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Semi-Infinite Assignment and Transportation Games

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Abstract

Games corresponding to semi-infinite transportation and related assignment situations are studied. In a semi-infinite transportation situation, one aims at maximizing the profit from the transportation of a certain good from a finite number of suppliers to an infinite number of demanders. An assignment situation is a special kind of transportation situation where the supplies and demands for the good all equal one unit. It is shown that the special structure of these situations implies that the underlying infinite programs have no duality gap and that the core of the corresponding game is nonempty.

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1 Introduction

In 1972 Shapley and Shubik introduced (finite) assignment games. These are games corresponding to an assignment situation where a (finite) set of agents has to be matched to another set of agents in such a way that the revenue obtained from these matchings is as large as possible. Since this introduction different generalizations related to these games have been developed. The paper of Llorca, Tijs and Timmer (1999) provides an infinite extension of these games. They introduce semi-infinite bounded assignment games in which one set of agents is finite and the other is countably infinite and prove that these games have a nonempty core. That is, there exists an allocation of the maximal profit over all the players such that any coalition of players cannot do better on its own. Sánchez-Soriano, López and García-Jurado (2000) introduce finite transportation games, which are based on transportation situations. Given a set of supply and demand points of a certain good, how much should be transported

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from each supply point to each demand point to maximize the revenue from this transportation plan? The arising transportation games can be seen as a finite extension of the finite assignment games. Fragnelli, Patrone, Sideri and Tijs (1999) and Timmer, Llorca and Tijs (2000) study games with a nonempty core arising from semi-infinite linear programming situations, where one of the factors involved is countably infinite, but the number of players is finite.

In this paper, we look at semi-infinite transportation problems where the number of suppliers is finite and the number of demanders is countably infinite. For each semi-infinite transportation situation we define a related assignment problem. With the help of the results of Llorca et al. (1999), we show that semi-infinite transportation problems have no duality gap and the corresponding semi-infinite transportation games have a nonempty core.

This work is organized in five sections. In the next section, we present finite transportation and assignment games. Section 3 summarizes the main results and concepts for semi-infinite bounded assignment games that are needed to study semi-infinite transportation situations. In section 4 we study transportation games arising from semi-infinite transportation problems in which there is a countably infinite number of players of one type, the matrix of benefits per unit is bounded and supplies and demands are natural numbers. We show that the corresponding primal and dual programs have no duality gap and prove that the games have a nonempty core. The proofs are based on an expansion-contraction procedure, which uses a semi-infinite assignment problem associated to the corresponding transportation problem generated by splitting the supply and demand points. In the final section, we add a remark about the idea of dropping the conditions that force the supplies and the demands to be natural numbers, in order to consider the transport of infinitely divisible goods.

2 Finite transportation and assignment games

A (finite) transportation problem describes a situation in which demands at several locations for a certain good need to be covered by supplies from other locations. The transportation of one unit of the good from a supply point to a demand point generates a certain profit. The goal of the cooperating suppliers and demanders is to maximize the total profit from transport. For an example one may consider a large supermarket that has to supply its stores at various locations with bottles of wine stored in several warehouses.

More formally, let P be the finite set of supply points and Q the finite set of demand points. The supply of the good at point $i \in P$ equals s_i units and the demand at point $j \in Q$ is d_j units. Both s_i and d_j are (positive) *integer* numbers for all $i \in P$ and $j \in Q$, we assume that the good is indivisible. The profit of sending one unit of the good from supply point i to demand point j is t_{ij}, a non-negative real number. All profits are gathered in the matrix $T = [t_{ij}]_{i\in P, j\in Q}$. Hence, a *transportation problem* can be described by the tuple (P, Q, T, s, d) where $s = \{s_i\}_{i\in P}$ and $d = \{d_j\}_{j\in Q}$ are the vectors containing respectively the supplies and demands of the good. For the sake of brevity we will use T to denote the transportation problem (P, Q, T, s, d).

