

Ambiguity Aversion and Heterogeneity in Financial Markets: An Empirical and Theoretical Perspective

PROEFSCHRIFT

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Chapter 1

Introduction

One of the main roles of the financial economic science is to provide plausible explanations on the dynamics of financial asset prices and to forecast their future movements. The complexity of the real factors which influence the pricing of the assets cannot be easily taken into account in a linear, static model; nonetheless, the role of a parsimonious model is to capture the key economic factors which may drive a substantial part of the variation of the asset prices. The current financial crisis has shown to modelers and policy makers that there may be factors which are essential to the understanding of financial facts and to the evaluation of the policies to be adopted: heterogeneity among investors, ambiguity on the true data generating model, nonlinear models and structural breaks, the interlinkages between macroeconomic risk and financial risk.

This thesis assesses the contribution of heterogeneity among investors and deviation from rational expectations frameworks, using simple consumption-based asset pricing models, where investors dislike multiple possible data generating models and/or distrust the probabilities associated to the different scenarios. The objective of the analysis is to test the relevance of the introduction of new preference modeling, such as the investors' pessimistic/optimistic attitude or the ambiguity adverse behavior.

Before the financial crisis, these features were not considered priorities to be taken into account in modeling and policy works. The recent financial turmoil, however, has caused an intense effort, from economists and financial observers, to look for plausible explanations and causes of the financial facts. The scientific literature has recently produced important steps

in this respect. Alternative ways of modeling preferences taking into account non rational behaviors have been proposed and more and more often integrated in the current models.

This thesis tries to push forward such attempts, testing the appropriateness of newly proposed assumptions. The aim is to provide an analysis to judge how well we can explain, not only in terms of moments comparison with observed variables under study, but also looking at the historical movements and fluctuations of the variables of interest, observed key macroeconomic variables, like asset returns and price-dividend ratio.

Despite the plethora of research efforts which have been made to contribute to the theoretical modeling of departures from rational expectation models, the empirical counterparts have been little and not always successful in the identification and the estimation of the key parameters characterizing the preferences attitudes. Motivated by the need to empirically test the introduction of the ambiguity adverse behavior into the investor's preference structure, Chapter 2, "Ambiguity in Asset Prices: An Empirical Investigation" provides an empirical analysis of a representative-investor consumption based asset pricing model with recursive ambiguity adverse preferences. Up to our knowledge, it constitutes the first attempt to empirically test the parameters of risk aversion, intertemporal substitution and ambiguity aversion. After deriving the testable model restrictions, we use Generalized Methods of Moments (GMM) to identify and estimate the degree of ambiguity aversion. We propose several sets of instrumental variables and, across them, we always estimate a positive degree of ambiguity aversion, compatible with existing experimental evidence.

Chapter 3 and Chapter 4 provide a theoretical assessment of the relevance of ambiguity aversion in the explanation of the distribution of asset prices and returns.

The paper "Ambiguity and Volatility: Asset Pricing Implications" assesses the importance of ambiguity aversion in the presence of switching between states characterized by different growth rates and different volatilities. The goal is to capture the effects of the decline of macroeconomic risk which has been witnessed from the 1990s (the so called the great moderation) in the pricing of risky assets. Apart from matching the moments of the risk-free and risky returns, the analysis focuses also on the ability of the model to replicate the historical observed series of price dividend ratios and on providing insights on the mechanisms which drive the relation between prices, macroeconomic risk and uncertainty. The calibration of the model shows that

ambiguity aversion is an essential ingredient to explain the high risky returns observed in the postwar sample because the aversion to uncertainty makes the investor pessimistic, down weighting high-mean states in favor of low-mean ones. With respect to different regimes in volatilities, the analysis shows that the pessimistic distortion appears much stronger in low-volatility regimes: high-volatility attenuates the distortion due to ambiguity concerns. This counter intuitive result and its implications are carefully analyzed. An important message that we convey is that the adoption of new preferences modeling must be considered and evaluated looking at a multitude of aspects and at the generality of implications that are derived and that more flexibility in modeling the perception of uncertainty and ambiguity aversion is needed.

The above result has provided the motivation for the last chapter of the thesis, “Waves of Optimism and Pessimism”. The model presents a consumption-based economy with simple CRRA utility function and with heterogeneous agents. The agents have access to the same observations but they use different updating rules to infer information about the growth state of the economy. In particular, we consider an optimistic and pessimistic group of agents who use distorted Bayesian updating rules. A key assumption of our model is that heterogeneity is not resolved with learning: pessimists and optimists are equally wrong about the probability of future states of nature; they both survive in the long run so that prices reflect the temporary deviation from the long-run persistence of heterogeneity. The aim of the work is to understand to what extent the interaction of such distorted Bayesian rules can explain the low and medium frequency fluctuations observed in price dividend ratios and to what extent it can give rise, endogenously, to waves of pessimism and optimism which are associated with sustained asset price booms and busts. We show that the interaction of the two groups is able to reproduce the waves of optimism and pessimism observed in the dynamics of asset prices. Low levels of risk aversion can determine a noticeable amplification effect due to heterogeneity: risk neutral investors tend to make more extreme investment decisions, which, if based on wrong beliefs, lead them to be driven out of the market. Such distortion is also amplified by the distortion imposed on the agents: the more agents are (equally) wrong about the probabilities of future realizations, the more price fluctuations are volatile. The chapter also presents an attempt to estimate the model with GMM technicalities. The analysis, despite its parsimonious characterization, is able to reproduce quite realistic price dynamics: heterogeneity in the form of persistent different

attitudes is relevant in understanding the main forces driving asset prices and can constitute a useful alternative to the existing models which consider agents heterogeneity in terms of robustness and validation.

This thesis constitutes an attempt to uncover important limitations in the current developments of asset pricing models. It considers deviations from expected utility theory to investigate the effects of ambiguity aversion in asset pricing, both from a theoretical and an empirical perspective. The thesis investigates the reasons of the poor performance of the theoretical and empirical implications of the existing frameworks. It tries to provide possible explanations which allow a deeper understanding of the capabilities of new preferences modeling in terms of explaining asset prices and equity premium puzzles. Understanding the mechanisms which link macroeconomic risk, uncertainty and investors' behavior to pricing is not only important to interpret the historical facts, but it is also necessary to give the researchers better tools to improve the understanding of the current financial instabilities, the forecasting of future scenarios and the contribution to the policy making process.

Chapter 2

Ambiguity in Asset Prices: An Empirical Investigation

2.1 Introduction

The rational expectation hypothesis is a workhorse assumption in macroeconomics. Under this hypothesis, there is one objective data generating process which coincides with the economic agents' subjective beliefs. However, more than two decades ago, macroeconomists started to question whether this assumption can explain financial facts. Mehra and Prescott (1985) demonstrate that the equity premium (i.e., the return earned by risky asset in excess to the riskfree T-bills as a premium for bearing risk) is an order of magnitude greater than could be rationalized in the context of the standard neoclassical paradigm with time-separable expected power utility functions, with not too high risk aversion. Resolving the equity premium puzzle would require a high risk aversion, but the assumption of a power utility function implies the identity between the degree of relative risk aversion and the inverse of the elasticity of intertemporal substitution (EIS). Despite the empirical evidence that the EIS is small, extremely small values of the EIS (corresponding to a high risk aversion) would make the investor willing to strongly prefer flat consumption paths and to borrow from the future. This would result in the interest rate puzzle, i.e., unrealistically high risk-free interest rates (Weil, 1989).

Epstein and Zin (1989) axiomatize the Generalized Expected Utility (GEU) preferences which allows a separation between risk aversion and the intertemporal substitution, so that, in

principle, it is possible to have high values of risk aversion without necessarily implying a too small EIS. Therefore, the GEU framework could potentially resolve the interest rate puzzle, in allowing for the possibility of preferences for early resolution of uncertainty (i.e., the degree of risk aversion being greater than the inverse of EIS). However, the empirical evidence (Epstein and Zin, 1991) shows that this potentiality is not corroborated by the data and that it is difficult to provide robust estimates of preferences for early resolution of uncertainty.

Among the plethora of research efforts to improve the understanding and the explanation of financial facts, starting from Gilboa and Schmeidler (1989) and Schmeidler (1989), an important growing body of literature has developed theoretical models of decision making under ambiguity, with multiple probability models describing the data generating process. As suggested by the Ellsberg's paradox (1961), the investors, as decision makers, may prefer an unambiguous choice with known probabilities over an ambiguous choice with the same possible outcomes but with unknown probabilities: agents dislike not knowing which is the true objective probability law and would, therefore, require a premium to bear this source of uncertainty. Following this evidence, many contributions have analyzed the asset pricing implications of the multiple-priors utility model, including, among others, Chen and Epstein (2002), Epstein and Wang (1994), and Routledge and Zin (2001).

Our work is inspired by the dynamic model of Ju and Miao (2012), who have developed a consumption-based asset pricing model that departs from rational expectations by proposing novel generalized recursive smooth ambiguity preferences, whose static setting has been axiomatized by Klibanoff et al. (2005). Their framework is based on two key ingredients. First, consumption (and dividends) follow a hidden Markov regime-switching model. Agents update their posterior beliefs of the hidden states based on past data. The presence of multiple probability laws formalizes the *perception of ambiguity*. Second, agents are *ambiguity averse*, they distort the probabilities of the hidden states in a pessimistic way: pricing kernels are modified in such a way that agents attach more weight to low continuation values in recessions. Klibanoff et al. (2005) show that the multiple-priors model of Gilboa and Schmeidler (1989) is a limiting case of the smooth ambiguity model with infinite ambiguity aversion.

The pessimistic distortion of multiple-priors recursive preferences has constituted an appealing feature in order to explain the high returns of risky assets. Despite this fact, there

has been little econometric work on estimating models with multiple priors compared to other utility specifications. The goal of this paper is to fill this gap: our work investigates the testable restrictions on the time-series behavior of consumption and returns implied by a representative-agent consumption based asset pricing model with ambiguity-averse preferences. It represents the empirical counterpart of Ju and Miao (2012). The preference modeling allows a separation among the degrees of risk aversion, ambiguity aversion, and intertemporal substitution. This framework includes as a particular case the class of GEU preferences, defined by Epstein and Zin (1989) in case of ambiguity neutrality. Indeed, our paper is in the tradition of Epstein and Zin (1991)'s empirical work: using consumption and stock market data, the orthogonality restrictions implied by the Euler equations of the agents' optimization problem are used to identify and estimate the parameters of the utility function using the generalized method of moments (GMM). We show that it is possible to derive an analytical expression for the sample analogue using an appropriate definition of posterior beliefs. The results can therefore be used to set up unconditional moment restrictions in order to apply the GMM method. To the best of our knowledge, this is the first contribution which attempts to empirically identify the parameters of intertemporal substitution, risk aversion, and ambiguity aversion.

Using several sets of instrumental variables, we always estimate a positive degree of ambiguity aversion, although not always very precisely. Our estimated values of ambiguity aversion are higher than the value calibrated in Ju and Miao (2012), but, as we will show, still compatible with experimental evidence. We also find that including ambiguity aversion allows to find evidence for preferences for early resolution of uncertainty, which could potentially help in explaining the risk-free rate puzzle.

The remainder of the paper is structured as follows. The next section describes the economy, the preferences, and the testable Euler equilibrium conditions. Section 2.3 defines the data set used in the estimation, Section 2.4 describes the details of the application of the GMM method and Section 2.5 presents the results of the estimation. Section 2.6 reports reasonable values of ambiguity aversion parameters based on experimental evidence and Section 2.7 concludes.

2.2 The Economy

In this section we briefly describe the economy, the smooth ambiguity preferences, and the derivation of the pricing kernel with the corresponding testable restrictions. Time is denoted by $t = 0, 1, 2, \dots$. The state space, as observed by the (representative) investor, is the same for each period and is denoted by S . The state space describing the outcomes of the economy will be denoted by \bar{S} , with $S \subset \bar{S}$, also assumed to be independent of t , but possibly partly unobserved by the investor. At time t the decision maker's information consists of the observed history $s^t = (s_0, s_1, \dots, s_t)$, with $s_0 \in S$ given and $s_t \in S$. The probability distribution over S^∞ , conditional on s^t , will be denoted by \mathbb{P}_t . However, we assume that the decision maker is unsure about \mathbb{P}_t and considers $\mathbb{P}_{jt}, j \in \mathcal{J}$, as possible probability distributions over future states, conditional upon s^t , where \mathcal{J} is a finite set, not depending on t , reflecting, for example, as in our empirical application, unobserved regimes, with $\bar{S} = \mathcal{J} \times S$. We denote by $\mathbb{E}_{jt}[\cdot]$ the expectation with respect to $\mathbb{P}_{jt}, j \in \mathcal{J}$, and we denote by $\pi_{jt} \geq 0$ the probability that the decision maker assigns to $j \in \mathcal{J}$ at time $t + 1$, conditional upon s^t , such that $\sum_{j \in \mathcal{J}} \pi_{jt} = 1$.

The decision maker chooses a consumption plan $C = (C_t)_{t \geq 0}$, with $C_t = C_t(s^t)$, a measurable function of s^t , using the following recursive smooth ambiguity preferences

$$V_t(C) = H(C_t, \mathcal{R}_t(V_{t+1}(C))), \quad (2.1)$$

with

$$H(c, y) = ((1 - \beta)c^{1-\rho} + \beta y^{1-\rho})^{\frac{1}{1-\rho}} \quad (2.2)$$

and

$$\mathcal{R}_t(V_{t+1}) = \left(\sum_{j \in \mathcal{J}} \pi_{jt} \left(\mathbb{E}_{jt} \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1}{1-\alpha}}. \quad (2.3)$$

Here, the parameter $\beta \in (0, 1)$ denotes the subjective discount factor, $1/\rho$ is the elasticity of intertemporal substitution (EIS), γ is the relative risk aversion parameter, while $\alpha - \gamma$ represents the degree of ambiguity aversion. If $\alpha = \gamma$, the preference specification reduces to Epstein and Zin (1989)'s preferences with ambiguity neutrality. The goal of this paper is to estimate these parameters (and to test the model to be presented in this section).

The decision maker chooses the consumption plan $C = (C_t)_{t \geq 0}$ to maximize $V_0(C)$, resulting from (2.1), subject to the budget constraints

$$W_{t+1} = (W_t - C_t) R_{w,t+1}, \quad t = 0, 1, 2, \dots, \quad (2.4)$$

with W_t the wealth at (the beginning of) time t , W_0 given, and with

$$R_{w,t+1} = \sum_{k=1}^K x_{kt} R_{k,t+1}, \quad (2.5)$$

where $R_{k,t+1}$ denotes the return on asset k at time $t+1$, with corresponding weight x_{kt} at time t , $k \in \{1, \dots, K\}$. We denote by R_{t+1} the K -dimensional vector of returns with components $R_{k,t+1}$, $k = 1, \dots, K$.

In our empirical application, we shall assume that ambiguity arises due to regime switching in the economy, unobserved by the investor. We set $\bar{S} = \mathcal{J} \times S$, where each $j \in \mathcal{J}$ represents a regime. We denote by \Pr the probability distribution over $\mathcal{J}^\infty \times S^\infty$. To avoid cumbersome notation, we shall write $\Pr(A)$ instead of $\Pr(\mathcal{J}^\infty \times A)$ for $A \subset S^\infty$, $\Pr(B)$ instead of $\Pr(B \times S^\infty)$ for $B \subset \mathcal{J}^\infty$, and similarly for conditional probabilities, as long as no ambiguity arises. Let $\xi_{t+1} \in \mathcal{J}$ denote the time $t+1$ (unobserved) regime indicator. In this setting, we write for $A \subset S^\infty$

$$\mathbb{P}_{\xi_{t+1}, t}(A) = \Pr(A \mid \xi_{t+1}, s^t), \quad (2.6)$$

so that \mathbb{P}_{jt} is the probability distribution over S^∞ , conditional on s^t and given that the economy is in state $\xi_{t+1} = j$ at time $t+1$.

We shall focus on an economy with two unobserved regimes, i.e., $\mathcal{J} = \{h, \ell\}$, where h represents a “high” unobserved state and ℓ a “low” unobserved state. Given this set-up, the growth rate of consumption follows –from the perspective of the decision maker– a regime switching model, where the mean of the process is assumed to capture the uncertainty. We follow the consumption growth model similar to Cecchetti et al. (2000), i.e., we postulate

$$\log\left(\frac{C_{t+1}}{C_t}\right) = c_{t+1} - c_t = \Delta c_{t+1} = \mu_{\xi_{t+1}} + \sigma \varepsilon_{t+1}, \quad t = 1, \dots, T, \quad (2.7)$$

where $\xi_{t+1} \in \mathcal{J} = \{h, \ell\}$ with $\mu_h > \mu_\ell$, so that ξ_{t+1} is a two-state Markov Chain, and where the driving process ε_{t+1} is a sequence of i.i.d. standard normal random variables. The transition probability matrix of the Markov Chain, denoted by P , is defined as follows

$$P = \begin{bmatrix} p_{hh} & 1 - p_{\ell\ell} \\ 1 - p_{hh} & p_{\ell\ell} \end{bmatrix}, \quad (2.8)$$

with p_{hh} and $p_{\ell\ell}$ denoting the transition probabilities.

