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The Hoffman program for mixed graphs

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ABSTRACT

We consider Hoffman's program about the limit points of the spectral radius of the Hermitian adjacency matrix of mixed graphs. In particular, we determine all such limit points. As an intermediate step, we determine all mixed graphs without negative 4-cycle whose spectral radius does not exceed $\sqrt{2+\sqrt{5}}$.

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1. Background

Let $\mathcal{G}$ be a class of graphs, and let $Q(G)$ be a square matrix associated to $G \in \mathcal{G}$, with Q-*spectral radius* $\rho(Q(G))$. A real number γ is said to be a *limit point* of Q-spectral radii (Q-*limit point* for short) of items in $\mathcal{G}$ if there exists a sequence $(G_k)_{k\in\mathbb{N}}$ in $\mathcal{G}$ such that

$$\rho(Q(G_i)) \neq \rho(Q(G_j)) \text{ whenever } i \neq j, \text{ and } \lim_{k\to\infty} \rho(Q(G_k)) = \gamma.$$

The two facets of the *Hoffman program* with respect to the matrix Q for the class $\mathcal{G}$ are posed by Hoffman [9] (see also [1]):

(∗) establishing all the possible Q-limit points;
(∗∗) characterizing the graphs in $\mathcal{G}$ whose Q-spectral radius does not exceed a fixed limit point.

Let $G = (V(G), E(G))$ be a simple graph with $(0, 1)$-*adjacency matrix* $A(G)$. When $\mathcal{G}$ is a class consisting of all simple graphs, Hoffman [9] determined all the A-limit points of items in $\mathcal{G}$ up to $\rho^* := \sqrt{2+\sqrt{5}}$; Shearer [14] proved γ is an A-limit point of items in $\mathcal{G}$ for all $\gamma \geq \rho^*$, which solved the first part of the Hoffman program with respect to the adjacency matrix. Smith [15] characterized all simple graphs whose A-spectral radius does not exceed 2 (the smallest A-limit point); Brouwer, Neumaier [3] and Cvetković, Doob, Gutman [4] determined all simple graphs with A-spectral radius between 2 and ρ^*.

Belardo and Brunetti [1] completely solved the first part of the Hoffman program with respect to the $(0, \pm 1)$-adjacency matrix of signed graphs. For the second part, they showed that 2 is also the smallest A-limit point for signed graphs. McKee and Smyth [11] identified all signed graphs whose A-spectral radius does not exceed 2; recently, Wang et al. [16] determined all signed graphs with A-spectral radius between 2 and ρ^*, which solved an open problem posed by Belardo et al. [2].

A *mixed graph* M is obtained from a simple graph G by orienting some of its edges; we call G the underlying graph of M, denoted by $G := \Gamma(M)$. Guo and Mohar [7] determined all mixed graphs whose H-spectral radius (the H-matrix of a mixed graph is defined in Definition 1 below) is less than 2, from which we know that 2 is also the smallest H-limit point for mixed graphs. Yuan et al. [21] determined all mixed graphs containing no mixed 4-cycle whose H-spectral radius is at most 2. Greaves [6] characterized all the gain graphs with gains from the Gaussian or Eisenstein integers whose adjacency eigenvalues are contained in $[-2, 2]$, but this does not determine all mixed graphs whose H-spectral radius is at most 2 (see [5]). Based on the result in [6], Gavrilyuk and Munemasa [5] completed the characterization of mixed graphs whose H-spectral radius is at most 2.

Motivated by the above, we study the Hoffman program with respect to the H-matrix of mixed graphs. We say that a mixed graph has no *negative* 4-*cycle* if it has none of the mixed graphs in Fig. 1 as a subgraph (not necessarily induced). In this paper, on one

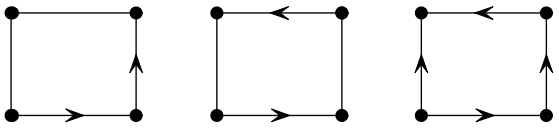

Fig. 1. Negative 4-cycles.

hand, we determine all mixed graphs without negative 4-cycle whose H-spectral radius is at most ρ^*. This solves the second part of the Hoffman program with respect to the H-matrix of $\mathcal{G}$ and the limit value ρ^*, where $\mathcal{G}$ is the class consisting of all mixed graphs without negative 4-cycle. On the other hand, based on this result and the results in [9,14,1], we identify all H-limit points of mixed graphs, which completely solves the first part of the Hoffman program.

The remainder of this paper is organized as follows: In Section 2, we give some preliminary results, which will be used in the subsequent sections. In Section 3, we determine all mixed graphs without negative 4-cycle whose H-spectral radius is at most ρ^*. In Section 4, we identify all H-limit points of mixed graphs.

2. Definitions and preliminaries

As usual, let P_n, C_n and K_n denote the path, cycle and complete graph on n vertices, respectively; and let $K_{m,n}$ denote the complete bipartite graph with the orders of partite sets being m and n. The diameter of a simple graph G is denoted by $\mathrm{diam}(G)$.

For a mixed graph M, the numbers of vertices $|V(M)|$ and edges $|E(M)|$ in M are called the *order* and *size* of M, respectively. We say that two vertices u and v are *adjacent* (or *neighbors*) if there is an arc or an undirected edge between them and we write it as $u \sim v$. We write an undirected edge as $\{u, v\}$ and a directed edge (or an arc) from u to v as $\overrightarrow{uv}$. Usually, we denote an edge of M by uv if we are not concerned whether it is directed or not. For a mixed graph M and a vertex v, define $d_M(v)$ as the *degree* of v in M, i.e., the number of vertices adjacent to v. The largest degree of M is denoted by $\Delta(M)$.

A mixed graph M' is an (induced) subgraph of M, denoted by $M' = M[\Gamma(M')]$, if $\Gamma(M')$ is an (induced) subgraph of $\Gamma(M)$ and the direction of each arc in M' coincides with that in M. For a vertex subset V' of $V(M)$, $M[V']$ is the subgraph of M induced on V'. By $M - u$, we denote the mixed graph obtained from M by deleting the vertex $u \in V(M)$. A mixed graph is called *connected* if its underlying graph is connected. The *girth* of a mixed graph M, denoted by $g(M)$, is the length of the shortest cycle contained in $\Gamma(M)$.

The *Hermitian adjacency matrix* (*H-matrix* for short) was proposed by Guo, Mohar [8] and Liu, Li [10], independently.

Definition 1 *([8,10]).* Let M be a mixed graph, then the H-matrix $H(M) = (h_{uv})$ for M is defined as

$$h_{uv} = \begin{cases} \mathbf{i} & \text{if } \overrightarrow{uv} \text{ is an arc for } M; \\ -\mathbf{i} & \text{if } \overrightarrow{vu} \text{ is an arc for } M; \\ 1 & \text{if } \{u, v\} \text{ is an undirected edge for } M; \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{i}$ is the imaginary unit.

Clearly, $H(M)$ is Hermitian, hence its eigenvalues (also called the *eigenvalues* of M) are all real. We denote the eigenvalues of M by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_{|V(M)|}(M)$. The collection of eigenvalues of M (with multiplicities) is called the *spectrum* of M. Two mixed graphs are called *cospectral* if they have the same spectrum. The *spectral radius* of M, written as $\rho(M)$, is defined to be the largest absolute value of its eigenvalues.

Given a mixed graph M, let M^T be its *converse* (the mixed graph obtained by reversing all the arcs of M). It is immediate from Definition 1 that $H(M^T)$ is the transpose of $H(M)$. This implies that M and M^T are cospectral.

Guo and Mohar [8] proposed *four-way switching* on mixed graphs, which can be summarized as follows (see also [19]).

Definition 2 *([8]).* Let M be a mixed graph. A four-way switching is the operation of changing M into the mixed graph M' by choosing an appropriate diagonal matrix D with $D_{ii} \in \{\pm 1, \pm\mathbf{i}\}$, $i = 1, \ldots, |V(M)|$, and $H(M) = D^{-1}H(M')D$.

By appending the four-way switching with vertex relabeling and taking the converse, the following is a natural notion of equivalence, which is motivated from Wissing and van Dam [20].

Definition 3. Two mixed graphs are said to be *switching isomorphic* if one may be obtained from the other by a sequence of four-way switchings and vertex permutations, possibly followed by taking the converse.

Clearly, two mixed graphs are cospectral if they are switching isomorphic.