A *transportation plan* $X = [x_{ij}]_{i\in P, j\in Q}$ is a matrix with integer entries where x_{ij} is the number of units of the good that will be transported from supply point i to demand point j. Of course, each supply point $i \in P$ cannot supply more than s_i units of the good, $\sum_{j \in Q} x_{ij} \leq s_i$. Similarly, each demand point $j \in Q$ wants to receive at most d_j units, $\sum_{i \in P} x_{ij} \leq d_j$. The maximal profit that the supply and demand points can achieve equals

$$v_p(T) = \max \left\{ \sum_{(i,j) \in P \times Q} t_{ij} x_{ij} : X \text{ is a transportation plan} \right\}.$$

A transportation plan X is also called a solution for T. Such a solution is an *optimal* solution if $\sum_{(i,j) \in P \times Q} t_{ij} x_{ij} = v_p(T)$.

Given a transportation problem T, the corresponding transportation game (N, w) is a cooperative TU game with player set $N = P \cup Q$. Let $S \subset N$, $S \neq \emptyset$, be a coalition of players and define $P_S = P \cap S$ and $Q_S = Q \cap S$. If $S = P_S$ then there are no demand points present in S and therefore the supply points in S cannot get rid of their goods. In this case the worth $w(S)$ of coalition S equals zero. Similarly, if $S = Q_S$ then the demand points in S cannot receive any units of the good and $w(S) = 0$. Otherwise, the worth $w(S)$ depends upon the possible transportation plans. A transportation plan $X(S)$ for coalition S is a transportation plan for the transportation problem $T_S = (P_S, Q_S, [t_{ij}]_{i \in P_S, j \in Q_S}, \{s_i\}_{i \in P_S}, \{d_j\}_{j \in Q_S})$. In this case

$$\begin{aligned} w(S) &= \max \left\{ \sum_{(i,j) \in P_S \times Q_S} t_{ij} x_{ij} : X(S) \text{ is a transportation plan for } S \right\} \\ &= v_p(T_S) \end{aligned}$$

is the worth of coalition S.

One of the main issues in cooperative game theory is how to divide the total profit derived from cooperation. One way to share this profit among the players in N is to do so according to an element in the core. The *core* of a transportation game (N, w) is the set

$$C(w) = \left\{ x \in \mathbb{R}^N : \begin{array}{l} \sum_{i \in N} x_i = w(N) \text{ and} \\ \sum_{i \in S} x_i \geq w(S) \text{ for all } S \subset N,\ S \neq \emptyset \end{array} \right\}.$$

When a core-element x is proposed as a distribution of the total profit $w(N)$, then each coalition S will get at least as much as it can obtain on its own because $\sum_{i \in S} x_i \geq w(S)$. So, no coalition has an incentive to disagree with this proposal.

A special case of transportation problems occurs when all supplies s_i and demands d_j equal 1. This kind of problem is called an *assignment problem* because in an optimal plan we either have that the whole supply of $i \in P$ is transported to one demand point or nothing is transported. This is like assigning supply points to demand points. For example, how should employees be assigned to jobs such that the total costs are minimized? Such an assignment problem is described by a tuple (M, W, A), where the sets M and W contain respectively the supply and demand points. The benefit of assigning $i \in M$ to $j \in W$ equals $a_{ij} \geq 0$, $A = [a_{ij}]_{i \in M, j \in W}$.

In the next section, based on Llorca et al. (1999), we summarize the most relevant results about semi-infinite assignment situations and corresponding games.

3 Semi-infinite assignment games

In assignment situations we are interested in how to match, for example, a finite set of machines to a set of jobs such that we achieve the highest possible benefit. Consider a firm with a finite number of glass-cutting machines that can be programmed to produce a vase. This firm can choose from an infinite number of patterns (their designers are very productive). The machines can produce all of these patterns, but with different (bounded) rewards. The marketing policy of the firm is to make unique vases. So, the firm has to tackle an assignment problem in which there is a finite number of one type (machines) and an infinite number of the other type (possible designs). It's goal is to achieve the 'maximal' total benefits from matching the machines with the patterns.

A semi-infinite (bounded) assignment problem is denoted by a tuple (M, W, A), where $M = \{1, 2, ..., m\}$ is a finite set, $W = \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, and the nonnegative rewards a_{ij} are bounded from above, for all $i \in M$, $j \in W$. We will use A to denote the assignment problem (M, W, A).