We assume that the decision maker starts with a prior distribution π_0 over \mathcal{J} , given by $\pi_0(\xi_1 = j) = \pi_{j,0}$, and constructs the posteriors π_t over \mathcal{J} , with $\pi_t(\xi_{t+1} = j) = \pi_{j,t}$, $j \in \mathcal{J}$, by Bayesian updating, after observing s_t , $t = 1, 2, \dots$. We shall write this Bayesian updating as

$$\pi_{t+1} = B_t(s_t, \pi_t). \quad (2.9)$$

Assuming that π_0 satisfies $\pi_{j,0} = \Pr(\xi_1 = j \mid s_0)$, we also will have for $A \subset S^\infty$ and $t = 1, 2, \dots$,

$$\mathbb{P}_t(A) = \Pr(A \mid s^t) = \sum_{j \in \mathcal{J}} \pi_{j,t} \mathbb{P}_{j,t}(A). \quad (2.10)$$

The posteriors over the unobserved regime variable $\xi_{t+1} \in \mathcal{J}$, given in (2.9), result from updating after observing s_t , yielding the history s^t . After observing in addition s_{t+1} , equation (2.9) represents the construction of a posterior at time $t+1$ over the unobserved regime variable $\xi_{t+2} \in \mathcal{J}$ in period $t+2$. But after observing s_{t+1} one might also construct at time $t+1$ a posterior over the unobserved regime variable $\xi_{t+1} \in \mathcal{J}$ at time $t+1$, i.e., in the same period as s_{t+1} . We shall denote this posterior by $\tilde{\pi}_{t+1}$, with $\tilde{\pi}_{t+1}(\xi_{t+1} = j) = \tilde{\pi}_{j,t+1}$, $j \in \mathcal{J}$, $t = 1, 2, \dots$. Starting with $\tilde{\pi}_0$, given by $\tilde{\pi}_0(\xi_0 = j) = \tilde{\pi}_{j,0}$, we shall write the corresponding Bayesian updating as

$$\tilde{\pi}_{t+1} = \tilde{B}_{t+1}(s_{t+1}, \tilde{\pi}_t). \quad (2.11)$$

Assuming that $\tilde{\pi}_0$ satisfies $\tilde{\pi}_{j,0} = \Pr(\xi_0 = j \mid s_0)$, we also will have for $A \subset S^\infty$ and $t+1 = 1, 2, \dots$,

$$\mathbb{P}_{t+1}(A) = \Pr(A \mid s^{t+1}) = \sum_{j \in \mathcal{J}} \tilde{\pi}_{j,t+1} \tilde{\mathbb{P}}_{j,t+1}(A), \quad (2.12)$$

with

$$\tilde{\mathbb{P}}_{j,t+1}(A) = \Pr(A \mid s^{t+1}, \xi_{t+1} = j). \quad (2.13)$$

Next, we derive the pricing kernel, using results presented in Ju and Miao (2012). We have the following result.

Lemma 1 *For any return $R_{k,t+1}$ at time $t + 1$, with $k = 1, \dots, K$ and $t = 0, 1, 2, \dots$, we have*

$$1 = \mathbb{E}_t \left[\left(\sum_{j \in \mathcal{J}} \tilde{\pi}_{j,t+1} M_{j,t+1} \right) R_{k,t+1} \right], \quad (2.14)$$

where $\tilde{\pi}_{j,t+1} = \tilde{\pi}_{t+1}(\xi_{t+1} = j)$, and where $\tilde{\pi}_{t+1}$ denotes the posterior over $\xi_{t+1} \in \mathcal{J}$, obtained by Bayesian updating after observing s_{t+1} , $t = 0, 1, 2, \dots$, starting with $\tilde{\pi}_0$ satisfying $\tilde{\pi}_{j,0} = \Pr(\xi_0 = j \mid s_0)$, see equation (2.11), and where

$$M_{j,t+1} = \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma} - 1}. \quad (2.15)$$

Proof. According to Ju and Miao (2012) the resulting pricing kernel in this economy is given by

$$M_{\xi_{t+1},t+1} = \sum_{j \in \{h,\ell\}} 1_{\{j\}}(\xi_{t+1}) M_{j,t+1}, \quad (2.16)$$

with

$$M_{j,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho-\gamma} \left(\frac{\left(\mathbb{E}_{j,t} \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(V_{t+1})} \right)^{-(\alpha-\gamma)} \quad (2.17)$$

where $V_{t+1} \equiv V_{t+1}(C_{t+1})$, see equation (2.1), and $\mathcal{R}_t(V_{t+1})$ is defined in (2.3). Using

$$\frac{W_t}{C_t} = \frac{1}{1-\beta} \left(\frac{V_t}{C_t} \right)^{1-\rho} \quad (2.18)$$

(see Ju and Miao, 2012, equation (18)) equation (2.17) can be rewritten as (2.15). Let \mathbb{E}_{Pr}

denote the expectation with respect to Pr . Then we have

$$\begin{aligned}
1 &= \mathbb{E}_{\text{Pr}} \left[M_{\xi_{t+1}, t+1} R_{k, t+1} \mid s^t \right] = \mathbb{E}_{\text{Pr}} \left[\sum_j 1_{\{j\}} (\xi_{t+1}) M_{j, t+1} R_{k, t+1} \mid s^t \right] \\
&= \mathbb{E}_{\text{Pr}} \left[\mathbb{E}_{\text{Pr}} \left[\sum_j 1_{\{j\}} (\xi_{t+1}) \mid s^{t+1} \right] M_{j, t+1} R_{k, t+1} \mid s^t \right] \\
&= \mathbb{E}_{\text{Pr}} \left[\sum_j \text{Pr} [\xi_{t+1} = j \mid s^{t+1}] M_{j, t+1} R_{k, t+1} \mid s^t \right] \\
&= \mathbb{E}_t \left[\sum_j \tilde{\pi}_{j, t+1} M_{j, t+1} R_{k, t+1} \right]. \tag{2.19}
\end{aligned}$$

The final step (replacing expectation with respect to Pr , conditional on s^t , by expectation with respect to \mathbb{P}_t) is possible since $M_{j, t+1}$ and $R_{k, t+1}$ are observed at time $t + 1$, i.e., they are measurable functions of s^{t+1} . ■

In the empirical application the result of this lemma will be used to set up unconditional moment restrictions, so that we can apply the Generalized Method of Moments (GMM). As follows from the Lemma, the pricing kernel is given by $\sum_{j \in \mathcal{J}} \tilde{\pi}_{j, t+1} M_{j, t+1}$. The posterior belief $\tilde{\pi}_{j, t+1}$ will be evaluated using Bayesian updating (2.11). In order to evaluate $M_{j, t+1}$, as defined by (2.15), we need to evaluate $\mathbb{E}_{j, t}(\cdot)$. We proceed as follows. First, we have

$$R_{w, t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{(W_{t+1} - C_{t+1}) + C_{t+1}}{W_t - C_t}, \tag{2.20}$$

so that we might consider $W_t - C_t$ as the “price” at time t and C_{t+1} as the “dividend” at time $t + 1$ in the return $R_{w, t+1}$. We can rewrite the right hand expression in equation (2.20) as

$$\frac{(W_{t+1} - C_{t+1}) + C_{t+1}}{W_t - C_t} = \frac{C_{t+1}}{C_t} \left(\frac{1 + \varphi_{t+1}^C}{\varphi_t^C} \right), \tag{2.21}$$

with φ_t^C the “price-dividend” ratio (i.e., $(W_t - C_t)/C_t$) at time t . Next, we assume that the price-dividend ratio is a function only depending on the beliefs, i.e., $\varphi_{t+1}^C = \varphi^C(\pi_{t+1})$, and that $\log(C_{t+1}/C_t)$ is “sufficient statistic” that can replace s_{t+1} in (2.9). Given these assumptions

and letting $y = \log\left(\frac{C_{t+1}}{C_t}\right)$, we obtain:

$$\begin{aligned}
\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] &= \mathbb{E}_{j,t} \left[\left(\frac{C_{t+1}}{C_t} \left(\frac{1 + \varphi^C(\pi_{t+1})}{\varphi^C(\pi_t)} \right) \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \\
&= \mathbb{E}_{j,t} \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{1 + \varphi^C(\pi_{t+1})}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right] \\
&= \int \exp((1-\gamma)y) \left(\frac{1 + \varphi^C(B(y, \pi_t))}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} f(y, j) dy, \quad (2.22)
\end{aligned}$$

with $f(y, j)$ being the density function of $\log(C_{t+1}/C_t)$ in state j . The function $\varphi^C(\pi_{t+1})$ will be approximated via the Chebyshev collocation method (see Judd (1998) and Fackler and Miranda (2002) for details). We approximate the function $\varphi^C(\pi_t)$ by the function

$$\Phi(\pi_t) = \sum_{i=0}^n c_i T_i(\pi_t),$$

where $T_i(\pi_t)$ is an i -th order Chebyshev function adjusted on $[-1, 1]$ and c_0, c_1, \dots, c_n coefficients to be determined. We set $n=20$. The density functions $f(y, j)$, $j \in \mathcal{J}$, will be estimated using (3.5).

This completes the description of the economy. In Section 2.4 we apply the Generalized Method of Moments (GMM) to estimate the parameters, after first presenting the data in the next section.

2.3 Data

We use postwar quarterly data from 1947:2 to 2009:1. Our measure of per capita consumption is data on expenditure on nondurable goods and services. Consumption series data is taken from the Personal Consumption Expenditure table published by the Bureau of Economic Analysis (BEA). Prices are measured with the implicit price deflator corresponding to the definition of consumption adopted, also taken from the BEA website.

The nominal return on the optimal portfolio is approximated with the value-weighted index of shares traded on the NYSE. Data on nominal returns are from the WRDS - Center

Table 2.1: Descriptive statistics of data. $R_{w,t}$ is the market return, $R_{ABC,t}$, $R_{E,t}$ and $R_{FG,t}$ are returns on stock portfolios. The consumption series C_t refers to non durables and services. Quarterly data.

Variable	Mean	St. Dev.
$\log(\frac{C_{t+1}}{C_t})$	0.0058	0.0055
$R_{w,t}$	1.0273	0.1001
$R_{ABC,t}$	1.0273	0.1144
$R_{E,t}$	1.0245	0.0931
$R_{FG,t}$	1.0269	0.1070

for Research in Security Prices database. We view the market portfolio as consisting of four individual stock return indices that are value-weighted and equally-weighted returns for broad groups of the standard industrial classification (SIC) of individual firms. The first broad group is composed by groups A, B, and C of the SIC code (i.e., agriculture, forestry, fishing, mining, and construction). The second asset return is a value-weighted index of stocks in category E of the SIC code (transportation and public utilities). The third asset return comprises groups F and G (wholesale trade and retail trade) and the fourth asset return comprises groups H and I (finance, insurance, real estate, and services). Since we include the equation for the market return, we omit group H–I which is redundant. All nominal asset returns are converted to real returns, using the appropriate consumption deflator. Descriptive statistics for our data is displayed in Table 2.1 which shows the typical pattern of a lower volatility of the consumption series compared to the returns series.

2.3.1 Background Estimation

In order to evaluate the conditional expectations term (2.22), we estimate the parameters of the switching process of the endowment, i.e., equations (3.5)–(3.6). We use Maximum Likelihood methods applying the EM algorithm (see Kim and Nelson, 1999). Table 2.2 presents the estimation results. The high-mean regime has average consumption growth equal to 0.72 percent per quarter, whereas the regime represented by the low mean μ_ℓ has average growth rate of -0.2 percent per quarter. The table shows that the economy spends more time in the high-growth state, which is also more persistent than the low-growth state: the probability that the high mean state will be followed by another high mean growth state is 0.9609, implying that the high mean state is expected to last on average 25 quarters, while the expected duration of the

Table 2.2: Maximum likelihood estimates for the Markov Switching model of consumption growth. Estimates in the columns μ_h , μ_ℓ , and σ are expressed in percentage. Standard errors are in parenthesis. Data are quarterly and span the period between 1947:2-2009:2. Prior probabilities: $p_{hh} = 0.9$, $p_{\ell\ell} = 0.1$. Results are robust to different choices of prior probabilities.

μ_h	μ_ℓ	σ	p_{hh}	$p_{\ell\ell}$	Log-Likelihood
0.72	-0.20	4.58	0.9609	0.8123	989.8832
(1.2187e-5)	(6.1154e-5)	(1.0819e-5)	(2.3120e-4)	(0.0094)	

low mean state is about 5 quarters. The EM algorithm is used to compute also the posterior probabilities which appear in equation (2.19).

2.4 Applying the Generalized Methods of Moments

In this section, we describe the application of the Generalized Methods of Moments (GMM) used to test our model. The objective of the analysis is the estimation of the key parameters which characterize the preferences of our investor:

$$\theta = \left[\alpha \quad \gamma \quad \rho \quad \beta \right]'. \quad (2.23)$$

The vector θ enters the asset pricing via the stochastic discount factor term $M_{j,t+1}$, $j = h, \ell$, see equation (2.15).

Using a q -dimensional vector of instrumental variable z_t (to be discussed in more detail below), included in the information set of the investor at time t , we turn the conditional moment restrictions (2.14) into unconditional moment restrictions, recalling that R_{t+1} stands for the K -dimensional vector of asset returns (in our case $K = 4$):

$$g(\theta) = \mathbb{E} \left[\left(\sum_{j \in \mathcal{J}} \pi_{j,t+1} M_{j,t+1} R_{t+1} - \iota_K \right) \otimes z_t \right] = 0, \quad (2.24)$$

with ι_k a vector of ones of the same dimension as R_{t+1} . Equation (2.24) represent $K \times q$ moment restrictions. We shall assume stationarity and ergodicity, which makes g time-independent. We define

$$f_t(\theta) \equiv \left(\sum_j \pi_{j,t+1} M_{j,t+1} R_{t+1} - \iota_k \right) \otimes z_t, \quad (2.25)$$

Table 2.3: Sets of instrumental variables. The variable y denotes the consumption growth rate. $z(-j)$ refers to the j -th lag of instrumental variable z .

INST 1	$constant, y(-1), y(-1)^2, y(-2), y(-2)^2, y(-3), y(-3)^2, y(-4), y(-4)^2, R_w(-1), R_w(-1)^2, R_w(-2), R_w(-2)^2, R_w(-3), R_w(-3)^2, R_w(-4), R_w(-4)^2, y(-1)R_w(-1).$
INST 2	$constant, R_w(-1), y(-1), y(-1)^2, y(-1)^3, y(-1)^4, y(-2), y(-2)^2, y(-3), y(-3)^2, y(-4), y(-4)^2.$
INST 3	$constant, y(-1), R_w(-1), R_w(-1)^2, R_w(-2), R_w(-2)^2, R_w(-3), R_w(-3)^2, R_w(-4), R_w(-4)^2.$

so that the sample analogue of $g(\theta)$ is given by

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^T f_t(\theta), \quad (2.26)$$

a $k \times q$ -dimensional vector. The GMM objective is then defined as

$$F_T(\theta) = g_T(\theta)' W_T g_T(\theta) \quad (2.27)$$

where W_T is a $(k \times q) \times (k \times q)$ positive-definite weighting matrix, a consistent estimator of the positive-definite weighting matrix W .

We shall consider three sets of instrumental variables, INST 1, INST 2, and INST 3, composed by different sets of observed lagged variables. Table 2.3 describes the composition of the instrumental vectors. All the instrument vectors are made by lagged values of consumption and return data, which we suppose belonging to the information set of the investor. The first instrument is a wider combination of lagged observed variables, while the second and the third sets are made of mainly lagged consumption data and lagged returns data, respectively. In order to test the robustness of our results, for each instrument set, we consider both value-weighted market returns (V) and equally-weighted market returns (E) so that, in total, we report six cases.

The GMM estimator $\hat{\theta}$ is found by minimizing (2.27) with respect to θ . Under appropriate regularity conditions (such as stationarity and ergodicity) we have that $\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} N(0, V_\theta)$,

where

$$\mathbf{V}_\theta = [d_\theta W d_\theta']^{-1} d_\theta W \Phi W d_\theta' [d_\theta W d_\theta']^{-1}, \quad (2.28)$$

with

$$d_\theta \equiv \frac{\partial g(\theta)'}{\partial \theta},$$

and where Φ is the long run covariance matrix of $f_t(\theta)$, defined as

$$\Phi = \mathbb{E} [f_t(\theta) f_t(\theta)'] + \sum_{j=1}^{\infty} [\mathbb{E} [f_t(\theta) f_{t+j}(\theta)'] + \mathbb{E} [f_{t+j}(\theta) f_t(\theta)']].$$

We use a two-stage GMM:

1. We find $\hat{\theta}_1$ which minimizes (2.27) using the identity weighting matrix ($W_T = I$).
2. To obtain asymptotically efficient estimates we set

$$W_T = \hat{\Phi}^{-1} \equiv \hat{\Phi}_T^{-1}$$

where $\hat{\Phi}$ is a consistent estimate of Φ . This choice of the weighting matrix secures the smallest asymptotic covariance matrix (Hansen, 1982). A popular consistent estimate of the spectral density matrix is the Newey-West estimator (Newey and West, 1987), given by:

$$\hat{\Phi} = \hat{\Phi}_0 + \sum_{j=1}^r \left(1 - \frac{j}{r+1}\right) (\hat{\Phi}_{1,j} + \hat{\Phi}_{2,j})$$

where

$$\hat{\Phi}_{1,j} = \frac{1}{T} \sum_{t=1}^{T-j} f_t(\hat{\theta}_1) f_{t+j}(\hat{\theta}_1)',$$

and

$$\hat{\Phi}_{2,j} = \frac{1}{T} \sum_{t=1}^{T-j} f_{t+j}(\hat{\theta}_1) f_t(\hat{\theta}_1)',$$

and $\hat{\Phi}_0 = \hat{\Phi}_{1,0} = \hat{\Phi}_{2,0}$. The current practice suggests using the smallest integer greater than or equal to $T^{1/4}$ for r . However, in order to ensure a positive definite covariance matrix, we need to use $r = 1$ in the specifications that we considered. In the second step, we again minimize (2.27) but using W_T determined as in the previous item, so to obtain

an optimal value for $\hat{\theta}$.

When the number of moments conditions $K \times q$ is larger than the number of parameters to be estimated, we can use Hansen's J -test: Hansen (1982) explains how, under the null hypothesis that the model is correctly specified, we have that, for $T \rightarrow \infty$,

$$TF_T(\hat{\theta}) \xrightarrow{d} \chi^2(K \times q - d), \quad (2.29)$$

with d the number of parameters included in the estimation (in our case $d = 3$ or $d = 4$), so that it is possible to test for the correct specification.

2.5 Estimation and Testing Results

In this section we present the estimation and test results. Table 2.4 shows the result under ambiguity neutrality (i.e., imposing $\alpha = \gamma$), which can be used as a baseline to understand the role of the ambiguity aversion parameter. The results with ambiguity aversion included are reported in Table 2.5. In both tables, we present the estimated values for the six cases detailed in Table 2.3 for both value-weighted (V) and equally weighted (E) market returns. Moreover, the χ^2 -test of the overidentifying restrictions is also displayed, together with the corresponding degrees of freedom and p -values.

The results show some systematic findings depending on the instrument sets used. For instance, in all cases with instrument set INST3 (i.e., including mainly lagged returns of different orders), we find estimates which result in a rejection (at the 5% significance level) of the null hypothesis that the corresponding moments are satisfied, as is evidenced by the low p -values of Hansen's J -test. On the other hand, when using instrument set INST2 (i.e., including mainly lagged consumption of different orders) or the wider set of instruments INST1, the null hypothesis that the corresponding moments are satisfied, cannot be rejected in any of the eight cases at the 5 or 10% (and higher) significance levels.

