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The Core and the τ-Value for Cooperative Games with Coalition Structures

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The Core and the τ-Value for Cooperative Games with Coalition Structures

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The paper is devoted to solution concepts for cooperative games with coalition structures. The τ-value concept for such games is introduced as an equitable compromise between a lower bound and an upper bound for the core. Several properties for the τ-value are presented. The determination of both the core and the τ-value is carried out for convex and 1-convex games with coalition structures.

Keywords: Cooperative game, coalition structure, core, τ-value, convex game.

AMS Subject Classification (1980): 90D12.

1. Introduction

A coalition structure in a cooperative game is defined to be a partition of the player set. The study of game theoretic solution concepts with respect to a given coalition structure was started during the development of the theory concerning the various bargaining sets (Aumann and Maschler, 1964; Davis and Maschler, 1967). The research on coalition structures was continued by a systematic study of six common solution concepts in Aumann and Drèze (1974). Their paper presented an analysis for the bargaining set, the kernel, the nucleolus, the core, the Von Neumann-Morgenstern solutions and the Shapley value with reference to an arbitrary coalition structure. A subsequent treatment of the nucleolus for cooperative games with coalition structures can be found in Owen (1977a) and Potters and Tijs (1989).

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In the second stage of the research on coalition structures, the focus was on the coalition structure value as introduced in Owen (1977b). Axiomatizations and/or extensions of Owen's value and related values can be found in Hart and Kurz (1983, 1984), Levy and McLean (1989), and Winter (1989). An essential feature in the approach of Owen (1977b) is overall efficiency which requires the division of the total earnings among all the players. It differs from the approach of Aumann and Drèze (1974), where the efficiency requirement is applied to each coalition in the coalition structure. We refer to the paper of Kurz (1988) for the exposition of these two approaches. Nowadays one can perceive a growing interest in the topic on solution concepts for cooperative games with coalition structures.

The main purpose of this paper is the introduction and the study of the τ-value for cooperative games with coalition structures. The value developed can be regarded as an extension of the classical τ-value concept with respect to the all-player coalition structure. The classical τ-value was introduced and axiomatized in Tijs (1981, 1987) and further, an overview of the classical τ-value was given in Tijs and Driessen (1987) and Driessen (1988). The development of the theory concerning the extended τ-value leads to the introduction of two interesting types of games (the so-called convex and 1-convex games with reference to a given coalition structure). The second purpose of this paper is to characterize these two types of games by means of the determination of the core for such games.

The organization of the paper is as follows. In Section 2 we recall the notion of a cooperative game with a coalition structure, or simply a c.s. game, and pay attention to the core of c.s. games. The main result of Section 2 provides the generalized balancedness conditions that are necessary and sufficient for the non-emptiness of the core of a c.s. game. Section 3 is devoted to the introduction of the τ-value concept for c.s. games as an equitable compromise between a lower bound and an upper bound for the core. An axiomatization of the τ-value on the class of quasibalanced c.s. games is included. In Section 4 it is established that the τ-value for 1-convex c.s. games occupies a central position within the core and moreover, it coincides with the nucleolus. As a matter of fact, another main result of Section 4 provides that the 1-convexity of a c.s. game is fully characterized by a specific structure of the core of the c.s. game. Section 5 deals with the generalized convexity notion for c.s. games. The class of convex c.s. games is characterized as the class of c.s. games for which the core is fully generated by the so-called marginal worth vectors. The τ-value concept on the class of convex c.s. games is determined by considering the enlarged class of semiconvex c.s. games.

2. Balancedness and bounds for the core

A <u>cooperative game in characteristic function form</u>, or simply a <u>game</u>, is an ordered pair (N,v), where N represents the finite player set and where the so-called characteristic function v is a real-valued function on the family of subsets of N. Any non-empty subset of the player set is called a coalition and let $|S|$ denote the cardinality of the coalition $S \subset N$. The worth $v(S)$ of coalition S is interpreted as the earnings obtainable from the cooperation between the members of S excluding the non-members of S. We always suppose $v(\emptyset) = 0$.

A coalition structure is a partition of the player set. Formally, a <u>coalition structure</u> $\underline{B}$ on N is a sequence $(B_1, B_2, \ldots, B_m)$ of coalitions such that $B_j \cap B_k = \emptyset$ for $j \neq k$ and $\bigcup_{j=1}^{m} B_j = N$. The classical coalition structure on N is given by $\underline{B} = (N)$.

A <u>cooperative game with a coalition structure</u>, or simply a <u>c.s. game</u>, is an ordered triple $(N,v,\underline{B})$ which is composed of the game (N,v) and the coalition structure $\underline{B}$ on N. Throughout the paper it is supposed that the player set N is fixed. A payoff vector is formally defined as a real-valued function x on N, but it is usually identified with the vector whose coordinates are indexed by the players. The set of all payoff vectors is denoted by $\mathbb{R}^N$. For the sake of notation, write $x(S) = \sum_{i \in S} x(i)$ for all $x \in \mathbb{R}^N$ and all $S \subset N$. Note that $x(\emptyset) := 0$.

In accordance with the approach of Aumann and Drèze (1974), it is supposed that the coalitions in the given coalition structure will be formed and that each one of these coalitions gets what it can assure itself in the game. Thus, payoff vectors are meant to describe possible divisions of the earnings obtainable from the cooperation between the members of each separate coalition in the coalition structure. Hence, it is always required that payoff vectors belong to the pre-imputation set $I^*(v,\underline{B})$ which is given by

$$I^*(v,\underline{B}) := \{x \in \mathbb{R}^N \mid x(B_j) = v(B_j) \text{ for all } 1 \leq j \leq m\}.$$

Generally speaking, the determination of one or more specific pre-imputations can be based on the opportunities that the players have outside their own coalition in the coalition structure. A pre-imputation is said to be a core-element if it can not be improved upon by any coalition. So, the <u>core</u> $C(v,\underline{B})$ of a c.s. game $(N,v,\underline{B})$ is defined by

$$C(v,\underline{B}) := \{x \in I^*(v,\underline{B}) \mid x(S) \geq v(S) \text{ for all } S \subset N,\ S \neq \emptyset\}.$$

In the classical case $\underline{B} = (N)$, the class of games with a non-empty core is characterized in Bondareva (1963) and Shapley (1967) as the class of balanced games-games that satisfy certain balancedness conditions. The purpose is to provide the generalized balancedness conditions that determine whether or not the game possesses a non-empty core with respect to an arbitrary coalition structure $\underline{B}$ on N. The next theorem expresses that the generalized balancedness conditions differ only from the classical balancedness conditions in that the term $v(N)$ is replaced by the sum $\sum_{j=1}^{m} v(B_j)$. For any $S \subset N$ we define the indicator function $e_S \colon N \to \{0,1\}$ by $e_S(i) := 1$ for all $i \in S$ and $e_S(i) := 0$ otherwise.

