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On Balanced Games and Games with Committee Control

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Summary. A generalization of flow games, namely, flow games with committee control is considered, to obtain a representation of non-negative balanced games. The committee control is modeled with the aid of simple games. Linear production games with committee control are also studied and results on the balancedness of such games are obtained.

Zusammenfassung. Eine Verallgemeinerung von Flußspielen, nämlich solchen mit Komitee-Kontrolle, wird in der vorliegenden Arbeit betrachtet, um eine Darstellung nicht-negativer ausgewogener Spiele zu erhalten. Die Komitee-Kontrolle wird mit Hilfe einfacher Spiele modelliert. Ebenso werden lineare Produktionsspiele mit Komitee-Kontrolle untersucht, und Ergebnisse zur Ausgewogenheit solcher Spiele werden hergeleitet.

1. Introduction

Kalai and Zemel [5] introduced flow games and proved that every flow game is totally balanced and every non-negative totally balanced game is a flow game. Our main aim in this paper is to derive a similar result for the larger class of balanced games and a generalization of flow games, namely, flow games with committee control. We will also look at a generalization involving committee control of the linear production game introduced by Owen [7]. In Sect. 2 we will give the necessary definitions and notations. In Sect. 3 the main result concerning balanced games and flow games with committee control is proved. Section 4 is dedicated to linear production games with committee control and to some remarks on related work.

2. Flow Games with Committee Control

In this section we will introduce and define the concepts that we will need throughout the paper.

A *cooperative game in characteristic function form* is an ordered pair $\langle N, v\rangle$ where $N = \{1, 2, \ldots, n\}$ is a finite set, the *set of players* and v is the *characteristic function* which assigns to every subset of N a real number with the restriction that v assigns 0 to the empty subset. When there can be no ambiguity about the set of players we identify $\langle N, v\rangle$ with the characteristic function v. Elements of 2^N, that is, the set of subsets of N, are called *coalitions* and for every coalition S the number $v(S)$ is regarded as the worth of S, i.e. that which the members of S can achieve if they work together. The question which arises is how to divide $v(N)$ once the grand coalition N is formed. A distribution among the players is represented by a *payoff vector* $x = (x_1, \ldots, x_n)$ with $x(N) := \sum_{i \in N} x_i = v(N)$. Here x_i is the amount that the i-th player obtains. In cooperative game theory several *solution concepts* which assign a payoff vector or a set of payoff vectors to a cooperative game are studied. In this paper we will only look at the core of a game. The core of a game $\langle N, v\rangle$ is denoted by $C(v)$ and is defined by

$$C(v) := \{x \in \mathbb{R}^n : x(N) = v(N), x(S) \geqslant v(S) \text{ for every } S \in 2^N\}.$$

Here and the remainder of this paper $x(S) := \sum_{i \in S} x_i$ for every $S \in 2^N$. If $v(N)$ is divided according to an element of $C(v)$ no coalition has an incentive to leave the grand coalition because no coalition can do better on its own. The core of a game can be empty. A game is called *balanced* if it has a non-empty core. For every coalition S we define the subgame $\langle S, v_S \rangle$ of $\langle N, v \rangle$ by $v_S(T) = v(T)$ for every $T \subset S$. A game is said to be *totally balanced* if all its subgames are balanced.

A game $\langle N, v \rangle$ is called $(0, 1)$-*normalized* if $v(i) = 0$ for each $i \in N$ and $v(N) = 1$.

