MATHEMATICS OF OPERATIONS RESEARCH
Vol. 22, No. 3, August 1997
Printed in U.S.A.

ON THE DETERMINATENESS OF $m \times \infty$-BIMATRIX GAMES

HENK NORDE AND JOS POTTERS

In this paper a positive answer is given to the question of whether every semi-infinite bimatrix game is weakly determined.

1. Introduction. A noncooperative game in normal form is called *determined* if there are (*Nash*) *equilibria*: outcomes from which unilateral deviation yields no profit for the deviating player. Von Neumann (1928) proved that mixed extensions of finite matrix games are determined. For finite bimatrix games the determinateness was shown by Nash (1951). Noncooperative games, in which at least one player has a large (e.g., noncompact) strategy space and where unbounded payoff functions are allowed, are not necessarily determined. For such games *approximate equilibria* become important. In Tijs (1977) and (1981) approximate equilibrium point theorems for two person games and n-person games can be found.

Lucchetti, Patrone and Tijs (1986) introduced different types of approximate equilibria for two-person games, leading to the notion of *weak determinateness*. In their paper, dealing with semi-infinite bimatrix games, various boundedness restrictions are provided, guaranteeing weak determinateness. However, they did not solve the general question of whether any semi-infinite bimatrix game is weakly determined. In this paper we answer this question in the affirmative. For $m = 2$ this result was already obtained by Jurg and Tijs (1993).

The question of weak determinateness for $\infty \times \infty$-(bi)matrix games has a negative answer. Wald (1945) provides an example of such a matrix game which is not weakly determined.

This paper is organized as follows. In §2 definitions are given concerning the notion of weak determinateness of semi-infinite bimatrix games. Some lemmas, dealing with the continuity of approximate optimal responses of the second player (with infinitely many pure strategies), are provided in §3. In §4 we derive the main result of this paper: Every semi-infinite bimatrix game is weakly determined.

NOTATION. The unit vectors in $\mathbb{R}^m$ are denoted by $e_1, \ldots, e_m$. The set of extreme points of a polytope K in $\mathbb{R}^m$ is denoted by $\mathrm{ext}(K)$ and the convex hull of a set $V \subset \mathbb{R}^m$ by $\mathrm{ch}(V)$. By $\mathbb{R}^{\mathbb{N}}$ we denote the set of sequences of real numbers and for $q = (q_1, q_2, \cdots) \in \mathbb{R}^{\mathbb{N}}$ the carrier of q is given by $C(q) := \{ j \in \mathbb{N}: q_j \neq 0 \}$. For $p \in \mathbb{R}^m$ the carrier $C(p)$ is defined in an analogous way. The Euclidean norm in $\mathbb{R}^m$ is denoted by $\|\cdot\|$.

2. Definitions. For two real $m \times \infty$-matrices $A = [a_{ij}]_{i=1;j=1}^{m\;\;\infty}$ and $B = [b_{ij}]_{i=1;j=1}^{m\;\;\infty}$ the *semi-infinite bimatrix game* (A, B) is the two-person noncooperative game, with strategy spaces

Received May 2, 1995; revised August 12, 1996 and December 17, 1996.
AMS 1991 subject classification. Primary: 90D10.
OR/MS subject classification. Primary: Games/Noncooperative.
Key words. Semi-infinite bimatrix game, approximate equilibria, determinateness.

0364-765X/97/2203/0631/$05.00

$$\Delta_m := \left\{ p \in \mathbb{R}^m \,|\, p \geq 0, \sum_{i=1}^{m} p_i = 1 \right\},$$

for player 1 (the row player),

$$\Delta_\infty := \left\{ q \in \mathbb{R}^{\mathbb{N}} \,|\, q \geq 0, \sum_{j=1}^{\infty} q_j = 1, q_j = 0 \text{ for large } j \right\},$$

for player 2 (the column player) and payoff matrices A and B. In a play of this game player 1 chooses a strategy $p \in \Delta_m$ and player 2 chooses a strategy $q \in \Delta_\infty$. Subsequently, player 1 obtains an (expected) payoff $pAq := \sum_{i=1}^{m}\sum_{j=1}^{\infty} p_i a_{ij} q_j$ and player 2 obtains $pBq := \sum_{i=1}^{m}\sum_{j=1}^{\infty} p_i b_{ij} q_j$.