Definition 2.1

The c.s. game $(N,v,\underline{B})$ is said to be <u>$\underline{B}$-balanced</u> if

$$\lambda_S \geq 0 \text{ for all } S \subset N, \quad \sum_{S \subset N} \lambda_S e_S = e_N \qquad \text{implies}$$

$$\sum_{S \subset N} \lambda_S v(S) \leq \sum_{j=1}^{m} v(B_j). \tag{2.1}$$

Theorem 2.2. The c.s. game $(N,v,\underline{B})$ is $\underline{B}$-balanced if and only if $C(v,\underline{B}) \neq \emptyset$.

Proof. Let $(N,v,\underline{B})$ be a c.s. game. Consider the following linear programming problem LP and its dual problem DLP.

$$\text{(LP)}\quad \text{Minimize} \sum_{j=1}^{m} y_j \qquad \text{subject to } y \in \mathbb{R}^m,\ x \in \mathbb{R}^N,$$
$$x(S) \geq v(S) \text{ for all } S \subset N,\ S \neq \emptyset,$$
$$y_j - x(B_j) \geq -v(B_j) \text{ for all } 1 \leq j \leq m.$$

$$\text{(DLP)}\quad \text{Maximize} \sum_{S\subset N} \lambda_S v(S) - \sum_{j=1}^{m} v(B_j) \qquad \text{subject to } \lambda_S \geq 0 \text{ for all } S \subset N,$$
$$\sum_{S\subset N} \lambda_S e_S = e_N .$$

Evidently, the core $C(v,\underline{B}) \neq \emptyset$ iff the value of the problem LP equals zero. This equivalence and the duality theorem for linear programs yield that $C(v,\underline{B}) \neq \emptyset$ if and only if the value of the problem DLP equals zero. From this we conclude that the non-emptiness of the core is equivalent to the $\underline{B}$-balancedness of the c.s. game. □

It turns out that an upper bound for the core is deducible from the marginal contributions of any player with respect to the formation of the corresponding coalition in the coalition structure. The upper bound involved induces a lower bound for the core by taking into account the differences of the earnings of any available coalition and the coalition's total payoff according to the upper bound involved.

Definition 2.3

The <u>marginal vector</u> $M(v,\underline{B}) \in \mathbb{R}^N$ and the <u>minimal right vector</u> $m(v,\underline{B}) \in \mathbb{R}^N$ of a c.s. game $(N,v,\underline{B})$ are given by

$$M_i(v,\underline{B}) := v(B_j) - v(B_j-\{i\}) \qquad \text{for all } i \in B_j,\ \text{all } 1 \leq j \leq m,$$
$$m_i(v,\underline{B}) := M_i(v,\underline{B}) - \min[\sum_{k\in S} M_k(v,\underline{B}) - v(S) \mid S \subset N,\ i \in S]$$
$$= \max[v(S) - \sum_{k\in S-\{i\}} M_k(v,\underline{B}) \mid S \subset N,\ i \in S] \quad \text{for all } i \in N.$$

Proposition 2.4. Let $(N,v,\underline{B})$ be a c.s. game. Then

$$m_i(v,\underline{B}) \leq x_i \leq M_i(v,\underline{B}) \qquad \text{for all } i \in N \text{ and all } x \in C(v,\underline{B}).$$

Proof. Let $x \in C(v,\underline{B})$ and $i \in N$, say $i \in B_j$. From the definitions of both the core and the marginal vector, we derive that

$$x_i = v(B_j) - x(B_j-\{i\}) \leq v(B_j) - v(B_j-\{i\}) = M_i(v,\underline{B}).$$

From the first part, it follows that for any coalition S containing player i

$$v(S) - \sum_{k \in S-\{i\}} M_k(v,\underline{B}) \le v(S) - x(S-\{i\}) \le x_i$$

and so, $m_i(v,\underline{B}) \le x_i$ as was to be shown. □

A detailed explanation of the marginal vector and the minimal right vector for c.s. games is similar to the one presented in Tijs (1987) for games with respect to the classical coalition structure. In fact, both vectors were already considered in Tijs and Lipperts (1982).

Consider a c.s. game $(N,v,\underline{B})$ with a non-empty core. In view of Proposition 2.4, the payoff to each player $i \in N$ according to a core-element is bounded below by the minimal right $m_i(v,\underline{B})$ as well as bounded above by the marginal contribution $M_i(v,\underline{B})$. Roughly speaking, the minimal rights $m_i(v,\underline{B})$, $i \in N$, are surely insufficient to meet the earnings $v(B_j)$, $1 \le j \le m$, because of the inequalities $\sum_{i \in B_j} m_i(v,\underline{B}) \le v(B_j)$, $1 \le j \le m$, whereas the marginal contributions $M_i(v,\underline{B})$, $i \in N$, are more than sufficient to meet these earnings. This observation gives rise to the treatment of a specific class of c.s. games in Section 3.

3. The τ-value of a quasibalanced c.s. game

Definition 3.1

The class $QB(N,\underline{B})$ of $\underline{B}$-quasibalanced c.s. games is given by

$$QB(N,\underline{B}) := \{(N,v,\underline{B}) \mid m_i(v,\underline{B}) \le M_i(v,\underline{B}) \text{ for all } i \in N \text{ and}$$
$$\sum_{i \in B_j} m_i(v,\underline{B}) \le v(B_j) \le \sum_{i \in B_j} M_i(v,\underline{B}) \text{ for all } 1 \le j \le m\}.$$

Due to Theorem 2.2 and Proposition 2.4, the class of $\underline{B}$-quasibalanced c.s. games includes the class of $\underline{B}$-balanced c.s. games. On the class of quasibalanced c.s. games, we aim to introduce a solution concept which prescribes somehow an equitable compromise between the minimal rights on the one hand and the marginal contributions on the other hand.