A *simple game* is a cooperative game $\langle N, v \rangle$ with $v(N) = 1$ and $v(S) \in \{0, 1\}$ for every $S \in 2^N$. Such a game describes a situation where a coalition S is either all powerful, $v(S) = 1$, or completely powerless, $v(S) = 0$. Simple games can be used to model voting situations and situations with committee control. A player i in a simple game is called a *veto player* if $v(S) = 1$ implies $i \in S$. Player i is called a dictator if $v(S) = 1$ iff $i \in S$. It is a well known fact that a simple game has a non-empty core iff the set of veto players of the game is non-empty. Let $\langle N, v \rangle$ be a simple game with set of veto players $V \neq \emptyset$. Then

$$C(w) = \{x \in \mathbb{R}^n : x(V) = 1, x_i \geqslant 0 \text{ for all } i \in N\}.$$

Flow games are defined as follows. Let G be a directed network with set of vertices P and set of arcs L. For every $p \in P$ let $B(p)$ denote the set of arcs which start in p and $E(p)$ the set of arcs which end in p. Every arc $l \in L$ has a certain capacity $c(l) \geqslant 0$ and belongs to a player $i \in N$. We distinguish two different vertices from the others, a *source* and a *sink*. For each $S \in 2^N$ let G_S be the network obtained from G by keeping all the vertices but removing all arcs which are not owned by a member of S. The new set of arcs is denoted by L_S. Note that $G_N = G$ and $L_N = L$. A *flow* from source to sink in such a network is a function f from L_S to $\mathbb{R}$ with $0 \leqslant f(l) \leqslant c(l)$ for each $l \in L_S$ and such that for every vertex p except the source and the sink

$$\sum_{l \in B(p)} f(l) = \sum_{l \in E(p)} f(l).$$

The *value* of such a flow is

$$\sum_{l \in B(\text{source})} f(l) - \sum_{l \in E(\text{source})}$$

$$= \sum_{l \in E(\text{sink})} f(l) - \sum_{l \in B(\text{sink})} f(l).$$

In the flow game corresponding to this network $v(S)$ is defined to be the value of a maximum flow from source to sink in G_S.

Let A be a subset of P such that the source is an element of A and the sink is not. By $(A, P \backslash A)_S$ we denote the subset of L_S consisting of all arcs which have as their starting point an element of A and as their endpoint an element of $P \backslash A$. Formally,

$$(A, P \backslash A)_S := \{l \in L_S : l \in B(p) \text{ for a } p \in A \text{ and } l \in E(p) \text{ for a } q \in P \backslash A\}.$$

Such a subset of L_S is called a *cut* of the network G_S. Note that a cut of G can be made into a cut of G_S by removing from it all arcs which are not owned by a member of S. The capacity of a cut is the sum of the capacities of the arcs that it contains. A well-known theorem of Ford and Fulkerson [3] states that the value of a maximum flow in a network is equal to the capacity of a minimum cut, i.e., a cut with minimum capacity. Kalai and Zemel [5] use this theorem to prove that a flow game is totally balanced.

Here we want to combine the notions of flow games and simple games to define *flow games with committee control*. In these games the arcs are not owned by the players but are controlled by committees consisting of subsets of players. This committee control is modeled with the aid of simple games. To every arc $l \in L$ a simple game w_l is assigned. A coalition S is said to control l iff $w_l(S) = 1$. The network G_S is obtained from G by keeping all the vertices and removing all arcs which are not controlled by S. Again L_S denotes the resulting set of arcs and $G_N = G$, $L_N = L$. Again $v(S)$ is defined to be the value of a maximum flow in G_S. The following example shows that a flow game with committee control can have an empty core.

Example. Let $N = \{1, 2\}$, $P = \{\text{source, sink}\}$, L consists of one arc directed from the source to the sink with $c(l) = 10$, $w_l(S) = 1$ for every $S \in 2^N \backslash \{\emptyset\}$. Then $v(S) = 10$

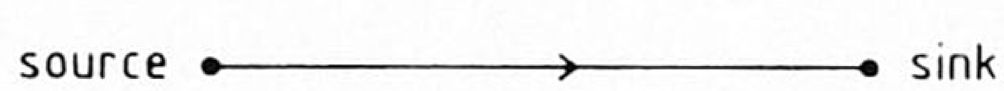

Fig. 1

for every $S \in 2^N \backslash \{\emptyset\}$ and it follows that $C(v) = \emptyset$.

