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# Egalitarianism in convex fuzzy games

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**Abstract**

In this paper, the egalitarian solution for convex cooperative fuzzy games is introduced. The classical Dutta–Ray algorithm for finding the constrained egalitarian solution for convex crisp games is adjusted to provide the egalitarian solution of a convex fuzzy game. For arbitrary fuzzy games, the equal division core is introduced. It turns out that both the equal division core and the egalitarian solution of a convex fuzzy game coincide with the corresponding equal division core and the constrained egalitarian solution, respectively, of the related crisp game.


*Keywords:* Convex games; Egalitarian solutions; Equal division core; Fuzzy coalitions; Fuzzy games

*JEL classification:* C71

## 1. Introduction

The concept of egalitarianism, mainly based on Lorenz domination, has generated several core-related solution concepts on the set of cooperative crisp games with transferable utility (cooperative TU-games): the constrained egalitarian solution (Dutta and Ray, 1989), the Lorenz solution (Hougaard et al., 2001), the Lorenz stable set and the egalitarian core (Arin and Inarra, 2001). The class of convex crisp games is the only standard class of cooperative TU-games for which the constrained egalitarian solution exists and, moreover, it belongs to the core and Lorenz dominates every other core allocation. It turns out that all the other egalitarian solutions mentioned above coincide for convex crisp games with the constrained egalitarian solution. On this class of cooperative TU-games, alternative axiomatic character-

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izations of the constrained egalitarian solution are provided by Dutta (1990), Hokari (2000), Klijn et al. (2000). This solution for a convex crisp game can be obtained using the algorithm proposed by Dutta and Ray (1989) or the formula suggested by Hokari (2000).

Another solution concept related to the norm of equity is the equal division core proposed by Selten (1972). He introduces it in order to explain outcomes in experimental cooperative games and notes that in 76% of 207 experimental games, the outcomes have a "strong tendency to be in the equal division core". Axiomatic characterizations of this solution concept on two classes of cooperative TU-games are provided by Bhattacharya (2002).

The main purpose of this paper is to introduce on one hand the egalitarian solution in the context of convex fuzzy games as proposed by Branzei et al. (2002, 2003), and on the other hand the equal division core for arbitrary fuzzy games.

Cooperative fuzzy games have proved to be suitable for modelling cooperative behavior of agents in economic situations (Billot, 1995; Nishizaki and Sakawa, 2001) and political situations (Butnariu, 1978; Lebret and Ziad, 2001) in which some agents do not fully participate in a coalition but only to a certain extent. For example, in a class of production games, partial participation in a coalition means to offer a part of the resources while full participation means to offer all the resources. A coalition including players who participate partially can be treated in the context of cooperative game theory as a so-called fuzzy coalition, introduced by Aubin (1974, 1981).

The theory of cooperative fuzzy games started with the cited work of Aubin where the notions of a fuzzy game and the core of a fuzzy game are introduced. In the meantime, many solution concepts have been developed (cf. Branzei et al., 2002a,b; Butnariu, 1978; Molina and Tejada, 2002; Nishizaki and Sakawa, 2001; Sakawa and Nishizaki, 1994; Tsurumi et al., 2001).

The outline of the paper is as follows. Sections 2 and 3 provide the necessary notions and facts for cooperative crisp and fuzzy games, respectively. Section 4 introduces an egalitarian solution for convex fuzzy games by adjusting the classical Dutta–Ray algorithm for convex crisp games. It starts by pointing out that computing the egalitarian solution for a supermodular fuzzy game is not a priori easy. It is proved that adding coordinate-wise convexity to supermodularity guarantees the existence of a maximal fuzzy coalition corresponding to a crisp coalition, at each step of the adjusted Dutta–Ray algorithm. It turns out that the introduced egalitarian solution lies in the core of the convex fuzzy game and coincides with the Dutta–Ray egalitarian solution of the corresponding crisp game. In Section 5, the equal division core of an arbitrary fuzzy game is introduced and it is shown that for any convex fuzzy game, the egalitarian solution is an allocation in the equal division core of the game, and the equal division core of a convex fuzzy game coincides with the equal division core of the corresponding crisp game. This leads straightforwardly to an axiomatic characterization of the egalitarian solution for convex fuzzy games. Section 6 concludes with some final remarks.

## 2. Cooperative crisp games

A *cooperative crisp game* $\langle N, w\rangle$ consists of a finite set of players $N$, $N=\{1,2,\ldots,n\}$ and a map $w$: $2^N \rightarrow \Re$ with $w(\emptyset)=0$. For $S \in 2^N$, $w(S)$ is called the worth of coalition $S$ and it is interpreted as the amount of money (utility) the coalition can obtain, when the players in $S$ work together. The class of crisp games with player set $N$ is denoted by $G^N$.