An *assignment plan* $Y = [y_{ij}]_{i \in M, j \in W}$ is a matrix with 0,1-entries where $y_{ij} = 1$ if i is assigned to j and $y_{ij} = 0$ otherwise. Each supply point will be assigned to at most one demand point and vice versa, therefore $\sum_{j \in W} y_{ij} \leq 1$ and $\sum_{i \in M} y_{ij} \leq 1$. Then

$$v_p(A) = \sup \left\{ \sum_{(i,j) \in M \times W} a_{ij} y_{ij} : Y \text{ is an assignment plan} \right\}$$

is the smallest upper bound of the benefit that the supply and demand points can achieve. An assignment plan Y is also called a solution for A. Such a solution is *optimal* if $\sum_{(i,j) \in M \times W} a_{ij} y_{ij} = v_p(A)$.

The corresponding assignment game (N, w) is the game with player set $N = M \cup W$. Let $M_S = M \cap S$ and $W_S = W \cap S$. Then the coalition S of players in N receives $w(S) = 0$ if $S = M_S$ or $S = W_S$ because in these cases there is nothing to be assigned to. Otherwise, $w(S) = v_p(A_S)$ where A_S is the (semi-infinite) assignment problem $(M_S, W_S, [a_{ij}]_{i \in M_S, j \in W_S})$.

Relaxing the 0,1-condition of y_{ij} to nonnegativity does not change the value of the program, as the next lemma shows.

Lemma 1

$$v_p(A) = \sup \left\{ \sum_{(i,j) \in M \times W} a_{ij} y_{ij} : \begin{array}{l} \sum_{j \in W} y_{ij} \leq 1, \ \sum_{i \in M} y_{ij} \leq 1, \\ y_{ij} \geq 0 \text{ for all } i \in M, \ j \in W \end{array} \right\}.$$

Proof. Define

$$v_p(A^*) = \sup \left\{ \sum_{(i,j) \in M \times W} a_{ij} y_{ij} : \begin{array}{l} \sum_{j \in W} y_{ij} \leq 1, \ \sum_{i \in M} y_{ij} \leq 1, \\ y_{ij} \geq 0 \text{ for all } i \in M, \ j \in W \end{array} \right\}.$$

Obviously, $v_p(A) \leq v_p(A^*)$. We will show that $v_p(A) \geq v_p(A^*) - \varepsilon$ for all $\varepsilon > 0$.

Let $\varepsilon > 0$ and take a solution Y' of A^* such that $\sum_{(i,j) \in M \times W} a_{ij} y'_{ij} \geq v_p(A^*) - \varepsilon/2$. Let $n \in W$ be such that $\sum_{i \in M} \sum_{j=1}^{n} a_{ij} y'_{ij} \geq v_p(A^*) - \varepsilon$. It is well known that there exists an *integer* optimal solution Y'' of the finite program

$$\max\left\{\sum_{i\in M}\sum_{j=1}^{n} a_{ij}y_{ij} : \begin{array}{l}\sum_{j=1}^{n} y_{ij} \le 1,\ \sum_{i\in M} y_{ij} \le 1,\\ y_{ij}\ge 0 \text{ for all } i\in M,\ j=1,\ldots,n\end{array}\right\} =: v_p(A_n^*)$$

and that

$$\begin{aligned}&\sum_{i\in M}\sum_{j=1}^{n} a_{ij}y''_{ij}\\ &= \max\left\{\sum_{i\in M}\sum_{j=1}^{n} a_{ij}y_{ij} : \begin{array}{l}\sum_{j=1}^{n} y_{ij} \le 1,\ \sum_{i\in M} y_{ij} \le 1,\\ y_{ij}\in\{0,1\} \text{ for all } i\in M,\ j=1,\ldots,n\end{array}\right\}\\ &=: v_p(A_n),\end{aligned}$$

where the finite assignment problem $(M,\{1,\ldots,n\},[a_{ij}]_{i\in M,j=1,\ldots,n})$ is denoted by A_n. Then

$$v_p(A_n) = \sum_{i\in M}\sum_{j=1}^{n} a_{ij}y''_{ij} \ge \sum_{i\in M}\sum_{j=1}^{n} a_{ij}y'_{ij} \ge v_p(A^*) - \varepsilon$$

where the first inequality follows from the fact that $[y'_{ij}]_{i\in M,j=1,\ldots,n}$ is a solution of A_n^*. Together with $v_p(A)\ge v_p(A_n)$ we conclude that $v_p(A)\ge v_p(A^*)-\varepsilon$. □