Flow games with ownership can be seen as flow games with committee control as well. The simple game which describes the control of an arc is then defined to be the game with the owner as dictator. For every $i \in N$ we denote the game with i as dictator by δ_i. Hence, $\delta_i(S) = 1$ iff $i \in S$.

It follows from Kalai and Zemel [5] that flow games with committee control, where all the simple games describing the control of the arcs, have dictators, are totally balanced. In the next section we will see that these are not the only flow games with committee control which are balanced.

3. Balanced Games

In this section we will give a representation of non-negative balanced games as flow games with committee control. First we will show that a flow game with committee control where all the controlling games have veto players is balanced.

Theorem 1. *Let $\langle N, v\rangle$ be a flow game with committee control such that for all $l \in L$ the simple games w_l which describe the control of l have a non-empty set of veto players V_l. Then $C(v)$ is not empty.*

Proof. Let $(A, P\backslash A)$ be a minimum cut in the network G. For each $l \in (A, P\backslash A)$ let $z^l \in C(w_l)$. We define an $x \in \mathbb{R}^n$ by

$$x_i = \sum_{l \in (A, P\backslash A)} c(l) z_i^l \quad \text{for each } i \in N.$$

Then

$$x(N) = \sum_{l \in (A, P\backslash A)} c(l) = v(N).$$

Here the last equality follows from the theorem of Ford-Fulkerson. Further,

$$x(S) = \sum_{l \in (A, P\backslash A)} c(l) \sum_{i \in S} z_i^l \geqslant \sum_{l \in (A, P\backslash A)} c(l) w_l(S)$$

$$= \sum_{l \in (A, P\backslash A)_S} c(l) \geqslant v(S).$$

Here the second equality follows from the fact that we can obtain the cut $(A, P\backslash A)_S$ in the network G_S by removing all $l \in (A, P\backslash A)$ for which $w_l(S) = 0$. The last inequality follows from the Ford-Fulkerson theorem. So x as defined above is an element of $C(v)$. □

Although the existence of veto players in all the controlling games is sufficient to guarantee the balancedness of a flow game with committee control, it is not a necessary condition. In the proof above we only used the fact that there exists a minimum cut such that all arcs belonging to it are controlled by games which have veto players. But even this is not necessary for the flow game with committee control to be balanced as the following example shows.

Example. Let $N = \{1, 2, 3, 4\}$. The network G is given below, the numbers denote the capacities of the arcs.