A game $\langle N, w\rangle \in G^N$ is called convex if for each $S, T \in 2^N$

$$w(S \cup T) + w(S \cap T) \geq w(S) + w(T).$$

The core of a game $\langle N, w\rangle \in G^N$ is the convex set

$$C(N, w) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = w(N), \sum_{i \in S} x_i \geq w(S) \text{ for each } S \in 2^N \right\},$$

consisting of efficient vectors $x$ with the property that no coalition $S$ can obtain more than $\sum_{i \in S} x_i$ in splitting off.

An interesting element of the core of a convex crisp game $\langle N, w\rangle$ is the Dutta–Ray egalitarian allocation $E(N, w)$ which can be described in a simple way and found easily in a finite number of steps. Let $|S|$ be the number of players in the coalition $S$, $S \in 2^N$. For any coalition $S$, we denote its average worth with respect to the characteristic function $w$ by $a(S, w) := w(S)/|S|$.

In Step 1 of the Dutta–Ray algorithm, one considers the game $\langle N_1, w_1\rangle$ with $N_1 := N$, $w_1 := w$, and the per capita value $a(T, w_1)$ for each non-empty subcoalition $T$ of $N_1$. Then the largest element $T_1 \in 2^{N_1} \setminus \{\emptyset\}$ in $\text{argmax}_{T \in 2^{N_1} \setminus \{\emptyset\}} a(T, w_1)$ is taken and $E_i(N, w) = a(T_1, w_1)$ for all $i \in T_1$ is defined. For a convex crisp game $\langle N_1, w_1\rangle$, it is well known that the finite set $\text{argmax}_{S \in 2^{N_1} \setminus \{\emptyset\}} a(S, w_1)$ is closed w.r.t. the union operation, that is if $S_1, S_2 \in \text{argmax}_{S \in 2^{N_1} \setminus \{\emptyset\}} a(S, w_1)$, then $S_1 \cup S_2 \in \text{argmax}_{S \in 2^{N_1} \setminus \{\emptyset\}} a(S, w_1)$. This implies that $\text{argmax}_{S \in 2^{N_1} \setminus \{\emptyset\}} a(S, w_1)$ has a largest element w.r.t. the partial order of inclusion on sets, namely $\cup \{T | T \in \text{argmax}_{S \in 2^{N_1} \setminus \{\emptyset\}} a(S, w_1)\}$. If $T_1 = N$, then we stop.

In case $T_1 \neq N$, then in Step 2 of the algorithm, one considers the convex game $\langle N_2, w_2\rangle$ where $N_2 := N_1 \setminus T_1$ and $w_2(S) = w_1(S \cup T_1) - w_1(T_1)$ for each $S \in 2^{N_2} \setminus \{\emptyset\}$, takes the largest element $T_2$ in $\text{argmax}_{T \in 2^{N_2} \setminus \{\emptyset\}} a(T, w_2)$ and defines $E_i(N, w) = a(T_2, w_2)$ for all $i \in T_2$. If $T_1 \cup T_2 = N$ we stop; otherwise we continue by considering the game $\langle N_3, w_3\rangle$ with $N_3 := N_2 \setminus T_2$ and $w_3(S) = w_2(S \cup T_2) - w_2(T_2)$ for each $S \in 2^{N_3} \setminus \{\emptyset\}$, etc. After a finite number of steps, the algorithm stops, and the obtained allocation $E(N, w)$ is called the constrained egalitarian solution of the game $\langle N, w\rangle$.

Since the constrained egalitarian solution is in the core of the corresponding convex game, it is interesting to study the interrelation between $E(N, w)$ and every other core allocation in terms of a special kind of domination which can be introduced as follows.

Consider a society of $n$ individuals with aggregate income fixed at $I$ units. For any $x \in \Re^n_+$ denoted by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, the vector obtained by rearranging its coordinates in a non-decreasing order, that is, $\hat{x}_1 \leq \hat{x}_2 \leq \ldots \leq \hat{x}_n$. For any $x, y \in \Re^n_+$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = I$, we say that $x$ Lorenz dominates $y$, and denote it by $x \succ_L y$, iff $\sum_{i=1}^p \hat{x}_i \geq \sum_{i=1}^p \hat{y}_i$ for all $p \in \{1, \ldots, n-1\}$, with at least one strict inequality.

As mentioned in the Introduction, Dutta and Ray (1989) prove that for convex crisp games, the constrained egalitarian solution Lorenz dominates every other core allocation.

Another core-like solution concept which is related to the norm of equity is the equal division core introduced by Selten (1972). Given a cooperative crisp game $\langle N, w\rangle$, the equal division core EDC $(N, w)$ is the set

$$\left\{x\in\Re^N \mid \sum_{i\in N} x_i = w(N), \nexists S\in 2^N\backslash\{\emptyset\} \text{ s.t. } a(S,w) > x_i \text{ for all } i\in S\right\},$$

consisting of efficient pay off vectors for the grand coalition which cannot be blocked by the equal division allocation of any subcoalition. It is clear that the core of a cooperative crisp game is included in the equal division core of that game.