If we replace the condition $y_{ij}\in\{0,1\}$ by $y_{ij}\ge 0$ then the dual program, with value $v_d(A)$, of the problem that determines $v_p(A)$ equals

$$\begin{aligned} v_d(A) = \inf\ & \sum_{i\in M} u_i + \sum_{j\in W} v_j\\ \text{s.t.}\ & u_i + v_j \ge a_{ij},\ \text{for all } i\in M,\ j\in W\\ & u_i, v_j \ge 0,\ \text{for all } i\in M,\ j\in W.\end{aligned}$$

Let $O_d(A)$ be the set of optimal solutions of this dual problem. Both the primal and the dual program have an infinite number of variables and an infinite number of restrictions. In general, $\infty\times\infty$-programs show a gap between the optimal primal and dual value. There is a large literature on the existence or absence of so-called duality gaps in (semi-)infinite programs. See for example the books by Glashoff and Gustafson (1983) and Goberna and López (1998).

These semi-infinite assignment problems can be analyzed by *finite approximation matrices* $A_n\in\mathbb{R}^{m\times n}$ where $A_n=[a_{ij}]_{i\in M,j=1,2,\ldots,n}$, and by means of the so-called *hard-choice number* of the matrix A. The following example illustrates this last concept.

Example 2 Let $M=\{1,2,3\}, W=\mathbb{N}$ and

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots\\ 10 & 3 & 8 & 2 & 9 & 2 & 2 & \ldots\\ \frac{7}{4} & 1 & \frac{3}{2} & \frac{5}{3} & 4 & \frac{9}{5} & \frac{11}{6} & \ldots\end{bmatrix}.$$

The *choice set* C_i of agent $i\in M$ consists of maximal $|M|$ agents in W that give the highest reward a_{ij} when assigned to $i\in M$, if they exist. If more than $|M|$ agents in W satisfy this criterion then the choice set contains only those $|M|$ agents with the the smallest ranking number in W.

In this example, no matter to whom agent $1 \in M$ is assigned, the resulting reward equals $a_{1j} = 3$. Hence, we take the three agents with the smallest ranking number and $C_1 = \{1, 2, 3\}$.

If agent $2 \in M$ is assigned to agent $1 \in W$ then they obtain the maximal reward of 10. The second largest value is $a_{25} = 9$ and $a_{23} = 8$ is the third largest value. Thus, $C_2 = \{1, 3, 5\}$.

Finally, assigning agent $3 \in M$ to agent $5 \in W$ results in the maximal reward $a_{35} = 4$. However, there is no second largest value because a_{3n} goes to 2 when n goes to infinity. So, this agent has $C_3 = \{5\}$.

The *hard-choice number* $n^*(A)$ is the smallest number in $N \cup \{0\}$ such that $\cup_{i \in M} C_i \subset \{1, 2, .., n^*(A)\}$. In this example we have $n^*(A) = 5$.

The following theorem establishes that the primal and the dual problem have the same value and there exists an optimal solution of the dual problem.

Theorem 3 *(Theorem 3.9 in Llorca et al. (1999)) Let (M, W, A) be a semi-infinite bounded assignment problem. Then $v_p(A) = v_d(A)$ and $O_d(A) \neq \emptyset$.*

A sketch of the proof of the latter statement goes as follows. Take for each $n \in \mathbb{N}$, $n > n^*(A)$, an element (u^n, v^n) of $O_d(A_n)$ and remove all coordinates of v^n with index larger than $n^*(A)$. The set of all those elements, which is in the finite dimensional space $\mathbb{R}^m \times \mathbb{R}^{n^*(A)}$, is bounded. Without loss of generality, suppose that the limit, when n goes to infinity, of such a sequence exists (otherwise take a subsequence) and denote this limit by $(\overline{u}, \overline{v})$. With the aid of $(\overline{u}, \overline{v})$ construct the vector $(\widehat{u}, \widehat{v})$ by taking $\widehat{u} = \overline{u}$ and $\widehat{v}$ is obtained from $\overline{v}$ by adding an infinite number of zeros. Then $(\widehat{u}, \widehat{v})$ is an optimal dual solution of the corresponding semi-infinite bounded assignment problem.