If player 1 chooses strategy $p \in \Delta_m$ the maximal obtainable payoff for player 2 is

$$v_B(p) := \sup_{q \in \Delta_\infty} pBq.$$

Obviously $v_B(p) \in \mathbb{R} \cup \{+\infty\}$. Strategy $p \in \Delta_m$ is called *bounded* if $v_B(p) \in \mathbb{R}$, otherwise p is called *unbounded*. The set of bounded strategies is denoted by C_B. For $p \in C_B$ a strategy $p' \in \Delta_m$ is called a *bounded direction with origin* p if there is a number $\epsilon \in (0, 1]$ with $(1 - \epsilon)p + \epsilon p' \in C_B$. The set of bounded directions with origin p is denoted by $D_B(p)$. In the sequel we omit the subscript B for notational convenience, since no confusion about the payoff matrix B can occur. We will need the following lemma.

LEMMA 1. *For every $m \times \infty$-matrix B the set C of bounded strategies is a convex subset of Δ_m and the restriction of v to C is a convex function. Moreover, for every $p \in C$, the set $D(p)$ of bounded directions with origin p is a convex subset of Δ_m as well.*

PROOF. Let $p_0, p_1 \in C$ and $t \in [0, 1]$. Put $p_t := (1 - t)p_0 + tp_1$. For every strategy $q \in \Delta_\infty$ we have $p_t Bq = (1 - t)p_0 Bq + tp_1 Bq \leq (1 - t)v(p_0) + tv(p_1)$, which proves that $p_t \in C$ and $v(p_t) \leq (1 - t)v(p_0) + tv(p_1)$.

Let $p \in C$, $p_0, p_1 \in D(p)$ and $t \in [0, 1]$. Then there are numbers $\epsilon_0, \epsilon_1 \in (0, 1]$ such that $(1 - \epsilon_i)p + \epsilon_i p_i \in C$ for every index $i \in \{0, 1\}$. Let $\epsilon := \min\{\epsilon_0, \epsilon_1\}$. By convexity of C we have $(1 - \epsilon)p + \epsilon p_i \in C$ for every index $i \in \{0, 1\}$. Writing $p_t := (1 - t)p_0 + tp_1$ we have, again by convexity of C,

$$(1 - \epsilon)p + \epsilon p_t = (1 - t)((1 - \epsilon)p + \epsilon p_0) + t((1 - \epsilon)p + \epsilon p_1) \in C.$$

Hence $p_t \in D(p)$. □

The previous lemma shows that the set C of bounded strategies is convex. The following example shows however, that this set need not be open or closed.

EXAMPLE. Consider the $3 \times \infty$-matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ -1 & -4 & -9 & -16 & \cdots \end{bmatrix}.$$

One easily verifies that $C = \Delta_3 \setminus \{te_1 + (1 - t)e_2 : t \in [0, 1)\}$, which is not open or closed. Moreover $D(e_1) = C$ and $D(p) = \Delta_3$ for every $p \in C \setminus \{e_1\}$.

For a semi-infinite bimatrix game (A, B) we introduce two types of partial equilibria.

Let $\epsilon > 0$. A strategy-pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_\infty$ is called an *ϵ-optimal partial equilibrium of* (A, B) if $\hat{p}A\hat{q} = \max_{p \in \Delta_m} pA\hat{q}$ and $\hat{p}B\hat{q} > v_B(\hat{p}) - \epsilon$. So, in an ϵ-optimal partial equilibrium, player 1 cannot gain by deviating unilaterally and player 2 can gain at most ϵ. The set of all ϵ-optimal partial equilibria will be denoted by $PE^{\epsilon}(A, B)$.