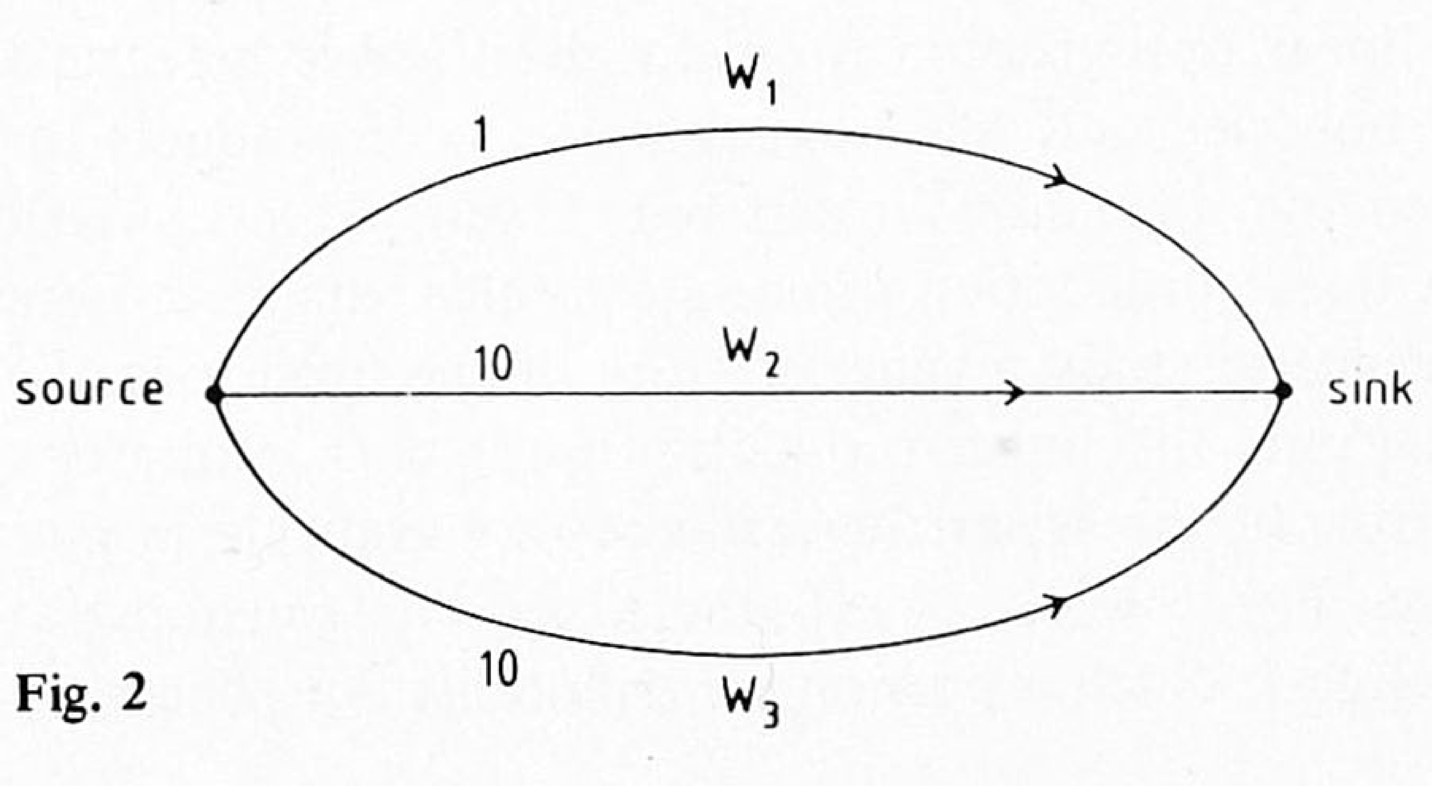

Fig. 2

The winning coalitions of w_1 are $\{1, 3\}$, $\{2, 4\}$ and N, those of w_2 are $\{1, 2\}$ and N and those of w_3 are $\{3, 4\}$ and N. Note that there is no minimum cut such that all arcs belonging to it have control games with veto players. The core of this game, however, is not empty. An element of the core, for example, is $(6, 5, 5, 5)$.

The remainder of this section will be devoted to the proof of the fact that every non-negative balanced game can be seen as a flow game with committee control where all arcs are controlled by simple games with veto players. In the following we will call such games veto rich flow games.

We will need the following theorem of Spinetto [8].

Theorem 2 (Spinetto). *Every $(0, 1)$-normalized balanced game $\langle N, v\rangle$ with $v \geqslant 0$ can be written as a convex combination of balanced simple games.*

The following lemma states that the veto rich flow games form a cone and that the minimum of two veto rich flow games is again a veto rich flow game.

Lemma 3. *Let $\langle N, v_1\rangle$ and $\langle N, v_2\rangle$ be two veto rich flow games. Then the games $\langle N, v_1 \wedge v_2\rangle$ and $\langle N, \alpha v_1 + \beta v_2\rangle$ with $\alpha, \beta \geqslant 0$, defined by $(v_1 \wedge v_2)(S) = v_1(S) \wedge v_2(S) = \min\{v_1(S), v_2(S)\}$ and $(\alpha v_1 + \beta v_2)(S) = \alpha v_1(S) + \beta v_2(S)$ are also veto rich flow games.*

Proof. Let G_1 be the network from which v_1 is obtained and G_2 the network from which v_2 is obtained. Then $v_1 \wedge v_2$ is obtained from the network which results by combining G_1 and G_2 in series, hereby identifying the sink of G_1 with the source of G_2.

