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REGULARITY AND THE GENERALIZED ADJACENCY SPECTRA OF GRAPHS

By Andrey A. Chesnokov, Willem H. Haemers

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Tilburg University

Regularity and the generalized adjacency spectra of graphs

Andrey A. Chesnokov

Moscow State University, Dept. Computational Mathematics and Cybernetics
e-mail: ache@mccme.ru

and

Willem H. Haemers

Tilburg University, Dept. Econometrics and Operations Research
e-mail: Haemers@uvt.nl

Abstract

For every rational number $x \in (0, 1)$, we construct a pair of graphs one regular and one nonregular with adjacency matrices A_1 and A_2, having the property that $A_1 - xJ$ and $A_2 - xJ$ have the same spectrum (J is the all-ones matrix). This solves a problem of Van Dam and the second author. For some values of x, we have generated the smallest examples (with respect to the number of vertices) by computer. (Keywords: Graphs, Matrices; JEL code: C0.)

1 Introduction

It is well known that with respect to the adjacency matrix, a regular graph cannot be cospectral to a nonregular graph. See for example [1, p.94]. If A is the adjacency matrix of a graph, any matrix of the form $M = \alpha A + \beta J + \gamma I$ with $\alpha \neq 0$ is called a *generalized adjacency matrix* (as usual, J denotes the all-ones matrix and I the identity matrix). For generalized adjacency matrices the following result was proved in [2].

Proposition 1 *With respect to the generalized adjacency matrix $M(\alpha, \beta, \gamma) = \alpha A + \beta J + \gamma I$, a regular graph cannot be cospectral with a nonregular one, except possibly when $-1 < \beta/\alpha < 0$.*

Note that two graphs are cospectral with respect to $M(\alpha, \beta, \gamma)$ if and only if they are cospectral with respect to $M(1, \beta/\alpha, 0)$. So the value of γ is irrelevant and we only need to consider the possible values of β/α.

The statement 'a regular graph cannot be cospectral to a nonregular one' is clearly not true if $\beta/\alpha = -1/2$. Then multiplication of some rows and the corresponding columns of M with -1 gives a cospectral matrix that in general corresponds to a different graph (the operation is called Seidel switching). For example the triangle (which is regular) and the graph on three vertices with one edge (which is nonregular) are cospectral with respect to $2A - J$. When [2] was written, it was an open problem whether the above statement is true or false for $-1 < \beta/\alpha < 0$, $\beta/\alpha \neq -1/2$. In this note we will show that for all these cases the statement is false, provided β/α is rational. For irrational values of $\beta/\alpha \in (-1, 0)$ we still don't know the answer.

It should be noted that the case $\beta/\alpha = -1/2$ is rather special. In the proof of the above proposition it was observed that for graphs with more than one vertex, the spectrum of M determines the number of edges if and only if $\beta/\alpha \neq -1/2$. An other relevant remark is that Johnson and Newman [6] (see also [2]) have shown that if two graphs are cospectral with respect to two generalized adjacency matrices with different values of β/α, then they are cospectral with respect to all generalized adjacency matrices, and therefore they are both regular or both nonregular. In other words, if one graph is regular and the other one not, the graphs can only be cospectral for one value of β/α. These remarks show some difficulties that should be dealt with in finding counterexamples to the above statement. Both graphs must have the same number of edges, and may not be cospectral for any other value of β/α.

The first counterexamples to the statement (with $\beta/\alpha \neq -1/2$) were found by computer. We will present them in the next section. These computer results motivated us to find a theoretic construction for counterexamples, which is presented in Section 3.

2 Computer results

If two graphs are cospectral with respect to $M(1, \beta/\alpha, 0)$, then their complements are cospectral with respect to $M(1, -1 - \beta/\alpha, 0)$. So one may use to following algorithm to construct (integral) counterexamples.