Let $k \in \mathbb{R}$. A strategy-pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_\infty$ is called a *k-guaranteeing partial equilibrium of* (A, B) if $\hat{p}A\hat{q} = \max_{p \in \Delta_m} pA\hat{q}$ and $\hat{p}B\hat{q} > k$. So, in a k-guaranteeing partial equilibrium, player 1 cannot gain by deviating unilaterally and player 2 receives at least k. The set of all k-guaranteeing partial equilibria will be denoted by $PE_k(A, B)$.

Clearly, a Nash equilibrium of a semi-infinite bimatrix game is an ϵ-optimal partial equilibrium for every $\epsilon > 0$. However, the following example shows that the set of Nash equilibria may be empty. For such games partial equilibria become important.

EXAMPLE. The $1 \times \infty$-bimatrix game (A_1, B_1) with

$$A_1 = [0\,0\,0\,0 \cdots] \quad \text{and} \quad B_1 = \left[\frac{1}{2}\,\frac{2}{3}\,\frac{3}{4}\,\frac{4}{5} \cdots\right]$$

has an ϵ-optimal partial equilibrium for every $\epsilon > 0$.

The $1 \times \infty$-bimatrix game (A_2, B_2) with

$$A_2 = [0\,0\,0\,0 \cdots] \quad \text{and} \quad B_2 = [1\,2\,3\,4 \cdots]$$

has a k-guaranteeing partial equilibrium for every $k \in \mathbb{R}$. Obviously, both games have no Nash equilibria.

The game (A, B) is called *weakly determined* if, at least, one of the following properties holds true:

$$PE^{\epsilon}(A, B) \neq \varnothing \quad \text{for every } \epsilon > 0,$$
$$PE_k(A, B) \neq \varnothing \quad \text{for every } k \in \mathbb{R}.$$

The concept of weak determinateness, introduced above, is somewhat stronger than the one introduced in Lucchetti et al. (1986), where this concept is based upon four types of equilibria: (ϵ_1, ϵ_2)-equilibria (strategy-pairs in which player 1 (player 2) can gain no more than $\epsilon_1(\epsilon_2)$ by deviating unilaterally), (ϵ, k)-equilibria (strategy-pairs in which player 1 can gain no more than ϵ and player 2 receives at least k), (k, ϵ)-equilibria (strategy-pairs in which player 1 receives at least k and player 2 can gain no more than ϵ) and (k_1, k_2)-equilibria (strategy-pairs in which player 1 (player 2) receives at least $k_1(k_2)$). Notice that the (ϵ_1, ϵ_2)-equilibria with $\epsilon_1 = 0$ coincide with the ϵ_2-optimal partial equilibria and that the (ϵ, k)-equilibria with $\epsilon = 0$ coincide with the k-guaranteeing partial equilibria.

In this paper we prove that every semi-infinite bimatrix game (A, B) is weakly determined. Notice that this is equivalent to

$$PE^{\epsilon}(A, B) \cup PE_k(A, B) \neq \varnothing$$

for every $\epsilon > 0$ and $k \in \mathbb{R}$.

For $q \in \Delta_\infty$ the set

$$B(q) := \{p \in \Delta_m \mid pAq = \max_{p' \in \Delta_m} p'Aq\}$$
$$= \operatorname{ch}\{e_i \mid e_iAq = \max_{j \in \{1, \ldots, m\}} e_jAq\}$$

is the set of *best responses* of player 1 to strategy q.