Definition 3.2

The $\underline{\tau\text{-value}}$ $\tau(v,\underline{B}) \in I^*(v,\underline{B})$ of a $\underline{B}$-quasibalanced c.s. game $(N,v,\underline{B})$ is given by

$$\tau_i(v,\underline{B}) := \alpha_j m_i(v,\underline{B}) + (1-\alpha_j)M_i(v,\underline{B}) \qquad (3.1)$$

for all $i \in B_j$, all $1 \le j \le m$,

where the real numbers $\alpha_j \in [0,1]$, $1 \le j \le m$, are (uniquely) determined by the equations $\sum_{i\in B_j} \tau_i(v,\underline{B}) = v(B_j)$, $1 \le j \le m$.

From the geometric viewpoint, the restriction of the τ-value vector to any coalition B_j in the coalition structure $\underline{B}$ may be regarded as the unique efficient vector lying on the straight line segment with end points the restrictions of the minimal right vector and the marginal vector to the coalition B_j. The next proposition expresses that the τ-value concept possesses several interesting standard properties for one-point solution concepts.

Proposition 3.3. The τ-value τ: $QB(N,\underline{B}) \to \mathbb{R}^N$ possesses the next five properties.

(i) $\underline{B}$-efficiency: $\tau(v,\underline{B}) \in I^*(v,\underline{B})$ for all $(N,v,\underline{B}) \in QB(N,\underline{B})$.

(ii) Individual rationality: $\tau_i(v,\underline{B}) \ge v(\{i\})$ for all $i \in N$ and all $(N,v,\underline{B}) \in QB(N,\underline{B})$.

(iii) Dummy player property: $\tau_i(v,\underline{B}) = v(\{i\})$ for all $(N,v,\underline{B}) \in QB(N,\underline{B})$ and any dummy player $i \in N$ in the game (N,v).
Here player i is called a dummy in the game (N,v) if
$v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subset N-\{i\}$.

(iv) Relative invariance under S-equivalence: $\tau(\beta v+c,\underline{B}) = \beta\tau(v,\underline{B}) + c$ for all $(N,v,\underline{B}) \in QB(N,\underline{B})$, all $\beta \in (0,\infty)$ and all $c \in \mathbb{R}^N$.
Here the game $(N,\beta v+c)$ is defined by $(\beta v+c)(S) := \beta v(S) + \sum_{i\in S} c(i)$ for all $S \subset N$.

(v) Substitution property: $\tau_i(v,\underline{B}) = \tau_k(v,\underline{B})$ for all $(N,v,\underline{B}) \in QB(N,\underline{B})$ and any substitutes $i,k \in N$, $i \ne k$, in the c.s. game $(N,v,\underline{B})$.
Here players i and k are called substitutes in the c.s. game $(N,v,\underline{B})$ if $\{i,k\} \subset B_j$ for a certain $1 \le j \le m$ and
$v(S \cup \{i\}) = v(S \cup \{k\})$ for all $S \subset N-\{i,k\}$.

Proof. Let $(N,v,\underline{B}) \in QB(N,\underline{B})$. Due to its construction, the τ-value $\tau(v,\underline{B})$ is a pre-imputation satisfying $m_i(v,\underline{B}) \le \tau_i(v,\underline{B}) \le M_i(v,\underline{B})$ for all $i \in N$.
(ii) The τ-value is individually rational since $\tau_i(v,\underline{B}) \ge m_i(v,\underline{B}) \ge v(\{i\})$ for all $i \in N$ where the last inequality follows from Definition 2.3.
(iii) Let $i \in N$ be a dummy player in the game (N,v). In particular, $M_i(v,\underline{B}) = v(B_j) - v(B_j-\{i\}) = v(\{i\})$ whenever $i \in B_j$. Now it follows that $\tau_i(v,\underline{B}) \le M_i(v,\underline{B}) = v(\{i\})$, while $\tau_i(v,\underline{B}) \ge v(\{i\})$ by applying part (ii). Hence, $\tau_i(v,\underline{B}) = v(\{i\})$ and so, the τ-value possesses the dummy player property.
(iv) Let $\beta \in (0,\infty)$ and $c \in \mathbb{R}^N$. By straightforward calculations, we deduce from Definition 2.3 that $M(\beta v+c,\underline{B}) = \beta M(v,\underline{B}) + c$ as well as $m(\beta v+c,\underline{B}) = \beta m(v,\underline{B}) + c$. From this and Definition 3.1 we conclude that $(N,\beta v+c,\underline{B}) \in QB(N,\underline{B})$ iff $(N,v,\underline{B}) \in QB(N,\underline{B})$. Further, $\tau(\beta v+c,\underline{B}) = \beta\tau(v,\underline{B}) + c$ by using formula (3.1). So, the τ-value is relatively invariant under S-equivalence.
(v) Let $i,k \in N$ be substitutes in the c.s. game $(N,v,\underline{B})$, say $\{i,k\} \subset B_j$. In particular, $v(B_j-\{i\}) = v(B_j-\{k\})$ and thus, $M_i(v,\underline{B}) = M_k(v,\underline{B})$. By straightforward calculations, we also obtain $m_i(v,\underline{B}) = m_k(v,\underline{B})$. Now it is evident from formula (3.1) that $\tau_i(v,\underline{B}) = \tau_k(v,\underline{B})$ and so, the τ-value possesses the substitution property. □

It is obvious from formula (3.1) that the τ-value vector of a quasibalanced c.s. game is sectionally proportional to the marginal vector of the c.s. game whenever the minimal right vector vanishes. The next theorem states that this property together with the efficiency and the relative invariance under S-equivalence fully characterize the τ-value concept on the class of quasibalanced c.s. games.

Theorem 3.4. The τ-value $\tau\colon QB(N,\underline{B}) \to \mathbb{R}^N$ is the unique value $\psi\colon QB(N,\underline{B}) \to \mathbb{R}^N$ with the following three properties.
(i) $\underline{B}$-efficiency,
(ii) Relative invariance under S-equivalence, and

(iii) Restricted proportionality:

for any $(N,v,\underline{B}) \in QB(N,\underline{B})$ satisfying $m(v,\underline{B}) = \underline{0} \in \mathbb{R}^N$, the value vector $\psi(v,\underline{B})$ is sectionally proportional to the marginal vector $M(v,\underline{B})$, i.e., there exist real numbers $\beta_j \in [0,1]$, $1 \leq j \leq m$, such that $\psi_i(v,\underline{B}) = \beta_j M_i(v,\underline{B})$ for all $i \in B_j$, all $1 \leq j \leq m$.