## 3. Cooperative fuzzy games

Given the set $N=\{1, 2, \ldots, n\}$ of players, a fuzzy coalition is a vector $s\in[0, 1]^N$. The $i$-th coordinate $s_i$ of $s$ is called the participation level of player $i$ in the fuzzy coalition $s$. Instead of $[0, 1]^N$, we will also write $\mathcal{F}^N$ for the set of fuzzy coalitions. A crisp coalition $S\in 2^N$ corresponds in a canonical way to the fuzzy coalition $e^s$, where $e^s\in\mathcal{F}^N$ is the vector with $(e^s)_i = 1$ if $i\in S$, and $(e^s)_i = 0$ if $i\in N\backslash S$. The fuzzy coalition $e^s$ corresponds to the situation where the players in $S$ fully cooperate (i.e. with participation level 1) and the players outside $S$ are not involved at all (i.e. they have participation level 0). We denote by $e^i$ the fuzzy coalition corresponding to the crisp coalition $S=\{i\}$. The fuzzy coalition $e^N$ is called the grand coalition, and the fuzzy coalition (the $n$-dimensional vector) $(0, 0, \ldots, 0)$ corresponds to the empty crisp coalition. We denote by $\mathcal{F}_0^N$ the set of non-empty fuzzy coalitions.

A fuzzy game $\langle N, v\rangle$ consists of the player set $N$ and a map $v$: $\mathcal{F}^N \rightarrow \Re$ with the property $v(0)=0$. The map $v$ assigns to each fuzzy coalition a number, telling what such a coalition can achieve in cooperation. In the following, the set of fuzzy games with player set $N$ will be denoted by $FG^N$ and in the next sections, we will consider the crisp operator $cr : FG^N\rightarrow G^N$. For a fuzzy game $\langle N, v\rangle\in FG^N$, the corresponding crisp game $\langle N, cr(v)\rangle\in G^N$ is given by $cr(v)(S)=v(e^s)$ for each $S\in 2^N$.

The core of a fuzzy game $\langle N, v\rangle$ (Aubin, 1974) is defined by

$$C(N,v) = \left\{x\in\Re^N \mid \sum_{i\in N} x_i = v(e^N), \sum_{i\in N} s_i x_i \geq v(s) \text{ for each } s\in\mathcal{F}^N\right\}.$$

So, $x\in C(N, v)$ can be seen as a distribution of the value of the grand coalition $e^N$, where for each fuzzy coalition $s$, the total payoff is not smaller than $v(s)$, if each player $i\in N$ with participation level $s_i$ is paid $s_i x_i$.

A special class of fuzzy games with a non-empty core is the class of convex fuzzy games introduced by Branzei et al., (2003). Here $\langle N, v\rangle\in FG^N$ is called convex iff $v$

satisfies the increasing average marginal return (IAMR) property, i.e. for each $s^1 s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$, each $i \in N$ and all $\varepsilon_1$, $\varepsilon_2 \in \Re_{++}$ with $s_i^1 + \varepsilon_1 \leq s_i^2 + \varepsilon_2 \leq 1$ it holds that

$$\varepsilon_1^{-1}(v(s^1 + \varepsilon_1 e^i) - v(s^1)) \leq \varepsilon_2^{-1}(v(s^2 + \varepsilon_2 e^i) - v(s^2)).$$

The IAMR property is equivalent to the following pair of properties (cf. Theorem 6 in Branzei et al. (2003)):

(*i*) Supermodularity (SM):

$$v(s \vee t) + v(s \wedge t) \geq v(s) + v(t) \text{ for all } s, t \in \mathcal{F}^N,$$

where $s \vee t$ and $s \wedge t$ are those elements of $[0\ 1]^N$ with the *i*-th coordinate equal to max $\{s_i, t_i\}$ and min $\{s_i, t_i\}$, respectively;

(*ii*) Coordinate-wise convexity (CwC):

For each $i \in N$ and each $s^{-i} \in [0, 1]^{N \setminus \{i\}}$ the function $g_{s^{-i}} : [0, 1] \rightarrow \Re$ with $g_{s^{-i}}(t) = v(s^{-i} \parallel t)$ is a convex function. Here $(s^{-i} \parallel t)$ is the element in $[0\ 1]^N$ with $(s^{-i} \parallel t)_j = s_j$ for each $j \in N \setminus \{i\}$ and $(s^{-i} \parallel t)_i = t$.

**Example 1.** Let $N = \{1, 2\}$ and

$$v(s_1, s_2) = \begin{cases} 1 & \text{if } s_1, s_2 \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that this fuzzy game is coordinate-wise convex but not supermodular.

**Example 2.** Let $N = \{1,2,3\}$ and

$$v(s_1, s_2, s_3) = \begin{cases} s_1 + s_2 + s_3 & \text{if } s_1, s_2, s_3 \in [0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the fuzzy game is supermodular but not coordinate-wise convex.

Hereafter, we will denote the class of convex fuzzy games with player set $N$ by $CFG^N$.