For $p \in \Delta_m$, $\epsilon > 0$ and $k \in \mathbb{R}$ the set

$$B^\epsilon(p) := \{q \in \Delta_\infty \mid pBq > v(p) - \epsilon\}$$

is the set of *ϵ-best responses* of player 2 to strategy p (notice that $B^\epsilon(p) \neq \emptyset$ if and only if $p \in C$) and the set

$$G_k(p) := \{q \in \Delta_\infty \mid pBq > k\}$$

is the set of *k-guaranteeing responses* of player 2 to strategy p. Clearly, for $(p, q) \in \Delta_m \times \Delta_\infty$, $\epsilon > 0$ and $k \in \mathbb{R}$ one has

$$(p, q) \in PE^\epsilon(A, B) \cup PE_k(A, B),$$

if and only if

$$p \in B(q) \quad \text{and} \quad q \in B^\epsilon(p) \cup G_k(p).$$

A set $V \subset \Delta_m$ is called a *neighborhood of $p \in \Delta_m$* if there is a number $\eta > 0$ such that $\{p' \in \Delta_m : \|p' - p\| < \eta\} \subset V$. In this paper we will make frequent use of the following lemma.

LEMMA 2. *For every strategy $p \in \Delta_m$ and every number $\delta \in (0, 1]$ the set*

$$(1 - \delta)\{p\} + \delta\Delta_m := \{(1 - \delta)p + \delta p' : p' \in \Delta_m\}$$

is a neighborhood of p.

PROOF. *Let $p \in \Delta_m$* and $\delta \in (0, 1]$. Define $M := \max\{p_i^{-1} : p_i > 0\}$. For every $p' \in \Delta_m$ with $\|p' - p\| < \delta/M$ one has $r := p + \delta^{-1}(p' - p) \in \Delta_m$ and hence $p' = (1 - \delta)p + \delta r \in (1 - \delta)\{p\} + \delta\Delta_m$. □

3. Continuity of approximate optimal responses. In this section we provide a theorem which is crucial for the proof of the weak determinateness of any semi-infinite bimatrix game. This theorem states that, for every $\epsilon > 0$, $k \in \mathbb{R}$, the strategy space Δ_m of player 1 can be covered by finitely many polytopes K, satisfying

$$\bigcap_{p \in \operatorname{ext}(K)} (B^\epsilon(p) \cup G_k(p)) \neq \emptyset, \tag{1}$$

i.e., by finitely many polytopes K, which are such that player 2 has a common approximate optimal (ϵ-best or k-guaranteeing) response to every extreme point of K. Polytopes, having property (1), are called *(ϵ, k)-polytopes*. In order to prove this theorem we need some lemmas, dealing with the continuity of approximate optimal responses of player 2. The first lemma is concerned with the continuity of ϵ-best responses of player 2 if player 1 changes from one bounded strategy to another. More precisely:

LEMMA 3. *Let B be an $m \times \infty$-matrix, $p_0 \in C$, $p_1 \in D(p_0)$, $\epsilon > 0$ and $q \in B^\epsilon(p_0)$. Then there is a number $\delta \in (0, 1]$ such that $q \in B^\epsilon((1 - t)p_0 + tp_1)$ for every $t \in [0, \delta]$.*

PROOF. Without loss of generality we may assume that $p_1 \in C$. Put $p_t := (1 - t)p_0 + tp_1$ for every number $t \in [0, 1]$. Suppose there is no number $\delta \in (0, 1]$ satisfying the required properties. Then there is a sequence $t_1, t_2, \cdots$ in $(0, 1]$ such that $\lim_{n\to\infty} t_n = 0$ and

$$p_{t_n}Bq \leq v(p_{t_n}) - \epsilon \tag{2}$$

for every index $n \in \mathbb{N}$. Since v is a convex function, we also have

$$v(p_{t_n}) \leq (1 - t_n)v(p_0) + t_n v(p_1) \tag{3}$$

for every index $n \in \mathbb{N}$. Combination of (2) and (3) gives

$$\begin{aligned} v(p_1) &\geq t_n^{-1}v(p_{t_n}) - (t_n^{-1} - 1)v(p_0) \\ &\geq t_n^{-1}p_{t_n}Bq + t_n^{-1}\epsilon - (t_n^{-1} - 1)v(p_0) \\ &= (t_n^{-1} - 1)p_0Bq + p_1Bq + t_n^{-1}\epsilon - (t_n^{-1} - 1)v(p_0) \\ &= t_n^{-1}(p_0Bq - (v(p_0) - \epsilon)) + p_1Bq - p_0Bq + v(p_0) \end{aligned}$$

for every index $n \in \mathbb{N}$. Since $p_0Bq - (v(p_0) - \epsilon) > 0$ and $\lim_{n\to\infty} t_n^{-1} = +\infty$ we get $v(p_1) = +\infty$. Contradiction. □