1. Choose any integer value of β and α such that $-1/2 < \beta/\alpha < 0$ and $\gcd(\alpha, \beta) = 1$.
2. Choose the number of vertices $n \geq 3$.
3. Generate all graphs on n vertices.
4. Compute the characteristic polynomial for each graph with respect to $M(\alpha, \beta, 0)$.
5. Look through the generated polynomials and find identical ones. See if one of graphs corresponding to one of such polynomials is regular and another graph isn't.

Graphs were generated using `nauty` package by B. McKay [7]. Then graphs were stored on a disc in a compressed form. The graphs generated were then fed to GAP [3] in such a way that strings were produced comprising the coefficients of the characteristic polynomials, and these were stored in a file, one to each line. To the end of each line we added suffix `true` or `false`, `true` stands for regular graphs and `false` for nonregular. This file was then sorted using the Linux utility `sort`, after which it was easy to look through this file and take identical polynomials such that one has suffix `true` and another has suffix `false`. Some of the enumeration ideas are taken from [5].

We first tried some obvious choices for β/α, like $-1/3$, $-1/4$ and $-1/5$ with $n = 3, \ldots, 9$. For $\beta/\alpha = -1/3$ and $-1/5$ there are no counterexamples on less than 10 vertices. However, for $\beta/\alpha = -1/4$ we found exactly two regular-nonregular pairs of cospectral graphs on less than 10 vertices. One pair is given in Figure 1. It is clear that one graph is regular, whilst the other one is not. For both graphs the characteristic polynomial with respect to $4A - J$ ($\alpha = 4$, $\beta = -1$) is

$$x^9 + 9x^8 - 72x^7 - 848x^6 + 19200x^4 + 38912x^3 - 110592x^2 - 393216x - 262144.$$

Figure 2 presents the second pair of such graphs (Libra and a hexagon with a triangle). Both graphs are disconnected. Their characteristic polynomial with respect to $4A - J$ is $x^9 + 9x^8 - 144x^7 - 1312x^6 + 5376x^5 + 54016x^4 - 40960x^3 - 581632x^2 + 262144x + 1376256$.

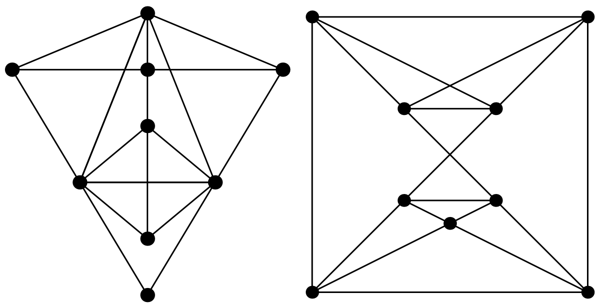

Figure 1: First pair of cospectral graphs wrt $4A - J$

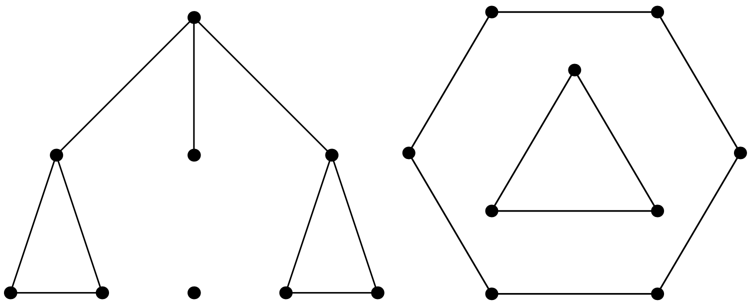

Figure 2: Second pair of cospectral graphs wrt $4A - J$

We arbitrarily tried a few more values of α and β and for $\alpha = 7$, $\beta = -3$ we found a regular graph cospectral with two nonisomorphic nonregular graphs. Figure 3 presents them. The characteristic polynomial with respect to $7A - 3J$ is

$$x^9 + 27x^8 - 126x^7 - 6762x^6 - 343x^5 + 545027x^4 - 67228x^3 - 13647284x^2 + 13176688x.$$