## 4. An egalitarian solution for convex fuzzy games

In this section, we are interested in introducing an egalitarian solution only for convex fuzzy games. We do this in a constructive way by adjusting the classical Dutta–Ray algorithm for a convex crisp game.

As mentioned in Section 2, at each step of the Dutta–Ray algorithm for convex crisp games, a largest element exists. Note that for the crisp case, supermodularity of the characteristic function is equivalent to convexity of the corresponding game.

Although the cores of a convex fuzzy game and its related (convex) crisp game coincide and the Dutta–Ray constrained egalitarian solution is a core element, finding the egalitarian solution of a convex fuzzy game is a task on itself. As we show in Lemma 3, supermodularity of a fuzzy game implies a semilattice structure of the corresponding (possibly infinite) set of fuzzy coalitions with maximal average worth, but it is not enough to ensure the existence of a maximal element. Different difficulties which can arise in fuzzy games satisfying only the supermodularity property are illustrated by means of three examples. According to Lemma 7, it turns out that adding coordinate-wise convexity to supermodularity is sufficient for the existence of such a maximal element; moreover, this element corresponds to a crisp coalition. Then, a simple method comes available to calculate the egalitarian solution of a convex fuzzy game.

For each $s\in\mathcal{F}^N$, let $\lfloor s\rfloor := \sum_{i=1}^{n} s_i$. Given $\langle N, v\rangle\in FG^N$ and $s\in\mathcal{F}_0^N$, we denote by $\alpha(s, v)$ the average worth of $s$ with respect to the aggregated participation level of players in $N$, that is

$$\alpha(s,v) := \frac{v(s)}{\lfloor s\rfloor}.$$

Note that $\alpha(s, v)$ can be interpreted as a per participation-level-unit value of coalition $s$.

**Lemma 3.** *Let $\langle N, v\rangle\in FG^N$ be a supermodular game. Then the set*

$$A(N,v) := \left\{ t\in\mathcal{F}_0^N \mid \alpha(t,v) = \sup_{s\in\mathcal{F}_0^N} \alpha(s,v) \right\}$$

*is closed w.r.t. the join operation $\vee$.*

**Proof**. Let $\bar{\alpha} = \sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$. If $\bar{\alpha}=\infty$, then $A(N, v)=\emptyset$, so $A(N, v)$ is closed w.r.t. the join operation.

Suppose now $\bar{\alpha}\in\Re$. Take $t^1$, $t^2\in A(N, v)$. We have to prove that $t^1\vee t^2\in A(N, v)$, that is $\alpha(t^1\vee t^2, v)=\bar{\alpha}$.

Since $v(t^1)=\bar{\alpha}\lfloor t^1\rfloor$ and $v(t^2)=\bar{\alpha}\lfloor t^2\rfloor$ , we obtain

$$\begin{aligned}\bar{\alpha}\lfloor t^1\rfloor+\bar{\alpha}\lfloor t^2\rfloor &= v(t^1)+v(t^2)\le v(t^1\vee t^2)+v(t^1\wedge t^2)\le\bar{\alpha}\lfloor t^1\vee t^2\rfloor+\bar{\alpha}\lfloor t^1\wedge t^2\rfloor\\ &=\bar{\alpha}\lfloor t^1\rfloor+\bar{\alpha}\lfloor t^2\rfloor,\end{aligned}$$

where the first inequality follows from the (SM) property and the second inequality follows from the definition of $\bar{\alpha}$ and the fact that $v(0)=0$. This implies that $v(t^1\vee t^2)=\bar{\alpha}\lfloor t^1\vee t^2\rfloor$, so $t^1\vee t^2\in A(N, v)$. □

We can conclude from the proof that in case $t^1$, $t^2\in A(N, v)$, not only $t^1$, $t^2\in A(N, v)$ but also $t^1\wedge t^2\in A(N, v)$ if $t^1\wedge t^2\neq 0$. Further, $A(N, v)$ is closed w.r.t. finite "unions", where $t^1\vee t^2$ is seen as the "union" of $t^1$ and $t^2$.

If we try to introduce in a way similar to that of Dutta and Ray (1989), an egalitarian rule for supermodular fuzzy games, then problems may arise since the set of fuzzy coalitions is infinite and it is not clear if there exists a maximal fuzzy coalition with "maximum value per unit of participation level". To be more precise, if $\langle N, v\rangle$ is a supermodular fuzzy game, then crucial questions are:

(1) Is $\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$ finite or not? Example 4 presents a fuzzy game for which $\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$ is infinite.