COROLLARY 1. *Let B be an $m \times \infty$-matrix, and let strategy $p \in C$ be such that $D(p) = \Delta_m$. Then there is an (ϵ, k)-polytope K which is a neighborhood of p.*

PROOF. Choose $q \in B^\epsilon(p)$. According to Lemma 3 there is, for every index $i \in \{1, \dots, m\}$, a number $\delta_i \in (0, 1]$ such that $q \in B^\epsilon((1 - t)p + te_i)$ for every $t \in [0, \delta_i]$. For the number $\delta := \min_i \delta_i$ the polytope $K := (1 - \delta)\{p\} + \delta\Delta_m$ satisfies the required properties. □

The following lemma provides a similar result for unbounded strategies.

LEMMA 4. *Let B be an $m \times \infty$-matrix, and let $p \in \Delta_m \backslash C$. Then there is an (ϵ, k)-polytope K which is a neighborhood of p.*

PROOF. Choose $q \in G_k(p)$. Since the map $p' \mapsto p'Bq$ $(p' \in \Delta_m)$ is continuous and $pBq > k$ there is a number $\delta \in (0, 1]$ such that $q \in G_k((1 - \delta)p + \delta e_i)$ for every index $i \in \{1, \dots, m\}$. For this number δ the polytope $K := (1 - \delta)\{p\} + \delta\Delta_m$ satisfies the required properties. □

So, every bounded strategy p with $D(p) = \Delta_m$, as well as every unbounded strategy, is an interior point of some (ϵ, k)-polytope. However, this statement need not be true for the last case: $p \in C$ and $D(p) \neq \Delta_m$. In Lemma 7 we prove that in this case there are *finitely many* (ϵ, k)-polytopes which cover a neighborhood of p. In order to prove this lemma we need two auxiliary lemmas. The first of these lemmas deals with the continuity of approximate optimal responses of player 2 if player 1 changes from a bounded strategy to an unbounded one (see also Proposition 3.1 in Jurg and Tijs (1993) where a similar result has been derived). More precisely:

LEMMA 5. *Let B be an $m \times \infty$-matrix, $p_0 \in C$, $p_1 \in \Delta_m \backslash D(p_0)$, $\epsilon > 0$ and $k \in \mathbb{R}$. Then $B^\epsilon(p_0) \cap G_k(p_1) \neq \emptyset$.*

PROOF. Suppose $B^\epsilon(p_0) \cap G_k(p_1) = \emptyset$. Choose $\hat{q} \in B^{\epsilon/2}(p_0) \subset B^\epsilon(p_0)$ and let $\delta := p_0B\hat{q} - (v(p_0) - \frac{1}{2}\epsilon)$ (>0). By assumption, we have $\gamma := k - p_1B\hat{q} \geq 0$. We will show that

$$\tilde{p} := \frac{\gamma}{\gamma + \delta} p_0 + \frac{\delta}{\gamma + \delta} p_1 \in C,$$

which leads to a contradiction, since $p_1 \notin D(p_0)$.