Proof. It remains to establish the uniqueness part. Suppose that ψ: $QB(N,\underline{B}) \to \mathbb{R}^N$ is a value with the three mentioned properties. We show that $\psi = \tau$ on $QB(N,\underline{B})$. Let $(N,v,\underline{B}) \in QB(N,\underline{B})$. Define the game (N,w) by $w := v - m(v,\underline{B})$. Since both values are relatively invariant under S-equivalence, we obtain $\psi(w,\underline{B}) = \psi(v,\underline{B}) - m(v,\underline{B})$ as well as $\tau(w,\underline{B}) = \tau(v,\underline{B}) - m(v,\underline{B})$. In order to show the equality $\psi(v,\underline{B}) = \tau(v,\underline{B})$, it suffices to prove $\psi(w,\underline{B}) = \tau(w,\underline{B})$. Clearly, we have $M(w,\underline{B}) = M(v,\underline{B}) - m(v,\underline{B})$, $m(w,\underline{B}) = \underline{0} \in \mathbb{R}^N$ and thus, $(N,v,\underline{B}) \in QB(N,\underline{B})$ implies $(N,w,\underline{B}) \in QB(N,\underline{B})$. It follows from $m(w,\underline{B}) = \underline{0}$ that the two value vectors $\psi(w,\underline{B})$ and $\tau(w,\underline{B})$ are sectionally proportional to the marginal vector $M(w,\underline{B})$. Together with the $\underline{B}$-efficiency property for ψ and τ, this yields directly that the equality $\psi(w,\underline{B}) = \tau(w,\underline{B})$ holds. This completes the proof. □

Finally, we remark that the content of this section generalizes the main topics in Tijs (1981, 1987). In Section 4 we treat a generalization of the topics in Driessen and Tijs (1983).

4. The core, the τ-value and the nucleolus of a 1-convex c.s. game

From Definition 2.3 we observe that each coordinate of the so-called concession vector $\lambda(v,\underline{B}) := M(v,\underline{B}) - m(v,\underline{B})$ is determined by the gap which is minimal among the gaps of coalitions containing the corresponding player. Here the notion of the gap refers to the difference of the earnings and the total payoff according to the marginal vector, on the understanding that the notion of the gap is considered for any coalition.

Definition 4.1

The <u>gap function</u> $g^{v,\underline{B}}: \mathcal{P}(N) \to \mathbb{R}$ of a c.s. game $(N,v,\underline{B})$ is given by

$$g^{v,\underline{B}}(S) := \sum_{i \in S} M_i(v,\underline{B}) - v(S) \qquad \text{for all } S \subset N.$$

Note that $g^{v,\underline{B}}(\emptyset) = 0$. In view of Proposition 2.4, the non-negativity of the gap function (i.e., $g^{v,\underline{B}}(S) \geq 0$ for all $S \subset N$) is a necessary condition for the non-emptiness of the core. Generally speaking, the non-negativity of the corresponding gap function is not a sufficient condition for the balancedness of the c.s. game. This section is devoted to the study of a specific type of c.s. games for which the gap of any coalition in the coalition structure is minimal among the gaps of coalitions intersecting the coalition involved. For any coalition structure $\underline{B}$ on N and any coalition $S \subset N$, we define the index set $I^{\underline{B}}(S) := \{j \mid 1 \leq j \leq m,\ B_j \cap S \neq \emptyset\}$.

Definition 4.2

The class $C^1(N,\underline{B})$ of $(\underline{B},1)$-<u>convex</u> <u>c.s.</u> <u>games</u> is given by

$$C^1(N,\underline{B}) := \{(N,v,\underline{B}) \mid g^{v,\underline{B}}(B_j) \geq 0 \text{ for all } 1 \leq j \leq m \text{ and}$$
$$g^{v,\underline{B}}(S) \geq \sum_{j \in I^{\underline{B}}(S)} g^{v,\underline{B}}(B_j) \text{ for all } S \subset N,\ S \neq \emptyset\}.$$

Theorem 4.3. Let $(N,v,\underline{B}) \in C^1(N,\underline{B})$.

(i) Let $x \in I^*(v,\underline{B})$. Then $x \in C(v,\underline{B})$ iff $x_i \leq M_i(v,\underline{B})$ for all $i \in N$.

(ii) $\tau_i(v,\underline{B}) = M_i(v,\underline{B}) - |B_j|^{-1} g^{v,\underline{B}}(B_j)$ for all $i \in B_j$, all $1 \leq j \leq m$.

(iii) In particular, $\tau(v,\underline{B}) \in C(v,\underline{B})$.

Proof. Fix $k \in N$, say $k \in B_j$, and let $S \subset N$ be such that $k \in S$. From $j \in I^{\underline{B}}(S)$ and the $(\underline{B},1)$-convexity of the c.s. game $(N,v,\underline{B})$, we deduce that $g^{v,\underline{B}}(S) \geq g^{v,\underline{B}}(B_j)$ for all $S \subset N$ with $k \in S$. This yields that $M_k(v,\underline{B}) - m_k(v,\underline{B}) = g^{v,\underline{B}}(B_j)$ whenever $k \in B_j$. Now we conclude from formula (3.1) that the τ-value is given by $\tau_i(v,\underline{B}) = M_i(v,\underline{B}) - \alpha_j g^{v,\underline{B}}(B_j)$ for all $i \in B_j$, all $1 \leq j \leq m$, where $\alpha_j = |B_j|^{-1}$ because of the $\underline{B}$-efficiency property for the τ-value. This proves part (ii), while part (iii) is a direct consequence of the parts (i)-(ii). So, it remains to establish part (i). By Proposition 2.4, the "only if part" holds for all c.s. games. In order to prove the "if part", suppose that $x \in I^*(v,\underline{B})$ satisfies $x_i \leq M_i(v,\underline{B})$ for all $i \in N$. Put $y := M(v,\underline{B})$. We obtain that for all $S \subset N$, $S \neq \emptyset$

$$x(S) = \sum_{j \in I_-^B(S)} x(B_j \cap S) = \sum_{j \in I_-^B(S)} [v(B_j) - x(B_j \cap (N-S))]$$

$$= \sum_{j \in I_-^B(S)} [-g^{v,\underline{B}}(B_j) + y(B_j \cap S) + (y-x)(B_j \cap (N-S))]$$

$$\geq y(S) - \sum_{j \in I_-^B(S)} g^{v,\underline{B}}(B_j) \geq v(S)$$

where the first inequality follows from $y_i - x_i \geq 0$ for all $i \in N$ and the second inequality from the $(\underline{B},1)$-convexity of the c.s. game $(N,v,\underline{B})$. Therefore, $x \in C(v,\underline{B})$ which completes the proof of part (i). □