(2) In case that $\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$ is finite, is there a $t\in\mathcal{F}_0^N$ s.t. $\alpha(t,v)=\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$? A fuzzy game for which the set $\mathrm{argsup}_{s\in\mathcal{F}_0^N}\alpha(s,v)$ is empty is given in Example 5. Note that if the set $\mathrm{argsup}_{s\in\mathcal{F}_0^N}\alpha(s,v)$ is non-empty then $\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)=\max_{s\in\mathcal{F}_0^N}\alpha(s,v)$.
(3) Let $\geq$ be the standard partial order on $[0,1]^N$. If $\max_{s\in\mathcal{F}_0^N}\alpha(s,v)$ exists, does the set $\mathrm{argmax}_{s\in\mathcal{F}_0^N}\alpha(s,v)$ have a maximal element in $\mathcal{F}_0^N$ w.r.t. $\geq$? That this does not always hold for a fuzzy game is shown in Example 6.

**Example 4.** Let $N=\{1,2\}$ and

$$v(s_1,s_2)=\begin{cases} s_2 tg\dfrac{\pi s_1}{2} & \text{if } s_1\in[0,1),\\ 0 & \text{otherwise}\end{cases}.$$

For this game $\sup_{s\in\mathcal{F}_0^{\{1,2\}}}\alpha(s,v)=\infty$.

**Example 5.** Let $N=\{1,2,3\}$ and

$$v(s_1,s_2,s_3)=\begin{cases} (s_1+s_2+s_3)^2 & \text{if } s_1,s_2,s_3\in[0,1),\\ 0 & \text{otherwise.}\end{cases}$$

For this game $\sup_{s\in\mathcal{F}_0^{\{1,2,3\}}}\alpha(s,v)=3$, and $\arg\sup_{s\in\mathcal{F}_0^{\{1,2,3\}}}\alpha(s,v)=\emptyset$.

**Example 6.** Let $N=\{1,2\}$ and

$$v(s_1,s_2)=\begin{cases} s_1+s_2 & \text{if } s_1,s_2\in[0,1),\\ 0 & \text{otherwise.}\end{cases}$$

For this game $\max_{s\in\mathcal{F}_0^{\{1,2\}}}\alpha(s,v)=1$, $\mathrm{argmax}_{s\in\mathcal{F}_0^{\{1,2\}}}\alpha(s,v)=[0,1)\times[0,1)\setminus\{0\}$, but this set has no maximal element w.r.t. $\geq$.

One can easily check that the games in Examples 4, 5 and 6 are supermodular, but not convex (the (CwC) property is not satisfied). For convex fuzzy games, all three questions mentioned above are answered affirmatively in Theorem 9. By using this theorem, the following additional problems can also be conquered: how to define the reduced games in the steps of the adjusted algorithm, and whether this algorithm has only a finite number of steps.

Lemma 7 plays a key role in obtaining our main results on egalitarianism in convex fuzzy games. In its proof, we will use the notion of degree of fuzziness of a coalition. For each $s\in\mathcal{F}^N$ this degree is defined by $\varphi(s)=|\{i\in N\,|\,s_i\in(0,1)\}|$. Note that $\varphi(s)=0$ implies that $s$ corresponds to a crisp coalition, and that in a coalition with $\varphi(s)=n$, no participation level equals 0 or 1. Note further that for $s\in\mathcal{F}_0^N$ with $\varphi(s)=0$, we have $\alpha(s,v)\leq\max_{s\in 2^N\setminus\{\emptyset\}}\alpha(e^S,v)$ because $s$ is equal to $e^T$, where $T=\{i\in N\,|\,s_i=1\}$.

**Lemma 7.** *Let $\langle N,v\rangle\in CFG^N$, $s\in\mathcal{F}_0^N$, and $supp(s)=\{i\in N\,|\,s_i>0\}$. If $\varphi(s)>0$, then there is a $t\in F_0^N$ with $\varphi(t)=\varphi(s)-1$, $supp(t)\subseteq supp(s)$, and $\alpha(t,v)\geq\alpha(s,v)$; if $\alpha(t,v)=\alpha(s,v)$ then $t\geq s$.*

**Proof.** Take $s \in \mathcal{F}_0^N$ with $\varphi(s) > 0$, and $i \in N$ such that $s_i \in (0,1)$. Consider $t^0 = (s^{-i}, 0)$ and $t^1 = (s^{-i}, 1)$. Note that $\varphi(t^0) = \varphi(t^1) = \varphi(s) - 1$ and $\text{supp}(t^0) \subset \text{supp}(t^1) = \text{supp}(s)$.

If $t^0 = 0$, then $t^1 = e^i$ and then $\alpha(e^i, v) \geq \alpha(s_i e^i, v) = \alpha(s, v)$ follows from (CwC). We then take $t = e^i$.

If $t^0 \neq 0$ and $\alpha(t^0, v) > \alpha(s, v)$, then we take $t = t^0$.