Multiply all capacities in G_1 by α and all capacities in G_2 by β. Then $\alpha v_1 + \beta v_1$ is obtained from the network which results by combining G_1 and G_2 with their new capacities parallel, hereby identifying the sink of G_1 with the sink of G_2 and the source of G_1 with the source of G_2. □

Theorem 2 and Lemma 3 together yield the proof of the following lemma.

Lemma 4. *Every $(0, 1)$-normalized non-negative balanced game is a veto rich flow game.*

Proof. Let $\langle N, v\rangle$ be a $(0, 1)$-normalized non-negative balanced game. From theorem 2 it follows that there exist simple games $w_1, w_2, \ldots, w_k$, each having a non-empty set of veto players and $\alpha_1, \ldots, \alpha_k$ with $\alpha_1 + \ldots + \alpha_k = 1$ and $\alpha_j \geqslant 0$ for all $j \in \{1, \ldots, k\}$ such that $v = \alpha_1 w_1 + \ldots + \alpha_k w_k$. Trivially, each w_j is a veto rich flow game with one arc with capacity 1 and with committee control described by w_j. With Lemma 3 it follows that v is a veto rich flow game. □

Now we can prove our main theorem.

Theorem 5. *Every non-negative balanced game is a veto rich flow game.*

Proof. Let $\langle N, v\rangle$ be a balanced game with $v \geqslant 0$.

We define a game $\langle N, u\rangle$ by

$$u(S) := \min\{\sum_{i\in S} v(i), v(S)\}.$$

Then $\langle N, u\rangle$ is balanced, $C(u) = \{(v(1), \ldots, v(n))\}$. For every $T \in 2^N\backslash\{\emptyset\}$ let w_T be the simple game defined by $w_T(T) = w_T(N) = 1$ and $w_T(S) = 0$ for $S \neq T, N$.

Then

$$u = \min\left\{\sum_{i=1}^{n} v(i)\delta_i, \sum_{T\in 2^N\backslash\{\emptyset\}} v(T)w_T\right. .$$

The games δ_i and w_T are veto rich flow games and with Lemma 3 it follows that u is a veto rich flow game. Hence if $v = u$ we are done. If $v \neq u$ we define a game $\langle N, w\rangle$ by

$$w(S) = \frac{v(S) - u(S)}{v(N) - \sum_{i\in N} v(i)}$$

Note that $v(N) - \sum_{i\in N} v(i) > 0$ because there exists an S with $\sum_{i\in S} v(i) < v(S)$ and $C(v) \neq \emptyset$. The game w is non-negative and (0, 1)-normalized as can be verified straightforwardly. Let $x \in C(v)$. We define $y \in \mathbb{R}^n$ by

$$y_i := \frac{x_i - v(i)}{v(N) - \sum_{i\in N} v(i)} \quad \text{for each } i \in N.$$

Then $y(N) = 1 = w(N)$ and

$$\begin{aligned} y(S) &= (v(N) - \sum_{i\in N} v(i))^{-1}(x(S) - \sum_{i\in S} v(i)) \\ &\geqslant (v(N) - \sum_{i\in N} v(i))^{-1}(v(S) - \sum_{i\in S} v(i)) \\ &= (v(N) - \sum_{i\in N} v(i))^{-1}(v(S) - u(S)) = w(S) \end{aligned}$$

$$\text{if } v(S) \geqslant \sum_{i\in S} v(i).$$

$$\begin{aligned} y(S) &= (v(N) - \sum_{i\in N} v(i))^{-1}(x(S) - \sum_{i\in S} v(i)) \geqslant 0 \\ &= (v(N) - \sum_{i\in N} v(i))^{-1}(v(S) - u(S)) = w(S) \end{aligned}$$

$$\text{if } v(S) < \sum_{i\in S} v(i).$$

Hence $y \in C(w)$ and w is balanced. It follows with Lemma 4 that w is a veto rich flow game. Further,

$$v = (v(N) - \sum_{i\in N} v(i))w + u$$

and with Lemma 3 it follows that v is a veto rich flow game.