In the ambiguity neutral case (Table 2.4) the estimated values of γ , the risk aversion parameter, and ρ , the inverse of the elasticity of intertemporal substitution (EIS), both are in the same range, between around 0.7 and around 6.0. Moreover, also given the corresponding standard errors, these parameter estimates are always rather close to each other. Indeed, when

Table 2.4: Instrumental Variable Estimates. Ambiguity Neutrality

<i>Ret</i>	<i>INST</i>	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\rho}$	$\hat{\beta}$	χ^2	<i>df</i>	<i>p</i> -value
<i>V</i>	1	– (–)	1.2011 (0.6851)	0.7265 (0.9289)	0.9953 (0.0062)	77.6488	69	0.2255
<i>V</i>	2	– (–)	5.0506 (1.75528)	5.6905 (2.3451)	0.9999 (0.0125)	38.9791	45	0.7238
<i>V</i>	3	– (–)	0.9602 (1.2521)	1.0460 (1.4480)	0.9931 (0.0092)	64.2854	37	0.0036
<i>E</i>	1	– (–)	0.7834 (0.3683)	1.6201 (1.0739)	0.9981 (0.0069)	74.7573	69	0.2968
<i>E</i>	2	– (–)	3.2288 (1.3759)	5.8875 (3.5560)	0.9999 (0.0206)	50.5282	45	0.2643
<i>E</i>	3	– (–)	0.7846 (1.5261)	1.2516 (1.8073)	0.9932 (0.0110)	59.5118	37	0.0109

Table 2.5: Instrumental Variable Estimates. Ambiguity Aversion

<i>Ret</i>	<i>INST</i>	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\rho}$	$\hat{\beta}$	χ^2	<i>df</i>	<i>p</i> -value
<i>V</i>	1	119.4916 (48.5793)	5.1335 (1.7637)	3.3701 (1.0788)	0.9990 (0.0004)	76.9518	68	0.2139
<i>V</i>	2	22.7978 (60.6939)	4.7116 (4.2105)	3.6746 (3.2310)	0.9999 (0.0050)	38.1339	44	0.7202
<i>V</i>	3	25.8487 (159.2178)	4.4713 (6.8475)	3.3766 (5.2153)	0.9958 (0.0083)	62.7839	36	0.0037
<i>E</i>	1	81.8586 (46.8041)	3.8834 (1.8496)	3.1724 (1.5053)	0.9954 (0.0004)	71.1444	68	0.3735
<i>E</i>	2	28.1777 (24.6582)	4.3586 (3.1969)	4.5519 (3.5969)	0.9999 (0.0042)	39.6766	44	0.6573
<i>E</i>	3	100.4623 (208.6326)	3.3748 (5.3612)	2.5923 (3.9530)	0.9896 (0.0017)	57.0270	36	0.0143

testing the null hypothesis $H_0^a : \gamma = \rho$, using a standard Wald test, we cannot reject this hypothesis in any of the cases at the conventional significance levels (with smallest p -value 0.2455 for instrument set INST2 with equally weighted returns (E)).

Next, we turn to the estimation results with ambiguity aversion included, see Table 2.5. The estimated value of α , the ambiguity aversion parameter, ranges from around 20 to around 120, depending on the types of returns and the instrument sets chosen. Taking into account the accuracy of these estimates, the ambiguity aversion parameter range might be even much wider. Ambiguity aversion is determined by $\alpha - \gamma$. In the context of our model, ambiguity neutrality corresponds to $\alpha - \gamma = 0$ and ambiguity aversion corresponds to $\alpha - \gamma > 0$. In all cases we estimate a positive ambiguity aversion ($\hat{\alpha} - \hat{\gamma} > 0$). However, when using instrument sets INST2 or INST3, we cannot reject the null hypothesis $H_0^b : \alpha = \gamma$. For instance, when we estimate $\alpha - \gamma$ using instrument set INST2, we find, using value-weighted returns (V), as estimate 18.0863, with corresponding standard error 63.8350, and we find, using equally-weighted returns (E), as estimate 23.8192, with corresponding standard error 26.4399. Thus, in these cases there is no significant empirical evidence for ambiguity aversion. On the other hand, when using instrument set INST1, we find, using the value-weighted returns (V), $\hat{\alpha} - \hat{\gamma} = 114.3581$, with corresponding standard error 46.8266, and we find, using equally-weighted returns (E), as estimate 77.9752, with corresponding standard error 44.9659. Thus, $\alpha - \gamma$ is estimated significantly different from zero at the 5% level using value-weighted returns, and significantly different from zero at the 10% level using equally-weighted returns, giving empirical evidence in favor of ambiguity aversion. In the next section we show that our estimates of $\alpha - \gamma$ are also reasonable, when using these estimates to calculate the ambiguity premium in a static context.

The inclusion of the ambiguity aversion parameter also modifies the estimation results of the other parameters characterizing the preferences. The estimated values of γ , the risk aversion parameter, are now in the range between around 3.3 and around 5.2. These values are in line with values typically reported as reasonable. The estimated values of ρ , the inverse of the elasticity of intertemporal substitution (EIS), are in five out of six cases smaller than the estimated values of γ , with as range around 3.3 to around 5.2. This allows for preferences for early resolution of uncertainty. There has been a wide discussion in the literature on the plausible values of the degree of elasticity of substitution, including independent evidence that

the elasticity of substitution should be small (Campbell, 2001). Our estimates are in line with this literature.

When testing the null hypothesis $H_0^a : \gamma = \rho$, using a standard Wald test, we cannot reject this hypothesis for the cases with instrument set INST2 and INST3 at the conventional confidence levels (p -values at least 0.34). However, for the instrument set INST1, we find as p -values 0.0134 and 0.0559 for the value-weighted (V) and equally-weighted (E) returns, respectively.¹ This suggests that extending the preferences by including ambiguity aversion, not only reveals that ambiguity aversion might play a role, but also, contrary to Epstein and Zin (1991)’s empirical work, might be helpful to estimate a robust preference for early resolution of uncertainty.

2.6 Reasonable Ambiguity Aversion

The estimated values of ambiguity aversions are, in all cases, much bigger than the value calibrated in Ju and Miao (2012): the authors calibrate a degree of ambiguity aversion equal to 8.864 with a risk aversion parameter of 2. In this section we discuss “reasonable” values for our parameters, focusing in particular on α , the parameter representing the ambiguity aversion. We will focus on a static environment, for which Camerer (1999) reports that the ambiguity premium is typically in the order of 10-20 percent of the expected value of a bet in the Ellsberg-style experiments. In order to quantify the ambiguity premium in our set up, we compute the difference between the utility level in terms of a certainty equivalent assuming $\alpha = \gamma$ and assuming $\alpha \neq \gamma$.

First, given that ξ (now without time index) has transition probability matrix P , given by (3.6), the unconditional probabilities of the possible states are given by:

$$\begin{aligned} p_h &= \frac{1 - p_{\ell\ell}}{2 - p_{hh} - p_{\ell\ell}}, \\ p_\ell &= 1 - p_h, \end{aligned}$$

Next, in a single-period setting, the continuation value in the recursive utility function of our

¹For the sake of completeness, we also tested $H_0^c : \alpha = \gamma = \rho$ for the cases with instrument sets INST1 and INST2, but we cannot reject this hypothesis at the conventional confidence levels (p -values at least 0.17).

model has the following expression:

$$\left(\sum_{j \in \mathcal{J}} p_j \left(\mathbb{E}_j \left[C_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1}{1-\alpha}}, \quad (2.30)$$

where

$$\log \left(\frac{C_{t+1}}{C_t} \right) \mid \xi_{t+1} = j \sim N(\mu_j, \sigma),$$

so that we have, with $\mu_j^* = \mu_j + \log C_t$,

$$\log(C_{t+1}) \mid \xi_{t+1} = j \sim N(\mu_j^*, \sigma) \quad \text{and} \quad C_{t+1} \mid \xi_{t+1} = j \sim \log N(\mu_j^*, \sigma).$$

When the agent is ambiguity neutral (i.e., $\alpha = \gamma$), the certainty equivalent is defined as:

$$CE_1 = \left(\sum_{j \in \mathcal{J}} p_j \mathbb{E}_j \left[C_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}. \quad (2.31)$$

If the agent distrusts the probabilities and she is averse to this uncertainty (i.e., $\alpha \neq \gamma$), the certainty equivalent of betting on the ambiguous urn becomes

$$CE_2 = \left(\sum_{j \in \mathcal{J}} p_j \left(\mathbb{E}_j \left[C_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1}{1-\alpha}}. \quad (2.32)$$

The ambiguity premium is the difference between (2.32) and (2.31).

The computation of the term $\mathbb{E}_{j,t} \left[C_{t+1}^{1-\gamma} \right]$ is presented next.

Lemma 2 *Given that $\log(C_{t+1}) \mid \xi_{t+1} = j \sim N(\mu_j^*, \sigma)$ we have*

$$\mathbb{E}_j \left[C_{t+1}^{1-\gamma} \right] = \exp \left((1-\gamma)\mu_j^* + (1/2)(1-\gamma)^2 \sigma^2 \right).$$

Table 2.6: Ambiguity premium as a percentage of the expected value of the bet.

γ	α								
	0.5	2	5	25	50	100	125	150	175
0.5	0	0.0030	0.0092	0.0541	0.1199	0.2797	0.3682	0.4573	0.5432
2	-0.0030	0	0.0062	0.0511	0.1168	0.2766	0.3652	0.4542	0.5401
3	-0.0051	-0.0020	0.0041	0.0490	0.1148	0.2746	0.3631	0.4522	0.5381
5	-0.0092	-0.0062	0	0.0499	0.1107	0.2704	0.3590	0.4480	0.5339
6	-0.0113	-0.0082	-0.0021	0.0428	0.1086	0.2683	0.3569	0.4459	0.5318

Proof. For $j \in \mathcal{J}$ it holds that

$$\begin{aligned}
 \mathbb{E}_j \left[C_{t+1}^{1-\gamma} \right] &= \exp \left(\log \mathbb{E}_j \left[\exp \left(\log \left(C_{t+1}^{1-\gamma} \right) \right) \right] \right) \\
 &= \exp \left(\mathbb{E}_j \left[\log \left(\exp \left(\log \left(C_{t+1}^{1-\gamma} \right) \right) \right) \right] + (1/2) \text{Var} \left[\log \left(\exp \left(\log \left(C_{t+1}^{1-\gamma} \right) \right) \right) \right] \right) \\
 &= \exp \left((1-\gamma) \mathbb{E}_j [\log (C_{t+1})] + (1/2) (1-\gamma)^2 \text{Var} [\log (C_{t+1})] \right) \\
 &= \exp \left((1-\gamma) \mu_j^* + (1/2) (1-\gamma)^2 \sigma^2 \right).
 \end{aligned}$$

■

We present values of $CE_2 - CE_1$ as fraction of the expected value of the bet (i.e., $\sum_{j \in \mathcal{J}} p_j \mathbb{E}_j (C_{t+1})$) in Table 2.6, for different values for some selected values of α and γ . Naturally, the ambiguity premium is null in case of ambiguity neutrality. Given our definition of the ambiguity premium, the ambiguity premia or penalties are symmetric with respect to the degree of ambiguity neutrality. Figure 2-1 plots the behavior of the ambiguity premium as a function of the degree of ambiguity aversion for selected values of the risk aversion degree. The figure suggests that the ambiguity premium is exponentially increasing in the degree of ambiguity aversion. Such behavior does not seem to be affected significantly by the degree of risk aversion: we can only distinguish a small complementarity effect between ambiguity aversion and risk aversion. The estimated parameters in Table 2.5 also show the complementarity effect: positive levels of ambiguity aversion are estimated together with higher levels of relative risk aversion, compared to the ambiguity neutrality case.

Given Table 2.6, according to Camerer's experiments, values of ambiguity aversion up to, at least, 100 seem reasonable: the estimates in Table 2.5 appear consistent with Camerer's evidence

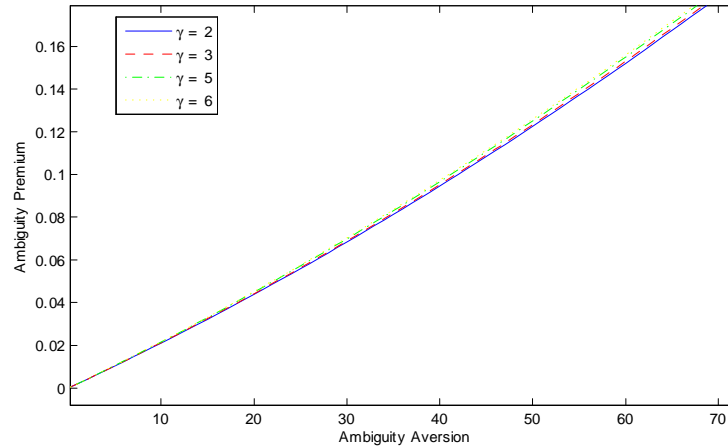


Figure 2-1: The ambiguity premium as a function of the degree of ambiguity aversion degree.

implying an ambiguity premium equal to about 20 percent of the expected bet. According to Table 2.6, the calibrated values in Ju and Miao (2012) would imply a quite small ambiguity premium, ranging from 0.6 to 5 percent of the expected value of the bet. Furthermore, the authors focus on a shorter sample ending on 1993, whereas we consider post-war data with observations until 2009, including the recent financial crisis, which registers a drastic drop in consumption data and a consequent increase in the uncertainty regarding the state representing the economy.

2.7 Summary and Conclusions

This paper provides a natural extension to the empirical analysis of Epstein and Zin (1991), in which the degree of relative risk aversion and the elasticity of substitution of a representative investor in a consumption based economy are estimated. We test the generalized recursive smooth ambiguity model which permits a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution. In our economy, consumption series is modeled as a hidden regime switching model and the investor is adverse to the uncertainty due to the hidden regimes. We define the testable restrictions on the behavior of consumption and asset returns series implied by the model and use the generalized methods of moments to estimate

the parameters characterizing the preferences and to test the overidentifying restrictions.

Results are quite sensitive to the choice of the returns and the instrumental variable set chosen. In all cases, however, we find a positive degree of ambiguity aversion which ranges from values of 20 to 120. These values are quite compatible with experimental evidence which, in static environments, suggests a degree of ambiguity aversion around 100 in order to imply an ambiguity premium equal to about 20 percent of the expected bet.

In all cases analyzed, we estimate a relative risk aversion parameter between 3 and 5 and a low (always less than one) degree of elasticity of substitution. These features are corroborated by independent empirical evidence (Campbell, 2001). We also present the estimates for the ambiguity neutrality case and show that extending the preference specification by including ambiguity aversion allows to estimate a robust preference for early resolution of uncertainty.

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Chapter 3

Ambiguity and Volatility: Asset Pricing Implications

3.1 Introduction

Recent years have witnessed a fast growing development in financial and macroeconomic contributions trying to explain the financial facts, in particular, the observed spread between risk free assets' returns and risky returns. According to standard representative agents models with time-separable preferences, an implausible high degree of risk aversion is needed to explain such higher returns required by the investors (Mehra and Prescott, 1985). Among those attempts, the literature on uncertainty aversion and robust decision making has certainly played an important role in this respect. Contributions have mainly focused on the maximin expected utility framework (or multiple priors) model of Gilboa and Schmeidler (1989) and the robustness theory developed by Hansen and Sargent (2001, 2007), according to which the agent acts with respect to a worst-case scenario. The experimental motivation of modeling preferences in this way is provided by the well known Ellsberg paradox (Ellsberg, 1961), showing that decision makers prefer the risky urn to the ambiguous urn, in a way that violates the expected utility framework. Ambiguity concerns have recently received a huge popularity in the macroeconomic and finance literature. Epstein and Schneider (2008) analyze the effects of ambiguous quality of intangible information on asset prices, Leippold et al. (2008) analyze a learning model under ambiguity

in a continuous time framework. This literature is also related to contributions which impose pessimism and beliefs' distortion in very specific ways without a formalized decision theoretic foundation (Cecchetti et al. 2000, Abel et al., 2002, and Brandt et al., 2004 among others). The general conclusion of these studies is that the pessimistic distortion imposed on the decision maker helps in explaining higher risky returns.

Among these contributions, Klibanoff et al.(2005,2008) provide an axiomatic foundations to the so called generalized recursive smooth ambiguity preferences. Ambiguity aversion is manifested through a pessimistic distortion of the pricing kernel so that agents attach more mass probability to the worse states. This framework nests the standard Epstein and Zin preferences framework (Epstein and Zin, 1989) and the maximin expected utility framework of Gilboa and Schmeidler (1989) as two special cases, respectively, with no and maximal ambiguity aversion. In this respect, this approach is more flexible than the maxmin approach according to which the decision maker conditions his decisions exclusively on the worst case model.

Despite the great effort made in order to propose plausible preferences or beliefs description which could help in resolving the equity premium puzzle, recent years have witnessed a decline in the risk premium: asset prices have soared to an unprecedented levels from the 1990s. Among the explanations that have been provided by macroeconomists, Lettau et al. (2008) propose the contingent decline of the volatility of the aggregate economy (the so called *great moderation*) as possible explanation. Rather than looking at stationary fluctuations in risk premia, the authors focus on the non stationary regime changes in asset prices that occurred in the late 1990s and use a representative agent model with Epstein-Zin preferences to investigate the role of a structural break in the volatility of the aggregate economy in order to explain assets valuations. Figure 3-1 and figure 3-2 depict, respectively, the post-war time series of the consumption growth rate and the CRSP Value-Weighted price-dividend ratios. The figures suggest a negative correlation between consumption volatility and the price-dividend ratios, documented also in Bansal and Yaron (2004).

The literature, however, has not documented in significant ways the effort to explain the dynamic behaviour of asset valuations and returns during the whole postwar sample. Specifically, there is no study on the implications of ambiguity averse preferences with time varying volatility of the endowment. This work tries to fill this gap, by studying the implications of a

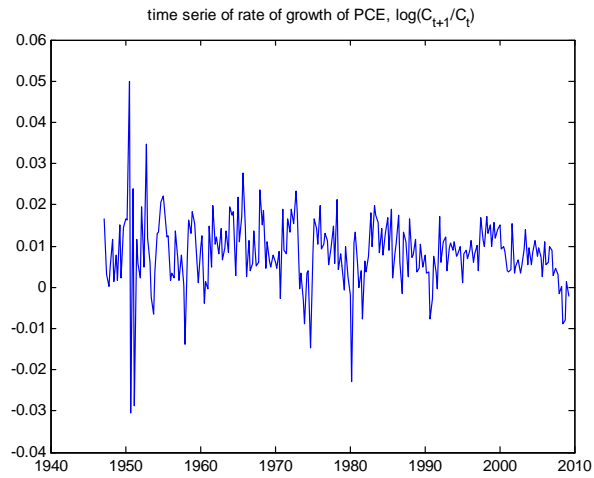


Figure 3-1: Time serie of the rate of growth of Total Personal Consumption Expenditure. Data are quarterly and span the period 1947:2-2009:2. Source: BEA.