A *mixed cycle* is a mixed graph whose underlying graph is a cycle. A mixed cycle is *even* (resp. *odd*) if its order is even (resp. odd). Let $\widetilde{C}$ be a mixed cycle with $\Gamma(\widetilde{C}) = v_1v_2v_3\cdots v_{l-1}v_lv_1$. Then the weight of $\widetilde{C}$ in a direction is defined by

$$wt(\widetilde{C}) = h_{12}h_{23}\cdots h_{(l-1)l}h_{l1},$$

where h_{jk} is the (v_j, v_k)-entry of $H(\widetilde{C})$. Note that $h_{jk} \in \{1, \pm\mathbf{i}\}$ if $v_j \sim v_k$, so we have $wt(\widetilde{C}) \in \{\pm 1, \pm\mathbf{i}\}$. Note that if, for one direction, the weight of a mixed cycle is α, then for the reversed direction its weight is $\bar{\alpha}$, the conjugate of α. For a mixed cycle $\widetilde{C}$, it is *positive* (resp. *negative*) if $wt(\widetilde{C}) = 1$ (resp. -1); it is *imaginary* if $wt(\widetilde{C}) \in \{\pm\mathbf{i}\}$. Further, we call a mixed cycle *real* if it is positive or negative.

Lemma 2.1 *([10]). If M is a mixed graph of order n without real mixed odd cycles, then its spectrum is symmetric about zero.*

Now we give an equivalent condition for switching isomorphism, which may be used to determine quickly whether or not a given pair of mixed graphs is switching isomorphic.

Lemma 2.2 *([18]). Let M and M' be two connected mixed graphs with the same underlying graph G. Then M and M' are switching isomorphic if and only if there is a mixed graph M'' isomorphic to M with $wt(M''[C]) = wt(M'[C])$ for every cycle C in G.*

By Lemma 2.2, the following lemma is clear, see also [7].

Lemma 2.3 *([7]). Let F be a forest. Then every mixed graph whose underlying graph is isomorphic to F is switching isomorphic to F.*

A *signed graph* $\dot{G} = (G, \sigma)$ is a pair (G, σ), where G is a simple graph (called the underlying graph of $\dot{G}$), and $\sigma : E(G) \to \{+1, -1\}$ is a sign function. The adjacency matrix of $\dot{G}$ is defined as a $|V(G)| \times |V(G)|$ matrix $A(\dot{G}) = (a_{ij})$ with $a_{ij} = \sigma(v_i v_j)$ if $v_i v_j \in E(G)$, and $a_{ij} = 0$ otherwise. For a signed graph $\dot{G} = (G, \sigma)$, the sign of a signed cycle $\dot{C}$ in $\dot{G}$ is $\sigma(\dot{C}) = \prod_{e \in \dot{C}} \sigma(e)$. The signed cycle $\dot{C}$ is *positive* (resp. *negative*) if $\sigma(\dot{C}) = 1$ (resp. -1).

Let M be a mixed graph with $\Gamma(M) = G$. We say M is *cycle-isomorphic* to a signed graph $\dot{G} = (G, \sigma)$, if there is a mixed graph M' isomorphic to M with $wt(M'[C]) = \sigma(\dot{C})$ for every cycle C in G. By comparing the coefficients of the characteristic polynomial of the adjacency matrix for a signed graph (see [2, Theorem 2.1]) and those of the characteristic polynomial of the Hermitian adjacency matrix for a mixed graph (see [10, Theorem 2.8]), it follows that if a mixed graph M is cycle-isomorphic to a signed graph $\dot{G}$, then $H(M)$ and $A(\dot{G})$ have the same characteristic polynomial, and so they are cospectral.

Remark 1. We note that by [12, Theorem 2.2] it can be shown also that if a mixed graph M is cycle-isomorphic to a signed graph $\dot{G}$, then it is also switching isomorphic to $\dot{G}$ in the sense that there is a diagonal $(\pm 1, \pm \mathbf{i})$-matrix D such that $H(M') = D^{-1} A(\dot{G}) D$, where M' is isomorphic to M.

Let G be a connected graph with a fixed spanning tree T. Then for each edge $e \in E(G) \backslash E(T)$, adding the edge e to T creates a unique cycle in $T \cup \{e\}$. This cycle is called a *fundamental cycle* of G associated with T. Let M' (resp. $\dot{G}$) be a mixed graph (resp. signed graph) with underlying graph G. For a cycle C in G, we denote $\dot{C}$ the signed cycle in $\dot{G}$ whose underlying graph is C. A result of Samanta and Kannan [12, Theorem 3.1] shows $wt(M'[C]) = \sigma(\dot{C})$ for every cycle C in G if and only if $wt(M'[C]) = \sigma(\dot{C})$ for every fundamental cycle C of G associated with T.

Lemma 2.4. *Let M be a connected mixed graph with $\Gamma(M) = G$. If M contains no imaginary mixed cycle, then M is cycle-isomorphic to a signed graph $\dot{G}$.*

Proof. If G is a tree, then the result is clear. Now let G be a connected graph with at least one cycle. Take a spanning tree T of G, let $\{C^1, \ldots, C^t\}$ be the family of all fundamental cycles of G associated with T, and let e_i be the edge in $E(C^i)\backslash E(T)$ for $i = 1, \ldots, t$. Since M contains no imaginary mixed cycle, $wt(M[C^i]) \in \{\pm 1\}$. Construct a sign function σ on $E(G)$ as follows: $\sigma(e) = 1$ for all $e \in E(T)$ and $\sigma(e_i) = wt(M[C^i])$ for all $i = 1, \ldots, t$. Then the signed graph $\dot{G} = (G, \sigma)$ has the property that $\sigma(\dot{C}^i) = wt(M[C^i])$ for all $i = 1, \ldots, t$, and so $\sigma(\dot{C}) = wt(M[C])$ for every cycle C in G. □

Denote the characteristic polynomial of an n-vertex ($n \geq 1$) mixed graph M by[2]: $\Phi(M, x) = \det(xI - H(M))$. Yuan et al. [21, Corollary 2.6] presented a recurrence relation for the characteristic polynomials of mixed graphs.

Lemma 2.5 *([21]). Let M be a mixed graph, and let u be a vertex of M. Then*

$$\begin{aligned}\Phi(M, x) =& x\Phi(M - u, x) - \sum_{v \sim u} \Phi(M - u - v, x) \\ & - 2 \sum_{\widetilde{C} \in \mathscr{C}^+(u)} \Phi(M - V(\widetilde{C}), x) + 2 \sum_{\widetilde{C} \in \mathscr{C}^-(u)} \Phi(M - V(\widetilde{C}), x),\end{aligned}$$

where $\mathscr{C}^+(u)$ (resp. $\mathscr{C}^-(u)$) is the set of positive (resp. negative) mixed cycles containing u.

The following is a direct corollary of Lemma 2.5.

Corollary 2.6. *Let $\widetilde{C}$ be an imaginary mixed cycle of length k, then*

$$\Phi(\widetilde{C}, x) = x\Phi(P_{k-1}, x) - 2\Phi(P_{k-2}, x).$$

Lemma 2.7 *([13,17]). The characteristic polynomial of the path satisfies the recurrence relation*

$$\Phi(P_0, x) = 1, \ \ \Phi(P_1, x) = x, \ \ \Phi(P_n, x) = x\Phi(P_{n-1}, x) - \Phi(P_{n-2}, x) \text{ for } n \geq 2. \quad (2.1)$$

Moreover, the recurrence relation gives

$$\Phi(P_n, x) = \frac{\varphi(x)^{2n+2} - 1}{\varphi(x)^{n+2} - \varphi(x)^n}, \text{ where } \varphi(x) := \frac{x + \sqrt{x^2 - 4}}{2}. \quad (2.2)$$

[2] $\Phi(M, x) = 1$ for a mixed graph M on 0 vertices.

Lemma 2.8. *If M is a mixed graph, then $\sqrt{\Delta(M)} \leq \rho(M) \leq \Delta(M)$.*

Proof. The upper bound for $\rho(M)$ follows by [8, Theorem 5.1]. To obtain the lower bound, let $V(M) = \{v_1, \ldots, v_n\}$ with $d_M(v_1) = \Delta(M)$, and let $\mathbf{e}$ be the vector of length n with $\mathbf{e}_1 = 1$ and $\mathbf{e}_i = 0$ for $2 \leq i \leq n$. Then $\mathbf{e}^T H^2(M)\mathbf{e} = d_M(v_1) = \Delta(M)$. On the other hand, by the min–max formula (see [8, Equation (1)]), $\rho^2(M) \geq \lambda_1^2(M) \geq \mathbf{e}^T H^2(M)\mathbf{e}$. Thus $\rho(M) \geq \sqrt{\Delta(M)}$. □

Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-t}$ (where $t \geq 1$) are two sequences of real numbers. We say that the sequence $\mu_i\,(1 \leq i \leq n-t)$ *interlaces* the sequence $\lambda_j\,(1 \leq j \leq n)$ if for every $s = 1, \ldots, n-t$, we have $\lambda_s \geq \mu_s \geq \lambda_{s+t}$.