Theorem 1 and Theorem 5 together state that the family of balanced games coincides precisely with the family of veto rich flow games.

4. Linear Production Games with Committee Control

Owen [7] introduced *linear production games*. In these games the players are producers each of which owns certain amounts of m different *resources*. The *resource vector* of player $i \in N$ is denoted by $b^i = (b_1^i, \ldots, b_m^i) \geqslant 0$. These resources can be used to produce r different *products*. To produce one unit of product $j \in \{1, \ldots, r\}$ one needs a_{jk} units of the k-th resource, where $k \in \{1, \ldots, m\}$. One unit of product j can be sold at a given *market price* p_j. Every producer wants to maximize his profit by producing that combination of products that will yield him most when sold. The producers can pool their resources in order to maximize their profits. A linear production game $\langle N, v\rangle$ is defined by

$$v(S) := \max p \cdot x \quad \text{subject to}$$

$$x \in \mathbb{R}^r,\ x \geqslant 0,\ xA \leqslant b(S). \tag{*}$$

Here $b(S) = \sum_{i\in S} b^i$ is the resource vector of coalition S and $A = [a_{jk}]_{j=1,}^{r}\,{}_{k=1}^{m}$ is the production matrix. We assume that $a_{jk} \geqslant 0$ for all $j \in \{1, \ldots, r\}$, $k \in \{1, \ldots, m\}$ and that for all j there is a k such that $a_{jk} > 0$. (It is not possible to produce something with nothing.) Then the linear optimization problem given above is feasible and bounded and there exists a vector of products for which the maximum is achieved. Owen [7] has proved that linear production games are totally balanced. Here we want to study a generalization of the linear production game, the linear production game with committee control. In the latter the resources are available in portions. For $k \in \{1, \ldots, m\}$ there are d_k portions of resource k. Such a portion we denote by B_k^q. The total amount of resource k which is available is $B_k = \sum_{q=1}^{d_k} B_k^q$. Each portion B_k^q is controlled by committees of players. For every B_k^q a simple game $\langle N, w_k^q \rangle$ describes the control. Coalition S can use B_k^q only if S is winning in w_k^q. Let $B_k^q(S) := B_k^q \cdot w_k^q(S)$. The total amount of the k-th resource available to coalition S is $B_k(S) := \sum_{q=1}^{d_k} B_k^q(S)$. The resource vector $B(S)$ of coalition S has coordinates $B_1(S), B_2(S), \ldots, B_m(S)$.

The characteristic function v of the *linear production game with committee control* is again given by (*) but with $b(S)$ replaced by $B(S)$. The following theorem shows that, just as in the case of flow games with committee control, if the simple games are balanced then the whole game will be balanced.

Theorem 6. *Linear production games with committee control where all the simple games which describe the controls are balanced, have a non-empty core.*

Proof. Let $\langle N, v \rangle$ be a linear production game with committee control which satisfies the condition of the theorem.

From the duality theorem of linear programming it follows that

$$v(S) = \min B(S) \cdot y \quad \text{subject to}$$

$$y \in \mathbb{R}^m,\ y \geqslant 0,\ Ay \geqslant p.$$

This is the dual problem of (*). Let $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_m)$ be a solution of the dual problem which determines $v(N)$. Let z_k^q be a core element of w_k^q. Let $z \in \mathbb{R}^n$ be defined by

$$z_i := \sum_{k=1}^{m} \hat{y}_k \sum_{q=1}^{d_k} B_k^q (z_k^q)_i$$

Then

$$\sum_{i \in N} z_i = \sum_{k=1}^{m} \hat{y}_k B_k = \sum_{k=1}^{m} \hat{y}_k \cdot B_k(N) = v(N).$$