Let $q \in B^\epsilon(p_0)$. Then, by assumption, we have $p_1 B q \leq k$. Together with $p_0 B q \leq v(p_0)$ we get

$$\tilde{p} B q \leq \max\{v(p_0), k\}. \tag{4}$$

Now let $q \in \Delta_\infty \backslash B^\epsilon(p_0)$. Then $p_0 B q \leq v(p_0) - \epsilon$ and hence $\kappa := v(p_0) - \frac{1}{2}\epsilon - p_0 B q > 0$. For $\tilde{q} = (\delta + \kappa)^{-1}(\kappa \hat{q} + \delta q)$ we have $p_0 B \tilde{q} = v(p_0) - \frac{1}{2}\epsilon$ and hence $\tilde{q} \in B^\epsilon(p_0)$. So, by assumption, we have $p_1 B \tilde{q} \leq k$, which implies

$$\tilde{p} B q \leq (\gamma + \delta)^{-1}(k\delta + (v(p_0) - \frac{1}{2}\epsilon)\gamma) \leq \max\{v(p_0), k\}. \tag{5}$$

Now (4) and (5) together yield $\tilde{p} \in C$. □

In the following lemma we prove that, given $p \in C$ with $D(p) \neq \Delta_m$, the simplex Δ_m can be covered by finitely many polytopes K of the following nature: All extreme points of K that are not elements of $D(p)$ admit a common k-guaranteeing response q, which is also an ϵ-best response to p. As the proof of this lemma does not use any game theoretical argument we formulate it as a geometric lemma.

LEMMA 6. *Let D be a convex subset of Δ_m, $D \neq \Delta_m$, and let O_p be a neighborhood of p for every $p \in \Delta_m \backslash D$. Then there are finitely many polytopes $M_1, \ldots, M_n$ and $p_1, \ldots, p_n \in \Delta_m \backslash D$ such that $\Delta_m = \bigcup_{j=1}^n M_j$ and $\text{ext}(M_j) \subset D \cup O_{p_j}$ for every $j \in \{1, \ldots, n\}$.*

PROOF. The proof is by induction on m. If $m = 1$, then Δ_1 consists of one point, $D = \emptyset$, and the lemma is trivial. Suppose the lemma has been proved for simplices with less than m extreme points. We distinguish between two cases.

Case 1. $D = \emptyset$. For every point $p \in \Delta_m$ there is a number $\delta(p) \in (0, 1]$ such that $K(p) := (1 - \delta(p))\{p\} + \delta(p)\Delta_m \subset O_p$. Since Δ_m is compact a finite subcollection of $\{K(p) | p \in \Delta_m\}$ satisfies the required properties.

Case 2. $D \neq \emptyset$. Choose $\tilde{p} \in D$. For every index $i \in \{1, \ldots, m\}$ we put $\Delta_m^{-i} := \text{ch}(\{e_j | j \neq i\})$ and $L_i := \text{ch}(\{\tilde{p}\} \cup \Delta_m^{-i})$. If $\Delta_m^{-i} \subset D$ then $\text{ext}(L_i) \subset D \subset D \cup O_p$ for any $p \in \Delta_m \backslash D$. Otherwise $D \cap \Delta_m^{-i} \neq \Delta_m^{-i}$ and we infer by induction hypothesis that there are polytopes $M_{i1}, \ldots, M_{ik_i}$ and points $p_{i1}, \ldots, p_{ik_i} \in \Delta_m^{-i} \backslash (D \cap \Delta_m^{-i})$ such that $\Delta_m^{-i} = \bigcup_{j=1}^{k_i} M_{ij}$ and $\text{ext}(M_{ij}) \subset (D \cap \Delta_m^{-i}) \cup (O_{p_{ij}} \cap \Delta_m^{-i})$ for every $j \in \{1, \ldots, k_i\}$. Now the polytope $\tilde{M}_{ij} := \text{ch}(\{\tilde{p}\} \cup M_{ij})$ satisfies $\text{ext}(\tilde{M}_{ij}) \subset \{\tilde{p}\} \cup \text{ext}(M_{ij}) \subset D \cup O_{p_{ij}}$ for every $j \in \{1, \ldots, k_i\}$. Moreover $L_i = \bigcup_{j=1}^{k_i} \tilde{M}_{ij}$. The observation that $\Delta_m = \bigcup_{i=1}^m L_i$ finishes the proof of the lemma. □