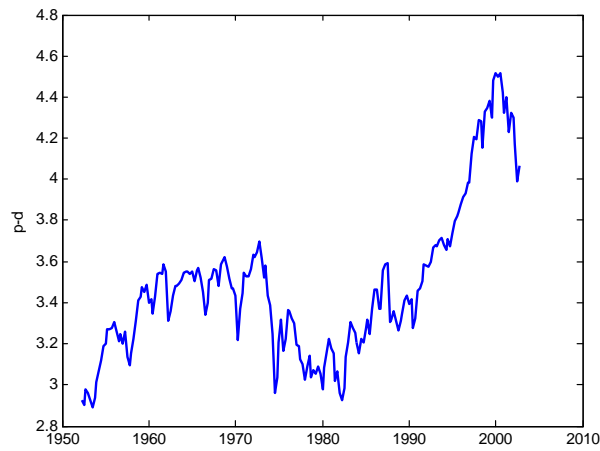


Figure 3-2: The observed log price-dividend ratio. Source: CRSP value weighted index.

model with an ambiguity averse investor in an endowment economy with time varying volatility and growth rate means. The preference side of our model is close to Ju and Miao (2012), who propose a generalized recursive smooth ambiguity model, where the representative agent, is ambiguous about the possible probability measures. As in Lettau et al. (2008), we model the endowment as a hidden regime switching process with time varying mean and volatilities regimes. Merging the preference modelling of Ju and Miao with the endowment description of Lettau et al. (2008), we want to investigate the implications of ambiguity aversion in terms of asset returns, not only as regards uncertainty about high and low state, but also the uncertainty about the hidden switching in volatility states. Does the perception of lower risk attenuate the pessimistic distortion due to ambiguity aversion? Could, as a consequence, smooth ambiguity averse preference explain both high asset returns during volatile periods and declining equity premia during stable aggregate economic condition?

We solve the model with dynamic optimization and show that the solution can be written as a function of the state beliefs only. We estimate the regime switching model's parameters and, feeding in the estimated filtered probabilities, we derive the implied time series of asset prices and expected returns. We show that, as expected, the ambiguity averse investor downweights high-mean states in favour of low-mean ones. However, such a distortion appears much stronger in low-volatility regimes. When the ambiguity averse investor believes to be in a high-mean state, he removes more probability mass from the stable state rather than from the unstable one. In good times, stable economic conditions are preferable to volatile ones. It follows that an ambiguity averse investor, being pessimistic, would consider the high-volatility state relatively more likely than a full bayesian learner would. Similarly, during bad times, the investor adds mass probability to the worst case, the stable scenario. We can give the following intuition: when the investor perceives a high-mean observation, the worst case is the one in which the quality of the observation is quite bad. Conversely, after a low-mean observation, the ambiguity averse investor adds mass probability to the low-volatility case ¹. In other words, ambiguity aversion has a first order effect among mean states. The distortion among volatility states is a second order effect.

¹The intuition is similar to Epstein and Schneider (2008) who consider a different framework where ambiguity is imposed on the quality of the intangible information.

We conclude that (i) ambiguity aversion always implies higher equity premia, but (ii) the effects of the distortion due to ambiguity aversion are stronger during stable economic conditions and, consequently, implied asset valuations failed to increase to the high levels observed during the 1990s. Nonetheless, we also present the implied moments of our exercise: ambiguity aversion helps in resolving the equity premium puzzle allowing to get high risky returns and low risk free rate. Therefore, we claim that is it important to look beyond the moment matching analysis and to investigate the implied dynamics of the financial series, especially if the study focuses on a sample during which important structural breaks take place, as the postwar sample.

The remainder of the paper is structured as follows. Section 3.2 describes the theoretical framework of the smooth ambiguity averse preferences. Section 3.3 presents the model and the theoretical asset pricing implications. Section 3.4 provides a comparative analysis of the effects of degree of ambiguity aversion, while Section 3.5 describes the quantitative implications of the model, both in terms of the analysis of the moments and the time series implications. Section 3.6 concludes.

3.2 Smooth Ambiguity Preferences

Recursive preferences have become a standard tool for studying economic behaviour in dynamic stochastic environments and for parameterizing risk aversion and intertemporal substitution. We assume that time is discrete, with dates $t = 0, 1, 2, \dots$. The state space in each period is denoted by S . At each time $t > 0$, let $s^t = \{s_0, s_1, \dots, s_t\}$ denote the decision maker's information with $s_0 \in S$ given and $s_t \in S$. The probability distribution over S^∞ , conditional on s^t , will be denoted by \mathbb{P}_t .

As a starting point, we use Epstein and Zin (1989) recursive preferences, which separate between risk aversion and the elasticity of intertemporal substitution such that consumers are not indifferent to the timing of the resolution of uncertainty. The decision maker chooses a consumption plan $C = (C_t)_{t>0}$ with $C_t = C_t(s^t)$, a measurable function of s^t , using the following constant elasticity of substitution - CES recursion:

$$V_t = H(C_t, [\mathcal{R}_t(V_{t+1})]) = \left[(1 - \beta) (C_t)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}}, \quad (3.1)$$

where H is a time aggregator and \mathcal{R}_t is a certainty equivalent function. The time aggregator is all that matters in deterministic settings. The rate of time preferences, $\beta \in (0, 1)$ is assumed to be built in and constant and $\frac{1}{\rho}$ denotes the intertemporal elasticity of substitution (IES) for deterministic consumption paths. Following Epstein and Zin, many recent applications in dynamic asset pricing models use the homothetic version of the utility function which combines the constant elasticity time aggregator in (3.1) with a linear homogenous (constant relative risk aversion) expected utility certainty equivalent:

$$\mathcal{R}_t(V_{t+1}) = \left(\mathbb{E}_t [V_{t+1}]^{1-\gamma} \right)^{\frac{1}{1-\gamma}}, \quad (3.2)$$

where γ is the constant Arrow Pratt coefficient for relative risk aversion. When $\gamma > \rho$ ($\gamma < \rho$), the consumer exhibits preference for early (late) resolution of uncertainty.

We use the smooth ambiguity model of Ju and Miao (2012) to model preferences. Suppose that the economy can be governed by a hidden \mathcal{J} -dimensional Markov chain. The representative investor attributes probabilities to the hidden models, but he is adverse to the uncertainty due to the existence of multiple model definition. Therefore, we assume that the decision maker is unsure about \mathbb{P}_t and considers $\mathbb{P}_{j,t}$, $j \in \mathcal{J}$, as possible probability distributions over future states, conditional upon s^t , where \mathcal{J} is a finite set. The consumption plan is chosen using the following recursive smooth ambiguity preferences:

$$V_t(C) = H(C_t, \mathcal{R}_t(V_{t+1}(C))),$$

with

$$\mathcal{R}_t(V_{t+1}) = \left[\sum_{j \in \mathcal{J}} \pi_{j,t} \left(\mathbb{E}_{j,t} [V_{t+1}]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1}{1-\alpha}} \quad (3.3)$$

where α represents the ambiguity aversion parameter. The recursive preferences then become:

$$V_t(C) = \left[(1-\beta) C_t^{1-\rho} + \beta \left\{ \sum_{j \in \mathcal{J}} \pi_{j,t} \left(\mathbb{E}_{j,t} [V_{t+1}]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right\}^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}}, \quad (3.4)$$

It is easy to see that if $\alpha = \gamma$, (3.4) reduces to (3.1) and the investor is ambiguity neutral. When $\alpha > \gamma$ the investor displays ambiguity aversion and no reduction is possible between the state beliefs and the conditional expectations.

3.3 The Model

3.3.1 Endowment

We now turn to the description of the endowment economy. In order to capture the effect of the switching of volatility regimes, we follow Lettau et al. (2008) and model the consumption growth rate process as a hidden Markov switching model of the form:

$$y_t = \log C_{t+1} - \log C_t = c_{t+1} - c_t = \Delta c_{t+1} = \mu(\xi_{t+1}^\mu) + \sigma(\xi_{t+1}^\sigma) \varepsilon_{t+1}, \quad (3.5)$$

where the independent Markov chains ξ_{t+1}^μ and ξ_{t+1}^σ can take the values h (high) or l (low) so that $\mu(h) = \mu_h > \mu(l) = \mu_l$ and $\sigma(h) = \sigma_h > \sigma(l) = \sigma_l$. The driving process ε_{t+1} is a simple i.i.d. standard normal. The transition probability matrices of the Markov chains, P^μ and P^σ are defined as follows:

$$P^\mu = \begin{bmatrix} p_{hh}^\mu & (1 - p_{ll}^\mu) \\ (1 - p_{hh}^\mu) & p_{ll}^\mu \end{bmatrix}, \quad (3.6)$$

$$P^\sigma = \begin{bmatrix} p_{hh}^\sigma & (1 - p_{ll}^\sigma) \\ (1 - p_{hh}^\sigma) & p_{ll}^\sigma \end{bmatrix}, \quad (3.7)$$

where p_{ii}^μ represents the probability of remaining in the mean-state i and p_{ii}^σ the probability of remaining on the volatility-state i . Therefore, we consider four states economy, $\mathcal{J} = 4$, whose transition matrix can be computed as follows:

$$P = P^\sigma \otimes P^\mu,$$

where \otimes denotes the Kronecker product operator so that we deal with a Markov chain, ξ_t , which switches between four states, i.e.,

$$\xi_t \in \left\{ \left(\begin{array}{l} \mu_1 = \mu_h \\ \sigma_1 = \sigma_h \end{array} \right), \left(\begin{array}{l} \mu_2 = \mu_l \\ \sigma_2 = \sigma_h \end{array} \right), \left(\begin{array}{l} \mu_3 = \mu_h \\ \sigma_3 = \sigma_l \end{array} \right), \left(\begin{array}{l} \mu_4 = \mu_l \\ \sigma_4 = \sigma_l \end{array} \right) \right\}.$$

3.3.2 Learning

The time t conditional probability of state j at time $t + 1$ is denoted with $\pi_t(j) = \pi_{j,t} = \Pr(\xi_{t+1} = j | s^t)$ and the vector of the conditional probabilities with $\pi_t = [\{\pi_t(j)\}_{j=1}^{\mathcal{J}}]'$. Agents update their posterior beliefs after each observation via Bayes' rule:

$$\pi_{t+1} = P \frac{(\pi_t \odot f_t)}{\mathbf{1}'(\pi_t \odot f_t)} \equiv B(\Delta c_{t+1}, \pi_t), \quad (3.8)$$

where $f_t = [\{f_t(j)\}_{j=1}^{\mathcal{J}}]$ and $f_t(j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{(\Delta c_{t+1} - \mu_j)^2}{2\sigma_j^2}\right]$ is the density function of the normal distribution with mean μ_j and σ_j , and the operator \odot represents element by element (or Hadamard) multiplication.

3.3.3 Asset pricing implications

We solve the investor's problem by dynamic programming methods. The state variables of our problem are the level of wealth and the state beliefs (W_t, π_t) . Supposing that the value function can be written in the following multiplicative way:

$$V_t(W_t, \pi_t) = W_t G(\pi_t), \quad (3.9)$$

we write the Bellman equation:

$$W_t G(\pi_t) = \max_{C_t, x_t} \left\{ (1 - \beta) C_t^{1-\rho} + \beta \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} [W_{t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

At each time t , the representative agent invests his disposable income $W_t - C_t$ such that:

$$W_{t+1} = (W_t - C_t) R_{w,t+1}, \quad (3.10)$$

where $R_{w,t+1}$ is the gross return on the wealth portfolio between period t and $t+1$, which is also the return on the consumption claim. If there are K traded assets, the return on the wealth portfolio, $R_{w,t+1}$, is equal to $\sum_{k=1}^K x_{kt} R_{k,t+1}$ with $\sum_{k=1}^K x_{kt} = 1$, where x_{kt} is the share of asset k that the investor decides to hold in period t . The optimization is subject to the following constraints:

$$\begin{aligned} W_{t+1} &= (W_t - C_t)(R_{w,t+1}), \\ \pi_{t+1} &= B(\Delta c_{t+1}, \pi_t), \\ R_{w,t+1} &= \sum_{k=2}^K x_{kt} R_{k,t+1} + \left(1 - \sum_{k=2}^K x_{kt}\right) R_{f,t+1} \end{aligned}$$

where $R_{f,t+1}$ is the gross return on riskless asset between period t and $t+1$.

Proposition 3 *The first order condition with respect to the consumption plan leads to the equilibrium condition:*

$$1 = \beta \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}}. \quad (3.11)$$

Proof. See Appendix 1 ■

Equation (3.11) can be manipulated in order to emphasize the effects of ambiguous beliefs in equilibrium. Suppose for the moment that the investor is ambiguity neutral, i.e. $\alpha = \gamma$. Then the equilibrium condition corresponding to (3.11) would become

$$\begin{aligned} 1 &= \beta \left(\sum_{j=1}^4 \pi_{j,t} E_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\rho}{1-\gamma}}, \\ &= \beta \left(\mathbb{E}_t \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\rho}{1-\gamma}}. \end{aligned} \quad (3.12)$$

In the Appendix 2, we show that eq. (3.11) can be rewritten as

$$1 = \beta \left[\sum_{j=1}^4 \hat{\pi}_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right) \right]^{\frac{1-\rho}{1-\gamma}}, \quad (3.13)$$

where

$$\hat{\pi}_{j,t} \equiv \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \quad (3.14)$$

can be interpreted as a *distorted* belief used by the investor who fears ambiguity over the transition probabilities. Compared to eq. (3.12), eq. (3.13) shows how the ambiguous averse investor behaves as an ambiguity neutral investor with distorted beliefs².

The first order condition with respect to the trading strategy, x_k , yields a characterization of the pricing kernel for our model, which is defined in the following proposition.

Proposition 4 *The pricing kernel is given by*

$$M_{\xi_{t+1},t+1} = \sum_{j \in \mathcal{J}} 1_{\{j\}} (\xi_{t+1}) M_{j,t+1}$$

with

$$M_{j,t+1} = \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma}-1}. \quad (3.15)$$

Proof. See Appendix 3. ■

The pricing kernel is also a hidden stochastic variable. Eq. (3.15) shows that when $\alpha = \gamma$, we get the usual pricing kernel for non time-separable preferences without ambiguity concerns

$$M_{t+1} = \beta^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}}.$$

Furthermore, imposing time separability, $\gamma = \rho$, we get the standard pricing kernel $M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}$ for CES utility functions. This equation reveals that optimal portfolio rules are affected by ambiguity.

²Note that the expected returns, being defined via the price-consumption and price-dividend ratios, depend on the degree of ambiguity aversion, in the general case in which utility functions are not logarithmic.

Next, we show the solutions for price-dividend ratios and expected returns, which can be computed as functions of the beliefs only. Let us define the dividend on equity with D_t . As in Campbell (1986) and Abel (1999), we define it as consumption raised to a power $\lambda > 1$:³

$$D_t = C_t^\lambda.$$

This specification implies that the volatility of dividends is proportional to the volatility of consumption and it is a convenient representation because, while keeping the number state variables limited, allows for higher volatility of the dividend dynamics compared to the consumption process. Let P_t^D denote the ex-dividend price of a claim to the dividend stream measured at the end of time t , and P_t^C denote the ex-dividend price of a share of a claim to the consumption stream. We conjecture that the price-dividend ratio is given by

$$P_t^D = \varphi(\pi_t) D_t,$$

and the price-consumption ratio

$$P_t^C = \varphi^C(\pi_t) C_t$$

where the functions $\varphi(\cdot)$ and $\varphi^C(\pi_t)$ have to be determined. In equilibrium, we have that on average, the return on equity, R_e , is defined as follows

$$R_{e,t+1} \equiv \frac{P_{t+1}^D + D_{t+1}}{P_t^D} = \frac{D_{t+1}}{D_t} \frac{1 + \varphi(\pi_{1,t+1})}{\varphi(\pi_{1,t})}. \quad (3.16)$$

Then, it is possible to rewrite eq. (3.11) as follows:

$$1 = \beta \left[\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[\left(\frac{1 + \varphi^C(\pi_{t+1})}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}. \quad (3.17)$$

³Abel (1999) shows that this specification is a good approximation to represent leverage equity.

The equilibrium condition (3.17) is a functional equation from which we can derive the price-consumption ratio $\varphi^C(\cdot)$.

Proposition 5 *The price-dividend ratio of a claim to the dividend stream satisfies*

$$\varphi(\pi_t) = \sum_{j=1}^4 \pi_{j,t} \mathbb{E}_{j,t} \left[M_{j,t+1} \left((1 + \varphi(\pi_{t+1})) \frac{D_{t+1}}{D_t} \right) \right]. \quad (3.18)$$

Proof. See Appendix 4 ■

We solve the functional equations (3.17) and (3.18) numerically on a grid of values for the state variables π_t .⁴

3.4 Comparative Analysis

In this section we present a comparative analysis of the behaviour of prices and the equity premium. Following similar comparative analyses (see, for instance, Ju and Miao, 2012, and Lettau et al., 2008), we set the parameters ρ and λ as 1/1.5 and 4.5, respectively. As in Veronesi (1999), we focus on the behaviour of the equilibrium prices as a function of the state beliefs.

Figure 3-3 represents the distorted beliefs as defined in eq. (4-1). It is represented the distortion of the belief (probability) of being in a high-mean state, given each plausible state of the economy. The first column refers to high-mean states, the second column to low-mean states. As expected and well known, ambiguity averse preferences imply a pessimistic distortion: the good states are considered less likely with respect to a full Bayesian perspective, the bad ones more likely. What is more surprising is the strength of distortion for different levels of risk (volatilities): the top row depicts the cases of low volatility, the bottom one the cases in which there is high volatility. It is clear that the distortion is more attenuated in presence of high risk, while it is stronger in conditions of low risk. The ambiguity averse agent considers the high-mean and low risk state as the best case. This is indeed the state whose probability is mostly reduced. On the contrary a low valuation state and low risk is perceived as the worst case (the stability decreases the perception of switching to the high-mean scenarios) and this

⁴In the Appendix 5 we describe the functional equations in more details. The Matlab codes and numerical algorithms are available under request.