According to the τ-value payoff for a $(\underline{B},1)$-convex c.s. game, the members of any coalition B_j in the coalition structure $\underline{B}$ contribute equally to the so-called joint concession amount $g^{v,\underline{B}}(B_j)$ (with respect to the inefficient marginal vector). In the remainder of the section we elucidate that the τ-value of a $(\underline{B},1)$-convex c.s. game occupies a central position within the core.

Definition 4.4

Let $(N,v,\underline{B})$ be a c.s. game. For any $1 \leq j \leq m$, let $\text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$ denote the convex hull of the set consisting of the vectors $f^i(v,\underline{B}) \in \mathbb{R}^{B_j}$, $i \in B_j$. Here the vector $f^i(v,\underline{B})$ is given by

$$f^i_k(v,\underline{B}) := M_k(v,\underline{B}) \qquad \text{for all } k \in B_j-\{i\},$$

$$f^i_i(v,\underline{B}) := M_i(v,\underline{B}) - g^{v,\underline{B}}(B_j).$$

The next theorem states that the Cartesian product of the above convex hulls is a core catcher. As a matter of fact, the core catcher in question coincides with the core merely for $(\underline{B},1)$-convex c.s. games.

Theorem 4.5

(i) $C(v,\underline{B}) \subset \prod_{j=1}^{m} \text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$ for any c.s. game $(N,v,\underline{B})$.

(ii) $C(v,\underline{B}) = \prod_{j=1}^{m} \text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$ iff $(N,v,\underline{B}) \in C^1(N,\underline{B})$.

Proof. (i) Let $(N,v,\underline{B})$ be a c.s. game and suppose $x \in C(v,\underline{B})$. Recall that both $x_i \le M_i(v,\underline{B})$ for all $i \in N$ and $g^{v,\underline{B}}(B_j) \ge 0$ for all $1 \le j \le m$ because of Proposition 2.4. Fix $1 \le j \le m$. In case $g^{v,\underline{B}}(B_j) = 0$, then $x_k = M_k(v,\underline{B}) = f_k^i(v,\underline{B})$ for all $i,k \in B_j$, and hence, the restriction of the core to $\mathbb{R}^{B_j}$ equals $\text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$. It remains to consider the case $g^{v,\underline{B}}(B_j) > 0$. Define the real numbers β_i, $i \in B_j$, by $\beta_i := [g^{v,\underline{B}}(B_j)]^{-1}[M_i(v,\underline{B}) - x_i]$. Obviously, $\beta_i \ge 0$ for all $i \in B_j$ as well as $\sum_{i\in B_j} \beta_i = 1$. Further, it is straightforward to verify that the equality $x_k = \sum_{i\in B_j} \beta_i f_k^i(v,\underline{B})$ holds for all $k \in B_j$. Hence, the restriction of the core-element x to $\mathbb{R}^{B_j}$ belongs to $\text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$. This completes the proof of the inclusion mentioned in (i).

(ii) To prove the inverse inclusion for any $(N,v,\underline{B}) \in C^1(N,\underline{B})$, suppose $z \in \prod_{j=1}^{m} \text{conv}\{f^i(v,\underline{B}) \mid i \in B_j\}$. Since $f_k^i(v,\underline{B}) \le M_k(v,\underline{B})$ for all $i,k \in B_j$, all $1 \le j \le m$, we also have that $z_k \le M_k(v,\underline{B})$ for all $k \in N$. Now it follows from Theorem 4.3(i) that $z \in C(v,\underline{B})$ which completes the proof of the "if part" in (ii).

To prove the "only if part" in (ii), suppose that the relevant equality holds. Fix $S \subset N$, $S \neq \emptyset$. For any $1 \le j \le m$, choose an arbitrary $i(j) \in B_j$ satisfying $i(j) \in B_j \cap S$ whenever $j \in I^{\underline{B}}(S)$. Define the vector $y \in \mathbb{R}^N$ by $y_k := f_k^{i(j)}(v,\underline{B})$ for all $k \in B_j$, all $1 \le j \le m$. Then we have $y \in C(v,\underline{B})$ as well as $y(S) = \sum_{k\in S} M_k(v,\underline{B}) - \sum_{j\in I^{\underline{B}}(S)} g^{v,\underline{B}}(B_j)$. From this we observe that the core constraint $y(S) \ge v(S)$ yields the inequality $g^{v,\underline{B}}(S) \ge \sum_{j\in I^{\underline{B}}(S)} g^{v,\underline{B}}(B_j)$ as was to be shown. Finally, we obtain that $g^{v,\underline{B}}(B_j) \ge 0$ for all $1 \le j \le m$ by applying Proposition 2.4 to $y \in C(v,\underline{B})$. This proves the $(\underline{B},1)$-convexity of the c.s. game $(N,v,\underline{B})$ whenever the relevant equality in (ii) holds. □

Consider a $(\underline{B},1)$-convex c.s. game $(N,v,\underline{B})$ and let $1 \le j \le m$. By Theorem 4.3(ii), the restriction of the τ-value vector $\tau(v,\underline{B})$ to the coalition B_j is equal to the weighted sum vector $|B_j|^{-1} \sum_{i\in B_j} f^i(v,\underline{B})$. From this and Theorem 4.5(ii), we conclude that the τ-value of a $(\underline{B},1)$-convex c.s. game coincides with the centre of gravity of the extreme points of the core regarded as a Cartesian product. Due to these extreme points of the core of a $(\underline{B},1)$-convex c.s. game, we say the marginal vector and the minimal right vector are sharp bounds for the core involved. We remark that the prefix 1- is used because each vector $f^i(v,\underline{B})$, $i \in B_j$, can be obtained from the restriction of the marginal vector to $\mathbb{R}^{B_j}$ by letting merely one coordinate decrease in such a way that the efficiency property on B_j is met. The term convexity will be discussed in the next section. The remainder of this section deals with the determination of the nucleolus for a $(\underline{B},1)$-convex c.s. game.