This theorem is of great importance for the next section. There we show through related semi-infinite assignment games that semi-infinite transportation problems have no duality gap and the corresponding games have a nonempty core.

4 Semi-infinite transportation problems and related games

In this section we extend finite transportation problems to semi-infinite transportation problems. These are transportation problems where the number of one type of agents (demanders or suppliers) is countably infinite. This kind of situations can appear in market models where the number of potential customers can be seen as infinite. For example, when a firm introduces a new product in the consumer market then each consumer in the infinite set of potential consumers has a finite demand for this new product. We assume that $Q = \mathbb{N}$, and that the profits t_{ij} are bounded from above, that is $\|T\|_\infty < \infty$.

Corresponding to such a semi-infinite transportation problem we define a semi-infinite transportation game (N, w) with player set $N = P \cup Q$. As before, the worth of coalition S equals zero, $w(S) = 0$, if $S = P_S$ or $S = Q_S$ and

$$\begin{aligned} w(S) &= \sup \left\{ \sum_{(i,j) \in P_S \times Q_S} t_{ij} x_{ij} : X(S) \text{ is a transportation plan for } S \right\} \\ &= v_p(T_S) \end{aligned}$$

otherwise.

Given a semi-infinite transportation problem T we construct a related semi-infinite assignment problem $A(T)$ in the following way:

- Each supply point $i \in P$ is split into s_i supply points named $i1$, $i2$, ..., is_i, each with a supply of 1 unit. Hence, $M = \{ir : i \in P, r \in \{1, ..., s_i\}\}$.
- Each demand point $j \in Q$ is split into d_j different players $j1$, $j2$, ..., jd_j, each with a demand of 1 unit. Therefore, $W = \{jc : j \in Q, c \in \{1, ..., d_j\}\}$. Notice that W is a countably infinite set of players because $Q = \mathbb{N}$.
- Define $a_{ir,\, jc} = t_{ij}$ for all $ir \in M,\ jc \in W$.

The next lemma deals with relations between solutions in T and $A(T)$.

Lemma 4 *Each solution for T determines a solution for $A(T)$, and conversely. These solutions have the same value.*

Before we prove the lemma, we give an example to illustrate a procedure that we use in the proof.

Example 5 Consider the transportation problem T with $P = \{1, 2, 3\}, Q = \mathbb{N}$ and

$$\begin{array}{c|ccccccc|l} & 2 & 1 & 2 & 1 & 1 & 1 & \dots & d_j \\ \hline 2 & 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \dots & \\ 1 & 0 & 2 & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{3}{4} & 1\frac{4}{5} & \dots & = T \\ 3 & 3 & 2 & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{3}{4} & 1\frac{4}{5} & \dots & \\ \hline s_i & & & & & & & & \end{array}$$

A solution for T is the transportation plan

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

with value $\sum_{(i,j) \in P \times Q} t_{ij} x_{ij} = 11$. The corresponding assignment problem $A(T)$ has supply points $M = \{11, 12, 21, 31, 32, 33\}$ and demand points $W = \{11, 12, 21, 31, 32, 41, 51, 61, ...\}$. From the solution X for T we construct a solution Y for $A(T)$ where each cell in X with $x_{ij} > 0$ will correspond to x_{ij} cells in Y with entry 1. The procedure goes as follows. We start with $i = j = 1$. If $x_{ij} \neq 0$ then we look for the smallest values for r and c such that both the points ir and jc are not assigned to any point, that is, row ir and column jc in Y contained no entry equal to 1 so far. Define $y_{ir,jc} = 1$. Continue searching for new values r and c until $\sum_{r=1}^{s_i} \sum_{c=1}^{d_j} y_{ir,jc} = x_{ij}$. Repeat this for all $(i, j) \in P \times Q$ with $x_{ij} \neq 0$, where you first consider the first row and first column in X, then the second row and second column, and so on. Set $y_{ir,jc} = 0$ for the remaining $(ir, jc) \in M \times W$. Following this procedure we obtain the assignment plan