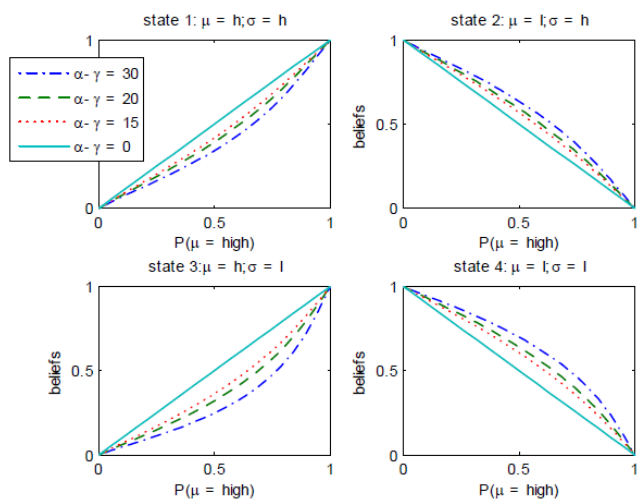


Figure 3-3: Ambiguity neutral and distorted beliefs. Each panel shows the ambiguity neutral belief (solid line) and distortions due to ambiguity for each state ($\beta = 0.9925$, $\rho = 1/1.5$, $\lambda = 4.5$).

is the state to which the highest mass probability is added to the ambiguity-neutrality belief. In unstable periods, with high risk, the perception of the distance between the two different growth levels is attenuated and this implies that the distortion is also smaller compared to stable times.

Figure 3-4 presents the implied dynamics of the price-dividend ratio as a function of the belief of being in the high-growth state conditioning in a volatility state (lower panel) and as a function of the belief of being in a low-volatility state conditioning in a mean state (upper panel). The solid line corresponds to the ambiguity neutrality case ($\alpha = \gamma = 25$), the dashed and dotted lines represent positive levels of ambiguity aversion, respectively, $\alpha - \gamma = 5$ and $\alpha - \gamma = 10$. We can notice that the implied price-dividend ratio is an increasing and convex function of the high-mean state belief, implying pro-cyclical prices, consistently with the empirical literature. To understand why, suppose we believe we are in the low growth state. When a positive news arrives, the probability of being in a high growth state increases. This causes the investor to expect a higher return in the future, to buy more assets increasing, therefore, the price. At the same time, however, closer is the probability of being in a high growth state to 0.5, higher is the uncertainty of the investor about which state is realizing. This uncertainty may cause the

investor to wait in order to get more information. The price-dividend ratio timidly increases, suggesting that the former effect dominates the latter.

Suppose, instead, that we believe to be in the high growth state and a bad news occurs. Again the effect is twofold. On one side, investors are encouraged to sell the asset, on the other the uncertainty increases, because our belief is now closer to 0.5. Therefore, both effects lower the equilibrium price and the decrease is more accentuated. This fact causes the price-dividend ratio to be a convex function. Lettau et al. (2008) also document an increasing and convex price-dividend ratio. Furthermore, higher levels of ambiguity aversion always imply lower prices since the investor requires a higher premium to hold risky and ambiguous asset. Similar findings have been documented by Ju and Miao (2012) who considers only ambiguity aversion with respect to multiple mean states. They show that ambiguity does not significantly modify the characteristics of the price-dividend ratio, except that it accentuates the curvature, helping to explain the high volatility of asset prices.

Our attention, however, is mainly focused on the interlinkages between risk (volatility) and ambiguity uncertainty. Comparing the two bottom panels horizontally, we see how the effect of ambiguity aversion is stronger in low-volatility regimes: as already shown by the analysis of the distortion placed by ambiguity aversion, high volatility attenuates the distortion due to ambiguity concerns. The investor perceives the increased risk, but his perception of ambiguity is reduced because the worst-case model is not too 'far', given some distance to measure closeness between data generating processes.

The upper panels show the implied price-dividend ratios as a function of the perception of being in low-risk states. As could be expected, a convex and increasing price-dividend ratio is also obtained. However, the curvature is flatter for higher levels of ambiguity aversion since, again, when the investor believes that the low-volatility state is very likely, distortions due to ambiguity aversion are more effective. This further implies that ambiguity aversion reduces the volatility of prices due to switching between volatility states, as if the ambiguity averse investor, being an insuring agent, would react less to news. In this respect, ambiguity aversion provides a rationale for price under-reaction to news about fundamentals. Our result is in contrast with Ju and Miao (2012) who claim that, in a world with perceived ambiguity (multiple plausible models), ambiguity aversion always increases the volatility of equity returns. Our more general

endowment economy model shows that their conclusion is not robust with respect to different definition of ambiguity perception.

Figure 3-5 compares the dynamics of the expected equity premium for different level of ambiguity. Again, not surprisingly, the equity premium is always higher for higher levels of ambiguity aversion. When the uncertainty about the mean state is higher, the ambiguity premium is also higher. Given that the beliefs of high-growth state is typically greater than 0.5, the figures also implies countercyclical conditional expected equity premia, as documented by the literature. When plotted against the belief of a low risk state, Figure 3-6 shows that, for a given level of ambiguity aversion, uncertainty about the volatility state also requires a higher expected equity premium. However, increasing ambiguity aversion implies lower levels of the equity premium. If the level of volatility is ambiguous, the ambiguity averse investor is not requiring a compensation for not knowing the true riskiness of the economy. This aspect of the model implies that switching among volatility states does not necessarily imply important effects on prices when the investor is ambiguity averse.

3.5 Quantitative Analysis

Since the state beliefs are the only state variables of the model, we proceed with the estimation of the filtered state probabilities in order to derive the implied time series of the economic variables of interest, prices and equity premia. Our goal is not only to engage in matching the moments of the risky returns and riskless return, as it is usually done in asset pricing studies, but we want also to compare the implied time series with real data and, in particular, we want to assess the ability of the model to replicate long and medium frequency fluctuations in prices.

The observed high risky returns and low risk-free returns are typically very hard to replicate via simple consumption based asset pricing models. Furthermore, from Shiller (1981), the literature has also documented very high volatility of the equity returns (see also Campbell, 1999 for a survey) which are hard to explain given the smoothness of the observed consumption series. Table 3.1 reproduces sample moments from annual US data.

We estimate the model (3.5)-(3.7) by the *Expectation-Maximization* (EM) algorithm. Estimated coefficients are reproduced in Table 3.2. The switching between mean states appears to

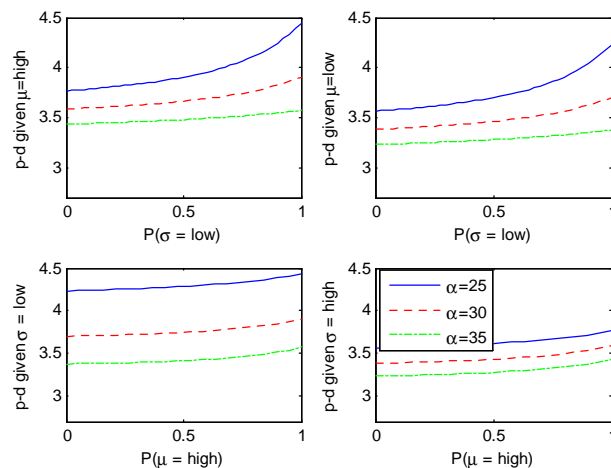


Figure 3-4: Price-dividend ratios ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

be a business-cycle phenomenon. The persistence of the high growth state is about 8 years, the persistence of the low growth state is almost 2 years, while the persistence of the high-volatility state and the low-volatility state, is 31 years and 125 years, respectively. This data confirms the common finding that the switching in volatility is a low frequency event. The postwar data, starting from the mid-1980s, has showed a notable decrease in volatility of consumption growth and it has been associated to the decline of *macroeconomic risk*, also termed *the great moderation*.

This pattern is confirmed by the estimated filtered probabilities. Figure 3-7 plots the filtered state probabilities of a low-volatility state along with the probabilities of a high-mean state. Consumption exhibits a considerable reduction in volatility starting from the 1990s and the filtered state probabilities approach one.

3.5.1 Matching the Moments

We start the analysis presenting in Table 3.3 the calibration results in case of ambiguity neutrality, $\gamma = 30$ and $\alpha = 30$, such that the model corresponds to the one calibrated in Lettau et al. (2008), who do not consider ambiguity aversion motives. Such calibration does not appear satisfactory because the risk aversion parameter is, however, much higher than values typically

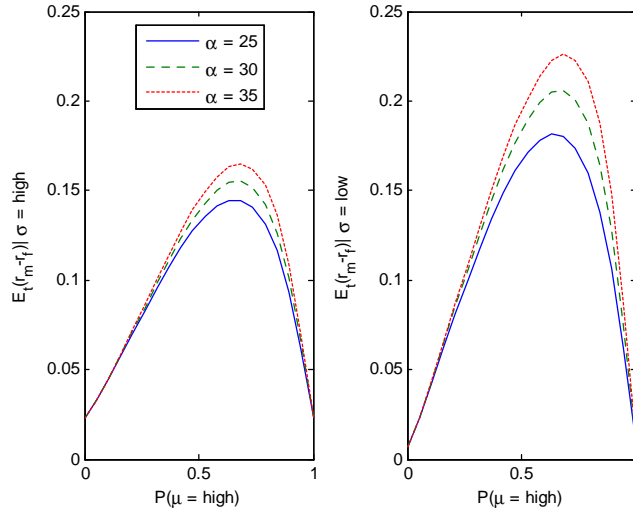


Figure 3-5: Conditional equity premium plotted as a function of the probability of high growth state. The left panel conditions on high volatility state and the right panel on low volatility state ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

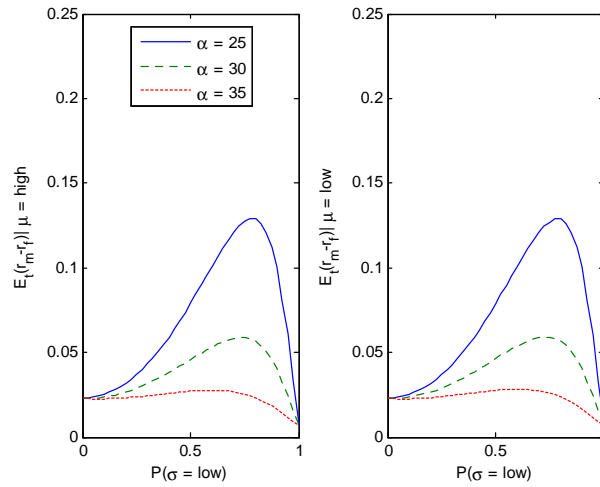


Figure 3-6: Conditional equity premium plotted as a function of the probability of low volatility state. The left panel conditions on high mean state and the right panel on low mean state ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

considered acceptable by macroeconomists (γ less than 10 and close to 2). Next, we consider the role of ambiguity motives ($\alpha > \gamma$). Tables 3.4, 3.5 and 3.6 collect the results from simulations for different values of the discount factor, β . We can notice that ambiguity aversion generally increases the equity return. Indeed, a more pessimistic investor requires more compensation to hold a risky asset in an ambiguous economy. Furthermore, increasing the values of ambiguity aversion lowers the risk free rate, helping explaining the equity premium puzzle.

Despite these results, the next Section shows that it is important to move beyond these exercises and to look at the implied time series of prices to understand whether ambiguity aversion helps in matching the observed assets valuations.

3.5.2 Time Series Implications

We consider a calibration which is quite satisfactory in matching the moments described above: $\beta = 0.9925$, $\gamma = 15$, and $\alpha = 35$. Figure 3-8 shows the comparison between the implied time series of price-dividend ratio with data. The comparison seems far from satisfactory. As already observed in the comparative study, due to ambiguity aversion, the switching from the high-volatility state to the low-volatility implies a moderation in prices which has not been observed in the data. Extensive simulations show that low values of risk aversion associated with high values of ambiguity aversion imply a similar qualitative dynamics. As we already noticed, the effects of ambiguity aversion is, indeed, much more effective in low-volatility states rather than in very volatile ones. Therefore, with a very high level of ambiguity aversion, the different intensity of the effect of distortion causes the prices to decrease during the last part of the sample. In Figure 3-9 we compare the observed time series of prices with the ones implied by our model with γ equal to 25 and α equal to 25 and 35, respectively. The figure makes clear that, while ambiguity aversion is very effective in lowering prices, high levels of ambiguity aversion produce attenuated effects during unstable times. Therefore, increased riskiness of the economy attenuates the premium due to ambiguity aversion. This effect worsens the performance of the model for the last part of the sample, the so called *great moderation* during which prices have increased considerably. In other words, ambiguity aversion with the respect to mean states reinforces risk aversion, but ambiguity aversion with the respect to volatility states does not. This tension was never emphasized previously in the literature and we believe it constitutes an

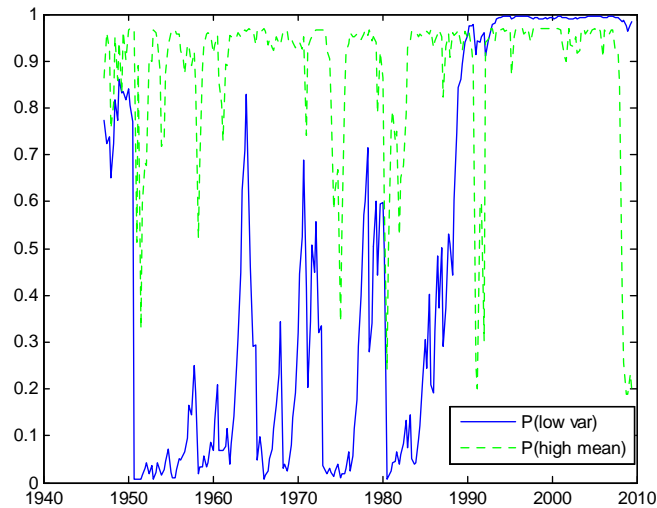


Figure 3-7: Time series of estimated state probabilities. $P(\text{low var})$ is the unconditional probability of being in a low consumption volatility state next period (solid line), calculated by summing the probability of being in a low volatility and high mean state and the probability of a low volatility and low mean state. $P(\text{high mean})$ is calculated analogously. The data are quarterly and span the period 1947:2-2009:2.

important warning for the growing literature on ambiguity aversion and asset prices.

3.6 Conclusions

In this paper we consider a smooth ambiguity averse representative investor in a standard consumption based asset pricing model. We model the consumption and dividend growth rate as a regime switching process. The switching is imposed both on the mean and on the volatility values. We estimate the Markov switching models and show that switching between volatility states is a low frequency event: the first part of the postwar sample is characterized by high volatility, while, starting from the 1990s we observe the so called *great moderation* period. The aim of our study is to understand how ambiguity concerns interact with beliefs about volatility and mean regimes. Our study shows that ambiguity aversion is certainly helpful in order to explain the observed high level of equity premia, even with low values of risk aversion. Indeed, higher ambiguity aversion lowers risk free rates, while increasing equity returns. Furthermore,

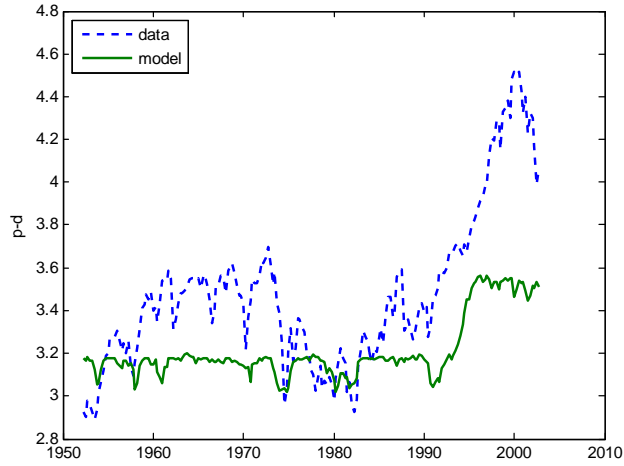


Figure 3-8: Time series of the log price-dividend ratio observed (dashed line) and implied by the model: $\beta = 0.9925$, $\rho = 1/1.5$, $\gamma = 15$, $\alpha = 35$ (solid line). Source: CRSP value-weighted index.

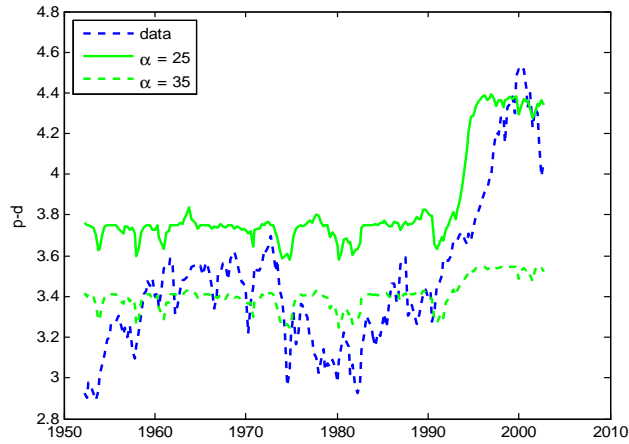


Figure 3-9: Time series of the log price-dividend ratio from the data and implied by the model for different ambiguity aversion values. In both simulations we set $\gamma = 25$, $\rho = 1/1.5$, $\beta = 0.9925$ and $\lambda = 4.5$. Source: CRSP value-weighted index.

this effect is more accentuated, the lower the level of risk aversion. Not surprisingly, ambiguity aversion motives have, indeed, had a great success on the asset pricing literature in recent years.

However, we also show some additional implications of ambiguity concerns that, up to our knowledge, have not been emphasized previously. We uncover the standard results reported in the exercises aimed at matching the moments and we analyze the time series implications of the smooth ambiguity averse preferences. Even if a successful calibration can match the observed moments of the equity premia and the risk free rate, the implied time series do not appear satisfactory for the levels of ambiguity aversion required to explain the equity premium puzzle. This is due to the second-order distortion that ambiguity aversion places among volatility states. Indeed, as has already been widely documented, the ambiguity averse investor behaves as a pessimistic agent with distorted state probabilities among good and bad states: she removes mass probabilities from good states to bad states. However, ambiguity aversion with respect to volatility states plays a second order effect: good stable economic conditions are preferred to states of high volatility. Being in bad and stable state is considered the worst case and, from a pessimistic perspective, more mass probability would be added to this case. Therefore, high levels of ambiguity aversion often reduce the volatility of the equity returns, due to the attenuated response of the investor to the new observation. This effect produces a counterfactual price-dividend ratio time series. Furthermore, following the switching to more stable economic conditions (*great moderation*), ambiguity aversion causes prices to decline. This produces counterfactual prices.