Now we treat the case $t^0 \neq 0$ and $\alpha(t^0, v) \leq \alpha(s, v)$. From the last inequality and from the fact that $v(s)/\lfloor s \rfloor$ is a convex combination of $v(t^0)/\lfloor t^0 \rfloor$ and $v(s) - v(t^0)/\lfloor s - t^0 \rfloor$, i.e.

$$\alpha(s, v) = \frac{v(s)}{\lfloor s \rfloor} = \frac{\lfloor t^0 \rfloor}{\lfloor s \rfloor} \cdot \frac{v(t^0)}{\lfloor t^0 \rfloor} + \frac{\lfloor s - t^0 \rfloor}{\lfloor s \rfloor} \cdot \frac{v(s) - v(t^0)}{\lfloor s - t^0 \rfloor},$$

we obtain

$$\frac{v(s) - v(t^0)}{\lfloor s - t^0 \rfloor} \geq \frac{v(s)}{\lfloor s \rfloor} = \alpha(s, v). \tag{1}$$

From the (IAMR) property of $\langle N, v \rangle$ (with $t^0$, $s$, $\lfloor s - t^0 \rfloor$, $\lfloor t^1 - s \rfloor$ in the roles of $s^1$, $s^2$, $\varepsilon_1$, $\varepsilon_2$, respectively) it follows

$$\frac{v(t^1) - v(s)}{\lfloor t^1 - s \rfloor} \geq \frac{v(s) - v(t^0)}{\lfloor s - t^0 \rfloor}. \tag{2}$$

Now from Eqs. (1) and (2), we have

$$\frac{v(t^1) - v(s)}{\lfloor t^1 - s \rfloor} \geq \frac{v(s)}{\lfloor s \rfloor} = \alpha(s, v). \tag{3}$$

Then by applying Eq. (3), we obtain

$$\alpha(t^1, v) = \frac{v(t^1)}{\lfloor t^1 \rfloor} = \frac{\lfloor t^1 - s \rfloor}{\lfloor t^1 \rfloor} \cdot \frac{v(t^1) - v(s)}{\lfloor t^1 - s \rfloor} + \frac{\lfloor s \rfloor}{\lfloor t^1 \rfloor} \cdot \frac{v(s)}{\lfloor s \rfloor} \geq \frac{\lfloor t^1 - s \rfloor}{\lfloor t^1 \rfloor} \cdot \frac{v(s)}{\lfloor s \rfloor} + \frac{\lfloor s \rfloor}{\lfloor t^1 \rfloor} \cdot \frac{v(s)}{\lfloor s \rfloor}$$
$$= \frac{v(s)}{\lfloor s \rfloor} = \alpha(s, v).$$

So, we can take $t = t^1$. □

From Lemma 7, it follows that for each $s \in \mathcal{F}_0^N$, there is a sequence $s^0, s^1, \ldots, s^k$ in $\mathcal{F}_0^N$, where $s^0 = s$ and $k = \varphi(s)$ such that $\varphi(s^{r+1}) = \varphi(s^r) - 1$, $\text{supp}(s^{r+1}) \subseteq \text{supp}(s^r)$, and $\alpha(s^{r+1}, v) \geq \alpha(s^r, v)$ for each $r \in \{0, 1, \ldots, k-1\}$. Since $\varphi(s^k) = 0$, $s^k$ corresponds to a crisp coalition, say $T$. So, we have proved

**Corollary 8.** *Let $\langle N, v \rangle \in CFG^N$. Then for all $s \in \mathcal{F}_0^N$, there exists $T \in 2^N \setminus \{\emptyset\}$ such that $T \subseteq supp(s)$ and $\alpha(e^T, v) \geq \alpha(s, v)$.*

From Corollary 8, it follows immediately

**Theorem 9.** *Let $\langle N, v \rangle \in CFG^N$. Then*

(i) $\sup_{s \in \mathcal{F}_0^N} \alpha(s, v) = \max_{T \in 2^N \setminus \{\emptyset\}} \alpha(e^T, v)$;

(ii) *$T^*=max(argmax_{T\in 2^N\setminus\{\emptyset\}}\alpha(e^T, v))$ generates the largest element in arg* $\sup_{s\in\mathcal{F}_0^N}\alpha(s,v)$ *namely $e^{T^*}$.*

In view of this result, it is easy to adjust the Dutta–Ray algorithm to a convex fuzzy game $\langle N, v\rangle$. In Step 1, one puts $N_1:=N$, $v_1:=v$ and considers $\text{argsup}_{s\in\mathcal{F}_0^N}\alpha(s,v_1)$. According to Theorem 9, there is a unique maximal element in $\text{argsup}_{s\in\mathcal{F}_0^{N_1}}\alpha(s,v)$, which corresponds to a crisp coalition, say $S_1$. Define $E_i(N, v)=\alpha(e^{S_1}, v_1)$ for each $i\in S_1$. If $S_1=N$, then we stop.

