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# On the solution set of the scalar feedback Nash algebraic Riccati equations

by

Jacob C. Engwerda

Tilburg University
Department of Econometrics
P.O. Box 90153
5000 LE Tilburg
The Netherlands

## Abstract

In this note we analyse the set of scalar algebraic Riccati equations (ARE) that play an important role in finding feedback Nash equilibria of the scalar multi-player linear-quadratic differential game. We show that in general there exists more than one solution of the (ARE) that gives rise to a Nash equilibrium. In particular we analyse the number of equilibria as a function of the state-feedback parameter.

**Keywords**: Linear quadratic games, feedback Nash equilibrium, solvability conditions, Riccati equations

## I. Introduction

During the last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Neck et al. (1995), Douven et al (1996)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with eachother. One of these scenarios is the so-called feedback Nash scenario. The scenario we will study here assumes that the involved parties act noncooperative, that is, each party is only interested in the minimization of his own cost. This bare fact, however, does not completely specify the game. The cost each party incurs depends on the decisions made by the other parties. So, there is still some freedom in specifying which decisions will be made by the parties. We will assume that the ultimate policies that will be played are such that no party can improve his outcome by altering his decision unilaterally. In literature this is known as the Nash equilibrium concept. Furthermore, it is assumed that the strategy used by the parties is either based on the complete history of the state of the game, or just the initial state and the current state of the game, or just the current state of the game. This requirement on the information structure the players have to determine their strategy leaves however, in general, still much freedom for the party in chosing his policy. Therefore, additional requirements have been formulated to restrict the class of possible policies. A requirement we make here is that the chosen policy must be strongly time consistent. That is, the parties have no reason at any future stage of the game, to deviate from the adopted policy even if there have been deviations in the past from the actions which are dictated by the original policy.

It turns out that it is possible to define a refinement of the Nash equilibrium concept which has the above stated requirements, the so-called feedback Nash equilibrium concept (see e.g. Başar and Olsder (1995) for a precise definition and survey of relevant literature). In fact, it can be shown that the class of strongly time consistent Nash equilibria coincides with the class of feedback Nash equilibria (see e.g. Weeren (1995, theorem 3.24)).

Note that, since according this scenario the participating parties can react to eachother's policies, its economic relevance is mostly larger than that of the open-loop Nash scenario. In particular the feedback Nash scenario is very popular in studying problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives, the parties are striving for, can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first analyzed by Starr and Ho in (1969) (see also Lukes (1971) for a result on uniqueness

within the class of affine memoryless strategies).
In this paper we study the infinite-planning horizon case and concentrate here on solving the with this problem associated algebraic Riccati equations. In Weeren et al. (1997) it was shown that in the two-player scalar case these equations have either one or three solutions which solve the optimization problem (see also Engwerda (1998) for a detailed study under which conditions on the system parameters these different situations occur).
In this paper we study the general n-player scalar case. We show that for any number $n$ of players there exists a positive number such that if the state-feedback parameter is smaller than this number, there exists exactly one solution for the (ARE) equations yielding a Nash equilibrium. Furthermore, there is a positive number such that whenever the state-feedback parameter exceeds this number there always exist $2^n - 1$ appropriate solutions for the (ARE) equations.
The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper. Section three analyzes the solutions of the algebraic Riccati equations. The paper ends with some concluding remarks.

## II. Problem statement

In this paper we consider the problem where $n$ parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x} = ax + \sum_{i=1}^{n} b_i u_i, \;\; x(0) = x_0. \tag{1}$$

Here $x$ is the state of the system, $u_i$ is a (control) variable player $i$ can manipulate, $x_0$ is the arbitrarily chosen initial state of the system, $a$ and $b_i$, $i = 1, .., n$ are constant system parameters, and $\dot{x}$ denotes the time derivative of $x$.
The performance criterion player $i = 1, .., n$ aims to minimize is:

$$J_i(u_1, .., u_n) := \frac{1}{2} \int_0^\infty \{x(t)^T q_i x(t) + u_i(t)^T r_{ii} u_i(t)\} dt.$$

We assume that both $q_i$ and $r_{ii}$ are positive and $b_i$ differs from zero.
In this paper we consider the existence of feedback Nash equilibria of this differential game (see Başar and Olsder (1995) for a precise definition of this equilibrium concept).
To that end we consider the following set of coupled algebraic Riccati equations (ARE):

$$(a - \sum_{j=1}^{n} k_j s_j) k_i + k_i (a - \sum_{j=1}^{n} s_j k_j) + q_i + k_i s_i k_i = 0, \;\; i = 1, .., n, \tag{2}$$

where $s_i := b_i r_{ii}^{-1} b_i$.
From Başar and Olsder (1995, proposition 6.8), we have:

Theorem 1:
Let $\bar{k_i} \geq 0$ solve the set of Riccati equations.
Then the strategies

$$u_i = -r_{ii}^{-1} b_i \bar{k_i} x \tag{3}$$

$i = 1,..,n$, provide a feedback Nash equilibrium, leading to the cost $J_i(u_1,..,u_n) := x_0 \bar{k_i} x_0$, for player $i$.
Moreover, the resulting system dynamics described by $\dot{x} = a_{cl}x$; $x(0) = x_0$, with $a_{cl} := a - \sum_{i=1}^{n} s_i \bar{k_i}$, is asymptotically stable. □