LEMMA 7. *Let B be an $m \times \infty$-matrix, $\hat{p} \in C$ with $D(\hat{p}) \neq \Delta_m$ and $\epsilon > 0$, $k \in \mathbb{R}$. Then there are finitely many (ϵ, k)-polytopes $K_1, \ldots, K_n$ such that $\bigcup_{j=1}^n K_j$ is a neighborhood of $\hat{p}$.*

PROOF. For every $p \in \Delta_m \backslash D(\hat{p})$ there is, according to Lemma 5, a strategy $q_p \in B^\epsilon(\hat{p}) \cap G_k(p)$. Since $pBq_p > k$ it is possible to choose a neighborhood O_p of p such that $p' B q_p > k$ for every $p' \in O_p$. So, q_p is an ϵ-best response to $\hat{p}$ and a k-guaranteeing response to every $p' \in O_p$.

Since $D(\hat{p})$ is a convex, proper subset of Δ_m there are, according to Lemma 6, finitely many polytopes $M_1, \ldots, M_n$ and $p_1, \ldots, p_n \in \Delta_m \backslash D(\hat{p})$ such that $\text{ext}(M_j) \subset D(\hat{p})$

$\cup O_{p_j}$ for every index $j \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, n\}$. Write $q_j := q_{p_j}$, $G_j := \text{ext}(M_j) \cap O_{p_j}$ and $H_j' := \text{ext}(M_j) \cap D(\hat{p})$. Since H_j' is a finite subset of $D(\hat{p})$ and q_j is an ϵ-best response to $\hat{p}$ there is, according to Lemma 3, a number $\delta_j \in (0, 1]$ such that q_j is an ϵ-best response to $(1 - \delta_j)\hat{p} + \delta_j p$ for every $p \in H_j'$. For this number δ_j we write $H_j := \{(1 - \delta_j)\hat{p} + \delta_j p \mid p \in H_j'\}$. Because q_j is an ϵ-best response to $\hat{p}$ and every strategy in H_j and a k-guaranteeing response to every strategy in G_j the polytope $K_j := \text{ch}(\{\hat{p}\} \cup G_j \cup H_j)$ is an (ϵ, k)-polytope.

Now we are left to prove that $\bigcup_{j=1}^{n} K_j$ is a neighborhood of $\hat{p}$. Let $\delta := \min\{\delta_1, \ldots, \delta_n\}$ and $j \in \{1, \ldots, n\}$. Then $(1 - \delta)\hat{p} + \delta p \in K_j$ for every strategy $p \in \text{ext}(M_j)$ (for $p \in G_j$ this statement is immediate, for $p \in H_j'$ one has $(1 - \delta)\hat{p} + \delta p \in \text{ch}(\{\hat{p}, (1 - \delta_j)\hat{p} + \delta_j p\}) \subset K_j$). Hence $(1 - \delta)\{\hat{p}\} + \delta M_j \subset K_j$. Therefore

$$(1 - \delta)\{\hat{p}\} + \delta\Delta_m = \bigcup_{j=1}^{n} ((1 - \delta)\{\hat{p}\} + \delta M_j) \subset \bigcup_{j=1}^{n} K_j,$$

which proves that $\bigcup_{j=1}^{n} K_j$ is a neighborhood of $\hat{p}$. □

Now we are able to prove the main theorem of this section.

THEOREM 1. *Let B be an $m \times \infty$-matrix, $\epsilon > 0$, $k \in \mathbb{R}$. Then there are finitely many (ϵ, k)-polytopes $L_1, \ldots, L_t$ such that $\Delta_m = \bigcup_{j=1}^{t} L_j$.*

PROOF. According to Corollary 1, Lemmas 4 and 7 there is, for every $p \in \Delta_m$, a neighborhood U_p of p which is the union of finitely many (ϵ, k)- polytopes. Since Δ_m is compact a finite subcollection of $\{U_p : p \in \Delta_m\}$ covers Δ_m, which proves the theorem. □