Lemma 2.9 *([8]). Let M be a mixed graph, and let M' be an induced subgraph of M. Then the eigenvalues of M' interlace the eigenvalues of M.*

Lemma 2.10. *Let $(M_k)_{k\in\mathbb{N}}$ be a sequence of connected mixed graphs such that*

$$\rho(M_i) \neq \rho(M_j) \text{ whenever } i \neq j, \text{ and } \lim_{k\to\infty} \rho(M_k) = \gamma$$

for some constant γ. Then $\lim_{k\to\infty} \operatorname{diam}(\Gamma(M_k)) = +\infty$.

Proof. By Lemma 2.8, $\sqrt{\Delta(M_k)} \leq \rho(M_k)$ for each $k \in \mathbb{N}$. By this inequality and the convergence of $(\rho(M_k))_{k\in\mathbb{N}}$, we know that there is a constant c such that $\sqrt{\Delta(M_k)} \leq c$ for each $k \in \mathbb{N}$, and so $\Delta(\Gamma(M_k)) = \Delta(M_k) \leq c^2$. Since $M_k\,(k \in \mathbb{N})$ are pairwise distinct, $\lim_{k\to\infty} |V(\Gamma(M_k))| = +\infty$. Now by the well-known inequality

$$|V(\Gamma(M_k))| \leq \Delta(\Gamma(M_k))^{\operatorname{diam}(\Gamma(M_k))} + 1 \leq c^{2\operatorname{diam}(\Gamma(M_k))} + 1,$$

one has $\lim_{k\to\infty} \operatorname{diam}(\Gamma(M_k)) = +\infty$. □

Finally, we extend Hoffman's classical tool to compute the limit of the spectral radius of a sequence of graphs obtained from a graph G by attaching a path of strictly increasing length to a fixed $v \in V(G)$, see [9, Lemma 3.4]. By the same discussion as the proof of [9, Lemma 3.4], one may obtain the following result.

Lemma 2.11. *Let v be a vertex of a mixed graph M, and let M_n be the mixed graph obtained by attaching the path P_{n+1} to v. If $\lim_{n\to\infty} \rho(M_n) = \lim_{n\to\infty} \lambda_1(M_n) > 2$, then $\lim_{n\to\infty} \rho(M_n)$ is the largest positive root of the function*

$$\varphi(x)\Phi(M, x) - \Phi(M - v, x), \text{ where } \varphi(x) = \frac{x + \sqrt{x^2 - 4}}{2}.$$

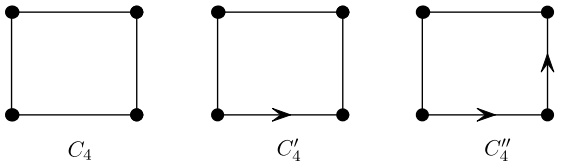

Fig. 2. Mixed graphs C_4, C_4' and C_4''.

3. Mixed graphs without negative 4-cycle whose spectral radius is at most $\sqrt{2+\sqrt{5}}$

In this section, we are to characterize all mixed graphs (up to switching isomorphism) without negative 4-cycle whose spectral radius is at most $\rho^* := \sqrt{2+\sqrt{5}} \approx 2.05817$. For convenience, if a mixed graph M' is switching isomorphic to an induced subgraph of M, then we say M' is an induced subgraph of M, or that M contains M'; if a mixed graph M' is not a subgraph of M, then we say M is M'*-free*.

In order to present our main result, we need the definitions for some classes of mixed graphs. Let $T_{a,b,c}$ be the tree of order $a+b+c+1$ consisting of three paths of orders $a+1$, $b+1$ and $c+1$, respectively, where these paths have one end vertex in common. Let $Q_{a,b,c}$ be the tree of order $a+b+c+3$ consisting of a path $v_1v_2\cdots v_{a+b+c+1}$ with two extra edges affixed at v_{a+1} and v_{a+b+1}, respectively.

We denote by C_n, C_n' and C_n'' the n-vertex mixed cycles having no arc, just one arc and just two consecutive arcs with the same direction, respectively; mixed graphs C_4, C_4' and C_4'' are depicted in Fig. 2. Then by Lemma 2.2, all positive, imaginary and negative mixed cycles of order n are switching isomorphic to C_n, C_n' and C_n'', respectively.

Given two positive integers $n > k \geq 3$, denote by $C_{k,n}$, $C_{k,n}'$ and $C_{k,n}''$ the mixed graphs obtained by affixing a path of order $n-k+1$ at a vertex in C_k, C_k' and C_k'', respectively.

Let C_n, C_n' and C_n'' be mixed cycles with underlying graph $v_1v_2\cdots v_nv_1$, then for $2 \leq k \leq n$, the mixed graphs $G_{n,k}$, $G_{n,k}'$ and $G_{n,k}''$ are obtained from C_n, C_n' and C_n'', respectively, by affixing an edge at v_1 and an edge at v_k; for $1 \leq k \leq n-7$, the mixed graph $U_{k,n-6-k}''$ is obtained from C_6'' by affixing a path of order $k+1$ at v_1 and a path of order $n-5-k$ at v_4; the mixed graph U_6'' is obtained from C_6'' by affixing a path of order 3 at v_1, an edge at v_3 and an edge at v_5; the mixed graph U_8'' is obtained from C_8'' by affixing a path of order 3 at v_1 and a path of order 3 at v_5.

Given positive integers $c \geq a$, define

$$b^*(a,c) = \begin{cases} a+c+2, & \text{for } a \geq 3; \\ c+3, & \text{for } a = 2; \\ c, & \text{for } a = 1. \end{cases}$$

Then the main result of this section is:

Theorem 3.1. *Let M be a connected C_4''-free mixed graph of order n. Then $\rho(M) \leq \sqrt{2+\sqrt{5}}$ if and only if M is an induced subgraph of one of the following mixed graphs:*

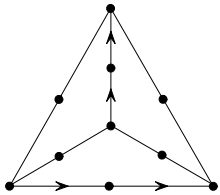

Fig. 3. Mixed graph M^* together with its spectral radius.

(i) $K_{1,4}$, $Q_{a,b,c}$ *for* $(a,b,c) \in \{(1,1,2),(2,4,2),(2,5,3),(3,7,3),(3,8,4)\}$*; or* $c \geq a > 0$ *and* $b \geq b^*(a,c)$*;*
(ii) C_n *for* $n \geq 3$*;*
(iii) C''_n *for odd* $n \geq 3$, $G''_{n-2,\frac{n}{2}}$ *for even* $n \geq 10$, $U''_{k,n-6-k}$ *for* $1 \leq k \leq n-7$, $G''_{8,4}$, $G''_{10,5}$, U''_6, U''_8*;*
(iv) $C'_{n-1,n}$ *for* $n \geq 6$, $C'_{3,n}$ *for* $n \geq 4$, $C'_{4,6}$.
(v) M^**, where* M^* *is depicted in Fig. 3.*

In order to prove this result, we will distinguish the mixed graphs according to the number of cycles.

3.1. Mixed trees

Smith [15] characterized all simple graphs whose spectral radius does not exceed 2; Brouwer, Neumaier [3] and Cvetković, Doob, Gutman [4] determined all simple graphs with spectral radius between 2 and ρ^*.

Proposition 3.2 *([3,4,15]). Let* G *be a connected simple graph of order* n*. Then* $\rho(G) \leq \sqrt{2+\sqrt{5}}$ *if and only if* G *is isomorphic to one of the graphs in* $\{C_n, K_{1,4}, P_n\} \cup \{T_{a,b,c} | a = 1, b \geq c \geq 1$*; or* $a = b = 2, c \geq 2$*; or* $a = 2, b = c = 3\} \cup \{Q_{a,b,c} | (a,b,c) \in \{(1,1,2),(2,4,2),(2,5,3),(3,7,3),(3,8,4)\}$*; or* $c \geq a > 0$, $b \geq b^*(a,c)\}$.

It follows from Lemma 2.3 that Proposition 3.2 provides all mixed trees (up to switching isomorphism) whose spectral radius is at most ρ^*. Each of these is an induced subgraph of one of the mixed graphs in (i) and (iii) of Theorem 3.1.

3.2. Unicyclic mixed graphs

A mixed graph is called *unicyclic* if it is connected and contains exactly one mixed cycle.

Lemma 3.3. *Let* $n \geq 4$*. Then* $\rho(C'_{n-1,n}) < \sqrt{2+\sqrt{5}}$.

Proof. Let u be the vertex of degree 1 in $C'_{n-1,n}$, and let v be the neighbor of u. By Lemma 2.5 and Corollary 2.6, one has

$$\begin{aligned}\Phi(C'_{n-1,n},x) &= x\Phi(C'_{n-1,n}-u,x)-\Phi(C'_{n-1,n}-u-v,x)\\ &= x[x\Phi(P_{n-2},x)-2\Phi(P_{n-3},x)]-\Phi(P_{n-2},x).\end{aligned}$$

Together with (2.2), one has

$$\begin{aligned}\Phi(C'_{n-1,n},x) &= (x^2-1)\frac{\varphi(x)^{2n-2}-1}{\varphi(x)^n-\varphi(x)^{n-2}}-2x\frac{\varphi(x)^{2n-4}-1}{\varphi(x)^{n-1}-\varphi(x)^{n-3}}\\ &= \frac{1}{\varphi(x)^{n-2}(\varphi(x)^2-1)}[(x^2-1)(\varphi(x)^{2n-2}-1)-2x\varphi(x)(\varphi(x)^{2n-4}-1)].\end{aligned} \tag{3.1}$$

Note that since $\rho^*=\sqrt{2+\sqrt{5}}$, we have that $\rho^*\varphi(\rho^*)=\frac{3+\sqrt{5}}{2}$ and $\varphi(\rho^*)^2=\frac{\sqrt{5}+1}{2}$. Hence,

$$\begin{aligned}\Phi(C'_{n-1,n},\rho^*) &= \frac{2}{(\sqrt{5}-1)\varphi(\rho^*)^{n-2}}[(1+\sqrt{5})((\frac{\sqrt{5}+1}{2})^{n-1}-1)\\ &\quad -(3+\sqrt{5})((\frac{\sqrt{5}+1}{2})^{n-2}-1)]\\ &= \frac{4}{(\sqrt{5}-1)\varphi(\rho^*)^{n-2}}\\ &> 0,\end{aligned}$$

and so $\rho^*>\lambda_1(C'_{n-1,n})$ or $\rho^*<\lambda_2(C'_{n-1,n})$. On the other hand, by Lemmas 2.8 and 2.9, $\lambda_2(C'_{n-1,n})\le\lambda_1(C'_{n-1,n}-u)<\rho^*$, and hence $\rho^*>\lambda_1(C'_{n-1,n})$. Finally, by Lemma 2.1, $\lambda_1(C'_{n-1,n})=\rho(C'_{n-1,n})$, which finishes the proof. □

Lemma 3.4. *Let $n\ge 4$. Then $\rho(C'_{3,n})\le\sqrt{2+\sqrt{5}}$.*

Proof. By Lemma 2.9, $\rho(C'_{3,n})$ is monotonically increasing, and by Lemma 2.8, $\rho(C'_{3,n})$ is bounded. Hence, $\lim_{m\to\infty}\rho(C'_{3,m})$ exists. On the other hand, by a direct computation, $\rho(C'_{3,6})=2.0285>2$, and so by Lemma 2.9, $\lim_{m\to\infty}\rho(C'_{3,m})\ge\rho(C'_{3,6})>2$. Note that $C'_{3,n}$ contains no real mixed odd cycle, by Lemma 2.1, $\lim_{m\to\infty}\rho(C'_{3,m})=\lim_{m\to\infty}\lambda_1(C'_{3,m})>2$. Then by Lemma 2.11, the value $\lim_{m\to\infty}\rho(C'_{3,m})$ is the largest positive root of

$$\varphi(x)\Phi(C'_3,x)-\Phi(P_2,x)=\frac{x+\sqrt{x^2-4}}{2}(x^3-3x)-(x^2-1),$$

which is ρ^*. By Lemma 2.9, for $n\ge 4$, $\rho(C'_{3,n})\le\lim_{m\to\infty}\rho(C'_{3,m})=\rho^*$, as desired. □

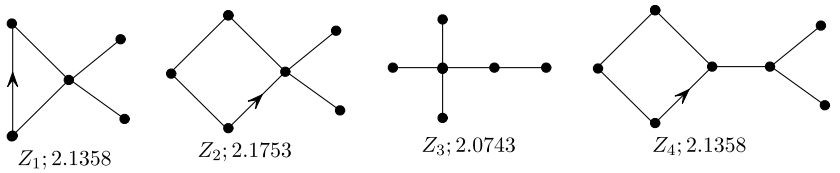

Fig. 4. Mixed graphs Z_1, Z_2, Z_3 and Z_4 together with their spectral radii.

Proposition 3.5. *Let M be a connected C_4''-free unicyclic mixed graph. Then $\rho(M) \leq \sqrt{2+\sqrt{5}}$ if and only if M is an induced subgraph of one of the following mixed graphs:*

(i) C_n *for* $n \geq 3$;
(ii) C_n'' *for odd* $n \geq 3$, $G''_{n-2,\frac{n}{2}}$ *for even* $n \geq 10$, $U''_{k,n-6-k}$ *for* $1 \leq k \leq n-7$, $G''_{8,4}$, $G''_{10,5}$, U_6'', U_8'';
(iii) $C'_{n-1,n}$ *for* $n \geq 6$, $C'_{3,n}$ *for* $n \geq 4$, $C'_{4,6}$.

Proof. We note that a direct computation shows that $\rho(C'_{4,6}) < \rho^*$. Together with Lemmas 3.3 and 3.4, this implies that all mixed graphs in (iii) have spectral radii at most ρ^*. Let M be a connected C_4''-free unicyclic mixed graph with $\rho(M) \leq \rho^*$. Let $\widetilde{C}$ be the unique mixed cycle (of length k) contained in M. If $\widetilde{C}$ is real, then by Lemma 2.4, M is cycle-isomorphic to a signed graph. A result of Wang et al. [16, Lemma 3.7] implies that $\rho(M) \leq \rho^*$ if and only if M is an induced subgraph of one of the mixed graphs in (i) and (ii).

In the following, we will consider the remaining case that $\widetilde{C}$ is imaginary. In this case, by Lemma 2.1, the spectrum of M is symmetric about 0, and so $\rho(M) = \lambda_1(M)$.

Claim 1. *For all $v \in V(\widetilde{C})$, one has $d_M(v) \leq 3$.*

Proof. Suppose there is a vertex $v \in V(\widetilde{C})$ such that $d_M(v) \geq 4$. Depending on whether $k = 3$, $k = 4$, or $k \geq 5$, it follows that Z_1, Z_2, or Z_3, respectively, is an induced subgraph of M, where Z_1, Z_2 and Z_3 are depicted in Fig. 4. Direct computations give $\rho(Z_1) = 2.1358 > \rho^*$, $\rho(Z_2) = 2.1753 > \rho^*$, $\rho(Z_3) = 2.0743 > \rho^*$. Thus, by Lemma 2.9, $\rho(M) > \rho^*$, a contradiction. □

Claim 2. *In M there is at most one vertex in $V(\widetilde{C})$ with degree 3.*

Proof. Suppose there are two vertices (say v_1, v_i) in $V(\widetilde{C})$ with degree 3. Then M contains M', an imaginary mixed k-cycle with 2 pendent edges at v_1 and v_i. Let u_1, u_2 be the pendent vertices in M' with $u_1 \sim v_1$ and $u_2 \sim v_i$, respectively. As M is unicyclic, M' is an induced subgraph of M.

By Lemma 2.5 and Corollary 2.6,

$$\Phi(M', x) = x\Phi(M' - u_1, x) - \Phi(M' - u_1 - v_1, x)$$

$$\begin{aligned}
&= x[x\Phi(M'-u_1-u_2,x)-\Phi(M'-u_1-u_2-v_i,x)]\\
&\quad -[x\Phi(M'-u_1-v_1-u_2,x)-\Phi(M'-u_1-v_1-u_2-v_i,x)]\\
&= x(x^2-2)\Phi(P_{k-1},x)-2x^2\Phi(P_{k-2},x)+\Phi(P_{i-2},x)\cdot\Phi(P_{k-i},x). \qquad (3.2)
\end{aligned}$$

Together with (2.2) and (3.2), one has

$$\begin{aligned}
\Phi(M',x) = x(x^2-2)\cdot\frac{\varphi(x)^{2k}-1}{\varphi(x)^{k+1}-\varphi(x)^{k-1}} - 2x^2\cdot\frac{\varphi(x)^{2k-2}-1}{\varphi(x)^{k}-\varphi(x)^{k-2}}\\
+\frac{\varphi(x)^{2i-2}-1}{\varphi(x)^{i}-\varphi(x)^{i-2}}\cdot\frac{\varphi(x)^{2(k-i)+2}-1}{\varphi(x)^{k-i+2}-\varphi(x)^{k-i}}.
\end{aligned}$$

Then

$$\begin{aligned}
\Phi(M',\rho^*) = \frac{2}{(3-\sqrt{5})\varphi(\rho^*)^k}\cdot[\frac{5+3\sqrt{5}}{2}((\frac{\sqrt{5}+1}{2})^{k+1}-(\frac{\sqrt{5}+1}{2})^{k}-\frac{\sqrt{5}-1}{2})\\
-(4+2\sqrt{5})((\frac{\sqrt{5}+1}{2})^{k+1}-(\frac{\sqrt{5}+1}{2})^{k}-1)\\
+(\frac{\sqrt{5}+1}{2})^{k+1}-(\frac{\sqrt{5}+1}{2})^{i}-(\frac{\sqrt{5}+1}{2})^{k-i+2}+\frac{\sqrt{5}+1}{2}]. \qquad (3.3)
\end{aligned}$$

Note that

$$(\frac{\sqrt{5}+1}{2})^{i}+(\frac{\sqrt{5}+1}{2})^{k-i+2}\geq 2(\frac{\sqrt{5}+1}{2})^{\frac{k+2}{2}}.$$

Using this and (3.3) gives

$$\Phi(M',\rho^*)\leq\frac{-2(\sqrt{5}+1)}{(3-\sqrt{5})\varphi(\rho^*)^k}\cdot[(\frac{\sqrt{5}+1}{2})^{\frac{k}{2}}-2]<0$$

for $k\geq 3$. Therefore, $\rho^*<\lambda_1(M')=\rho(M')\leq\rho(M)$, a contradiction. □

From Claims 1 and 2, we know that there is at most one vertex in $V(M)\backslash V(\widetilde{C})$ adjacent to some vertex in $V(\widetilde{C})$. If $V(M)\backslash V(\widetilde{C})=\emptyset$, then M is switching isomorphic to C'_k.

If $|V(M)\backslash V(\widetilde{C})|=1$, then M is switching isomorphic to $C'_{k,k+1}$.

If $|V(M)\backslash V(\widetilde{C})|=2$, then M is switching isomorphic to $C'_{k,k+2}$. By Lemma 2.5,

$$\Phi(M,x)=(x^2-1)\Phi(\widetilde{C},x)-x\Phi(P_{k-1},x).$$

By using Corollary 2.6 and (2.2), one has

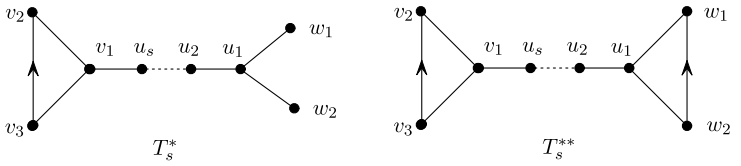

Fig. 5. Mixed graphs T_s^* and T_s^{**}.

$$\begin{aligned}\Phi(M,x) &= x(x^2-2)\Phi(P_{k-1},x) - 2(x^2-1)\Phi(P_{k-2},x)\\ &= x(x^2-2)\frac{\varphi(x)^{2k}-1}{\varphi(x)^{k+1}-\varphi(x)^{k-1}} - 2(x^2-1)\frac{\varphi(x)^{2k-2}-1}{\varphi(x)^k-\varphi(x)^{k-2}}.\end{aligned}$$

Then

$$\Phi(M,\rho^*) = \frac{2}{(\sqrt{5}-1)\varphi(\rho^*)^k}[-(\frac{\sqrt{5}+1}{2})^{k-1} + \frac{\sqrt{5}+7}{2}].$$

Hence for $k \geq 5$, $\Phi(M,\rho^*) < 0$, and so $\rho^* < \lambda_1(M) = \rho(M)$, a contradiction.

Because $\rho(C'_{k,k+2}) > \rho^*$ for $k \geq 5$, we only need to consider the case $k < 5$ when $|V(M)\backslash V(\widetilde{C})| \geq 3$. For $k = 4$, consider the mixed graphs $C'_{4,7}$ and Z_4 (see Fig. 4). By a direct computation, $\rho(C'_{4,7}) = 2.0743 > \rho^*$ and $\rho(Z_4) = 2.1358 > \rho^*$, a contradiction.

For $k = 3$, if M is not switching isomorphic to $C'_{3,n}$ for some $n \geq 6$, then M contains T_s^* for some $s \geq 1$, see Fig. 5. By Lemma 2.5,

$$\begin{aligned}\Phi(T_s^*,x) &= x\Phi(T_s^*-v_2,x) - \Phi(T_s^*-v_2-v_1,x) - \Phi(T_s^*-v_2-v_3,x)\\ &= x[x\Phi(T_s^*-v_2-w_1,x) - \Phi(T_s^*-v_2-w_1-u_1,x)]\\ &\quad - [x\Phi(T_s^*-v_2-v_1-w_1,x) - \Phi(T_s^*-v_2-v_1-w_1-u_1,x)]\\ &\quad - [x\Phi(T_s^*-v_2-v_3-w_1,x) - \Phi(T_s^*-v_2-v_3-w_1-u_1,x)]\\ &= x^2\Phi(P_{s+3},x) - x\Phi(P_{s+2},x) - 2x^2\Phi(P_{s+1},x) + x\Phi(P_s,x) + x^2\Phi(P_{s-1},x).\end{aligned}$$

Using (2.2), one has

$$\begin{aligned}\Phi(T_s^*,x) = x^2\frac{\varphi(x)^{2s+8}-1}{\varphi(x)^{s+5}-\varphi(x)^{s+3}} - x\frac{\varphi(x)^{2s+6}-1}{\varphi(x)^{s+4}-\varphi(x)^{s+2}} - 2x^2\frac{\varphi(x)^{2s+4}-1}{\varphi(x)^{s+3}-\varphi(x)^{s+1}}\\ + x\frac{\varphi(x)^{2s+2}-1}{\varphi(x)^{s+2}-\varphi(x)^s} + x^2\frac{\varphi(x)^{2s}-1}{\varphi(x)^{s+1}-\varphi(x)^{s-1}}.\end{aligned}$$

Then it follows that

$$\Phi(T_s^*,\rho^*) = \frac{-(1+\sqrt{5})^2}{2\varphi(\rho^*)^{s+3}} < 0.$$

Thus, $\rho^* < \lambda_1(T_s^*) \leq \lambda_1(M) = \rho(M)$, a contradiction. Therefore, M is an induced subgraph of $C'_{3,n}$ for $n \geq 6$, which completes the proof. □

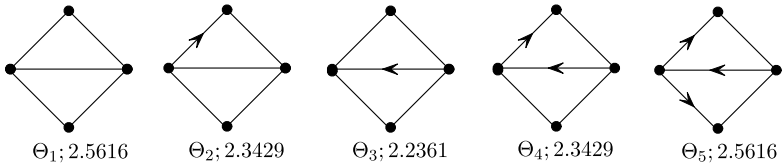

Fig. 6. Mixed graphs $\Theta_1, \ldots, \Theta_5$ together with their spectral radii.

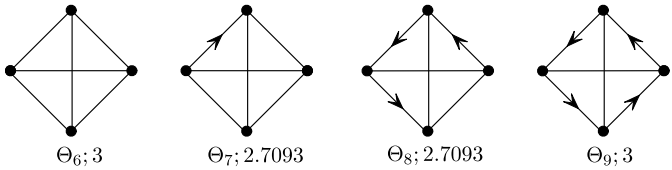

Fig. 7. Mixed graphs $\Theta_6, \ldots, \Theta_9$ together with their spectral radii.

3.3. Mixed graphs with at least two mixed cycles

Proposition 3.6. *If M is a connected C_4''-free mixed graph with at least two mixed cycles, then $\rho(M) > \sqrt{2+\sqrt{5}}$, unless M is an induced subgraph of M^*, see Fig. 3.*

Proof. Let M be a connected C_4''-free mixed graph with at least two mixed cycles. By a direct computation, $\rho(M^*) = 2 < \rho^*$. We only need to show that M contains a mixed graph M' with $\rho(M') > \rho^*$ when M is not an induced subgraph of M^*. We consider 6 cases depending on the girth of M.

Girth 3. Let $\widetilde{C}_3$ be a mixed 3-cycle contained in M. If $\widetilde{C}_3$ is real and there is a vertex in $V(M)\backslash V(\widetilde{C}_3)$ adjacent to exactly one vertex in $V(\widetilde{C}_3)$, then M contains $C_{3,4}$ or $C''_{3,4}$. By Proposition 3.5, $\rho(C_{3,4}) > \rho^*$ and $\rho(C''_{3,4}) > \rho^*$, as desired. In the following, if there is a vertex in $V(M)\backslash V(\widetilde{C}_3)$ adjacent to exactly one vertex in $V(\widetilde{C}_3)$, we only need to consider the case that $\widetilde{C}_3$ is imaginary. We now distinguish cases 3.1, 3.2 and 3.3.

3.1). Suppose there is an induced mixed k-cycle sharing one edge with $\widetilde{C}_3$.

For $k = 3$, let v be a vertex in $V(M)\backslash V(\widetilde{C}_3)$ adjacent to two or three vertices in $V(\widetilde{C}_3)$. Let $\widetilde{C}_3$ be positive (resp. negative, imaginary) and fix it. Since M is C_4''-free, consider all the possibilities of the edges between v and $V(\widetilde{C}_3)$ such that $M[V(\widetilde{C}_3) \cup \{v\}]$ is C_4''-free, count the numbers of positive (resp. negative, imaginary) triangles and positive (resp. imaginary) quadrangles contained in $M[V(\widetilde{C}_3) \cup \{v\}]$. By using Lemma 2.2, it follows that $M[V(\widetilde{C}_3) \cup \{v\}]$ is switching isomorphic to one of Θ_i, $i = 1, 2, \ldots, 9$, see Fig. 6 and Fig. 7. By direct computations, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

For $k = 4$, we may assume there is no mixed 3-cycle sharing one edge with $\widetilde{C}_3$. Since M is C_4''-free, M contains one of Θ_{10}, Θ_{11} and Θ_{12}, see Fig. 8. By direct computations, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

For $k \geq 5$, we may assume there is no mixed 3-cycle and no mixed 4-cycle sharing one edge with $\widetilde{C}_3$. Then M contains $G'_{3,2}$. By Proposition 3.5, $\rho(G'_{3,2}) > \rho^*$, as desired.

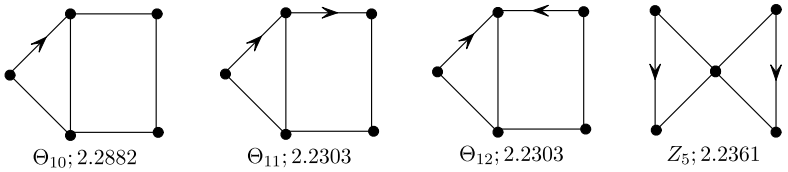

Fig. 8. Mixed graphs $\Theta_{10}, \Theta_{11}, \Theta_{12}$ and Z_5 together with their spectral radii.

3.2). Suppose there is no induced mixed cycle sharing an edge with $\widetilde{C}_3$, but an induced mixed k-cycle sharing one vertex with $\widetilde{C}_3$.

For $k = 3$, M contains Z_5, see Fig. 8. By a direct computation, $\rho(Z_5) = 2.2361 > \rho^*$, as desired. For $k \geq 4$, M contains Z_1, see Fig. 4. By Proposition 3.5, $\rho(Z_1) > \rho^*$, as desired.

3.3). Suppose there is no induced mixed cycle sharing a vertex with $\widetilde{C}_3$.

Now M contains T_s^* or T_s^{**}, see Fig. 5. By Proposition 3.5, $\rho(T_s^*) > \rho^*$, as desired. For T_s^{**}, by Lemma 2.5, one has

$$\begin{aligned}
\Phi(T_s^{**}, x) &= x\Phi(T_s^{**} - w_1, x) - \Phi(T_s^{**} - w_1 - u_1, x) - \Phi(T_s^{**} - w_1 - w_2, x) \\
&= x[x\Phi(T_s^{**} - w_1 - v_2, x) - \Phi(T_s^{**} - w_1 - v_2 - v_1, x) \\
&\quad - \Phi(T_s^{**} - w_1 - v_2 - v_3, x)] \\
&\quad - [x\Phi(T_s^{**} - w_1 - u_1 - v_2, x) - \Phi(T_s^{**} - w_1 - u_1 - v_2 - v_1, x) \\
&\quad - \Phi(T_s^{**} - w_1 - u_1 - v_2 - v_3, x)] - [x\Phi(T_s^{**} - w_1 - w_2 - v_2, x) \\
&\quad - \Phi(T_s^{**} - w_1 - w_2 - v_2 - v_1, x) - \Phi(T_s^{**} - w_1 - w_2 - v_2 - v_3, x)] \\
&= x^2\Phi(P_{s+3}, x) \\
&\quad - 2x\Phi(P_{s+2}, x) - (2x^2 - 1)\Phi(P_{s+1}, x) + 2x\Phi(P_s, x) + x^2\Phi(P_{s-1}, x).
\end{aligned}$$

Together with (2.2),

$$\begin{aligned}
\Phi(T_s^{**}, x) = x^2 \cdot \frac{\varphi(x)^{2s+8} - 1}{\varphi(x)^{s+5} - \varphi(x)^{s+3}} - 2x \cdot \frac{\varphi(x)^{2s+6} - 1}{\varphi(x)^{s+4} - \varphi(x)^{s+2}} \\
- (2x^2 - 1) \cdot \frac{\varphi(x)^{2s+4} - 1}{\varphi(x)^{s+3} - \varphi(x)^{s+1}} \\
+ 2x \cdot \frac{\varphi(x)^{2s+2} - 1}{\varphi(x)^{s+2} - \varphi(x)^s} + x^2 \cdot \frac{\varphi(x)^{2s} - 1}{\varphi(x)^{s+1} - \varphi(x)^{s-1}}.
\end{aligned}$$

Now it follows that

$$\Phi(T_s^{**}, \rho^*) = \frac{-(\sqrt{5}+1)^2}{\varphi(\rho^*)^{s+3}} < 0.$$

Thus, $\rho^* < \lambda_1(T_s^{**}) \leq \rho(T_s^{**})$, as desired.

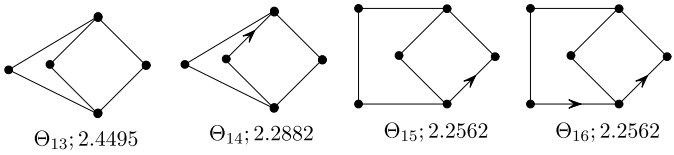

Fig. 9. Mixed graphs $\Theta_{13}, \ldots, \Theta_{16}$ together with their spectral radii.

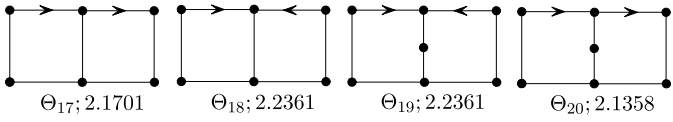

Fig. 10. Mixed graphs $\Theta_{17}, \ldots, \Theta_{20}$ together with their spectral radii.

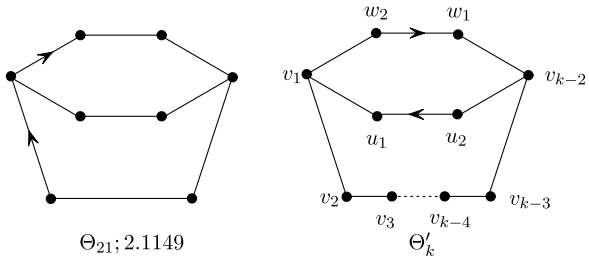

Fig. 11. Mixed graphs Θ_{21} and Θ'_k.

Girth 4. Let $\widetilde{C}_4$ be a mixed 4-cycle contained in M. Since M is C''_4-free, $\widetilde{C}_4$ is positive or imaginary.

If $\widetilde{C}_4$ is positive, then M contains $C_{4,5}$, Θ_{13}, or Θ_{14}, where Θ_{13} and Θ_{14} are depicted in Fig. 9. By Proposition 3.5 and direct computations, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

If $\widetilde{C}_4$ is imaginary, then M contains $C'_{4,7}$, $G'_{4,2}$, $G'_{4,3}$, Z_2, Z_4, Θ_{14}, Θ_{15}, Θ_{16}, Θ_{17}, or Θ_{18}, where Z_2 and Z_4 are depicted in Fig. 4, Θ_{14}, Θ_{15}, Θ_{16} are depicted in Fig. 9, Θ_{17} and Θ_{18} are depicted in Fig. 10. By Proposition 3.5 and direct computations, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

Girth 5. Let $\widetilde{C}_5$ be a mixed 5-cycle contained in M. Depending on whether $\widetilde{C}_5$ shares two adjacent edges with another induced mixed k-cycle, M contains one of $C_{5,6}$, $C''_{5,6}$, Θ_{19}, Θ_{20} $(k = 5)$ and $G_{5,3}$, $G'_{5,3}$, $G''_{5,3}$ $(k \geq 6)$, where Θ_{19} and Θ_{20} are depicted in Fig. 10, or one of $C_{5,6}$, $C''_{5,6}$ and $C'_{5,7}$. By Proposition 3.5 and direct computations, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

Girth 6. Let $\widetilde{C}_6$ be a mixed 6-cycle contained in M. We now distinguish cases 6.1, 6.2 and 6.3.

6.1). Suppose there is an induced mixed k-cycle sharing a path of length 3 with $\widetilde{C}_6$.

For $k = 6$, M contains one of $C_{6,7}$ and Θ_{21}, where Θ_{21} is as depicted in Fig. 11. By Proposition 3.5 and a direct computation, both $C_{6,7}$ and Θ_{21} have spectral radii larger than ρ^*, as desired.

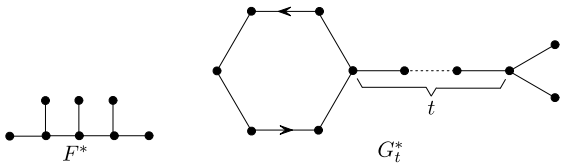

Fig. 12. Mixed graphs F^* and G_t^*.

For $k \geq 7$, M contains one of $C_{6,7}$, $C'_{6,8}$, $C_{k,k+1}$ and Θ'_k, where Θ'_k is as depicted in Fig. 11. By Proposition 3.5, all of $C_{6,7}$, $C'_{6,8}$ and $C_{k,k+1}$ have spectral radii larger than ρ^*. For the mixed graph Θ'_k, by Lemma 2.5, one has

$$\begin{aligned}
\Phi(\Theta'_k, x) &= x\Phi(\Theta'_k - u_1, x) - \Phi(\Theta'_k - u_1 - v_1, x) \\
&\quad - \Phi(\Theta'_k - u_1 - u_2, x) + 2\Phi(\Theta'_k - V(\widetilde{C}_6), x) \\
&= x[x\Phi(\Theta'_k - u_1 - w_1, x) - \Phi(\Theta'_k - u_1 - w_1 - w_2, x) \\
&\quad - \Phi(\Theta'_k - u_1 - w_1 - v_{k-2}, x)] \\
&\quad - [x\Phi(\Theta'_k - u_1 - v_1 - u_2, x) - \Phi(\Theta'_k - u_1 - v_1 - u_2 - v_{k-2}, x)] \\
&\quad - \Phi(\Theta'_k - u_1 - u_2, x) + 2\Phi(\Theta'_k - V(\widetilde{C}_6), x) \\
&= x^2\Phi(P_k, x) - 2x\Phi(P_{k-1}, x) - x^2\Phi(P_{k-2}, x) \\
&\quad + (x^2+1)\Phi(P_{k-4}, x) - \Phi(\widetilde{C}'_k, x). \qquad (3.4)
\end{aligned}$$

By Corollary 2.6, $\Phi(\widetilde{C}'_k, x) = x\Phi(P_{k-1}, x) - 2\Phi(P_{k-2}, x)$. Together with (2.2) and (3.4), one has

$$\begin{aligned}
\Phi(\Theta'_k, x) &= x^2 \cdot \frac{\varphi(x)^{2k+2} - 1}{\varphi(x)^{k+2} - \varphi(x)^k} - 3x \cdot \frac{\varphi(x)^{2k} - 1}{\varphi(x)^{k+1} - \varphi(x)^{k-1}} \\
&\quad - (x^2 - 2)\frac{\varphi(x)^{2k-2} - 1}{\varphi(x)^k - \varphi(x)^{k-2}} + (x^2+1)\frac{\varphi(x)^{2k-6} - 1}{\varphi(x)^{k-2} - \varphi(x)^{k-4}}.
\end{aligned}$$

Now it follows that

$$\Phi(\Theta'_k, \rho^*) = \frac{-(1+\sqrt{5})^2}{\varphi(\rho^*)^k} < 0.$$

Thus, $\rho^* < \lambda_1(\Theta'_k) \leq \rho(\Theta'_k)$, as desired.

6.2). Suppose there is no induced mixed cycle sharing a path of length 3 with $\widetilde{C}_6$, but an induced mixed k-cycle sharing two adjacent edges with $\widetilde{C}_6$.

For $k = 6$, if M is not an induced subgraph of M^*, then M contains one of $C_{6,7}$, $C'_{6,8}$, $G''_{6,2}$, $C_{8,9}$ and F^* (see Fig. 12). By Propositions 3.2 and 3.5, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

For $k \geq 7$, M contains one of $C_{k,k+1}$, $C'_{k,k+2}$ and $G''_{k,3}$. By Proposition 3.5, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

6.3). Suppose there is no induced mixed cycle sharing a path of length 2 or 3 with $\widetilde{C}_6$.

Now M contains one of $C_{6,7}$, $C'_{6,8}$, $G''_{6,2}$ and G^*_t (see Fig. 12) for some $t \geq 1$. By Proposition 3.5, all of these mixed graphs have spectral radii larger than ρ^*, as desired.

Girth 8. Let $\widetilde{C}_8$ be a mixed 8-cycle contained in M. If $\widetilde{C}_8$ is positive or imaginary, then M contains $C_{8,9}$ or $C'_{8,10}$. By Proposition 3.5, both $C_{8,9}$ and $C'_{8,10}$ have spectral radii larger than ρ^*, as desired. In the following, we consider the case that $\widetilde{C}_8$ is negative, and distinguish 2 cases.

8.1). Suppose there is a mixed k-cycle sharing a path of length 4 with $\widetilde{C}_8$.

For $k = 8$, M contains 3 mixed 8-cycles. It is not possible that all of these mixed 8-cycles are negative. Hence, M contains $C_{8,9}$ or $C'_{8,10}$. By Proposition 3.5, both $C_{8,9}$ and $C'_{8,10}$ have spectral radii larger than ρ^*, as desired. For $k \geq 9$, M contains $C''_{8,11}$. By Proposition 3.5, $\rho(C''_{8,11}) > \rho^*$, as desired.

8.2). Suppose there is no mixed cycle sharing a path of length 4 with $\widetilde{C}_8$.

Now M contains $C''_{8,11}$. By Proposition 3.5, $\rho(C''_{8,11}) > \rho^*$, as desired.

Girth 7 or at least 9. M contains one of $C_{k,k+1}$, $C'_{k,k+2}$ for $k = 7$ or $k \geq 9$, $C''_{k,k+1}$ for $k = 7$ and $C''_{k,k+3}$ for $k \geq 9$. By Proposition 3.5, all of these mixed graphs have spectral radii larger than ρ^*, as desired. □

Proof of Theorem 3.1. The proof follows by Propositions 3.2, 3.5 and 3.6. □

4. Limit points for the spectral radius of the Hermitian adjacency matrix of mixed graphs

In this section, based on the results in [9,14,1] and Theorem 3.1, we are to establish all the H-limit points for mixed graphs, thereby completely solving the first part of the Hoffman program with respect to the Hermitian adjacency matrix of mixed graphs.

For every integer $k > 0$, let β_k be the largest positive root of $x^k(x^2 - x - 1) + 1$. Define

$$\eta_k = \beta_k^{\frac{1}{2}} + \beta_k^{-\frac{1}{2}}. \tag{4.1}$$

Hoffman [9] and Shearer [14] determined all A-limit points for simple graphs.