In fact, one can immediately deduce from Weeren (1995, p.96) that when the players are restricted at the outset to memoryless strategies (cf. Lukes (1971)) then existence of a positive solution to the above scalar Riccati equations is a both necessary and sufficient condition for existence of a feedback Nash equilibrium.
A natural question which arises is how many solutions the above set of algebraic Riccati equations (ARE) have. To analyze this question we introduce (for notational convenience) the variables:

$$\sigma_i := s_i q_i \text{ and } \kappa_i := s_i k_i,\ i = 1,..,n, \text{ and } \kappa_{n+1} = -a_{cl}.$$

Using this notation (2) can be rewritten as

$$\kappa_i^2 - 2\kappa_{n+1}\kappa_i + \sigma_i = 0,\ i = 1,..,n. \tag{4}$$

The above question can therefore be reformulated as under which conditions the above $n$ kwadratic equations and the equation

$$\kappa_{n+1} = -a + \sum_{j=1}^{n} \kappa_j \tag{5}$$

have a positive solution $\kappa_i,\ i = 1,..,n+1$.
In the next section we will study this problem in detail.

## III. The intersection points

To simplify the analysis we will consider in this section the case of three players, i.e. $n = 3$. From the analysis it will be clear that the obtained results can be generalized straightforwardly for the general $n$-player case. Furthermore we will assume, without loss of generality, that the $\sigma_i$'s satisfy $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Provided that $\sqrt{\sigma_1} \leq \kappa_{n+1}$ we have that the first n equations always have two positive solutions. From (4) we have that either $\kappa_i = \kappa_{n+1} + \sqrt{\kappa_{n+1}^2 - \sigma_i}$ or $\kappa_i = \kappa_{n+1} - \sqrt{\kappa_{n+1}^2 - \sigma_i}$, $i = 1,..,n$. Substitution of this into (5) shows that $\kappa_4 > 0$ must satisfy the following equation

$$2\kappa_4 \pm \sqrt{\kappa_4^2 - \sigma_1} \pm \sqrt{\kappa_4^2 - \sigma_2} \pm \sqrt{\kappa_4^2 - \sigma_3} = a. \tag{6}$$

Now, first consider the function

$$f(x) := 2x - \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2} - \sqrt{x^2 - \sigma_3}. \tag{7}$$

Straightforward differentiation of this function shows that

$$f'(x) := \frac{-\sigma_2}{\sqrt{x^2 - \sigma_2}(\sqrt{x^2 - \sigma_2} + x)} + \frac{-\sigma_3}{\sqrt{x^2 - \sigma_3}(\sqrt{x^2 - \sigma_3} + x)} + \frac{-x}{\sqrt{x^2 - \sigma_1}}. \tag{8}$$

So, $f(x)$ is a monotonically decreasing function with $\lim_{x\to\infty} f(x) = -\infty$. Therefore, we conclude that the equation $f(x) = a$ has for every $a \leq f(\sqrt{\sigma_1})$ a unique solution.
Next, consider e.g.

$$g_1(x) := 2x + \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2} - \sqrt{x^2 - \sigma_3}. \tag{9}$$

It is easily verified that $g_1(x)$ can be rewritten as

$$g_1(x) = \frac{\sigma_2}{\sqrt{x^2 - \sigma_2} + x} + \frac{\sigma_3}{\sqrt{x^2 - \sigma_3} + x} + \sqrt{x^2 - \sigma_1}. \tag{10}$$

From this it is obvious that $g_1(x)$ is always strictly positive for $x \geq \sqrt{\sigma_1}$. It will be clear that this conclusion holds also for all other $g_i(x)$, $i = 2,..,2^3 - 1$ (the $g_i(x)$ are obtained from (6) by considering all the remaining $2^3 - 2$ different sign patterns). So, combining this result with the monotonicity property of $f(x)$ we conclude

Theorem 2:
There exists always a positive number $a^*$ such that for every state feedback parameter $a \leq a^*$ the set of algebraic Riccati equations (2) has a unique (positive) solution. □

Next, we show that the functions $g_i(x)$ do not intersect if $x$ becomes large. To prove this property we need that fact that all $\sigma_i$ differ. Therefore, we assume from now on that $\sigma_1 > \sigma_2 > \sigma_3$.
This property will be usefull in the derivation of the next theorem.
Lemma 3:
There exists a constant $x_1$ such that functions the $g_i(x)$, $i = 1..,2^3 - 1$ do not intersect on the interval $(x_1, \infty)$.