These observations constitute an important warning for the growing literature on ambiguity aversion and asset prices and suggest the necessity to deal with such preferences with caution. Our possible development of our critique can be the modelling of time varying degrees of ambiguity aversion, which could be higher during periods of high uncertainty perception (typically during recessions). A second line of research could focus on the time variation of the perception of ambiguity perception (rather than the variation in the ambiguity aversion), exploiting the peculiar feature of the smooth ambiguity averse preferences, which allows to distinguish between ambiguity aversion (beliefs) and ambiguity perception (multiple probability measures). We leave such extensions for future research.

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3.7 Appendix

3.7.1 Appendix 1: Euler equation

Substituting the first line of the budget constraint into the Bellman equation, we get as maximization problem

$$W_t G(\pi_t) = \max_{C_t, x_{k,t}} \left\{ \begin{aligned} & (1 - \beta) C_t^{1-\rho} + \beta (W_t - C_t)^{1-\rho} \times \\ & \times \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}} \end{aligned} \right\}^{\frac{1}{1-\rho}}. \quad (3.19)$$

Taking the first order condition with the respect to consumption leads to the following:

$$(1 - \beta) C_t^{-\rho} = \beta (W_t - C_t)^{-\rho} \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}}. \quad (3.20)$$

We conjecture a consumption rule $C_t = a_t W_t$ and we substitute it in (3.19) and (3.20) to get, respectively,

$$G(\pi_t) = \left\{ \begin{aligned} & (1 - \beta) a_t^{1-\rho} + \beta (1 - a_t)^{1-\rho} \times \\ & \times \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}} \end{aligned} \right\}^{\frac{1}{1-\rho}} \quad (3.21)$$

and

$$(1 - \beta) a_t^{-\rho} = \beta (1 - a_t)^{-\rho} \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_t [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}}. \quad (3.22)$$

From this it follows that

$$\frac{G(\pi_t)^{1-\rho} - (1-\beta)a_t^{1-\rho}}{1-a_t} = (1-\beta)a_t^{-\rho}.$$

Therefore,

$$G(\pi_t) = (1-\beta)^{\frac{1}{1-\rho}} (a_t)^{-\frac{\rho}{1-\rho}}, \quad (3.23)$$

or, alternatively,

$$a_t = \left[\frac{G(\pi_t)^{1-\rho}}{1-\beta} \right]^{-\frac{1}{\rho}}. \quad (3.24)$$

Substituting (3.24) into (3.21), we get the following equilibrium condition for $G(\cdot)$:

$$\begin{aligned} G(\pi_t)^{1-\rho} &= (1-\beta) \left(\frac{G(\pi_t)^{1-\rho}}{1-\beta} \right)^{-\frac{1-\rho}{\rho}} + \\ &+ \beta \left(1 - \left(\frac{G(\pi_t)^{1-\rho}}{1-\beta} \right)^{-\frac{1}{\rho}} \right)^{1-\rho} \times \\ &\times \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}} \end{aligned} \quad (3.25)$$

Using the definition of $G(\pi_t)$ in (3.23) into (3.22), we get

$$a_t^{-\rho} = \beta (1-a_t)^{-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{1-\gamma} \left(\frac{C_{t+1}}{W_{t+1}} \right)^{-\frac{\rho(1-\gamma)}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}. \quad (3.26)$$

From the budget constraint, we derive

$$\begin{aligned} W_{t+1} &= (W_t - C_t) R_{w,t+1} = W_t (1-a_t) R_{w,t+1} = \\ &= C_t \frac{1-a_t}{a_t} R_{w,t+1} \end{aligned} \quad (3.27)$$

which, substituted in (3.26), and after a few simplifications, yields condition (3.11).

3.7.2 Appendix 2: Distorted Beliefs

Let us go back to eq. (3.11) and multiply and divide the expression inside the inner bracket by the term $\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right]$ to get

$$1 = \beta \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma}-1} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right) \right)^{\frac{1-\rho}{1-\alpha}}.$$

We can further raise both terms to a power of $\frac{1-\alpha}{1-\gamma}$ and get:

$$\begin{aligned} 1 &= \beta^{\frac{1-\alpha}{1-\gamma}} \left(\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right) \right)^{\frac{1-\rho}{1-\gamma}} \\ &= \beta \left(\sum_{j=1}^4 \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right) \right)^{\frac{1-\rho}{1-\gamma}} \\ &= \beta \left(\sum_{j=1}^4 \hat{\pi}_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right) \right)^{\frac{1-\rho}{1-\gamma}}. \end{aligned}$$

where

$$\hat{\pi}_{j,t} \equiv \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}}.$$

3.7.3 Appendix 3: The pricing kernel

Substituting the expression for $R_w = \sum_{k=1}^K x_k R_k$ in (3.19) and calculating the foc with respect to the trading strategy, we have

$$0 = \sum_{j=1}^4 \pi_{j,t} \left\{ \begin{aligned} &\left(\mathbb{E}_{j,t} [W_{t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \times \\ &\times \mathbb{E}_{j,t} \left[(R_{w,t+1})^{-\gamma} G(\pi_{t+1})^{1-\gamma} (R_{k,t+1} - R_{f,t+1}) \right] \end{aligned} \right\}$$

$k = 2, \dots, K.$

Substituting (3.23) for $G(\cdot)$, we get:

$$0 = \sum_{j=1}^4 \pi_{j,t} \left\{ \begin{aligned} & \left(\mathbb{E}_{j,t} \left[R_{w,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \times \\ & \times \mathbb{E}_{j,t} \left[(R_{w,t+1})^{-\gamma} G(\pi_{t+1})^{1-\gamma} (R_{k,t+1} - R_{f,t+1}) \right] \end{aligned} \right\}, k = 2, \dots, K, \quad (3.28)$$

From eq. (3.28),

$$\begin{aligned} & \sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \mathbb{E}_{j,t} \left[(R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] R_{f,t+1} = \\ & = \sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \times \\ & \quad \times \mathbb{E}_{j,t} \left[(R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{k,t+1} \right] \end{aligned} \quad (3.29)$$

$k = 2, \dots, K$

so that, multiplying by x_k and summing over the k , we have:

$$\begin{aligned} & \sum_{j=1}^4 \pi_{j,t} \left[\mathbb{E}_{j,t} \left(R_{w,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{\gamma-\alpha}{1-\gamma}} \times \mathbb{E}_{j,t} \left[(R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] R_{f,t+1} = \\ & = \sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \mathbb{E}_{j,t} \left[(R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \end{aligned}$$

and, recalling (3.11), we have

$$\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{\gamma-\alpha}{1-\gamma}} \mathbb{E}_{j,t} \left[(R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] R_{f,t+1} = \left(\frac{1}{\beta} \right)^{\frac{1-\alpha}{1-\rho}} \quad (3.30)$$

Therefore, since

$$\frac{1}{R_{f,t+1}} = \sum_{j=1}^4 \pi_{j,t} (\mathbb{E}_{j,t} [M_{j,t+1}]),$$

eq. (3.15) must hold.

3.7.4 Appendix 4: Price-dividend ratio

Substituting eq. (3.30) in eq.(3.29), the equilibrium condition (3.18) can be derived.

3.7.5 Appendix 5: Numerical Methods

Eq.(3.18) can be rewritten

$$\begin{aligned} \varphi(\pi_t) &= \sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[M_{j,t+1} \left((1 + \varphi(\pi_{t+1})) \frac{D_{t+1}}{D_t} \right) \right] \right) \\ &= \sum_{j=1}^4 \pi_{j,t} \left(\int_{-\infty}^{\infty} M_{j,t+1} ((1 + \varphi(B(\pi_t, y))) \exp((1-\gamma)\lambda y)) f(y, j) dy \right) \end{aligned}$$

where $y = \log\left(\frac{C_{t+1}}{C_t}\right)$ and $f(y, j)$ is the density function of a normal distribution with mean μ_j and variance σ_j . The posterior probabilities π_t are the only state variables in this framework, so the price-dividend ratio is a function only of the vector π_t . We solve these functional equations numerically on a grid of values for the state variables π_t . In order to solve for the price-dividend ratio we first need to solve for the price-consumption ratio using eq. (3.17), which can be rewritten as

$$\begin{aligned}
\varphi^C(\pi_t) &= \beta \left[\sum_{j=1}^4 \pi_{j,t} \left(\mathbb{E}_{j,t} \left[\left(1 + \varphi^C(\pi_{t+1}) \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \\
&= \beta \left[\sum_{j=1}^4 \pi_{j,t} \left[\int_{-\infty}^{\infty} \left(1 + \varphi^C(B(\pi_t, y)) \right)^{\frac{1-\gamma}{1-\rho}} \exp((1-\gamma)y) f(y, j) dy \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}
\end{aligned}$$

which, again, is solved numerically on a grid of values for the state variables π_t .

3.8 Tables

<i>Return</i>	<i>Mean</i>	<i>Std</i>
r_e	9.08	15.36
r_f	1.28	1.22

Table 3.1: Asset Market data: Annualized sample moments from quarterly US data 1948:II-2005:IV, r_e is the return on the value-weighted NYSE portfolio and r_f is the return on the three-months Treasury bill. Returns are measured in percent per quarter. Source: Hansen and Sargent (2008), page 311.

μ_h	μ_l	σ_h^2	σ_l^2	p_{hh}^μ	p_{ll}^μ	p_{hh}^σ	p_{ll}^σ
0.926	-0.047	0.890	0.208	0.970	0.813	0.992	0.998
(0.075)	(0.233)	(0.087)	(0.034)	(0.023)	(0.134)	(0.004)	(0.015)

Table 3.2: Maximum likelihood estimates of model (7). Numbers in the first four columns are in percentage. Data are quarterly and span the period between 1947:2-2009:2. Standard errors are in parentheses. Estimation by EM algorithm.

$\mathbb{E}_t(r_{e,t+1} - r_{f,t+1})$	$Std(r_{e,t+1} - r_{f,t+1})$	$\mathbb{E}_t(r_{f,t+1})$	$Std(r_{f,t+1})$
9.91	11.89	3.09	0.33

Table 3.3: Standard deviation of returns, average premium, average risk free rate and standard deviation in the absence of ambiguity case. Numbers are in percentage.

Comparative Statistics: $\beta = 0.9925$				
α	r_f	$\sigma(r_f)$	$\mathbb{E}(r_m)$	$\sigma(r_m)$
$\gamma = 30$				
30	3.02	0.33	14.06	11.89
35	2.97	0.34	14.10	11.62
40	2.91	0.35	14.31	11.49
$\gamma = 28$				
28	3.14	0.33	13.29	12.07
35	3.07	0.35	13.41	11.66
40	3.01	0.35	13.70	11.57
$\gamma = 25$				
25	3.32	0.32	12.20	12.57
30	3.27	0.34	12.15	11.87
35	3.21	0.35	12.40	11.70
$\gamma = 20$				
25	3.55	0.32	10.28	13.09
30	3.49	0.33	10.33	11.62
35	3.43	0.35	10.78	11.91
$\gamma = 15$				
30	3.69	0.33	6.74	18.95
35	3.63	0.34	9.19	16.62
40	3.56	0.36	10.54	16.68
45	3.49	0.38	10.62	16.93

Table 3.4: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$ and $\beta = 0.9925$ in all cases.

Comparative Statistics: $\beta = 0.994$				
α	r_f	$\sigma(r_f)$	$\mathbb{E}(r_m)$	$\sigma(r_m)$
$\gamma = 30$				
30	2.40	0.32	14.08	12.05
35	2.37	0.34	13.85	11.70
$\gamma = 25$				
25	2.72	0.32	12.34	13.17
30	2.68	0.33	11.84	12.01
35	2.64	0.35	11.97	11.74
$\gamma = 20$				
20	3.00	0.31	9.35	59.01
25	2.96	0.32	9.47	22.37
30	2.92	0.33	9.83	11.66
$\gamma = 15$				
15	3.12	0.33	4.56	22.70
35	3.07	0.35	6.64	25.28
$\gamma = 10$				
25	3.34	0.30	4.54	22.74
30	3.29	0.32	3.42	27.55
40	3.23	0.34	3.09	33.15

Table 3.5: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$ and $\beta = 0.994$ in all cases.

Comparative Statistics: $\beta = 0.996$				
α	r_f	$\sigma(r_f)$	$\mathbb{E}(r_m)$	$\sigma(r_m)$
$\gamma = 25$				
25	1.81	0.31	14.63	15.50
30	1.81	0.32	11.83	12.38
35	1.80	0.34	11.41	11.81
$\gamma = 20$				
20	2.14	0.30	-0.4	59.17
25	2.12	0.32	4.98	42.04
30	2.10	0.33	9.14	11.70
$\gamma = 15$				
15	2.42	0.28	-0.02	50.68
20	2.39	0.30	6.95	28.98
25	2.36	0.31	6.96	16.16
30	2.31	0.33	7.74	25.53
$\gamma = 10$				
10	2.64	0.26	2.45	38.52
20	2.57	0.29	6.53	18.25
40	2.21	0.42	9.75	55.70

Table 3.6: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$ and $\beta = 0.996$ in all cases.

Chapter 4

Waves of optimism and pessimism

4.1 Introduction

Standard consumption-based asset pricing models have difficulty in explaining many characteristics of financial markets. The literature has documented typical puzzles implied by standard representative agent model with time separable utility: the observed higher returns of risky assets compared to risk-free investment, which could be rationalize only with an implausible high degree of relative risk aversion (Mehra and Prescott, 1985), the unrealistic implication of high risk free returns (Weil, 1989) and the excess volatility observed in stock prices, which is difficult to replicate in models (Shiller, 1981). Such failures have motivated an important effort in the financial literature to provide plausible explanations. One direction has focused on the representative investor who fears uncertainty about the model used and/or tries to learn to resolve such uncertainty. The idea of knightian uncertainty, precautionary behaviors and the need of robustness against model misspecification, have been proposed as a possible explanation for high returns of risky assets (Hansen and Sargent, 2005; Maenhout, 2004; Anderson, Hansen and Sargent, 2003; Leippold, Trojani and Vanini, 2008). Another stream has explored the implication of the interaction of heterogeneous agents and beliefs and focused on the conditions for long run survival of the agents with the most correct beliefs (Blume and Easley, 1992; Sandroni, 2000). It is known that the interaction of agents with different beliefs is able to capture some chaotic pricing behavior observed in data, such as clustering volatility, bubbles and fat tails. From Brock and Hommes (1997, 1998) an increasing stream of literature has focused on

a computationally intensive approach based on the "artificial stock market", which consists of simulating many interacting agents who can exchange assets repeatedly in time. One drawback of this literature is the high parameterization on which it relies. This naturally casts problems of estimation, validation and robustness (Amilon, 2008, Li, Donkers and Melenberg 2010).

In order to overcome this problem, we consider a very simple model in line with Sandroni (2000) who proposes a model where agents have different beliefs about consumption growth's realizations. He investigates the conditions for the convergence to rational expectations stating that if agents have the same intertemporal discount factor and the same utility function, those who make wrong predictions are driven out of the market. Our goal is different. We do not investigate long-run rational expectation conditions, rather we want to study the implications of a long-run persistence of heterogeneity of beliefs where all the agents have access to the same information. Such simple framework allows us to study the effect of heterogeneity of beliefs due to the disagreement in the interpretation of new information.

Disagreement caused by psychological attitudes has been disregarded by traditional economics who fails to take into account the extent to which people are also guided by noneconomic motivations. Akerlof and Shiller (2009) reasserts the necessity to consider the role of "animal spirits" in economic analysis. *"The idea that economic crises [...] are mainly caused by changing thought patterns goes against standard economic thinking. But the current crisis bears witness to the role of such changes in thinking. It was caused precisely by our changing confidence [...] and especially by changing stories of the nature of the economy"* (Akerlof and Shiller, 2009, p.4). The uncertainty about the unstable economic conditions after the severe 2008 financial crisis has been an example of the relevance of this aspect: investors were not wondering about the access of information or measurement issues, rather they have tried to understand the underlying characteristics of the economy, discussing about whether and when the recession period would be over. Temporary positive observations do not necessarily resolve the uncertainty: some analysts have interpreted them as a clear sign of recovery, others were more cautious. In this work, we want to analyze such disagreement in the interpretation of information and we study its effects on asset prices. The heterogeneity in our context is characterized by the attitude toward optimistic and pessimistic interpretation of the information.

Changing confidence in our model is represented by time-varying fraction of optimists and

pessimists. We impose distortions in a way such that all the agents are *equally wrong* about future consumption growth realizations. In this way we preserve heterogeneity in the long run and focus on the implications of the dynamics of prices. The consumption shares, and their effects on prices, inversely depend on how wrong investors' forecasts have been compared to consumption realizations. With this simple framework, we want to model the idea that investors do not simply have difficulties in accessing information, rather they can interpret it differently, depending on their personal attitudes. In particular, we ask ourselves whether this simple framework is able to capture the main observed low-medium frequency movements in asset prices. Indeed, contrary to the hypothesis of informationally efficient markets (Fama 1970, 1991), which implies stock returns close to being unpredictable and prices close to a random walk, a central fact driving forecastable long-horizon returns is that the price-dividend ratio is far from being a random walk: it has persistent fluctuations and is excessively volatile. Apart from the macroeconomic idea that level and movement of risk premia are important for understanding the business cycle, the attention to price-dividend ratio is also motivated by the empirical evidence that the price-dividend ratios appear to be able to predict substantial amounts of stock return variation.

We show that introducing a very simple form of distortion of beliefs in the way described above, can have a notable effect on the implied price dynamics: heterogeneity of beliefs amplifies prices and fluctuations so to capture and reproduce the medium-frequency waves of prices observed in the data.

In order to analyze the sole effect of heterogeneity and to make the interpretation of the results easy, we assume that the consumption growth is modeled as a simple discrete regime switching process. Even in such an unrealistic case, the implied price-dividend ratio is very volatile compared to the distortion-neutral case and the dynamics of the implied fluctuations are able to match, qualitatively, the observed fluctuations.

Finally, we show that testable Euler moment conditions can be defined introducing only one additional degree of freedom and model parameters can be estimated using a Generalized Method of Moments technique. We find that estimates of the distortion are typically positive, but present very high standard errors. Such failure, however, cannot be attributed solely to our model as it is a common feature of simple consumption based asset pricing models (Campbell

and Cochrane, 2000).