$M \backslash W$	11	12	21	31	32	41	51	61	71	$\cdots$
11	0	0	0	1	0	0	0	0	0	$\cdots$
12	0	0	0	0	0	0	0	1	0	$\cdots$
21	0	0	0	0	1	0	0	0	0	$\cdots$
31	1	0	0	0	0	0	0	0	0	$\cdots$
32	0	1	0	0	0	0	0	0	0	$\cdots$
33	0	0	1	0	0	0	0	0	0	$\cdots$

$= Y$

with value $\sum_{(ir,jc)\in M\times W} a_{ir,jc} y_{ir,jc} = 11$. Conversely, given a solution Y for $A(T)$, a solution X for T is given by $x_{ij} = \sum_{r=1}^{s_i} \sum_{c=1}^{d_j} y_{ir,jc}$ for all $i \in P$, $j \in Q$.

Proof of lemma 4. Let X be a solution for T. Define the matrix $Y \in \{0,1\}^{M\times W}$ by

$$y_{ir,jc} = \begin{cases} & (i) \quad r \in \left(\sum_{q<j} x_{iq}, \sum_{q\le j} x_{iq}\right], \\ 1 \quad \text{if} & (ii) \quad c \in \left(\sum_{p<i} x_{pj}, \sum_{p\le i} x_{pj}\right] \text{ and} \\ & (iii) \quad r - \sum_{q<j} x_{iq} = c - \sum_{p<i} x_{pj}, \\ 0 \quad \text{otherwise.} & \end{cases}$$

We show that Y is a solution of $A(T)$. By definition $y_{ir,jc} \in \{0,1\}$.

Assume that $y_{ir,jc} = 1$ and $c < d_j$, that is, there exists a $jc' \in W$ with $c' > c$. Then

$$r - \sum_{q<j} x_{iq} = c - \sum_{p<i} x_{pj} < c' - \sum_{p<i} x_{pj}$$

where the equality follows from $y_{ir,jc} = 1$ and the inequality from $c < c'$. Hence, condition (iii) is not satisfied for (ir, jc') and therefore $y_{ir,jc'} = 0$ for all $c' > c$.

Next, consider $(ir, j'c')$ with $j' > j$. If $x_{ij'} = 0$ then $\sum_{q<j'} x_{iq} = \sum_{q\le j'} x_{iq}$, condition (i) cannot be satisfied and therefore $y_{ir,j'c'} = 0$. Otherwise, if $x_{ij'} > 0$ then $r \le \sum_{q\le j} x_{iq} \le \sum_{q<j'} x_{iq}$, where the first inequality follows from $y_{ir,jc} = 1$. But then $r \not> \sum_{q<j'} x_{iq}$, condition (i) is not satisfied for $(ir, j'c')$ and so, $y_{ir,j'c'} = 0$.

We conclude that if $y_{ir,jc} = 1$ then the remainder of row ir in Y (as of column jc) contains only entries equal to zero. Similarly, we can show that the remainder of column jc (as of row ir) also consists of entries equal to zero. Hence, $\sum_{jc\in W} y_{ir,jc} \le 1$ and $\sum_{ir\in M} y_{ir,jc} \le 1$. The matrix Y is a solution of $A(T)$.

Finally, let Y be a solution of $A(T)$. Define $x_{ij} = \sum_{r=1}^{s_i} \sum_{c=1}^{d_j} y_{ir,jc}$ for all $i \in P$, $j \in Q$. Then, x_{ij} is a non-negative integer and for all $j \in Q$ we have

$$\begin{aligned} \sum_{i\in P} x_{ij} &= \sum_{i\in P} \sum_{r=1}^{s_i} \sum_{c=1}^{d_j} y_{ir,jc} = \sum_{c=1}^{d_j} \sum_{i\in P} \sum_{r=1}^{s_i} y_{ir,jc} \\ &= \sum_{c=1}^{d_j} \sum_{ir\in M} y_{ir,jc} \le \sum_{c=1}^{d_j} 1 = d_j. \end{aligned}$$

The inequality holds because Y is a solution of $A(T)$. Analogously, we can show that $\sum_{j\in Q} x_{ij} \leq s_i$ for all $i \in P$. Hence, X is a solution of T. It is a trivial exercise to show that both solutions have the same value. □

The following result is an immediate consequence of lemma 4.

Lemma 6 *Let T be a semi-infinite transportation problem and $A(T)$ the corresponding assignment problem. Then $v_p(T) = v_p(A(T))$.*

Recall that $v_p(T)$ is the value of the problem

$$\sup\left\{\sum_{(i,j)\in P\times Q} t_{ij}x_{ij} : X \text{ is a transportation plan}\right\}.$$

Similarly to lemma 1 for semi-infinite assignment problems, we can show that relaxing the condition $x_{ij} \in \mathbb{N}$ to $x_{ij} \geq 0$ will not change the value of the problem. The dual problem D corresponding to this program is

$$\inf\left\{\sum_{i\in P} s_i u_i + \sum_{j\in Q} d_j v_j : u_i + v_j \geq t_{ij},\ u_i, v_j \geq 0 \text{ for all } i \in P,\ j \in Q\right\}.$$

We denote the value of this program by $v_d(T)$ and $O_d(T)$ is the set of optimal solutions of D. Similarly, we define for the related assignment problem $A(T)$

$$v_d(A(T)) = \inf\left\{\sum_{ir\in M} u_{ir} + \sum_{jc\in W} v_{jc} : \begin{array}{l} u_{ir} + v_{jc} \geq a_{ir,jc},\ u_{ir}, v_{jc} \geq 0 \\ \text{for all } ir \in M,\ jc \in W \end{array}\right\}.$$

Let $O_d(A(T))$ be the set of optimal solutions of this infimum problem. As is the case for semi-infinite assignment problems, semi-infinite transportation problems have no duality gap, that is, $v_p(T) = v_d(T)$ and $O_d(T)$ is nonempty.

Theorem 7 *Let T be a semi-infinite transportation problem. Then*

1. $v_p(T) = v_d(T)$ *and*
2. $O_d(T) \neq \emptyset$.

Proof. Theorem 3 states that $O_d(A(T)) \neq \emptyset$, so, let $(u, v) \in O_d(A(T))$. Then, $u_{ir} + v_{jc} \geq a_{ir,jc} = t_{ij}$ for all $ir \in M$, $jc \in W$. Thus for all $i \in P$, $j \in Q$,

$$\sum_{r=1}^{s_i}\sum_{c=1}^{d_j}(u_{ir} + v_{jc}) = d_j \sum_{r=1}^{s_i} u_{ir} + s_i \sum_{c=1}^{d_j} v_{jc} \geq \sum_{r=1}^{s_i}\sum_{c=1}^{d_j} t_{ij} = s_i d_j t_{ij}.$$

Dividing both sides by $s_i d_j$ gives

$$\sum_{r=1}^{s_i} u_{ir}/s_i + \sum_{c=1}^{d_j} v_{jc}/d_j \geq t_{ij}.$$

Define $\bar{u}_i := \sum_{r=1}^{s_i} u_{ir}/s_i$ and $\bar{v}_j := \sum_{c=1}^{d_j} v_{jc}/d_j$. Then $\bar{u}_i \geq 0$, $\bar{v}_j \geq 0$, and $\bar{u}_i + \bar{v}_j \geq t_{ij}$ for all $i \in P$, $j \in Q$. Hence,

$$v_p(T) = v_p(A(T)) = v_d(A(T)) = \sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \bar{v}_j \geq v_d(T)$$

where the first equality follows from lemma 6, the second one from theorem 3, the third one from $(u,v) \in O_d(A(T))$, and the last inequality follows from the definition of $v_d(T)$. From duality theory we know that $v_p(T) \leq v_d(T)$ and therefore

$$v_p(T) = \sum_{i \in P} s_i \bar{u}_i + \sum_{j \in Q} d_j \bar{v}_j = v_d(T).$$

We conclude that $v_p(T) = v_d(T)$ and $(\bar{u}, \bar{v}) \in O_d(T)$. □

A concept related to the core is the so-called *Owen set*[4], which is defined by

$$\text{Owen}(T) = \left\{ x \in \mathbb{R}^N : \begin{array}{l} \exists (u,v) \in O_d(T) \text{ such that } x_k = s_k u_k \\ \text{if } k \in P \text{ and } x_k = d_k v_k \text{ if } k \in Q \end{array} \right\}.$$

This set is not empty because $O_d(T) \neq \emptyset$. An element of the Owen set is easy to find and it turns out to be an element of the core of the corresponding transportation game as well.