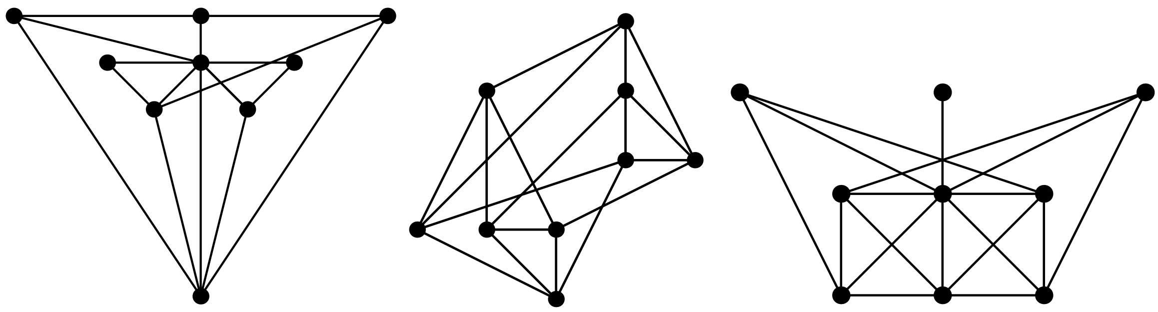

Figure 3: Triplet of cospectral graphs wrt $7A - 3J$

There is exactly one more cospectral regular-nonregular pair on nine vertices with respect to $7A - 3J$ (and none on fewer vertices).

3 A construction

In the theorem below we restrict to generalized adjacency matrices of the form $A - xJ$. As remarked before, two graphs are cospectral with respect to such a matrix, if and only if they are cospectral with respect to $M(\alpha, \beta, \gamma)$ with $\beta/\alpha = -x$.

Theorem 1 *For every rational value of $x \in (0, 1)$, there exist a pair of graphs, one regular and one not, that is cospectral with respect to $A - xJ$.*

Proof. Write $x = p/q$, such that p and q are integers and q is even. We will construct two cospectral generalized adjacency matrices M and $\overline{M}$ of size $4q + q^2$ with entries $-p$ and $r = q - p$. Define

$$M = \begin{bmatrix} K & B \\ B^\top & C \end{bmatrix} \text{ and } \overline{M} = \begin{bmatrix} K & \overline{B} \\ \overline{B}^\top & C \end{bmatrix}.$$

The matrices $K, B, \overline{B}$ and C are built with $q \times q$ blocks with constant row and column sums. The construction is as follows:

$$K = \begin{bmatrix} -pJ & rJ & rJ & -pJ \\ rJ & -pJ & -pJ & -pJ \\ rJ & -pJ & -pJ & -pJ \\ -pJ & -pJ & -pJ & -pJ \end{bmatrix},$$

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & \cdots & B_{1,q-1} & B_{1,q} \\ B_{2,1} & \overline{B}_{2,2} & B_{2,3} & \overline{B}_{2,4} & \cdots & B_{2,q-1} & \overline{B}_{2,q} \\ \overline{B}_{3,1} & B_{3,2} & \overline{B}_{3,3} & B_{3,4} & \cdots & \overline{B}_{3,q-1} & B_{3,q} \\ \overline{B}_{4,1} & \overline{B}_{4,2} & \overline{B}_{4,3} & \overline{B}_{4,4} & \cdots & \overline{B}_{4,q-1} & \overline{B}_{4,q} \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} \overline{B}_{4,1} & \overline{B}_{4,2} & \overline{B}_{4,3} & \overline{B}_{4,4} & \cdots & \overline{B}_{4,q-1} & \overline{B}_{4,q} \\ \overline{B}_{3,1} & B_{3,2} & \overline{B}_{3,3} & B_{3,4} & \cdots & \overline{B}_{3,q-1} & B_{3,q} \\ B_{2,1} & \overline{B}_{2,2} & B_{2,3} & \overline{B}_{2,4} & \cdots & B_{2,q-1} & \overline{B}_{2,q} \\ B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & \cdots & B_{1,q-1} & B_{1,q} \end{bmatrix},$$

where $B_{i,j}$ is any $q \times q$ matrix with $p - 1$ times r and $r + 1$ times $-p$ in each row and column, and $\overline{B}_{i,j}$ is any $q \times q$ matrix with $p + 1$ times r and $r - 1$ times $-p$ in each row and column. So $B_{i,j}$ has row sums $-q$ and $\overline{B}_{i,j}$ has row sums q. Notice that the first $4q$ rows of M all have row sum $q(q - 4p)$, whilst the first $4q$ row sums of $\overline{M}$ take three different values: $q(3q - 4p)$, $q(q - 4p)$ and $q(-q - 4p)$. Also observe that $\overline{B}$ can be obtained from B by reversing the order of the block rows. The matrix

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,q} \\ \vdots & & \vdots \\ C_{q,1} & \cdots & C_{q,q} \end{bmatrix}$$

should be taken such that C is symmetric with diagonal entries $-p$ and all row and column sums equal to $q(q - 4p)$ (which makes all row sums of M equal). All blocks $C_{i,j}$ must have constant row and column sums. There are many ways to establish this. For instance, take $C_{1,1} = C_{1,2} = C_{1,q} = -pJ$, $C_{1,q/2+1} = rJ$ and for the remaining values of i take for $C_{1,i}$ any $q \times q$ matrix with p times r and r times $-p$ in each row and column. Then put $C = \text{circulant}(C_{1,1}, \ldots, C_{1,q})$.

It is clear that M represents a regular graph and $\overline{M}$ represents a nonregular graph. What remains to be proved is that M and $\overline{M}$ are cospectral. First observe that the given partition of M (and $\overline{M}$) into $q+4$ blocks of size $q \times q$ is an equitable partition, that is, all blocks have constant row and column sum. The quotient matrix of such a partitioned matrix is the $(q+4)\times(q+4)$ matrix whose entries are the row sums of the blocks. For an equitable partition it is well known (see for example [4, p.78]) that the eigenvalues of the quotient matrix are also eigenvalues of the original matrix and that the corresponding eigenvectors are constant over each partition class, that is, the eigenvectors span the column space V of $I \otimes J$. Note that the quotient matrix of $\overline{M}$ can be obtained from the quotient matrix of M by multiplying the first four rows and columns by -1. Hence these quotient matrices are cospectral. The remaining eigenvalues of M and $\overline{M}$ have eigenvectors in $V^{\perp}$. This implies that these eigenvalues are not changed if any block $M_{i,j}$ of M is replaced by $M_{i,j} + cJ$ for some constant c. Define

$$M' = \begin{bmatrix} O & B \\ B^{\top} & C \end{bmatrix} \text{ and } \overline{M}' = \begin{bmatrix} O & \overline{B} \\ \overline{B}^{\top} & C \end{bmatrix}.$$

Then for the eigenvectors in $V^{\perp}$, M' and M have the same eigenvalues, and so do $\overline{M}'$ and $\overline{M}$. But since $\overline{B}$ can be obtained from B by a row permutation, M' and $\overline{M}'$ are cospectral. The conclusion is that M and $\overline{M}$ have the same eigenvalues for the eigenvectors in V and for the eigenvectors in $V^{\perp}$. Therefore M and $\overline{M}$ have the same spectrum. □

References

[1] D.M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs,* third edition, Johann Abrosius Barth Verlag, 1995. (First edition: Deutscher Verlag der Wissenschaften, Berlin 1980; Academic Press, New York 1980.)

[2] E.R. van Dam and W.H. Haemers, Which graphs are determined by their spectrum? *Linear Alg. Appl.* **373** (2003), 241–272.

[3] *GAP — Groups, Algorithms and Programming*, Version 4.3, 2002.

[4] C.D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, 1993.

[5] W.H. Haemers and E. Spence, Enumeration of cospectral graphs, *European J. Combin.* **25** (2004), 199-211.

[6] C.R. Johnson and M. Newman, A note on cospectral graphs, *J. Combin. Theory B* **28** (1980) 96-103.

[7] B.D. McKay, `nauty` Users' Guide (Version 1.5). Technical Report TR-CS-90-02, Australian National University, Department of Computer Science, 1990.