We conclude that the ingredient of heterogeneity among economic agents remains fundamental to understand and forecast price fluctuations. The additional degree of freedom is due to the fact that we allow agents to have a personal and heterogeneous attitude in interpreting information and exchanging assets. The agents neither learn nor do they imitate others' behavior: they are just different in the way in which they interpret reality even if they can share the same degree of risk aversion or the same discount factor.

The remainder of this paper is structured as follows. Section 4.2 describes the model and the agents' optimization problem. Section 4.3 presents a quantitative analysis based on the model estimates, focusing, especially on the historical implied dynamics of the price-dividend ratio for different distortion degrees. In Section 4.4, the testable model restrictions are derived and Section 4.5 concludes.

4.2 The model

We study a simple dynamically complete market with two long-lived groups of agents, two assets, two states of nature and one single consumption good. Let T be the set of natural numbers. At period $t \in T$, the agents observe the state of nature of the consumption growth process, $y_t = \log \frac{C_t}{C_{t-1}}$. The set of states of nature is given by $Y \equiv \{y_h, y_l\}$, with $y_h > y_l$. The true stochastic process of the states of nature is given by a Markovian structure so that

$$\log \frac{C_{t+1}}{C_t} = y_{t+1} = y(S_{t+1}), \quad (4.1)$$

where S_{t+1} is a two-state Markov chain with associated transition matrix $P = \{p_{ij}\}_{2 \times 2}$ where $p_{ij} = \text{prob}[S_{t+1} = j | S_t = i]$, with $i, j = h, l$. In the remainder of the paper, we denote $y(h)$ with y_h and $y(l)$ with y_l .

The investors can choose among two assets: asset 1 pays the aggregate dividend D_t if y_h realizes, and zero otherwise. Asset 2 pays D_t if y_l realizes and zero dividends if y_h realizes. As standard, we assume that aggregate dividends correspond to aggregate consumption, $D_t = C_t$. The assumption of two states can provide a simple, yet realistic, description of how economic agents process information in reality: agents are, by nature, are mostly influenced by the

qualitative aspects (good versus bad) of new information more than by the precise quantitative data.

We consider two groups of agents who have different psychological attitudes: they can be optimistic or pessimistic. They have access to the same information y_t , but they bias their beliefs so that the optimistic agents tend to believe that good periods are more likely and the pessimistic ones bias their beliefs so to increase the probability of the economy being in a low-mean state.

Let T be the set of natural numbers. At time $t \in T$, the history of realizations of the consumption growth, $y^t = \{y_0, y_1, \dots, y_t\}$, is a commonly available information: agents observed the state and update their beliefs about the next observation. The subjective beliefs of agent i are defined by the vector $\pi_t^i = \begin{bmatrix} \pi_t^i(h) & \pi_t^i(l) \end{bmatrix}$, where $\pi_t^i(l) = 1 - \pi_t^i(h) = \text{prob}^i(S_{t+1} = l | y^t)$ is the prior probability of state l in period $t + 1$ according to agent i after she observes y_t . The full-bayesian prior belief of state s in time $t + 1$ is defined as

$$\pi_t(s) = \sum_{j \in \{h, l\}} p_{js} 1_{\{j\}}.$$

Brandt et al. (2004) provide a simple way to model subjective attitudes: the optimistic belief of the high-mean state, π_t^o , is defined as:

$$\pi_t^o(h) = (1 - \omega^o) \pi_t(h) + \omega^o$$

where $\omega^o \in (0, 1)$ is a parameter which measures the degree of the optimistic distortion. Similarly, the pessimistic belief is described as

$$\pi_t^p(h) = (1 - \omega^p) \pi_t(h)$$

where $\omega^p \in (0, 1)$. The definitions of $\pi_t^o(l)$ and $\pi_t^p(l)$ follow immediately. It is also easy to check that $\pi_t^o(h)$ can be obtained by distorting the belief about the transition probabilities:

$$\pi_t^o(s) = \sum_{j \in \{h, l\}} p_{js}^o 1_{\{j\}},$$

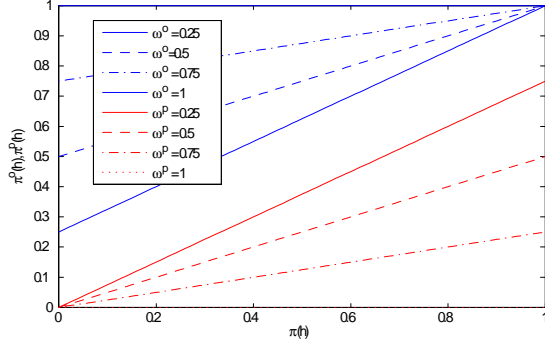


Figure 4-1: Optimistic (blue lines) and pessimistic (red lines) distortions of beliefs.

where

$$p_{hh}^o = (1 - \omega^o)p_{hh} + \omega^o$$

and

$$p_{lh}^o = (1 - \omega^o)(1 - pu) + \omega^o.$$

We can, therefore, think of the optimistic (pessimistic) agents also as those who consider the good (bad) states more persistent. Figure 4-1 shows the effect of the distortion of the belief of the high growth state, plotted on the x -axis. The optimistic (pessimistic) beliefs are always above (below) the 45-degree line and the greater are the distortion parameters, ω^o and ω^p , the farther are the subjective beliefs from the unbiased ones.

Agents' optimization problem We can find the equilibrium allocation by posing a Pareto problem for a fictitious social planner, who attaches nonnegative Pareto weights λ_i , $i = o, p$ on the consumers and maximizes the social utility function W :

$$W = \lambda_o U_o + \lambda_p U_p \tag{4.2}$$

where U_i is the agent i 's utility functional:

$$\begin{aligned}
U_i &= \mathbb{E}_0^i \sum_{t=0}^{\infty} \beta^t u(C_t^i) \\
&= \sum_{t=0}^{\infty} \sum_{y^t} \beta^t \text{prob}^i(y^t|y_0) u(C_t^i(y^t)),
\end{aligned} \tag{4.3}$$

where \mathbb{E}_0^i is the mathematical expectation operator conditioned on y_0 , and $\text{prob}^i(y^t|y_0)$ represents the agent i 's conditional probability of observing the realized history y^t prior to any observation. We assume that trading occurs after observing y_1 , and we set $\text{prob}^i(y^0|y_0) = 1$. The utility function $u(\cdot)$ is supposed to be identical among agents. Furthermore, $u(\cdot)$ is an increasing concave function of consumption $C \geq 0$ and satisfies the Inada conditions.

The maximization is subject to the time t , history y^t budget constraint:

$$\begin{aligned}
\sum_{i \in \{o,p\}} C_t^i(y^t) &= \\
&= \sum_{i \in \{o,p\}} \left\{ \sum_{m=1}^2 \left[(P_t^m(y^t) + D_t^m(y^t)) x_{t-1}^{i,m}(y^{t-1}) - P_t^m(y^t) x_t^{i,m}(y^t) \right] \right\},
\end{aligned} \tag{4.4}$$

where $x_t^{i,m}(y^t)$ denotes the quantity of asset m chosen by the investor i at time t , $P_t^m(y^t)$ and $D_t^m(y^t)$ are the time t price and the paid dividend of asset m , respectively. Markets clear when

$$\begin{aligned}
C_t^o(y^t) + C_t^p(y^t) &= C_t(y^t) = D_t(y^t), \forall t \in T, \\
x_t^{o,m}(y^t) + x_t^{p,m}(y^t) &= 1, \forall t \in T, m = 1, 2,
\end{aligned} \tag{4.5}$$

where $C_t^o(y^t)$ and $C_t^p(y^t)$ are the consumption allocations among the optimists and pessimists, respectively, while $C_t(y^t)$ and $D_t(y^t)$ denote the aggregate consumption and the aggregate dividend.

Let $\mu_t(y^t)$ be a nonnegative multiplier on the budget constraint. The Lagrangian of the

optimization problem (4.2)-(4.3) can then be formed:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{y^t} \left[\begin{aligned} & \sum_{i \in \{o,p\}} \{ \lambda_i \beta^t \text{prob}^i(y^t|y_0) u(C_t^i(y^t)) \} + \\ & + \mu_t(y^t) \left(\sum_{i \in \{o,p\}} \left\{ \sum_{m=1}^M \left[P_t^m(y^t) x_{t-1}^{i,m}(y^{t-1}) + D_t^m(y^t) x_{t-1}^{i,m}(y^{t-1}) \right] \right. \right. \\ & \quad \left. \left. - P_t^m(y^t) x_t^{i,m}(y^t) \right] - C_t^i(y^t) \right) \end{aligned} \right].$$

The first order conditions of maximizing \mathcal{L} with respect to $C_t^i(y^t)$ and $x_t^{i,m}(y^t)$ are

$$\lambda^i \beta^t \text{prob}^i(y^t|y_0) u'(C_t^i(y^t)) = \mu_t(y^t), \quad (4.6a)$$

$$\mu_t(y^t) P_t^m(y^t) = \sum_{y_{t+1}} \mu_{t+1}(y_{t+1}) (P_{t+1}^m(y_{t+1}) + D_{t+1}^m(y_{t+1})), \quad (4.6b)$$

$\forall m, \forall i, \forall t$. Substituting eq. (4.6a) into eq. (4.6b) we have:

$$P_t^m(y^t) = \beta \sum_{y_{t+1}} \text{prob}^i(y_{t+1}|y^t) \frac{u'(C_{t+1}^i(y_{t+1}))}{u'(C_t^i(y^t))} (P_{t+1}^m(y_{t+1}) + D_{t+1}^m(y_{t+1})). \quad (4.7)$$

In order to solve for prices we need to determine the equilibrium consumption shares or the ratio of the multipliers. Indeed, from eq. (4.6a), we derive the following condition on the ratio of multipliers:

$$\frac{\text{prob}^o(y^t|y_0) u'(C_t^o(y^t))}{\text{prob}^p(y^t|y_0) u'(C_t^p(y^t))} = \frac{\lambda^p}{\lambda^o}.$$

At time $t = 0$, the above condition implies that $\frac{\lambda^p}{\lambda^o} = \frac{u'(C_0^o(y_0))}{u'(C_0^p(y_0))}$. If we assume that the initial consumption shares of the two groups of agents are equal, we have:

$$\text{prob}^o(y_{t+1}|y_t) \frac{u'(C_{t+1}^o(y_{t+1}))}{u'(C_t^o(y^t))} = \text{prob}^p(y_{t+1}|y_t) \frac{u'(C_{t+1}^p(y_{t+1}))}{u'(C_t^p(y^t))},$$

or

$$\frac{u'(C_{t+1}^o(y_{t+1}))}{u'(C_{t+1}^p(y_{t+1}))} = \frac{\text{prob}^p(y_{t+1}|y_t) u'(C_t^o(y^t))}{\text{prob}^o(y_{t+1}|y_t) u'(C_t^p(y^t))}, \quad (4.8)$$

so that the optimal allocations are characterized by the property that, between any two com-

modities, all consumers share a common marginal rate of substitution. By backward substitution, we derive the fundamental equation for characterizing long-run equilibrium consumption:

$$\frac{u'(C_t^o(y^t))}{u'(C_t^p(y^t))} = \frac{u'(C_0^o(y_0)) \text{prob}^p(y^t|y_0)}{u'(C_0^p(y_0)) \text{prob}^o(y^t|y_0)}. \quad (4.9)$$

When equal initial consumption shares are considered,¹ eq. (4.9) simplifies to:

$$\frac{u'(C_t^o(y^t))}{u'(C_t^p(y^t))} = \frac{\text{prob}^p(y^t|y_0)}{\text{prob}^o(y^t|y_0)}. \quad (4.10)$$

Then we can write:

$$\begin{aligned} u'(C_t^o(y^t)) &= \frac{\text{prob}^p(y^t|y_0)}{\text{prob}^o(y^t|y_0)} u'(C_t^p(y^t)) \\ &= \frac{\text{prob}^p(y^t|y_0)}{\text{prob}^o(y^t|y_0)} u'(C_t(y^t) - C_t^o(y^t)). \end{aligned}$$

In order to facilitate the comparison with similar studies and for its tractability, we consider the well known isoelastic CRRA utility functional:

$$u(C_t^o(y^t)) = \frac{(C_t^o(y^t))^{1-\alpha} - 1}{1-\alpha},$$

where α denotes the constant Arrow-Pratt measure of relative risk aversion. The equilibrium condition on consumption shares then becomes:

$$C_t^o(y^t) = \left[\frac{\text{prob}^p(y^t|y_0)}{\text{prob}^o(y^t|y_0)} \right]^{-\frac{1}{\alpha}} [C_t(y^t) - C_t^o(y^t)]. \quad (4.11)$$

Denoting the ratio of beliefs of the representative agents of each group at time t , $\frac{\text{prob}^p(y^t|y_0)}{\text{prob}^o(y^t|y_0)}$, with $\eta_t(y^t)$, we can rewrite eq. (4.11) as follows:

¹Robustness studies have been conducted with different initial consumption endowment share. Results do not change significantly.

$$\begin{aligned}
C_t^o(y^t) &= \eta_t^{-\frac{1}{\alpha}}(y^t) [C_t(y^t) - C_t^o(y^t)] \\
&= \frac{1}{\eta_t^{\frac{1}{\alpha}}(y^t) + 1} C_t(y^t)
\end{aligned} \tag{4.12}$$

and

$$C_t^p(y^t) = C_t(y^t) - C_t^o(y^t) = \frac{\eta_t^{\frac{1}{\alpha}}(y^t)}{\eta_t^{\frac{1}{\alpha}}(y^t) + 1} C_t(y^t), \tag{4.13}$$

which define the equilibrium shares of consumptions. As a direct consequence of Pareto optimality, the allocation of resources is optimal: each type of agents consumes more in the state which is considered more likely. The ratio of beliefs, η_t , indeed, summarizes the relative performance of the two representative agents: it is the ratio of the product of the probabilities that the two groups have attributed to the real observations. Furthermore, we observe that the consumption share depends exclusively on the ratio of beliefs and on the risk aversion parameter. Figure 4-2 plots the consumption share of the optimistic agent, $x_t^o = \frac{C_t^o(y^t)}{C_t(y^t)}$, as a function of the ratio of beliefs for different values of risk aversion. Higher risk aversion parameters smooth the consumption share as a function of the ratio of beliefs. When traders are nearly risk neutral, they take more extreme asset positions, so those with incorrect beliefs will be driven out of the market soon. In the extreme case of risk neutrality, the optimistic agent would consume everything for values of η_t less than one, while he would get nothing for values of η_t greater than unity, when all the aggregate consumption would be allocated among the pessimistic ones.

The equilibrium conditions (4.7), (4.12), and (4.13) allow us to determine the equilibrium prices and allocations. From the perspective of the optimistic agents, the equilibrium conditions for the price-dividend ratio of the asset m , $\frac{P_t^m}{D_t}$, is given by the following fixed-point recursive equations, where $\varphi_t^m = \frac{P_t^m}{D_t}$:²

²Calculus details can be found in the Appendix 1.

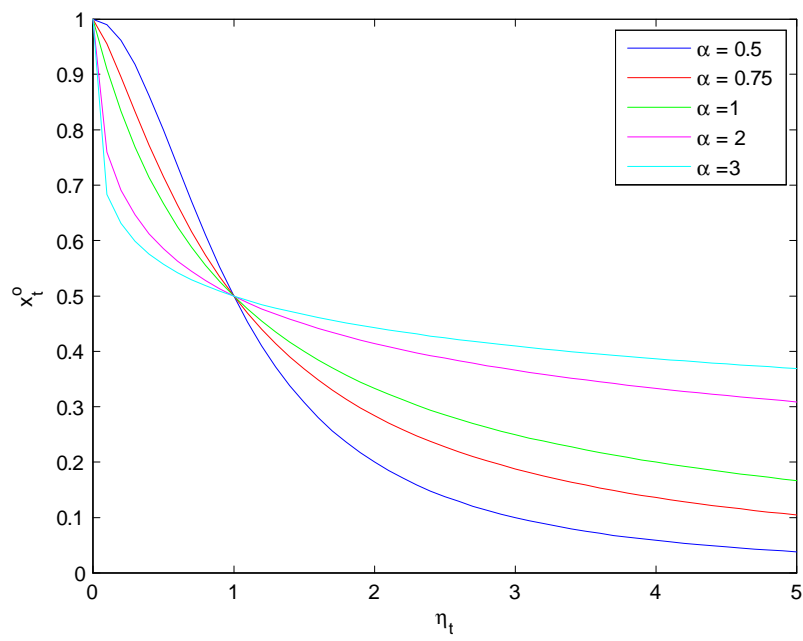


Figure 4-2: Share of the optimistic agent as a function of η_t .

$$\varphi_t^1 = \beta \left(\eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^{-\alpha} \times \left[\begin{array}{l} \pi_t^o(h) \left(\left[\frac{\pi_t^p(h)}{\pi_t^o(h)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^\alpha \exp((1-\alpha) y_h) [1 + \varphi_{t+1}^1] + \\ \pi_t^o(l) \left(\left[\frac{\pi_t^p(l)}{\pi_t^o(l)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^\alpha \exp((1-\alpha) y_l) \varphi_{t+1}^1 \end{array} \right]$$

and

$$\varphi_t^2 = \beta \left(\eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^{-\alpha} \times \left[\begin{array}{l} \pi_t^o(l) \left(\left[\frac{\pi_t^p(l)}{\pi_t^o(l)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^\alpha \exp((1-\alpha) y_l) [1 + \varphi_{t+1}^2] + \\ \pi_t^o(h) \left(\left[\frac{\pi_t^p(h)}{\pi_t^o(h)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}} (y^t) + 1 \right)^\alpha \exp((1-\alpha) y_h) \varphi_{t+1}^2 \end{array} \right].$$

If agents did not distort their beliefs or if they had homogeneous beliefs ($\eta_t = 1, \forall t$), π_t would be the only state variable needed to solve the dynamics of the model. In this case, the equilibrium price-dividend function would be a time constant function of the state beliefs:

$$\varphi_t^m = \varphi^m(\pi_t).$$

Heterogeneity of beliefs, on the contrary, implies that in each period t , a new value of η_t realizes so that the equilibrium price-dividend function is now a function of the stochastic variable η_t :

$$\varphi_t^m = \varphi^m(\eta_t, \pi_t; \omega^o, \omega^p)$$

Numerically, this requires solving the functional equations for φ_t^m for the state π_t for each time t , taking η_t as given. Numerical solutions such as Chebyshev collocation methods are quite fast to use and, therefore, suitable also in case of heterogeneity³.