Let $(N,v,\underline{B})$ be a c.s. game. For any payoff vector $x \in \mathbb{R}^N$, let $\theta(x)$ be the $2^{|N|}$-tuple whose components are the excesses $e^v(S,x) := v(S) - x(S)$, $S \subset N$, arranged in non-increasing order. Now the lexicographic order $\leq_L$ on $\mathbb{R}^{2^{|N|}}$ is used to order the "complaint" vectors $\theta(x)$ induced by any imputation x. Here the imputation set $I(v,\underline{B})$ is given by

$$I(v,\underline{B}) := \{x \in I^*(v,\underline{B}) \mid x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

Definition 4.6

The <u>nucleolus</u> $N(v,\underline{B})$ of a c.s. game $(N,v,\underline{B})$ with $I(v,\underline{B}) \neq \emptyset$ is given by

$$N(v,\underline{B}) := \{x \in I(v,\underline{B}) \mid \theta(x) \leq_L \theta(y) \text{ for all } y \in I(v,\underline{B})\}.$$

It is known that the nucleolus $N(v,\underline{B})$ consists of a unique point (Aumann and Drèze, 1974). The unique element of the nucleolus is denoted by $\eta(v,\underline{B})$. According to the next theorem, the nucleolus of a $(\underline{B},1)$-convex c.s. game coincides with the τ-value.

Theorem 4.7. $\eta(v,\underline{B}) = \tau(v,\underline{B})$ for all $(N,v,\underline{B}) \in C^1(N,\underline{B})$.

Proof. Let $(N,v,\underline{B}) \in C^1(N,\underline{B})$. In case $|B_j| = 1$ for all $1 \leq j \leq m$, then the imputation set $I(v,\underline{B})$ reduces to a singleton and so, $\eta(v,\underline{B}) = \tau(v,\underline{B})$ because the τ-value is individually rational. It remains to consider the case for which there exists at least one coalition B_j with $|B_j| \geq 2$. Put $x := \tau(v,\underline{B})$ and fix an arbitrary $y \in I(v,\underline{B})$, $y \neq x$. In order to prove $x = \eta(v,\underline{B})$, we establish $\theta(x) \leq_L \theta(y)$.

Define the index set $J := \{j \mid 1 \leq j \leq m,\ y_i = x_i \text{ for all } i \in B_j\}$. Notice that $y \neq x$ implies the existence of an index $j \notin J$. Further, let the set W of coalitions be given by

$$W := \{S \subset N \mid \text{there exists } j \notin J \text{ such that } B_j \cap S \neq \emptyset, B_j\}.$$

We may ignore the excesses of all $S \in \mathcal{P}(N)-W$ in the lexicographic comparison between $\theta(x)$ and $\theta(y)$ since $e^v(S,x) = e^v(S,y)$ for all $S \in \mathcal{P}(N)-W$. Firstly, we have that for all $i \in B_j$, all $j \notin J$

$$e^v(B_j-\{i\},x) = v(B_j-\{i\}) - x(B_j-\{i\}) = x_i - M_i(v,\underline{B}) = -|B_j|^{-1}g^{v,\underline{B}}(B_j)$$

where the last equality follows from Theorem 4.3(ii). Secondly, we obtain that for any $S \in W$ satisfying $B_{\hat{j}} \cap S \neq \emptyset, B_{\hat{j}}$ for a certain $\hat{j} \notin J$

$$e^v(S,x) = v(S) - x(S) = -g^{v,\underline{B}}(S) + \sum_{j \in I^{\underline{B}}(S)} |B_j|^{-1}|B_j \cap S|g^{v,\underline{B}}(B_j)$$

$$\leq \sum_{j \in I^{\underline{B}}(S)} |B_j|^{-1}[|B_j \cap S| - |B_j|]g^{v,\underline{B}}(B_j) \leq -|B_{\hat{j}}|^{-1}g^{v,\underline{B}}(B_{\hat{j}})$$

where the inequalities follow from the $(\underline{B},1)$-convexity of the c.s. game $(N,v,\underline{B})$. Choose $k \notin J$ such that $|B_k|^{-1}g^{v,\underline{B}}(B_k) \leq |B_j|^{-1}g^{v,\underline{B}}(B_j)$ for all $j \notin J$. The above reasonings yield that the maximal excess at x among the excesses of $S \in W$ is given by $\theta_1(x) = e^v(B_k-\{i\},x) = -|B_k|^{-1} g^{v,\underline{B}}(B_k)$ for any $i \in B_k$. From $k \notin J$ and $y(B_k) = x(B_k)$, we derive that there exists $\hat{i} \in B_k$ with $y_{\hat{i}} > x_{\hat{i}}$. Now it follows that

$$e^v(B_k-\{\hat{i}\},y) = y_{\hat{i}} - M_{\hat{i}}(v,\underline{B}) > x_{\hat{i}} - M_{\hat{i}}(v,\underline{B}) = -|B_k|^{-1}g^{v,\underline{B}}(B_k) = \theta_1(x).$$

From this we conclude that $\theta_1(y) > \theta_1(x)$ and hence, $\theta(x) \leq_L \theta(y)$ as was to be shown. □

5. The core and the τ-value of a convex c.s. game

Shapley (1971) introduced the convexity notion for a game (N,v) and described the structure of the classical core C(v,(N)) for a convex game (N,v) by means of the so-called marginal worth vectors. Ichiishi (1981) established that all the marginal worth vectors belong to the core merely for convex games. The purpose is to provide the generalized convexity notion for a c.s. game in such a way that the core of a convex c.s. game is generated by the adapted marginal worth vectors. For any coalition B_j, $1 \leq j \leq m$, in a coalition structure $\underline{B}$ on N, let Θ^{B_j} denote the set of all one-to-one mappings from B_j onto $\{1,2,\ldots,|B_j|\}$.

Definition 5.1

Let $(N,v,\underline{B})$ be a c.s. game. For any $1 \le j \le m$ and $\theta \in \Theta^{B_j}$, the <u>marginal worth vector</u> $x^{\theta}(v,\underline{B}) \in \mathbb{R}^{B_j}$ is given by

$$x_i^{\theta}(v,\underline{B}) := v(P_i^{\theta} \cup \{i\}) - v(P_i^{\theta}) \qquad \text{for all } i \in B_j$$

where $P_i^{\theta} := \{k \in B_j \mid \theta(k) < \theta(i)\}$ represents the set of players in B_j who precede player i with respect to the mapping θ.

Fix $1 \le j \le m$. Although the marginal worth vectors $x^{\theta}(v,\underline{B})$, $\theta \in \Theta^{B_j}$, are associated with the c.s. game $(N,v,\underline{B})$, they may also be interpreted as the marginal worth vectors obtainable from the subgame $(B_j, v|B_j)$ with respect to the classical coalition structure on B_j. Here the function $v|B_j$ denotes the restriction of the function v to the family of subsets of B_j. The first part of the next result is due to Weber (1988, page 117) and the second part is due to Shapley (1971) and Ichiishi (1981).

Proposition 5.2. Let $(N,v,\underline{B})$ be a c.s. game. Then

$$C(v|B_j,(B_j)) \subset \text{conv}\{x^{\theta}(v,\underline{B}) \mid \theta \in \Theta^{B_j}\} \qquad \text{for all } 1 \le j \le m.$$

In particular, the inclusion is an equality iff the subgame $(B_j, v|B_j)$ is a convex game.

The next theorem states that the Cartesian product of the above convex hulls is a core catcher. In fact, the core catcher involved coincides with the core merely for $\underline{B}$-convex c.s. games. The $\underline{B}$-convexity of a c.s. game is defined in terms of the classical convexity of some subgames and an additional requirement.

Definition 5.3

The class $C(N,\underline{B})$ of <u>$\underline{B}$-convex c.s. games</u> is given by

$$C(N,\underline{B}) := \{(N,v,\underline{B}) \mid (B_j, v|B_j) \text{ is a convex game for all } 1 \le j \le m$$
$$\text{and } v(S) \le \sum_{j \in I^{\underline{B}}(S)} v(B_j \cap S) \text{ for all } S \subset N,\ S \ne \emptyset\}. \qquad (5.1)$$

Theorem 5.4

(i) $C(v,\underline{B}) \subset \prod_{j=1}^{m} \text{conv}\{x^{\theta}(v,\underline{B}) \mid \theta \in \Theta^{B_j}\}$ for any c.s. game $(N,v,\underline{B})$.

(ii) $C(v,\underline{B}) = \prod_{j=1}^{m} \text{conv}\{x^{\theta}(v,\underline{B}) \mid \theta \in \Theta^{B_j}\}$ iff $(N,v,\underline{B}) \in C(N,\underline{B})$.

Proof. Let $(N,v,\underline{B})$ be a c.s. game. The statement in (i) is derived from the following chain of inclusions:

$$C(v,\underline{B}) \subset \prod_{j=1}^{m} C(v|B_j,(B_j)) \subset \prod_{j=1}^{m} \mathrm{conv}\{x^{\theta}(v,\underline{B}) \mid \theta \in \Theta^{B_j}\}$$

where the first inclusion is trivial and the second inclusion is valid because of Proposition 5.2. This proves part (i).

Concerning the statement in (ii), we first recall that the second inclusion in the above chain is an equality if and only if the subgames $(B_j, v|B_j)$, $1 \le j \le m$, are convex games.

To prove the "if part" in (ii), suppose $(N,v,\underline{B}) \in C(N,\underline{B})$. For any $x \in \prod_{j=1}^{m} C(v|B_j,(B_j))$ and any $S \subset N$, $S \neq \emptyset$, we obtain

$$x(S) = \sum_{j\in I^{B}_{-}(S)} x(B_j \cap S) \ge \sum_{j\in I^{B}_{-}(S)} v(B_j \cap S) \ge v(S) \tag{5.2}$$

where the last inequality follows from the $\underline{B}$-convexity of the c.s. game $(N,v,\underline{B})$. Therefore, $x \in C(v,\underline{B})$ which completes the proof of the "if part" in (ii).

To prove the "only if part" in (ii), suppose that the relevant equality holds. It remains to show that (5.1) holds. Fix $S \subset N$, $S \neq \emptyset$. For any $1 \le j \le m$, choose an arbitrary $\theta^j \in \Theta^{B_j}$ satisfying $\theta^j(B_j \cap S) = \{1,2,\ldots,|B_j \cap S|\}$ whenever $j \in I^{B}_{-}(S)$ and consequently, we have $\sum_{k\in B_j\cap S} x_k^{\theta^j}(v,\underline{B}) = v(B_j \cap S)$. Define the vector $y \in \mathbb{R}^N$ by $y_k = x_k^{\theta^j}(v,\underline{B})$ for all $k \in B_j$, all $1 \le j \le m$. Then we have $y \in C(v,\underline{B})$ and the core constraint $y(S) \ge v(S)$ yields the inequality $\sum_{j\in I^{B}_{-}(S)} v(B_j \cap S) \ge v(S)$ as was to be shown. This proves the $\underline{B}$-convexity of the c.s. game $(N,v,\underline{B})$ whenever the relevant equality in (ii) holds. □

The partial $\underline{B}$-convexity condition (5.1) for a c.s. game $(N,v,\underline{B})$ is weaker than the decomposition condition presented in Aumann and Drèze (1974). The game (N,v) is called <u>decomposable</u> with respect to the coalition structure $\underline{B}$ on N if $v(S) = \sum_{j\in I^{B}_{-}(S)} v(B_j \cap S)$ for all $S \subset N$, $S \neq \emptyset$. Note that each of the two conditions is sufficient for the decomposition property of the core, i.e., $C(v,\underline{B}) = \prod_{j=1}^{m} C(v|B_j,(B_j))$ (see (5.2)).

Moreover, Aumann and Drèze (1974) treated an axiomatic generalization of the classical Shapley value and their main Theorem 3 states that the generalized Shapley value for c.s. games possesses the "restriction property". To be exact, the restriction of the generalized Shapley value for a c.s. game $(N,v,\underline{B})$ to any coalition B_j is the classical Shapley value for the subgame $(B_j, v|B_j)$. Without going into details, we conclude from the restriction property involved and Theorem 5.4(ii) that the Shapley value of a $\underline{B}$-convex c.s. game coincides with the centre of gravity of the extreme points of the core regarded as a Cartesian product.

We end up by determining the τ-value of a $\underline{B}$-convex c.s. game. For that purpose, we first show that the class of $\underline{B}$-convex c.s. games is included in a specific class of games for which the gap of any single player is minimal among the gaps of coalitions containing the player involved. As usual, the gap function is also required to be non-negative.

Definition 5.5

The class $SC(N,\underline{B})$ of $\underline{B}$-semiconvex c.s. games is given by

$$SC(N,\underline{B}) := \{(N,v,\underline{B}) \mid g^{v,\underline{B}}(T) \geq g^{v,\underline{B}}(\{i\}) \geq 0 \text{ whenever } i \in T \subset B_j \text{ and } g^{v,\underline{B}}(S) \geq \sum_{j \in I^{\underline{B}}(S)} g^{v,\underline{B}}(B_j \cap S) \text{ for all } S \subset N,\ S \neq \emptyset\}.$$

Proposition 5.6. $C(N,\underline{B}) \subset SC(N,\underline{B})$.

Proof. Let $(N,v,\underline{B}) \in C(N,\underline{B})$. Put $y := M(v,\underline{B})$. We obtain that for all $S \subset N$, $S \neq \emptyset$,

$$g^{v,\underline{B}}(S) = y(S) - v(S) \geq y(S) - \sum_{j \in I^{\underline{B}}(S)} v(B_j \cap S)$$

$$= \sum_{j \in I^{\underline{B}}(S)} [y(B_j \cap S) - v(B_j \cap S)] = \sum_{j \in I^{\underline{B}}(S)} g^{v,\underline{B}}(B_j \cap S)$$

where the inequality follows from the $\underline{B}$-convexity of the c.s. game $(N,v,\underline{B})$. Fix $1 \leq j \leq m$ and let $k \in T \subset B_j$. The convexity of the game $(B_j, v|B_j)$ implies $v(T) - v(T-\{k\}) \leq v(B_j) - v(B_j-\{k\}) = y_k$ or equivalently, $g^{v,\underline{B}}(T-\{k\}) \leq g^{v,\underline{B}}(T)$. In other words, the gap function $g^{v,\underline{B}}$ is monotonic on the family of subsets of B_j and as such, $0 = g^{v,\underline{B}}(\emptyset) \leq g^{v,\underline{B}}(\{i\}) \leq g^{v,\underline{B}}(T)$ whenever $i \in T \subset B_j$. So, $(N,v,\underline{B}) \in SC(N,\underline{B})$. □

Theorem 5.7. Let $(N,v,\underline{B}) \in SC(N,\underline{B})$. Then

(i) $m_i(v,\underline{B}) = v(\{i\})$ for all $i \in N$.

(ii) $(N,v,\underline{B}) \in QB(N,\underline{B})$ iff $v(B_j) \geq \sum_{i \in B_j} v(\{i\})$ for all $1 \leq j \leq m$.

(iii) If $v(\{i\}) = 0$ for all $i \in N$ and $v(B_j) \geq 0$ for all $1 \leq j \leq m$, then $(N,v,\underline{B}) \in QB(N,\underline{B})$ and $\tau_i(v,\underline{B}) = v(B_j)[\sum_{k \in B_j} M_k(v,\underline{B})]^{-1} M_i(v,\underline{B})$ for all $i \in B_j$, all $1 \leq j \leq m$.

Proof. (i) Let $i \in N$, say $i \in B_j$. The $\underline{B}$-semiconvexity of the c.s. game $(N,v,\underline{B})$ yields that $g^{v,\underline{B}}(S) \geq g^{v,\underline{B}}(B_j \cap S) \geq g^{v,\underline{B}}(\{i\})$ for all $S \subset N$ with $i \in S$. From this it follows immediately that $m_i(v,\underline{B}) = M_i(v,\underline{B}) - g^{v,\underline{B}}(\{i\}) = v(\{i\})$.

(ii) Evidently, $M_i(v,\underline{B}) - m_i(v,\underline{B}) = g^{v,\underline{B}}(\{i\}) \geq 0$ for all $i \in N$ and further, $\sum_{i \in B_j} M_i(v,\underline{B}) - v(B_j) = g^{v,\underline{B}}(B_j) \geq 0$ for all $1 \leq j \leq m$. Hence, the $\underline{B}$-quasibalancedness condition for the c.s. game $(N,v,\underline{B})$ reduces to $v(B_j) \geq \sum_{i \in B_j} m_i(v,\underline{B}) = \sum_{i \in B_j} v(\{i\})$ for all $1 \leq j \leq m$. The statement in (iii) is a direct consequence of the parts (i)-(ii) and formula (3.1). □

Theorem 5.7(i) expresses that the minimal right vector of a semiconvex c.s. game is determined by the worths of the single players. Roughly speaking, the τ-value of a zero-normalized semiconvex c.s. game is sectionally proportional to the marginal vector. A detailed study of the τ-value on the class SC(N,(N)) can be found in Driessen and Tijs (1985).

Consider a $\underline{B}$-convex c.s. game $(N,v,\underline{B})$. For any mapping $\theta \in \Theta^{B_j}$ satisfying $\theta(i) = 1$ and $\theta(k) = |B_j|$, the corresponding coordinates of the marginal worth vector $x^{\theta}(v,\underline{B}) \in \mathbb{R}^{B_j}$ are given by $x_i^{\theta}(v,\underline{B}) = v(\{i\}) = m_i(v,\underline{B})$ and $x_k^{\theta}(v,\underline{B}) = v(B_j) - v(B_j - \{k\}) = M_k(v,\underline{B})$. Since the extreme points of the core of a convex c.s. game are composed of the marginal worth vectors, we say the minimal right vector and the marginal vector are sharp bounds for the core involved. Although the τ-value is constructed as an efficient compromise between these two sharp core bounds, the τ-value of a convex c.s. game may fall outside the core.

The terms convexity and 1-convexity are related to each other because Theorem 5.4 concerning convex c.s. games can be seen as the analogue to Theorem 4.5 concerning 1-convex c.s. games. For a detailed elucidation of the relationship between the notions of convexity and 1-convexity, we refer to Driessen (1988, Chapter VII).

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