4. Weak determinateness of semi-infinite bimatrix games. In this section we prove the main result of this paper:

THEOREM 2. *Every semi-infinite bimatrix game is weakly determined.*

PROOF. Let $A = [a_{ij}]_{i=1;j=1}^{m\ \ \infty}$ and $B = [b_{ij}]_{i=1;j=1}^{m\ \ \infty}$ be two $m \times \infty$-matrices, $\epsilon > 0$ and $k \in \mathbb{R}$. It suffices to prove that $PE^{\epsilon}(A, B) \cup PE_k(A, B) \neq \emptyset$. According to Theorem 1 there are finitely many (ϵ, k)-polytopes $L_1, \ldots, L_t$ such that $\Delta_m = \bigcup_{j=1}^{t} L_j$. Choose, for every index $j \in \{1, \ldots, t\}$, a strategy $q_j \in \bigcap_{p \in \text{ext}(L_j)} (B^{\epsilon}(p) \cup G_k(p))$ and let $K := \bigcup_{j=1}^{t} C(q_j)$. The set K is a finite set. Therefore, the bimatrix game, with payoff matrices $[a_{ij}]_{i \in \{1,\ldots,m\}, j \in K}$ and $[b_{ij}]_{i \in \{1,\ldots,m\}, j \in K}$, is a finite bimatrix game. Nash (1951) proved that such a game has a Nash-equilibrium. In other words, there is a strategy-pair $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_\infty$ such that

(1) $\hat{p}A\hat{q} \geq pA\hat{q}$ for every $p \in \Delta_m$,

(2) $\hat{p}B\hat{q} \geq \hat{p}Bq$ for every $q \in \Delta_\infty$ with $C(q) \subset K$.

There is a $j^* \in \{1, \ldots, t\}$ such that $\hat{p} \in L_{j^*}$. Write $\hat{p} = \sum_{l=1}^{s} \alpha_l p_l$ for some $p_1, \ldots, p_s \in \text{ext}(L_{j^*})$ and $\alpha_1, \ldots, \alpha_s > 0$ with $\sum_{l=1}^{s} \alpha_l = 1$. Since $C(q_{j^*}) \subset K$ we have $\hat{p}B\hat{q} \geq \hat{p}Bq_{j^*}$. This last inequality implies the existence of an index $r \in \{1, \ldots, s\}$ such that $p_r B\hat{q} \geq p_r Bq_{j^*}$. Since, by construction, $q_{j^*} \in B^{\epsilon}(p_r) \cup G_k(p_r)$ this last inequality gives

$$\hat{q} \in B^{\epsilon}(p_r) \cup G_k(p_r). \tag{6}$$

On the other hand we have $C(p_r) \subset C(\hat{p})$ and $\hat{p} \in B(\hat{q})$. Hence,

$$p_r \in B(\hat{q}). \tag{7}$$

Now (6) and (7) yield

$$(p_r, \hat{q}) \in PE^{\epsilon}(A, B) \cup PE_k(A, B).$$

□

References

Jurg, P., S. Tijs (1993). On the determinateness of semi-infinite bimatrix games. *Internat. J. Game Theory* **21** 361–369.

Lucchetti, R., F. Patrone, S. H. Tijs (1986). Determinateness of two-person games. *Boll. Un. Mat. Ital.* **6** 907–924.

Nash, J. F. (1951). Noncooperative games. *Ann. of Math.* **54** 286–295.

Tijs, S. H. (1977). ϵ-equilibrium points for two-person games. *Methods Oper. Res.* **26** 755–766.

——— (1981). Nash-equilibria for noncooperative n-person games in normal form. *SIAM Rev.* **23** 225–237.

Von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Math. Ann.* **100** 295–320.

Wald, A. (1945). Generalization of a theorem by von Neumann concerning zero sum two-person games, *Ann. of Math.* **46** 281–286.

H. Norde: Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

J. A. M. Potters: Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands.