CHAPTER 10

Risk sensitivity and related properties for bargaining solutions

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10.1 Introduction

In this chapter, we consider *n-person bargaining games* ($n \geq 2$), that is, pairs (S,d) where

(G.1) The *space S of feasible utility payoffs* is a compact and convex subset of $\mathbb{R}^n$,
(G.2) The *disagreement outcome d* is an element of S.

Furthermore, for mathematical convenience, we will also assume that

(G.3) $x \geq d$ for all $x \in S$,
(G.4) There is an $\hat{x} \in S$ with $\hat{x}_i > d_i$ for each $i \in N = \{1,2, \ldots, n\}$,
(G.5) For all $y \in \mathbb{R}^n$ with $d \leq y \leq x$ for some $x \in S$, we have $y \in S$.

Such a game (S,d) corresponds to a situation involving n bargainers (players) $1, 2, \ldots, n$, who may cooperate and agree upon choosing a point $s \in S$, which has utility s_i for player i, or who may not cooperate. In the latter case, the outcome is the point d, which has utility d_i for player $i \in N$. The family of all such bargaining games, satisfying (G.1) through (G.5), is denoted by G^n.

For a bargaining game $(S,d) \in G^n$, the *Pareto-set* $P(S)$ is defined by

$$P(S) := \{x \in S; \forall_{y \in S}[y \geq x \Longrightarrow y = x]\}$$

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and the *utopia point* $h(S) = (h_1(S), \ldots, h_n(S)) \in \mathbb{R}^n$ by

$$h_i(S) := \max\{x_i; (x_1, \ldots, x_i, \ldots, x_n) \in S\} \quad \text{for all } i \in N.$$

(The letter h is the first letter of *heaven.*)

We call a map $\phi: G^n \to \mathbb{R}^n$ an *n-person bargaining solution.* If, additionally, the following two properties hold, we call ϕ a *classical bargaining solution*:

PO: For each $(S,d) \in G^n$, we have $\phi(S,d) \in P(S)$ (*Pareto-optimality*).

IEUR: For each $(S,d) \in G^n$ and each transformation $A: \mathbb{R}^n \to \mathbb{R}^n$ of the form

$A(x_1,x_2, \ldots, x_n) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots, a_nx_n + b_n)$ for all $x \in \mathbb{R}^n$,

where $b_1, b_2, \ldots, b_n$ are real numbers and $a_1, a_2, \ldots, a_n$ are positive real numbers, we have $\phi(A(S),A(d)) = A(\phi(S,d))$ (*independence of equivalent utility representations*).

The PO property can be considered a basic property (see also our remark with regard to this point in Section 10.6). One of the arguments for taking the IEUR property as basic is a theorem in Kihlstrom, Roth, and Schmeidler (1981), which states that every (2-person) bargaining solution that is Pareto-optimal and risk sensitive satisfies also IEUR; so this latter property is, under the assumption of PO, a necessary condition for risk sensitivity.

Since, in this chapter, we will consider only classical solutions, we may without loss of generality restrict our attention here to n-person bargaining games with disagreement outcome 0. From now on, we assume that every $(S,d) \in G^n$, besides (G.1) through (G.5), satisfies

(G.6) $d = 0$,

and we will write S instead of (S,d).

The purpose of this chapter is to establish relations between some well-known and some new properties of bargaining solutions, where a central position is given to the risk-sensitivity property. Special attention will be paid to the risk aspects of solutions satisfying the following property, which was introduced by J. F. Nash (1950) in his fundamental paper.

IIA: A solution $\phi: G^n \to \mathbb{R}^n$ is called *independent of irrelevant alternatives* if for all S and T in G^n, we have $\phi(S) = \phi(T)$ if $S \subset T$ and $\phi(T) \in S$.

In addition, risk aspects of individually monotonic solutions (see Kalai and Smorodinsky (1975), Roth (1979*b*)) will be discussed.

IM: A bargaining solution $\phi: G^n \to \mathbb{R}^n$ is called *individually monotonic* if, for every $i \in N$, the following condition holds:

IM$_i$: For all $S,T \in G^n$ with $S \subset T$ and $h_j(S) = h_j(T)$ for each $j \in N - \{i\}$ we have $\phi_i(S) \leq \phi_i(T)$.

The organization of the chapter is as follows. Section 10.2 is devoted to the risk-sensitivity property of bargaining solutions and contains an overview of known results. In Section 10.3, the relation between risk sensitivity and twist sensitivity of two-person classical solutions is studied. We prove in Section 10.4 that all two-person classical IIA solutions and also all individually monotonic classical solutions are risk sensitive, using the result of Section 10.3, which states that all 2-person twist-sensitive classical solutions are risk sensitive. Section 10.5 discusses the risk profit opportunity and its relation to risk sensitivity and twist sensitivity for n-person classical solutions. Section 10.6 summarizes the results, and some concluding remarks are made.

10.2 Risk sensitivity of bargaining solutions

Pioneering work on the problem of how to compare the aversion to risk of decision makers was done by Pratt (1964), Yaari (1969), Arrow (1971), and Kihlstrom and Mirman (1974). Let A be the set of riskless alternatives for decision makers and $L(A)$ the set of finite lotteries over A. Let u and v be von Neumann–Morgenstern utility functions on $L(A)$ for two decision makers. Closely following Yaari (1969) and Kihlstrom and Mirman (1974), the following result was derived.

Theorem 1. (Peters and Tijs (1981)): The following two assertions are equivalent:

1. For all $a \in A$: $\{\ell \in L(A); v(\ell) > v(a)\} \subset \{\ell \in L(A); u(\ell) > u(a)\}$.
2. There exists a nondecreasing concave function k: $u(A) \rightarrow \mathbb{R}$ such that $v(a) = k \circ u(a)$ for all $a \in A$.

In view of this result, we have the following.

Definition 2. If u and v satisfy one of the equivalent conditions in theorem 1, then we say that the decision maker with utility function v is *more risk averse* than the decision maker with utility function u.

In the recent literature, the effects of increasing or decreasing risk aversion in bargaining situations have been studied in two directions. In the first direction, attempts have been made to answer the question of whether it is advantageous or disadvantageous for bargainers to have more risk averse opponents. This question was raised for the first time by Kihlstrom, Roth, and Schmeidler (1981). It also represents the approach followed in this chapter. In the second direction, investigators have tried to determine whether it is advantageous for a bargainer to *pretend* to be more (or less) risk averse in a bargaining situation. Interesting, in this context, is a remark by Kannai (1977, p. 54) – that in a resource-allocation problem of two agents, where the Nash bargaining solution is used, and where each bargainer knows only the preferences of the other bargainer and not the utility function, it is advantageous to announce that one's utility function is a minimally concave utility function, corresponding to the preferences. Other contributions in this second direction can be found in Kurz (1977, 1980), Crawford and Varian (1979), and Sobel (1981).

Let us first introduce some notation and then give the definition of risk sensitivity that we will use here. In this definition, it is implicitly assumed that all Pareto-optimal elements in a game S correspond to riskless alternatives (i.e., elements of A).

For $S \in G^n$ and $i \in N$, let $C_i(S)$ be the *family of nonconstant, nondecreasing continuous concave functions* on the closed interval $[0,h_i(S)]$ that have value 0 on 0; and for each $k_i \in C_i(S)$, let

$$K_i(x) = (x_1, \ldots, x_{i-1}, k_i(x_i), x_{i+1}, \ldots, x_n) \text{ for each } x \in S,$$
$$K_i(S) = \{K_i(x); x \in S\}.$$

Lemma 3. Let $S \in G^n$, $i \in N$, and $k_i \in C_i(S)$. Then, $K_i(S) \in G_n$.

Proof. We have to prove (G.1) through (G.6) for $K_i(S)$.

1. $0 = K_i(0) \in K_i(S)$, and so (G.2) and (G.6) hold.
2. Let $y \in \mathbb{R}^n$ *and* $x \in S$ such that $0 \leq y \leq K_i(x)$. Then, $0 \leq y_i \leq k_i(x_i)$, and so since k_i is continuous, there exists $z_i \in [0,x_i]$ such that $k_i(z_i) = y_i$. Hence, $y = (y_1, \ldots, y_{i-1}, k_i(z_i), y_{i+1}, \ldots, y_n) \in K_i(S)$ since $0 \leq (y_1, \ldots, z_i, y_{i-1}, y_{i+1}, \ldots, y_n) \leq x$. Thus, (G.5) holds.
3. $K_i(x) \geq K_i(0) = 0$ for all $x \in S$ since k_i is nondecreasing. Therefore, (G.3) holds.
4. Since S is compact and K_i is continuous, $K_i(S)$ is compact. Let $x,y \in S$ and $\lambda \in (0,1)$. Then, $\lambda K_i(x) + (1-\lambda)K_i(y) \leq K_i[\lambda x + (1-\lambda)y] \in K_i(S)$ since k_i is concave and S is convex. So $K_i(S)$ is convex, in view of (2) and (3). Hence, (G.1) holds
5. Let $\hat{x} \in S$, with $\hat{x} > 0$. Then, $k_i(\hat{x}_i) > 0$ since k_i is nondecreasing, nonconstant, and concave. So $K_i(\hat{x}) > 0$, and (G.4) holds.

The bargaining game $K_i(S)$ can be seen as the game that arises from the bargaining game S, if the ith player there (with utility function u) is replaced by a more risk averse player (with utility function $k_i \circ u$).

Definition 4. A bargaining solution is called *risk sensitive* (RS) if for all $S \in G^n$, $i \in N$, and $k_i \in C_i(S)$, we have for all $j \in N\backslash\{i\}$,

$$RS_i;\ \phi_j(K_i(S)) \geq \phi_j(S).$$

We can interpret risk sensitivity of a bargaining solution as follows: The solution assigns higher utilities to all bargainers in $N - \{i\}$, if bargainer i is replaced by a more risk averse opponent.

Our risk-sensitivity property is fairly strong. It is even stronger than, for example, the risk-sensitivity property in Roth (1979*a*), since we allow k_i to be nondecreasing in view of theorem 1. The difference, however, is only a minor one and it allows us the advantage of simplifying the proof of theorem 9, to follow. For more information on risk sensitivity for $n > 2$, see the end of this section, and Section 10.5.

In their fundamental paper, Kihlstrom, Roth, and Schmeidler (1981) prove that the symmetric 2-person bargaining solutions proposed by Nash (1950), Kalai and Smorodinsky (1975), and Perles and Maschler (1981) are all risk sensitive. Peters and Tijs (1981) prove that every nonsymmetric two-person classical IIA solution, as proposed in Harsanyi and Selten (1972), is also risk sensitive. In de Koster, Peters, Tijs, and Wakker (1983), it is shown that all two-person classical IIA solutions are risk sensitive; moreover, this class is described there. The class consists of the Harsanyi–Selten solutions and two other dictatorial bargaining solutions, D^1 and D^2. Here, $D^i(S)$ is the point of the Pareto-set $P(S)$ with maximal ith coordinate ($i = 1, 2$). As is well known, the Harsanyi–Selten solutions are the solutions F^t: $G^2 \to \mathbb{R}^2$ ($t \in (0,1)$) where, for every $S \in G^2$ and $t \in (0,1)$, $F^t(S)$ maximizes the product $x_1^t x_2^{1-t}$ on S. In Peters and Tijs (1982*a*), it is proved that every individually monotonic two-person classical solution is risk sensitive. Moreover, all of these solutions are described in that paper, as follows. In view of IEUR, it is sufficient to look at games S in G^2 with $h(S) = (1,1)$. A *monotonic curve* is a map λ: $[1,2] \to \text{conv}\{(1,0),(0,1),(1,1)\}$ with $\lambda(s) \leq \lambda(t)$ and $\lambda_1(s) + \lambda_2(s) = s$ for all $s,t \in [1,2]$ with $s \leq t$. For every monotonic curve λ, the classical solution π^λ: $G^2 \to \mathbb{R}^2$ is defined as follows: $\pi^\lambda(S)$ is the unique point in $P(S) \cap \{\lambda(t); t \in [1,2]\}$, for every $S \in G^2$ with $h(S) = (1,1)$. Then, $\{\pi^\lambda; \lambda$ is a monotonic curve$\}$ is the family of all individually monotonic 2-person classical solutions. In particular, the Kalai–Smorodinsky solution is the solution π^{λ_*} with $\lambda_*(T) := (\frac{1}{2}t, \frac{1}{2}t)$ for every $t \in [1,2]$, and $D^1 = \pi^{\underline{\lambda}}$, $D^2 = \pi^{\bar{\lambda}}$ where $\underline{\lambda}(t) := (1, t-1)$ and $\bar{\lambda}(t) := (t-1, 1)$ for every $t \in [1,2]$. So D^1 and D^2 also satisfy IM.

For all results with regard to risk sensitivity mentioned thus far, it is assumed implicitly that all points of the Pareto-set correspond to riskless alternatives. For results in the case of risky Pareto-points, we refer to Peters and Tijs (1981, 1983) and Roth and Rothblum (1982). In Peters and Tijs (1984), for a subclass of G^n, all individually monotonic n-person classical solutions are described, and it is proved that all of these bargaining solutions are risk sensitive. (It is well known (Roth (1979*b*)) that there does not exist a symmetric n-person classical solution on the whole class G^n that is individually monotonic.) In Section 10.5, we will see that the n-person Nash solution is not risk sensitive. This is one of the indications that for $n > 2$, risk sensitivity is a rather strong property. Two suitable weaker properties, risk profit opportunity and the worse-alternative property, will be introduced in that section. All of the n-person classical IIA solutions possess these properties (see Peters and Tijs (1983)). In the next section, we compare for two-person classical solutions the risk-sensitivity property and the twist-sensitivity property, and in Section 10.4 we present new short proofs of the fact that all two-person classical IIA solutions and all classical IM solutions are risk sensitive.

10.3 Risk sensitivity and twist sensitivity

Let $S, T \in G^n$, $i \in N$, and $\alpha_i \in [0,h_i(S)]$. Then, we say that T is a *favorable twisting of S for player i at level* α_i if

$$x_i > \alpha_i \text{ for all } x \in T \backslash S, \tag{10.1}$$

$$x_i < \alpha_i \text{ for all } x \in S \backslash T, \tag{10.2}$$

and an *unfavorable twisting of S for player i at level* α_i if

$$x_i < \alpha_i \text{ for all } x \in T \backslash S, \tag{10.3}$$

$$x_i > \alpha_i \text{ for all } x \in S \backslash T. \tag{10.4}$$

Definition 5. A bargaining solution $\phi: G^n \to \mathbb{R}^n$ is called *twist sensitive* (TW) if for each S and $T \in G^n$, with $\phi(S) \in P(T)$, we have for each $i \in N$,

TW_1: $\phi_i(T) \geq \phi_i(S)$, if T is a favorable twisting of S for player i at level $\phi_i(S)$.
TW_2: $\phi_i(T) \leq \phi_i(S)$, if T is an unfavorable twisting of S for player i at level $\phi_i(S)$.

Twist-sensitive bargaining solutions respond with a better payoff for a player in case a favorable twisting for that player is made at his payoff level in the solution point. Note that if $n = 2$, for Pareto-optimal solutions this notion TW is equal to the twisting property Tw, introduced by Thomson and Myerson (1980, p. 39). In general, Tw $\Rightarrow$ TW for $n = 2$.

The following theorem is one of the main results of the present discussion.

Theorem 6. Each twist-sensitive two-person classical solution is risk sensitive.

Proof. Let $\phi: G^2 \to \mathbb{R}^2$ be a twist-sensitive classical solution. We have to prove RS_1 and RS_2. We show only RS_2. Let $S \in G^2$ and $k_2 \in C_2(S)$. We have to prove that

$$\phi_1(K_2(S)) \geq \phi_1(S). \tag{10.5}$$

If $\phi_2(K_2(S)) = 0$, then by PO,

$$\phi_1(K_2(S)) = h_1(K_2(S)) = h_1(S) \geq \phi_1(S),$$

and thus (10.5) holds. Suppose now that $\phi_2(K_2(S)) > d_2$. Since ϕ satisfies the IEUR property, it is no loss of generality to suppose that

$$k_2(q_2) = q_2, \tag{10.6}$$

where $q = (q_1, q_2)$ is the point of $P(S)$ with first coordinate $\phi_1(K_2(S))$. By the concavity of k_2, we then have

$$k_2(x) \geq x \text{ for all } x \in [0, q_2] \qquad k_2(x) \leq x \text{ for all } x \geq q_2. \tag{10.7}$$

From (10.6) and (10.7) it follows that S is an unfavorable twisting of $K_2(S)$ for player 1 at level $\phi_1(K_2(S))$. From TW_2, we may conclude that (10.5) holds.

The converse of theorem 6 does not hold, as example 10 at the end of this section shows. We introduce now another property for two-person bargaining solutions, which, for classical solutions, is also implied by twist sensitivity.

Definition 7. A bargaining solution $\phi: G^2 \to \mathbb{R}^2$ is said to have the *slice property* (SL), if for all $S, T \in G^2$, with $h(S) = h(T)$ and $T \subset S$, we have

SL_1: $\phi_1(T) \geq \phi_1(S)$ if $x_2 > \phi_2(S)$ for all $x \in S \backslash T$,
SL_2: $\phi_2(T) \geq \phi_2(S)$ if $x_1 > \phi_1(S)$ for all $x \in S \backslash T$.

Thus, a bargaining solution $\phi: G^2 \to \mathbb{R}^2$ is said to have the slice property if it favors the opponent of a player i when a piece of the set of utility payoffs, preferred by i over $\phi(S)$, is sliced off, the utopia point remaining the same. For $n = 2$, the slice property resembles the cutting axiom of Thomson and Myerson (1980). The difference (for Pareto-optimal solutions) is that in the cutting axiom, there is no condition on the utopia point. Therefore, SL is considerably weaker than cutting. Theorem 9, to follow, shows that risk-sensitive classical solutions are twist sensitive if they additionally satisfy SL.

Theorem 8. Each twist-sensitive two-person classical solution has the slice property.

Proof. Let $\phi: G^2 \to \mathbb{R}^2$ be twist sensitive. We prove only that SL_1 holds. Therefore, let $S,T \in G^2$, with $h(S) = h(T)$, $T \subset S$, and $x_2 > \phi_2(S)$ for all $x \in S\backslash T$. We must show that

$$\phi_1(T) \geq \phi_1(S). \tag{10.8}$$

Note that $\phi(S) \in P(T)$ and $x_1 < \phi_1(S)$ for all $x \in S\backslash T$ because $\phi(S) \in P(S)$. Since $T\backslash S = \phi$, we may conclude that T is a favorable twisting of S for player 1 at level $\phi_1(S)$. Thus, (10.8) follows from TW_1.

Example 11 will show that the converse of theorem 8 does not hold.

The following theorem gives a characterization of twist sensitivity of two-person classical solutions.

Theorem 9. A 2-person classical solution is twist sensitive if and only if it has the slice property and is risk sensitive.

Proof. (See Figure 10.1.) In view of theorems 6 and 7, we have only to show the "if" part of the theorem. Hence, let $\phi: G^2 \to \mathbb{R}^2$ be a risk-sensitive classical solution having the slice property. We demonstrate only that TW_2 for $i = 1$ holds. Suppose that TW_2 does not hold for $i = 1$. Then, there are S and $T \in G^2$ with $\phi(S) \in P(T)$ and

$$\phi_1(T) > \phi_1(S) \qquad \phi_2(T) < \phi_2(S), \tag{10.9}$$

whereas

$$x_1 < \phi_1(S) \text{ for all } x \in T\backslash S, \tag{10.10}$$

$$x_1 > \phi_1(S) \text{ for all } x \in S\backslash T. \tag{10.11}$$

Let k_1: $[0,h_1(S)] \to \mathbb{R}$ be the function defined by $k_1(\lambda) = \lambda$ if $0 \leq \lambda \leq h_1(T)$ and $k_1(\lambda) = h_1(T)$ if $h_1(T) \leq \lambda \leq h_1(S)$. Then, $k_1 \in C_1(S)$, and

$$K_1(S) = \{x \in S; x_1 \leq h_1(T)\}. \tag{10.12}$$

Since ϕ is risk sensitive, we have

$$\phi_2(K_1(S)) \geq \phi_2(S). \tag{10.13}$$

Formula (10.13) and $P(K_1(S)) \subset P(S)$ imply that

$$\phi_1(K_1(S)) \leq \phi_1(S). \tag{10.14}$$

Let $D = S \cap T$. Then,

$$h(D) = h(K_1(S)) = (h_1(T),h_2(S)). \tag{10.15}$$

By (10.11), (10.12), and (10.14), we have

$$x_1 > \phi_1(S) \geq \phi_1(K_1(S)) \text{ for all } x \in K_1(S)\backslash D. \tag{10.16}$$

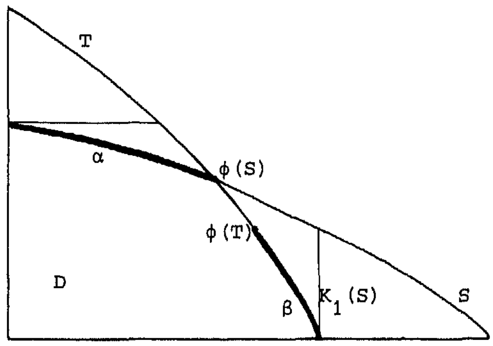


Figure 10.1 $\phi(D) \in \alpha$ as well as $\phi(D) \in \beta$, a contradiction

Since $D \subset K_1(S)$, we have, by (10.15), (10.16), and the slice property of ϕ,

$$\phi_2(D) \geq \phi_2(K_1(S)). \quad \textbf{(10.17)}$$

From (10.14), (10.11), and PO, it follows that $\phi(K_1(S)) \in P(D)$. Then, (10.17) implies that

$$\phi_1(K_1(S)) \geq \phi_1(D). \quad \textbf{(10.18)}$$

By (10.18) and (10.14), we obtain

$$\phi_1(D) \leq \phi_1(S). \quad \textbf{(10.19)}$$

Since, by (10.9), $\phi_1(T) > \phi_1(S)$, we can apply the same line of reasoning for T instead of S, interchanging the roles of players 1 and 2, to finally obtain

$$\phi_1(D) \geq \phi_1(T). \quad \textbf{(10.20)}$$

Now, (10.9), (10.19), and (10.20) yield a contradiction. Hence, TW_2 holds for $i = 1$.

We now discuss some examples of bargaining solutions with respect to the three properties that play a role in this section.

Example 10. The superadditive solution of Perles and Maschler (1981) is risk sensitive but not twist sensitive and does not have the slice property. See counter-example 7.1, p. 189, in Perles and Maschler (1981).

Example 11. Let the classical solution ϕ: $G^2 \rightarrow \mathbb{R}^2$ be defined by the following: For all $S \in G^2$ with $h(S) = (1,1)$, $\phi(S)$ is the point of intersection of $P(S)$ with γ with maximal second coordinate, where γ is the curve depicted in Figure 10.2. Let $\alpha := \frac{3}{2} - \frac{1}{2}\sqrt{3}$, then between (1,1) and (α,α),