In case $S_1\neq$ N, then in Step 2, one considers the convex fuzzy game $\langle N_2, v_2\rangle$ with $N_2:=N_1\setminus S_1$ and, for each $s\in[0, 1]^{N\setminus S_1}$,

$$v_2(s) = v_1(e^{S_1}\frown s) - v_1(e^{S_1}),$$

where $(e^{S_1}\frown s)$ is the element in $[0, 1]^N$ with

$$(e^{S_1}\frown s)_i = \begin{cases} 1 & \text{if } i\in S_1, \\ s_i & \text{if } i\in N\setminus S_1. \end{cases}$$

Once again, by using Theorem 9, one can take the largest element $e^{S_2}$ in $\text{argmax}_{S\in 2^{N_2}\setminus\{\emptyset\}}\alpha(e^S, v_2)$ and defines $E_i(N, v)=\alpha(e^{S_2}, v_2)$ for all $i\in S_2$. If $S_1\cup S_2=N$ we stop; otherwise we continue by considering the convex fuzzy game $\langle N_3, v_3\rangle$, etc. After a finite number of steps, the algorithm stops, and the obtained allocation $E(N, v)$ is called the egalitarian solution of the convex fuzzy game $\langle N, v\rangle$.

**Theorem 10.** *Let $\langle N, v\rangle\in CFG^N$. Then*

(i) *$E(N, v)= E(N, cr(v))$;*

(ii) *$E(N, v)\in C(N, v)$;*

(iii) *$E(N, v)$ Lorenz dominates every other allocation in $C(N, v)$.*

**Proof.**

(i) This assertion follows directly from Theorem 9 and the adjusted Dutta–Ray algorithm given above.

(ii) Note that $E(N, v)=E(N, cr(v))\in C(N, cr(v))=C(N, v)$, where the first equality follows from (i), the second equality follows from Theorem 7(iii) of Branzei et al. (2003), and the relation $E(N, cr(v))\in C(N, cr(v))$ is a main result of Dutta and Ray (1989) for convex crisp games.

(iii) $E(N, cr(v))$ Lorenz dominates every other element of $C(N, cr(v))$ according to Dutta and Ray (1989). Since $E(N, v)=E(N, cr(v))$ and $C(N, cr(v))=C(N, v)$, our assertion (iii) follows. □

Theorem 10 should be interpreted as strengthening Dutta and Ray's result. One can also think that the egalitarian solution for a convex crisp game will keep the Lorenz domination property in any fuzzy extension satisfying IAMR.

Further, if one is interested in studying egalitarian solutions for general fuzzy games, then a change of the starting point of the analysis is needed.[1] One has to consider the egalitarian allocations as the Lorenz undominated imputations in the Lorenz core. Since

[1] As suggested to us by one of the referees.

the latter is recursively defined by using the cardinality of the (crisp) coalitions, the task of introducing the notion of the Lorenz core of a general fuzzy game will not be a trivial one.

## 5. The equal division core for convex fuzzy games

Given a cooperative fuzzy game $\langle N, v\rangle$, we define the equal division core $EDC\,(N, v)$ as the set

$$\left\{x\in\Re^N \mid \sum_{i\in N} x_i = v(e^N),\ \nexists s\in\mathcal{F}_0^N \text{ s.t. } \alpha(s,v) > x_i \text{ for all } i\in\text{supp}(s)\right\}.$$

So $x\in EDC(N, v)$ can be seen as a distribution of the value of the grand coalition $e^N$, where for each fuzzy coalition $s$, there is a player $i$ with a positive participation level for which the pay off $x_i$ is at least as good as the equal division share $\alpha\ (s, v)$ of $v\ (s)$ in $s$.

Some interesting facts w.r.t. the equal division core for convex fuzzy games are collected in

**Theorem 11.** *Let* $\langle N, v\rangle\in FG^N$. *Then*
(i) $C(N, v)\subseteq EDC(N, v)$;
(ii) $E(N, v)\in EDC(N, v)$;
(iii) $EDC(N, v)= EDC(N, cr(v))$.

**Proof.**
(i) Suppose $x\notin EDC(N, v)$. Then there exists an $s\in\mathcal{F}_0^N$ s.t. $\alpha(s,v) > x_i$ for all $i\in\text{supp}(s)$. Then

$$\sum_{i=1}^{n} s_i x_i < \sum_{i=1}^{n} \alpha(s,v)s_i = v(s)$$

which implies that $x\notin C(N, v)$. So $C(N, v)\subseteq EDC(N, v)$.
(ii) According to (i) and Theorem 10(ii), we have $E(N, v)\in C(N, v)\subseteq EDC(N, v)$.
(iii) Suppose $x\in EDC(N, v)$. Then by the definition of $EDC(N, v)$ there is no $e^S\neq 0$ s.t. $\alpha(e^s, v)>x_i$ for all $i\in\text{supp}(e^S)$. Taking into account that $cr(v)(S)=v(e^S)$ for all $S\in 2^N$, there is no $S\neq\emptyset$ s.t. $(cr(v)(S))/|S|>x_i$ for all $i\in S$. Hence, $x\in EDC(N, cr(v))$.

Let $x\in EDC(N, cr(v))$. We prove that for each $s\in\mathcal{F}_0^N$, there is an $i\in\text{supp}(s)$ s.t. $x_i \geq \alpha(s, v)$.