Further,

$$\sum_{i \in S} z_i \geqslant \sum_{k=1}^{m} \hat{y}_k B_k(S) \geqslant v(S).$$

Here the first inequality follows from the fact that

$$B_k(S) = \sum_{q=1}^{d_k} B_k^q(S) \leqslant \sum_{q=1}^{d_k} B_k^q z_k^q(S)$$

and the second inequality follows from the fact that $\hat{y}$ is feasible for the dual problem which determines $v(S)$ because the constraints in the dual problem are independent of S. So $z \in C(v)$. □

That linear production games with committee control are indeed generalizations of the classical linear production games can be seen as follows. Let $\langle N, v \rangle$ be a linear production game where each $i \in N$ has a resource vector $b^i = (b_1^i, \ldots, b_m^i)$. For every $k \in \{1, \ldots, m\}$ we take $d_k = n$ and $B_k^q = b_k^q$ for $q \in \{1, \ldots, d_k\}$. The simple game which describes the control of B_k^q is taken to be the dictator game δ_q. Straightforward verification shows that the linear production game with committee control that we obtain in this way is equal to $\langle N, v \rangle$. So Theorem 6 implies Owens [7] theorem. But Theorem 6 also follows from Owens theorem. Let $\langle N, u \rangle$ be a linear production game with committee control. For every $k \in \{1, \ldots, m\}$ we consider the game $\langle N, B_k \rangle$ defined by

$$B_k(S) := \sum_{q=1}^{d_k} B_k^q(S) = \sum_{q=1}^{d_k} B_k^q \cdot w_k^q(S).$$

Let z_k^q be a core element of w_k^q for every $k \in \{1, \ldots, m\}$, $q \in \{1, \ldots, d_k\}$. We define $y^k \in \mathbb{R}^n$ by

$$y^k = \sum_{q=1}^{d_k} B_k^q z_k^q$$

Then y^k is an element of $C(B_k)$ for every $k \in \{1, \ldots, m\}$. Let $\langle N, v \rangle$ be the linear production game where every $i \in N$ has resource vector $b^i = (y_i^1, y_i^2, \ldots, y_i^m)$. Then

$$b(S) = \Big(\sum_{i \in S} y_i^1, \sum_{i \in S} y_i^2, \ldots, \sum_{i \in S} y_i^m\Big)$$
$$\geqslant (B_1(S), B_2(S), \ldots, B_m(S))$$

because $y^k \in C(B_k)$ for every $k \in \{1, \ldots, m\}$. Hence $b(S) \geqslant B(S)$ for every $S \in 2^N$ and $b(N) = B(N)$. It follows that $v(S) \geqslant u(S)$ for every $S \in 2^N$ and $v(N) = u(N)$. Thus we can construct for every linear production game with committee control a classical linear production game such that in both games the value of the grand coalition is the same and the values of the other coalitions are not less in the latter than in the former. From the balancedness of the linear production game the balancedness of the linear production game with committee control follows and we see that Theorem 6 follows from Owens theorem.

If all the simple games which describe the controls are monotonic, i.e., $v(S) \geqslant v(T)$ if $S \supset T$ and not balanced then the linear production game with committee control is not balanced too.

Theorem 7. *Linear production games with committee control where all the simple games which describe the controls are monotonic and not balanced, have an empty core.*

Proof. Because w_k^g has no veto players and is monotonic we have $w_k^g(N\backslash\{i\}) = w_k^g(N)$ for each $i \in N$. Hence $B(N\backslash\{i\}) = B(N)$ for each $i \in N$ which implies that $v(N\backslash\{i\}) = v(N)$ for each $i \in N$. For x to be an element of $C(v)$ it is required that $x(N\backslash\{i\}) = x(N)$ for every $i \in N$, but then $x_i = 0$ for every $i \in N$ and $x(N) = 0 < v(N)$. So $C(v) = \emptyset$. □

In Curiel et al. [1] linear production games with committee control which are not balanced are studied. Granot [4] also studies a generalization of linear production games. Although the two generalizations look different it can be shown that they are equivalent.

The concept of committee control can also be found in Dubey and Shapley [2]. The result on the nonemptiness of the core of a linear production game with committee control follows also from their result but contrary to their proof, the proof here is constructive.

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