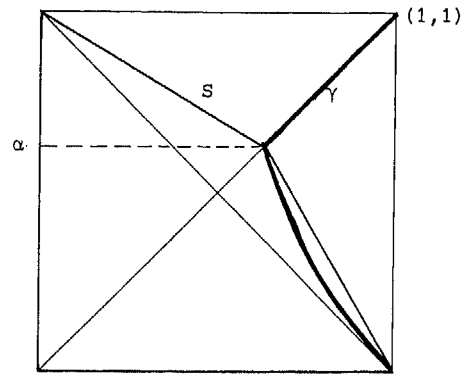


Figure 10.2 Curve γ, and game S

$\gamma = \text{conv}\{(1,1),(\alpha,\alpha)\}$, and between (α,α) and $(1,0)$, γ is an arc of the circle $(x_1 - 2)^2 + (x_2 - 1)^2 = 2$. By IEUR, ϕ is determined for all $S \in G^2$. It is easy to see that ϕ has the slice property. However, ϕ is not twist sensitive. Let S: $=$ $\text{conv}\{(0,0),(1,0),(\alpha,\alpha),(0,1)\}$ and T: $=$ $\text{conv}\{(0,0),(1,0),(0,\alpha(1-\alpha)^{-1})\}$. Then, T is an unfavorable twisting of S for player 1 at level $\alpha = \phi_1(S)$, but $\phi_1(T) = 1 > \alpha = \phi_1(S)$, and so ϕ is not twist sensitive.

Example 12. Let α: $G^2 \to \mathbb{R}^2$ be the *equal area split solution;* that is, for every $S \in G^2$, $\alpha(S)$ is that point of $P(S)$ such that the area in S lying above the line through 0 and $\alpha(S)$ equals half the area of S. Then, α is a classical solution, which is twist sensitive, and consequently is also risk sensitive and has the slice property.

In the next section, we investigate the classical IIA and IM solutions with respect to the three properties in this section.

10.4 Two-person classical IIA solutions and IM solutions

We start with considering the family $\{D^1,D^2,F^t;t \in (0,1)\}$ of two-person classical IIA solutions (compare with Section 10.2). In de Koster, Peters, Tijs, and Wakker (1983), it was proved that all elements in this family are risk sensitive. A new proof of this fact is given now, using theorem 9. See also Thomson and Myerson (1980, lemma 5).

Theorem 13. All two-person classical IIA solutions are risk sensitive, twist sensitive, and have the slice property.

Proof. Let $\phi: G^2 \to \mathbb{R}^2$ be a classical IIA solution. In view of theorem 9, it is sufficient to show that ϕ is twist sensitive. We prove only TW_1 for player 1. Let S and T be elements of G^2 and suppose that T is a favorable twisting of S for player 1 at level $\phi_1(S)$, that is, $\phi(S) \in T$ and

$$x_1 > \phi_1(S) \text{ for all } x \in T \backslash S, \tag{10.21}$$

$$x_1 < \phi_1(S) \text{ for all } x \in S \backslash T. \tag{10.22}$$

We have to prove that

$$\phi_1(T) \geq \phi_1(S). \tag{10.23}$$

Let $D = S \cap T$. Since $D \subset S$ and $\phi(S) \in T$, the IIA property implies that

$$\phi(D) = \phi(S). \tag{10.24}$$

Since $D \subset T$, the IIA property implies

$$\phi(D) = \phi(T) \quad \text{or} \quad \phi(T) \notin D.$$

In the case where $\phi(D) = \phi(T)$, we have $\phi(T) = \phi(S)$ in view of (10.24), and so (10.23) holds. If $\phi(T) \notin D$, then $\phi(T) \in T \backslash S$, and then (10.23) follows from (10.21).

Now, we want to look at the family of two-person individually monotonic classical solutions, that is, the family $\{\pi^\lambda; \lambda \text{ is a monotonic curve}\}$ as described in Section 10.2. In Peters and Tijs (1985), it was proved that all classical two-person IM solutions are risk sensitive. A new proof is presented now, using theorem 6.

Theorem 14. All classical two-person IM solutions are risk sensitive, twist sensitive, and have the slice property.

Proof. Let $\phi: G^2 \to \mathbb{R}^2$ be a classical IM solution. In view of theorems 6 and 8, we only have to show that ϕ is twist sensitive. We prove only that TW_1 holds for player 1. Let $S,T \in G^2$ and suppose that $\phi(S) \in T$ and that (10.21) and (10.22) hold. We have to show that

$$\phi_1(T) \geq \phi_1(S). \tag{10.25}$$

Let $D = S \cap T$. Since $D \subset S$ and $h_1(D) = h_1(S)$ by (10.22), the IM_2 property implies that $\phi_2(S) \geq \phi_2(D)$. Then, since $\phi(S) \in D$ and $\phi(D) \in P(D)$, $\phi_2(S) \geq \phi_2(D)$ implies that

$$\phi_1(S) \leq \phi_1(D). \tag{10.26}$$

From $D \subset T$, $h_2(D) = h_2(T)$, and IM_1, we may conclude that

$$\phi_1(D) \leq \phi_1(T). \tag{10.27}$$

Now, (10.26) and (10.27) imply (10.25).

Thomson and Myerson (1980) show (lemma 9, for $n = 2$) that their property WM (which is somewhat stronger than IM) together with WPO (for all $S \in G^2$ and $x \in \mathbb{R}^2$, if $x > \phi(S)$, then $x \notin S$) implies Tw.

In this section, we have proved that the family of two-person twist-sensitive classical solutions T contains the families of classical IIA solutions and IM solutions. The family T also contains solutions that are neither IIA or IM, as example 12 shows.

10.5 New risk properties for *n*-person bargaining solutions

In this section, we want to extend some of the results of Sections 10.3 and 10.4 to n-person bargaining solutions. Not all results can be extended, as the following example illustrates.

Example 15. Let S be the three-person bargaining game with S the convex hull of the points (0,0,0), (1,0,0), (0,1,0), (0,0,1), and (1,0,1). Let N be the three-person IIA Nash solution, assigning to S the unique point at which the function $(x_1, x_2, x_3) \rightarrow x_1 x_2 x_3$ takes its maximum. Let $k_3 \in C_3(S)$ be the function with $k_3(x) = \sqrt{x}$. Then, $P(S) = \{(\alpha, 1-\alpha, \alpha) \in \mathbb{R}^3; 0 \leq \alpha \leq 1\}$,

$$P(K_3(S)) = \{(\alpha, 1-\alpha, \sqrt{\alpha}) \in \mathbb{R}^3;\ 0 \leq \alpha \leq 1\},$$
$$N(S) = (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}),\ N(K_3(S)) = (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}\sqrt{15}).$$

Note that $N_2(K_3(S)) > N_2(S)$ but $N_1(K_3(S)) < N_1(S)$. Hence, N is not risk sensitive. However, $O_{-3}(S,N) \subset O_{-3}(K_3(S),N)$, where

$$O_{-3}(S,N) := \{(x_1, x_2) \in \mathbb{R}^2; (x_1, x_2, N_3(S)) \in S\},$$
$$O_{-3}(K_3(S),N) := \{(x_1, x_2) \in \mathbb{R}^2; (x_1, x_2, N_3(K_3(S)) \in S\}.$$

Nielsen (1984) also shows, using the same example, that the three-person IIA Nash solution is not risk sensitive. In addition, he proves that the n-person IM Kalai–Smorodinsky solution is risk sensitive (see Peters and Tijs (1984)), and that both the n-person IIA Nash and IM Kalai-Smorodinsky solutions satisfy a weaker property, the worse-alternative property, which can also be found in Peters and Tijs (1983), in definition 17, to follow. In Peters and Tijs (1983), it is shown that none of the nonsymmetric strongly individually rational n-person classical IIA solutions (see Roth (1979*a*, p. 16)), for $n > 2$, satisfy the risk-sensitivity property. Of all n-person classical IIA solutions (described in Peters (1983)), only a small subclass is risk sensitive, and all of these solutions are dictatorial (see Peters and Tijs (1983)). This motivated us to look for weaker risk properties. Before introducing two such weaker properties, we provide some notation.

For $i \in N$, the map $\pi_{-i}: \mathbb{R}^n \to \mathbb{R}^{n-1}$ assigns to a vector $x \in \mathbb{R}^n$ the vector $\pi_{-i}(x) = (x_1, \ldots, x_{i-1} x_{i+1}, \ldots, x_n)$, which is obtained from x by deleting the ith coordinate. Let ϕ be an n-person bargaining solution, $S \in G^n$ and $i \in N$. Then, the *opportunity set* $O_{-i}(S,\phi)$ for the bargainers in $N - \{i\}$ with respect to S and ϕ, is defined by

$$O_{-i}(S,\phi) := \pi_{-i}\{x \in S; x_i = \phi_i(S)\}.$$

The opportunity set $O_{-i}(S,\phi)$ consists of those payoff $(n-1)$-tuples, available for the collective $N - \{i\}$, if bargainer i obtains $\phi_i(S)$. We are interested in bargaining solutions where, if one of the players is replaced by a more risk averse player, the opportunity set of the other players increases. Formally, we have the following.

Definition 16. We say that a bargaining solution $\phi: G^n \to \mathbb{R}^n$ has the *risk profit opportunity* (RPO) *property* if for all $S \in G^n$, $i \in$ N, and $k_i \in C_i(S)$, we have

$$O_{-i}(S,\phi) \subset O_{-i}(K_i(S),\phi).$$

Peters and Tijs (1983) show that, for classical solutions, risk profit opportunity is equivalent to another property, which here we call the worse-alternative property. We first state this property informally. Let $S \in G^n$ and $i \in N$, and suppose that player i is replaced by a more risk averse player, $\hat{\imath}$. If, in such a situation, both player i and player $\hat{\imath}$ prefer the alternative (in the underlying set of alternatives A; see Section 10.2) assigned by an n-person classical solution ϕ in the game with player i to the alternative assigned by ϕ in the game with player $\hat{\imath}$, then we say that ϕ has the worse-alternative property (WA). Peters and Tijs (1983) show that, for games with risky outcomes, the equivalence between RPO and WA breaks down, and argue there that WA is the most elementary property. We think, however, that the RPO property is attractive because it says something about the possible benefits for the other players, if one player is replaced by a more risk averse one. We will now give the formal definition of the worse-alternative property. To avoid having to introduce many additional notations, we will state it in terms of utilities rather than alternatives.

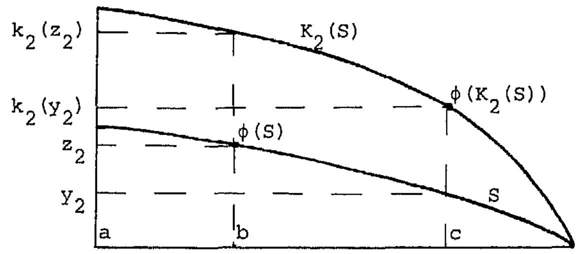


Figure 10.3 $z = \phi(S)$, $y_1 = \phi_1(K_2(S))$, $ab = O_{-2}(S,\phi)$, $ac = O_{-2}(K_2(S),\phi)$

Definition 17. We say that a bargaining solution ϕ: $G^n \rightarrow \mathbb{R}^n$ has the *worse-alternative* (WA) *property* if all $S \in G^n$, $i \in N$, and $k_i \in C_i(S)$, we have

$$z_i \geq y_i,$$

where $z = \phi(S)$, and $y \in S$ such that $K_i(y) = \phi(K_i(S))$.

Of course, we also have $k_i(z_i) \geq k_i(y_i)$ in definition 17, since k_i is nondecreasing. For a two-person classical solution ϕ: $G^2 \rightarrow \mathbb{R}^2$ and $S \in G$, we have

$$O_{-1}(S,\phi) = [0,\phi_2(S)] \qquad O_{-2}(S,\phi) = [0,\phi_1(S)].$$

From this and (essentially only the) Pareto-optimality (of a classical solution) follows immediately.

Theorem 18. For two-person classical solutions, the properties RS, RPO, and WA are equivalent.

Example 15 shows that there are RPO solutions that are not risk sensitive. In general, the RPO property is weaker than the RS property, as the following theorem demonstrates.

Theorem 19. Each risk-sensitive classical solution has the RPO property.

Proof. Let ϕ: $G^n \rightarrow \mathbb{R}^n$ be a risk-sensitive classical solution. Take $S \in G^n$, $k_i \in C_i(S)$. Let y be the point in $P(S)$ for which

$$K_i(y) = (y, \ldots, y_{i-1}, k_i(y_i), y_{i+1}, y_n) = \phi(K_i(S)). \qquad \textbf{(10.28)}$$

By risk sensitivity, we have

$$y_j \geq \phi_j(S) \text{ for all } j \in N - \{i\}. \tag{10.29}$$

Since $y \in P(S)$ and $\phi(S) \in P(S)$, the PO property and (10.29) imply

$$y_i \leq \phi_i(S). \tag{10.30}$$

But then by (10.30), G.5, and (10.28),

$$\begin{aligned} O_{-i}(S,\phi) &= \pi_{-i}\{x \in S; x_i = \phi_i(S)\} \subset \pi_{-i}\{x \in S; x_i = y_i\} \\ &= \pi_{-i}\{u \in K_i(S); u_i = k_i(y_i)\} \\ &= \pi_{-i}\{u \in K_i(S); u_i = \phi_i(K_i(S))\} = O_{-i}(K_i(S),\phi). \end{aligned}$$

Hence, ϕ has the RPO property.

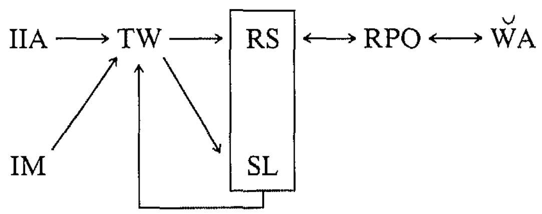
In Peters and Tijs (1983), it is proved that all n-person classical IIA solutions have the RPO property. Another proof of this result can be given by looking at twist sensitivity. Then, by modifying in a trivial way the proofs of theorems 13 and 6, we obtain the following.

Theorem 20.

1. Each n-person classical IIA solution is twist sensitive;
2. Each n-person twist-sensitive classical solution has the RPO property.

10.6 Summary and remarks

In the foregoing, we have shown that for two-person classical solutions, the following logical implications between the discussed properties hold:



For n-person classical solutions, we have the following implications:

$$\text{IIA} \longrightarrow \text{TW} \longrightarrow \text{RPO}, \qquad \text{RS} \longrightarrow \text{RPO} \longleftrightarrow \text{WA}.$$

In an obvious way, many results in this chapter can be extended to WPO (weak Pareto-optimal; see text immediately following theorem 14) solutions. Similar results can be derived for bargaining multisolutions (i.e., correspondences μ: $G^n \rightarrow \mathbb{R}^n$ such that $\mu(S) \subset S$ for every $S \in G^n$). See Peters, Tijs, and de Koster (1983) for a description of (two-person) weak (multi-)solutions with the IIA or the IM property.

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