Proof:
Two different situations can occur. For the first case, we consider without loss of generality e.g. $g_1(x)$ and $g_2(x) := 2x + \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2} + \sqrt{x^2 - \sigma_3}$. Then $g_1(x) = g_2(x)$ implies that $\sqrt{x^2 - \sigma_2} + \sqrt{x^2 - \sigma_3} = 0$. So, we conclude that both $\sqrt{x^2 - \sigma_2}$ and $\sqrt{x^2 - \sigma_3}$ must be zero. Since $x > \sqrt{\sigma_i}$, $i = 2, 3$, it follows that both curves do not intersect on the whole interval $(\sqrt{\sigma_1}, \infty)$.
For the second case that may occur, we consider $g_1(x)$ and $g_3(x) := 2x - \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2} - \sqrt{x^2 - \sigma_3}$. Then, the equality $g_1(x) = g_2(x)$ implies that $\sqrt{x^2 - \sigma_2} -$

$\sqrt{x^2-\sigma_1} = 0$. More in general, we obtain an equality of the form $\sum_i \sqrt{x^2-\sigma_i} - \sum_j \sqrt{x^2-\sigma_j} = 0$. Since both terms consist of sums of parabolic functions and all $\sigma$'s differ, they can intersect only a finite number of times, which shows the claim. $\square$

Next, for simplicity, we concentrate on $g_1(x)$ again. By differentiating $g_1(x)$ it is easily verified that $g_1(x)$ will be monotonically increasing for all $x \geq x_1^*$ for some number $x_1^* > \sqrt{\sigma_1}$. Furthermore, since $\lim_{x\to\infty} g_1(x) = \infty$ and $g_1(x)$ is bounded from above on the interval $(\sqrt{\sigma_1}, x_1^*)$, it follows that there exists a positive number $a_1^{**}$ such that for all $a \geq a_1^{**}$ the equation $g_1(x) = a$ has exactly one solution. Note that we always have that $a_1^{**} \geq g_1(\sqrt{\sigma_1})$, a property which may be helpfull if one is interested in finding this number $a_1^{**}$. A similar reasoning holds for all the other $g_i(x),\ i = 2,..,2^3-1$. To visualize the situation, we sketched in figure 1 all $g_i(x),\ i = 1,..,7$ as well as $f(x)$

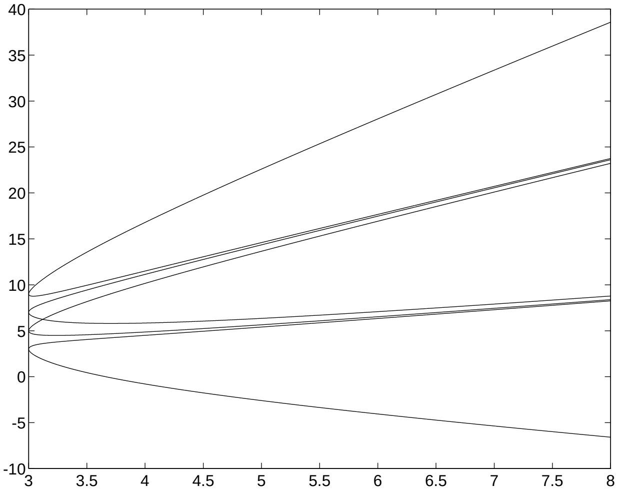

Figure 1: The curves $g_i$ and $f$ for $\sigma_1 = 9;\ \sigma_2 = 8;\ \sigma_3 = 5$.

Next, take the maximum over all $a_i^{**}$. Since according lemma 3 for a fixed $a$ the solutions for $g_i(x) = a$ differ for all $i$ if $a$ is chosen large enough, it is easily verified that the corresponding solutions $(\kappa_1, \kappa_2, \kappa_3)$ will also differ. So, it is clear then that the next conclusion holds

Theorem 4:
There exists a positive number $a^{**}$ such that for every state feedback parameter $a \geq a^{**}$ the set of algebraic Riccati equations (2) has $2^n - 1$ (positive) solutions. $\square$

Remark 5:
Like $\max_i g_i(\sqrt{\sigma_1})$ is a lowerbound for $a_1^{**}$, there exists also an upperbound for the number $a^*$. Due to the monotonicity property of $f(x)$ it is clear that $a^* \leq f(\sqrt{\sigma_1})$. However, note that since $g_i(x)$ is a decreasing function for some $i$ on the first part of the interval

$(\sqrt{\sigma_1}, x_i^*)$, in general $a^*$ will be smaller than this number $f(\sqrt{\sigma_1})$. $\square$

## IV. Concluding remarks

In this note we studied feedback Nash equilibria in the n-player linear quadratic scalar differential game. We showed that the corresponding set of algebraic Riccati equations has either one or more different positive solutions. In particular we saw that there exists a unique solution if and only if the state feedback parameter is smaller than some positive constant. In general, this constant depends on the other system parameters.
On the other hand we showed that in case the state feedback parameter exceeds some threshold, there will always exist $2^n - 1$ positive solutions.
These results make clear that for the study of infinite horizon feedback game problem, a closer study is needed to find an answer to the question which equilibrium one should choose. That this is a non-trivial problem was already indicated in Engwerda (1998).
We hope that the obtained results may be helpfull in analyzing the general multi-dimensional case.

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