Theorem 8 *Let T be a semi-infinite transportation problem and (N,w) the corresponding game. Then,* $\text{Owen}(T) \subset C(w)$.

Proof. Let $x \in \text{Owen}(T)$ and let $(u,v) \in O_d(T)$ be such that $x_k = s_k u_k$ if $k \in P$ and $x_k = d_k v_k$ if $k \in Q$. Then

$$\sum_{i \in N} x_i = \sum_{i \in P} s_i u_i + \sum_{j \in Q} d_j v_j = v_d(T) = v_p(T) = w(N),$$

where the third equality follows from theorem 7. Next, let $S \subset N$, $S \neq \emptyset$. If $S = P_S$ or $S = Q_S$ then $\sum_{k \in S} x_k \geq 0 = w(S)$ because $x_k \geq 0$ for all $k \in N$. Otherwise, we know that $u_i + v_j \geq t_{ij}$ for all $i \in P$, $j \in Q$, and this holds in particular for all $i \in P_S$, $j \in Q_S$. Thus

$$\sum_{i \in S} x_i = \sum_{i \in P_S} s_i u_i + \sum_{j \in Q_S} d_j v_j \geq v_d(T_S) = v_p(T_S) = w(S).$$

We conclude that $x \in C(w)$. □

In general, the Owen set does not coincide with the core of a transportation game, as the following example shows.

[4] Owen (1975) presents a method to find a nonempty subset of the core of a linear production game. Gellekom et al. (2000) names this set the 'Owen set'.

Example 9 Let T be a transportation problem with $P = \{1\}$, $Q = \mathbb{N}$, and

$$\begin{array}{cc|ccccc|l} & & 2 & 1 & 1 & 1 & \cdots & d_j \\ \cline{3-7} 4 & & 3 & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{3}{4} & \cdots & = T \\ \cline{3-7} s_i & & & & & & & \end{array}$$

In this problem the Owen set equals $\text{Owen}(T) = \{(8; 2, 0, 0, ...)\}$. However, the point $(10; 0, 0, ...)$ is an element of the core of the corresponding transportation game. Hence, $\text{Owen}(T)$ is strictly contained in the core $C(w)$.

5 Final remark

In our future research we will study semi-infinite transportation problems where supplies and demands are positive *real* numbers. The underlying idea is to consider *infinitely divisible* goods. One can think of using pipelines instead of containers for the transportation of petrol. In this framework we consider two semi-infinite transportation situations. The first one is such that the total demand for the good is infinite and the individual demands are bounded from below, and in the second one the total demand is finite. In both cases, we will show that the corresponding semi-infinite transportation games have a nonempty core.

References

Fragnelli, V., F. Patrone, E. Sideri, and S. Tijs. (1999). "Balanced Games Arising from Infinite Linear Models," Mathematical Methods of Operations Research, 50:385–397.

Gellekom, J.R.G. van, J.A.M. Potters, J.H. Reijnierse, S.H. Tijs, and M.C. Engel. (2000). "Characterization of the Owen Set of Linear Production Processes," to appear in Games and Economic Behavior.

Glashoff, K., and S-A. Gustafson. (1983). *Linear Optimization and Approximation.* New York: Springer-Verlag.

Goberna, M., and M.A. López. (1998). *Linear Semi-Infinite Optimization*. Chichester: Wiley.

Llorca, N., S. Tijs, and J. Timmer. (1999). "Semi-Infinite Assignment Problems and Related Games," CentER Discussion Paper no. 9974, Tilburg University.

Owen, G. (1975). "On the Core of Linear Production Games," Mathematical Programming, 9:358–370.

Sánchez-Soriano, J., M.A. López, and I. García-Jurado. (2000). "On the Core of Transportation Games," to appear in Mathematical Social Sciences.

Shapley, L.S., and S. Shubik. (1972). "The Assignment Game I: The Core," International Journal of Game Theory, 1:111–130.

Timmer, J., N. Llorca, and S. Tijs. (2000). "Games Arising from Infinite Production Situations," to appear in International Game Theory Review.