Proposition 4.1 *([9]). The numbers $2 = \eta_1 < \eta_2 < \cdots$ are precisely the limit points for the adjacency spectral radii of simple graphs smaller than $\sqrt{2+\sqrt{5}}$. Moreover, $\lim_{n\to\infty} \rho(T_{1,k,n}) = \eta_k$.*

Proposition 4.2 *([14]). Each real number $\lambda \geq \sqrt{2+\sqrt{5}}$ is a limit point for the adjacency spectral radii of a suitable sequence of trees.*

For every integer $k \geq 0$, let γ_k be the largest positive root of $x^{k+2}(x^2 - x - 1) + x^2 - 1$, and let ϑ be the largest positive root of $x^6 - 2x^5 + x^4 - x^2 + x - 1$. Define

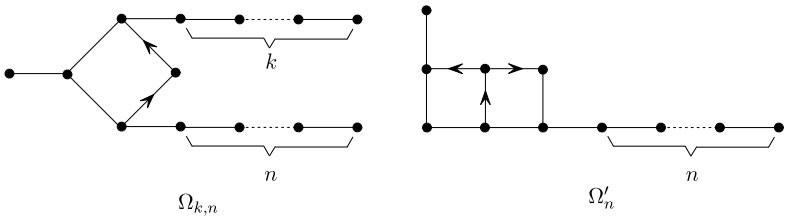

Fig. 13. Mixed graphs $\Omega_{k,n}$ and Ω'_n.

$$\zeta_k = \gamma_k^{\frac{1}{2}} + \gamma_k^{-\frac{1}{2}} \text{ and } \xi = \vartheta^{\frac{1}{2}} + \vartheta^{-\frac{1}{2}}. \tag{4.2}$$

Belardo and Brunetti [1] determined all A-limit points for signed graphs.

Proposition 4.3 *([1]). The numbers ξ and $\zeta_0 < \zeta_1 < \cdots$ are precisely the limit points for the adjacency spectral radii of signed graphs which cannot be obtained from sequences of simple graphs.*

Proposition 4.4. *For $k \geq 0$, let $\Omega_{k,n}$ and Ω'_n be the mixed graphs depicted in Fig. 13. Then*

$$\lim_{n\to\infty} \rho(\Omega_{k,n}) = \zeta_k \text{ and } \lim_{n\to\infty} \rho(\Omega'_n) = \xi.$$

Proof. The result follows from Lemma 2.4 and [1, Proposition 4.2]. □

Lemma 4.5. *It holds that*

$$\lim_{n\to\infty} \rho(C'_n) = 2 \text{ and } \lim_{n\to\infty} \rho(C'_{n-1,n}) = \lim_{n\to\infty} \rho(C'_{3,n}) = \sqrt{2+\sqrt{5}}.$$

Proof. By Lemma 2.9 and [8, Theorem 5.7], $\rho(P_{n-1}) \leq \rho(C'_n) \leq \rho(C_n)$, and so

$$2 = \lim_{n\to\infty} \rho(P_{n-1}) \leq \lim_{n\to\infty} \rho(C'_n) \leq \lim_{n\to\infty} \rho(C_n) = 2.$$

Hence, $\lim_{n\to\infty} \rho(C'_n) = 2$.

By Theorem 3.1, we have that $\rho(C'_{n-1,n}) \leq \rho^*$. On the other hand, $C'_{n-1,n}$ contains $T_{1,\lfloor\frac{n-3}{2}\rfloor,\lfloor\frac{n-3}{2}\rfloor}$ as an induced subgraph. By Lemma 2.9 and [9, Proposition 3.6], one has

$$\rho^* = \lim_{n\to\infty} \rho(T_{1,\lfloor\frac{n-3}{2}\rfloor,\lfloor\frac{n-3}{2}\rfloor}) \leq \lim_{n\to\infty} \rho(C'_{n-1,n}) \leq \rho^*,$$

which gives $\lim_{n\to\infty} \rho(C'_{n-1,n}) = \rho^*$.

Finally, in the proof of Lemma 3.4, we have shown $\lim_{n\to\infty} \rho(C'_{3,n}) = \rho^*$. □

Theorem 4.6. *A real number ζ is a limit point for the spectral radii of mixed graphs if and only if $\zeta \in \{\eta_k | k > 0\} \cup \{\xi\} \cup \{\zeta_k | k \geq 0\} \cup [\sqrt{2+\sqrt{5}}, +\infty)$, where η_k $(k > 0)$ are defined in (4.1), ξ and ζ_k $(k \geq 0)$ are defined in (4.2).*

Proof. One direction follows by Propositions 4.1, 4.2 and 4.4. To show that there are no other limit points, let ζ be a limit point for the spectral radii of mixed graphs, and let $(M_k)_{k\in\mathbb{N}}$ be a sequence of connected mixed graphs such that

$$\rho(M_i) \neq \rho(M_j) \text{ whenever } i \neq j, \text{ and } \lim_{k\to\infty} \rho(M_k) = \zeta.$$

If there is an infinite subsequence $(M_{k_i})_{i\in\mathbb{N}} \subseteq (M_k)_{k\in\mathbb{N}}$ such that each mixed graph in $(M_{k_i})_{i\in\mathbb{N}}$ contains no imaginary mixed cycle, then it follows from Lemma 2.4 and Propositions 4.1-4.3 that ζ is one of the stated limit points. In the following, assume there are finitely many mixed graphs in $(M_k)_{k\in\mathbb{N}}$ containing no imaginary mixed cycles.

If there is an infinite subsequence $(M_{k_i})_{i\in\mathbb{N}} \subseteq (M_k)_{k\in\mathbb{N}}$ such that each mixed graph in $(M_{k_i})_{i\in\mathbb{N}}$ contains an imaginary mixed triangle, then by Lemma 2.10, $\lim_{i\to\infty} \operatorname{diam}(\Gamma(M_{k_i})) = +\infty$. Then for each $s \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that M_{k_i} contains $C'_{3,s}$ as an induced subgraph, and so $\rho(M_{k_i}) \geq \rho(C'_{3,s})$ whenever $i \geq N$. Hence by Lemma 4.5,

$$\zeta = \lim_{k\to\infty} \rho(M_k) = \lim_{i\to\infty} \rho(M_{k_i}) \geq \lim_{s\to\infty} \rho(C'_{3,s}) = \sqrt{2+\sqrt{5}}.$$

If there is an infinite subsequence $(M_{k_i})_{i\in\mathbb{N}} \subseteq (M_k)_{k\in\mathbb{N}}$ such that each mixed graph in $(M_{k_i})_{i\in\mathbb{N}}$ contains an induced imaginary mixed t-cycle for some fixed $t \geq 4$. Then by Proposition 3.5, $\rho(M_{k_i}) > \sqrt{2+\sqrt{5}}$ for all M_{k_i} with $\operatorname{diam}(\Gamma(M_{k_i})) \geq \lceil \frac{t}{2} \rceil + 3$. From Lemma 2.10, it then follows that

$$\zeta = \lim_{k\to\infty} \rho(M_k) = \lim_{i\to\infty} \rho(M_{k_i}) > \sqrt{2+\sqrt{5}}.$$

If for all fixed $t \geq 3$, there are finitely many mixed graphs in $(M_k)_{k\in\mathbb{N}}$ containing an induced imaginary mixed t-cycle, then either there is an infinite subsequence $(M_{k_i})_{i\in\mathbb{N}} \subseteq (M_k)_{k\in\mathbb{N}}$ such that each mixed graph in $(M_{k_i})_{i\in\mathbb{N}}$ is an imaginary mixed cycle, or there is an infinite subsequence $(M_{k_i})_{i\in\mathbb{N}} \subseteq (M_k)_{k\in\mathbb{N}}$ such that for each $N \in \mathbb{N}$, there is a mixed graph in $(M_{k_i})_{i\in\mathbb{N}}$ containing $C'_{n-1,n}$ as an induced subgraph for some $n \geq N$. In the former case, by Lemma 4.5,

$$\zeta = \lim_{k\to\infty} \rho(M_k) = \lim_{i\to\infty} \rho(M_{k_i}) = 2.$$

In the latter case, by Lemmas 2.9 and 4.5,

$$\zeta = \lim_{k\to\infty} \rho(M_k) = \lim_{i\to\infty} \rho(M_{k_i}) \geq \lim_{n\to\infty} \rho(C'_{n-1,n}) = \sqrt{2+\sqrt{5}}.$$

Hence, if ζ is a limit point for the spectral radii of mixed graphs, then ζ must be one of the stated limit points. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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