Take $T$ as in Corollary 8. Since $x\in EDC(N, cr(v))$, there is an $i\in T$ s.t. $x_i\geq\alpha(e^T, v)$. Now, from Corollary 8, it follows that $x_i\geq\alpha(s, v)$ for $i\in T\subseteq\text{supp}(s)$. □

**Remark 12.** From the proof of Theorem 11(iii), it follows that for each arbitrary fuzzy game $\langle N, v\rangle$, we have $EDC(v)\subseteq EDC(cr(v))$. But these sets are not necessarily equal, as the following example shows.

**Example 13.** Let $N = \{1,2,3\}$ and $v(s_1,s_2,s_3) = \sqrt{s_1+s_2+s_3}$ for each $s_1, s_2, s_3\in[0, 1]$. For this game, $EDC(N,cr(v)) = \left\{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\}$ and $EDC(N, v)=\emptyset$.

Our last example is meant to illustrate the various interrelations among the egalitarian solution, the core, and the equal division core for convex fuzzy games as stated in Theorems 10 and 11.

**Example 14.** Let $N=\{1,2,3\}$ and $T=\{1,2\}\subset N$. Consider the unanimity fuzzy game $\langle N, u_{e^T}\rangle$ *with*

$$u_{e^T}(s)=\begin{cases}1 & \text{if } s_1=s_2=1,\\ 0 & \text{otherwise.}\end{cases}$$

In the work of Branzei et al. (2003), it is proved (Proposition 9) that a fuzzy game of this type is convex. Its core is given by

$$C(N,u_{e^T})=\text{conv}\{e^1,e^2\}=\text{conv}\{(1,0,0),(0,1,0)\},$$

and the egalitarian allocation is given by

$$E(N,u_{e^T})=\left(\frac{1}{2},\frac{1}{2},0\right)\in C(N,u_{e^T}).$$

It is easy to see that $E(N, u_{e^T})$ Lorenz dominates every other allocation in $C(N, u_{e^T})$. Moreover, the equal division core $EDC(N, u_{e^T})$ is the set

$$\text{conv}\left\{e^1,\frac{1}{2}(e^1+e^2),\frac{1}{2}(e^1+e^3)\right\}\cup\text{conv}\left\{\frac{1}{2}(e^1+e^2),e^2,\frac{1}{2}(e^2+e^3)\right\}.$$

It is clear that $C(N, u_{e^T})\subset EDC(N, u_{e^T})=EDC(N, cr(u_{e^T}))$.

Given Theorems 10 and 11, it is not difficult to provide an axiomatic characterization of the egalitarian solution on the class of convex fuzzy games. Inspiring here is the paper of Klijn et al. (2000) where there are five axiomatizations of the classical Dutta–Ray egalitarian solution. In order to be self-contained, we present the fuzzy counterparts of some of these axioms which will lead us to the analogue of Theorem 3.3 of Klijn et al. (2000) for the class of convex fuzzy games.

Given a solution $\psi$: $CFG^N\rightarrow\Re^N$, we will denote by $S^m$ the set $\{k\in N|\psi_k(N,v)=\max_{i\in N}\psi_i(N,v)\}$.

We will say that $\psi$ satisfies
efficiency, if for all $\langle N, v\rangle\in CFG^N$:

$$\sum_{i\in N}\psi_i(N,v)=v(e^N);$$

equal division stability, if for all $\langle N, v\rangle\in CFG^N$;

$$\psi(N,v)\in EDC(N,v);$$

max-consistency, if for all $\langle N, v\rangle\in CFG^N$ and all $i\in N\backslash S^m$:

$$\psi_i(N,v)=\psi_i(N\backslash S^m,v^{e^{-S^m}}),$$

where $v^{e-S^m}$ is the game defined by

$$v^{e-S^m}(s) = v(e^{S^m} \curlyvee s) - v(e^{S^m})$$

for all $s \in [0,1]^{N \setminus S^m}$.

**Theorem 15.** *There is a unique solution on $CFG^N$ satisfying the properties efficiency, equal division stability and max-consistency, and it is the egalitarian solution.*

## 6. Final remarks

In this paper, we introduce the equal division core for fuzzy games and the egalitarian solution for convex fuzzy games. With the aid of the key result in Lemma 7, we prove the coincidence of the egalitarian solution and the equal division core for a convex fuzzy game with the corresponding solution concepts for its related crisp game. This implies that an easy method comes available to calculate the egalitarian solution of a convex fuzzy game just by considering the corresponding crisp game, and applying on it the classical Dutta–Ray algorithm.

It would be interesting to develop egalitarian solution concepts also for non-convex fuzzy games. Inspiring in this could be the original constrained egalitarian solution of Dutta and Ray (1989), the Lorenz solution (Hougaard et al., 2001), the Lorenz stable set and the egalitarian core (Arin and Inarra, 2001) for cooperative crisp games.

Also, other systems of axioms for the egalitarian solution than the one indicated at the end of Section 5 could be developed (cf. Klijn et al. (2000)).

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