with the elements of Z_2 containing them, and interchanging types 0 and 2 we find the description of $\mathcal{O}$.)

If we join our system $\mathcal{B}$ to the set of lines of this partial geometry, we get a Steiner system $S(2,8,120)$ with automorphism group (at least) Alt(8), when both systems were constructed starting from the same spread S. (No doubt several nonisomorphic $S(2,8,120)$'s arise in this way, but we have not investigated the details. Several nonisomorphic $S(2,8,120)$'s were known already - obtained as the exterior lines and the points off a hyperoval in some projective plane of order 16, cf. [5].)

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Andries E. Brouwer, Department of Mathematics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.
e-mail: aeb@cwi.nl

Willem H. Haemers, Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.
e-mail: haemers@kub.nl

Vladimir D. Tonchev, Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931, USA.
e-mail: tonchev@mtu.edu