³Matlab code is available upon request.

4.2.1 Long-Run Dynamics

In this section we are interested in understanding the long-run dynamics of the ratio of beliefs, and, therefore, of the consumption share. We impose that heterogeneity among agents persists in the long run to formalize the idea that different opinions and attitudes among the agents are always present and do not characterize only a short-run condition of the economy. In order to model such persistence, we need to impose that both representative agents are *equally wrong* about the realization of the consumption observations. In the long run, with $t \rightarrow \infty$, we can take advantage of the following implications:

1. The full-Bayesian belief of the high-growth state, $\pi_t(h)$, converges to the unconditional probability, $\pi(h) = \frac{1-p_{ll}}{2-p_{hh}-p_{ll}}$;
2. Individual beliefs can be expressed in terms of the stationary probability:

$$\begin{aligned}\pi^o(h) &= [(1 - \omega^o) \pi(h) + \omega^o], \\ \pi^p(h) &= [(1 - \omega^p) \pi(h)],\end{aligned}$$

so that we can think of $\pi^o(h)$ and $\pi^p(h)$ as the individual unconditional probabilities of the high-growth state;

3. The evolution of the ratio of beliefs takes the following form:

$$\begin{aligned}\eta_t &= \frac{\text{prob}^p(y^t|y^0)}{\text{prob}^o(y^t|y^0)} = \frac{[\text{prob}^p(y_h)]^{\text{prob}(y_h)t} [\text{prob}^p(y_l)]^{\text{prob}(y_l)t}}{[\text{prob}^o(y_h)]^{\text{prob}(y_h)t} [\text{prob}^o(y_l)]^{\text{prob}(y_l)t}} \\ &= \frac{[\pi^p(h)]^{\pi(h)t} [\pi^p(l)]^{\pi(l)t}}{[\pi^o(h)]^{\pi(h)t} [\pi^o(l)]^{\pi(l)t}} = (\eta^*)^t.\end{aligned}\tag{4.14}$$

Imposing long run survival of both groups implies $\eta^* = 1$ and allows us to compute ω^p as a function of ω^o , so to study the implications of the persistent deviation from the full rationality framework while, technically, keeping limited the number of parameters in the model: for each value of the distortion of the optimistic agent, ω^o , we can derive the corresponding pessimistic distortion such that, in the long run, both types of agents survive. This implies that our model involves only one additional degree of freedom to the standard expected utility framework.

4.3 Quantitative Analysis

In this section, we estimate a simple two-states mean Markov switching model for the consumption growth process observed in the postwar sample:

$$\log\left(\frac{C_{t+1}}{C_t}\right) = \mu_{\xi_t} + \epsilon_t,$$

with

$$\epsilon_t \sim N(0, \sigma^2),$$

and ξ_t being a 2-states Markov chain whose transition matrix has diagonal terms p_{hh} and p_{ll} .

We then use these values to characterize the dynamics of our simplified two-state discrete version model (4.1) in a way that is made precise below. Table 4.1 shows the Maximum-Likelihood estimates for the consumption growth process using quarterly US economy data from the Bureau of Economic Analysis (*BEA*) website⁴. The high-growth state is more persistent and the unconditional probability that the economy finds itself in the high-growth state is higher ($\pi(h) = 0.845$).

μ_h	μ_l	σ	p_{hh}	p_{ll}
0.7244 (0.08)	-0.1999 (0.18)	0.0021 ($9.2e-06$)	0.9611 (0.0022)	0.8103 (0.0059)

Table 4.1: Maximum-Likelihood Estimates. Personal Consumption Expenditure (PCE) quarterly data from 1947:1 to 2009:4. Columns 1-3 are expressed in percentage terms. Standard errors are in parentheses.

Such asymmetry among the states is reflected in the long-run relation between the distortions of the individual beliefs which makes sure that η_t is equal to 1 in the long run: Figure 4-3 depicts such relation and shows that for a given value of ω^o , a considerably lower value of the pessimistic distortion is found. In other words, since the economy finds itself in the high-growth state more times, the optimistic believers distort more their beliefs compared to the distortion of the pessimists. If the pessimistic distortion was higher, those beliefs would be driven out

⁴A summary of data is available in the Appendix 2.

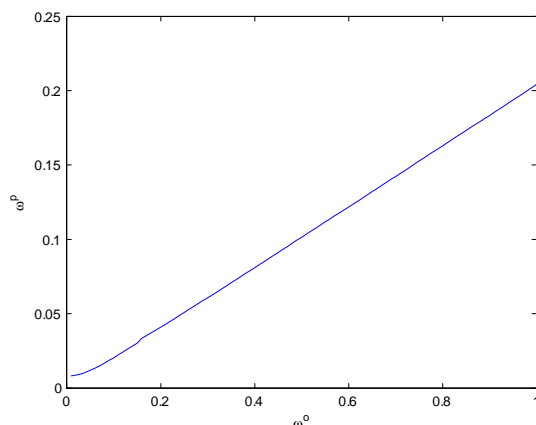


Figure 4-3: Locus of combinations of ω^o and ω^p such that $\eta^* = 1$ in the long run.

of the market because they would be wrong more times than the optimists and the long term ratio of the beliefs, η^* , would become smaller than unity, a possibility that we rule out.

In order to derive the implied price series, we need to discretize the real observations so to make them compatible with our simple discrete model. A natural choice is to consider the observation y_t as y_h (y_l) if $y_t < y_\mu$ ($y_t > y_\mu$), where y_μ is the value corresponding to the inverse cumulative function such that

$$prob(y_t < y_\mu) = \pi(l), \forall t.$$

In each time t , given η_t and π_t , the model can generate the implied consumption shares and equilibrium prices. Figure 4-4 presents a comparative graph where the dynamics of the optimistic share is plotted for different values of the distortion ω^o .

Higher levels of distortion imply more significant switching among consumption shares. In the extreme case of no distortion, consumption shares are constant and equal to 0.5. Figure 4-5 shows a similar comparative exercise, where equilibrium shares and prices are plotted for different values of distortions (vertically) and risk aversion (inside each panel).

As already noted above, risk aversion smooths the dynamics of the switching of the consumption share for different levels of risk aversion while higher levels of distortions have the opposite effect. Price-dividend ratios appear to be very sensitive to the switching of shares if

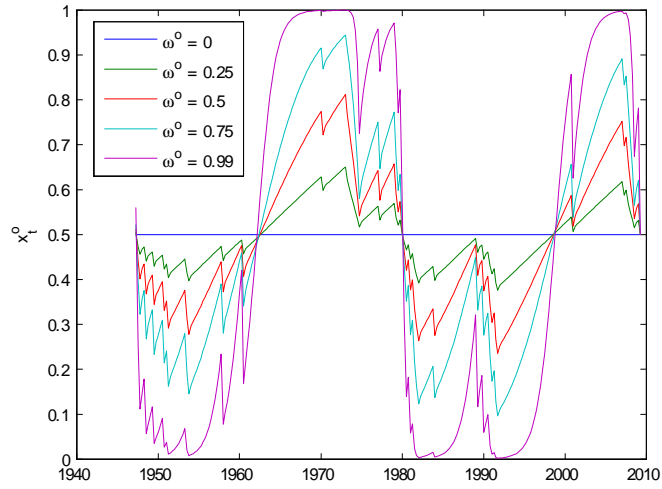


Figure 4-4: Time series of share of the optimistic agent for different intensities of distortion. ($\alpha = 4.5, \beta = 0.995$).

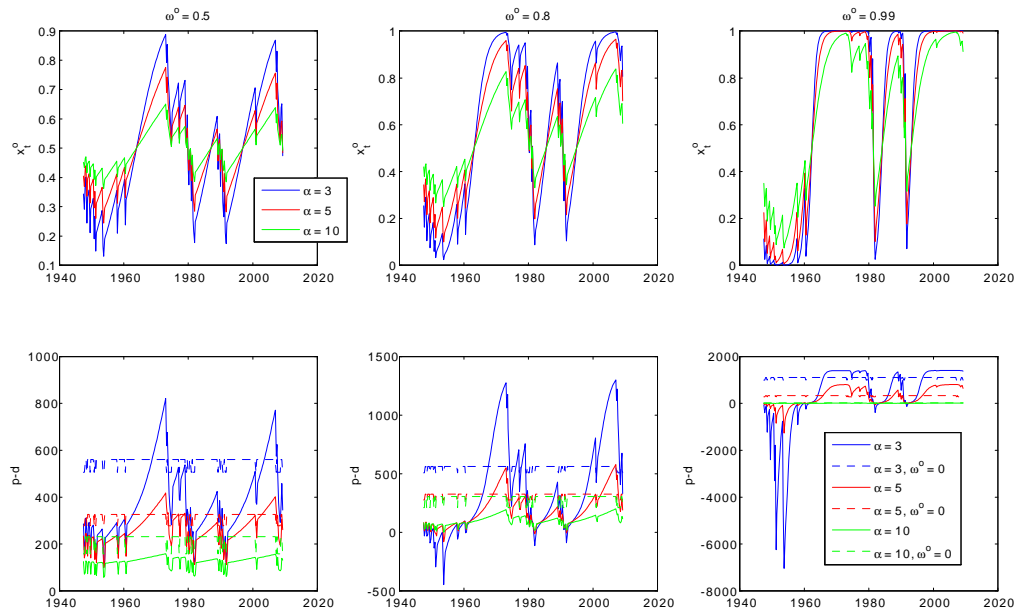


Figure 4-5: Comparative analysis of consumption shares and price-dividend ratios for different risk aversion and intensity of distortion parameters ($\beta = 0.995$).

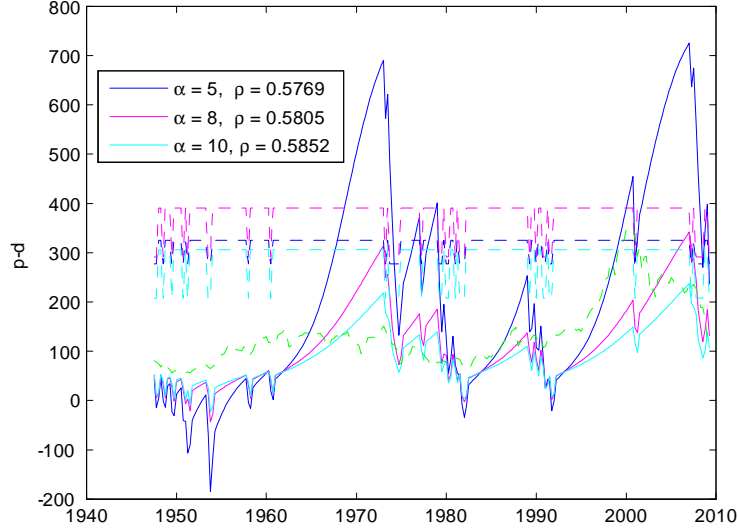


Figure 4-6: Implied price-dividend ratio for different values of risk aversion. The dashed green line represents the observed series. ($\omega^o = 0.9, \beta = 0.995$).

risk aversion is low. High biases in beliefs and low risk aversion are able to imply very volatile prices, which appear very low (even implausible negative for some combinations of the parameters) in periods in which pessimists get the biggest consumption share and very high in growth periods.

The implications of the model in terms of prices for different values of risk aversion and distortions can be directly compared with real data. Low levels of risk aversion can produce very volatile prices in presence of heterogeneity. This suggests that heterogeneity plays an important role towards the explanation of the equity premium puzzle because even with small risk aversion the model is able to generate price fluctuations close to the observed ones. Figures 4-6 and 4-7 show such comparisons and present the correlation coefficient between data and the implied series for several values of risk aversion and make clear that the presence of heterogeneity of beliefs amplifies the effect of low values of risk aversion, helping in matching the medium term fluctuations. Such effect would not be captured in a similar simple model with homogeneous beliefs.

We also observe that there is substitutability between risk aversion and distortion: higher

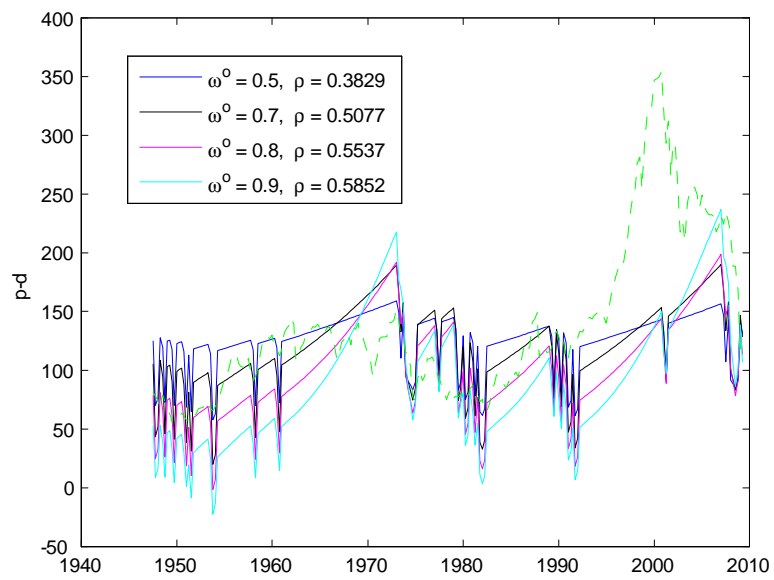


Figure 4-7: Implied price-dividend ratio for different values of distortions. The green dashed line represents the observed series. ($\alpha = 10, \beta = 0.995$).

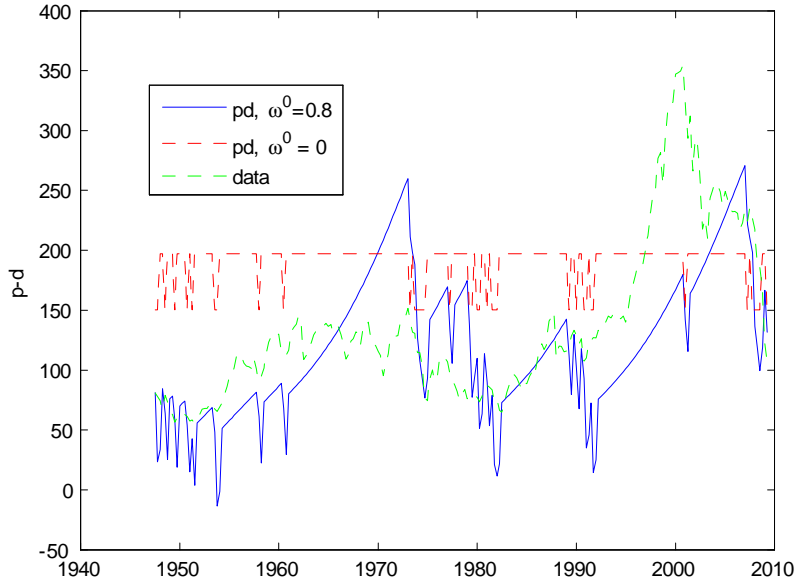


Figure 4-8: Implied and observed price-dividend ratios. The red dashed line represents the implied series with no distortion in beliefs. The green dashed line represents the observed series. ($\alpha = 8, \omega^o = 0.8, \gamma = 1, \beta = 0.995$).

risk aversion produces lower but less volatile price-dividend ratios, while higher distortion in beliefs causes prices to be lower and more volatile. This fact suggests that high values of distortion in beliefs can substitute risk aversion so that a good matching of the series can be obtained with low values of risk aversion. Heterogeneity in this sense can help in resolving the equity premium puzzle: low price-dividend ratios usually imply expected higher returns.

As a baseline case, we choose a calibration which appears quite satisfactory in matching the level and the medium-long term fluctuations in prices, where we set α equal to 8 and ω^o equal to 0.8. Figure 4-8 shows the comparison between the data (green dashed line) and the implied prices (blue solid line). We also show the implied series with the same level of risk aversion without heterogeneity in beliefs (red dashed line). The correlation coefficient, ρ , is 0.5580, that can be considered quite satisfactory given the simplistic structure of the model.

The corresponding dynamics of the share of the optimists is represented in Figure 4-9. Clearly, the dynamics of the shares among agents determines the waves of prices: when the

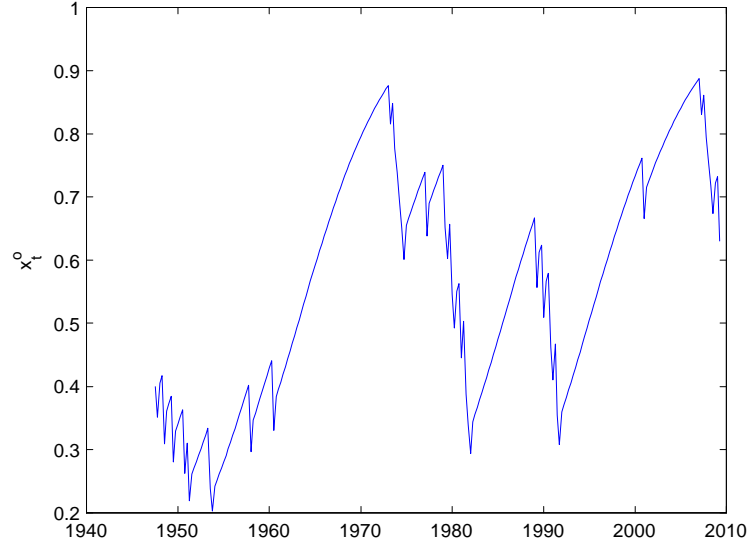


Figure 4-9: Share of the optimistic agents. ($\alpha = 8, \omega^o = 0.8, \gamma = 1, \beta = 0.995$).

optimistic group is the majority, their consumption shares cause prices to increase and the reverse happens when the pessimistic share is the greatest. At the beginning of the year 2010, the share of the optimistic is still greater than the pessimistic one, but it is rapidly decreasing. We know that during the current prolonged financial distress, in the first half of 2011, there has been a lot of uncertainty regarding the consumers' sentiment and the possibility of a new fast recovery. The figures suggests, however, that fast recovery in the economy and in consumers' sentiment has been experienced after years of extreme pessimism, as if it is likely to observe still unstable periods of recession before experiencing a new stable optimism view of the economy.

We conclude that this simple model is able to imply persistent effects of changing confidence levels and can be thought as a simple formalized counterpart of the idea of confidence multiplier of Akerlof and Shiller (2009): "*Changes in confidence will result in changes in income and confidence in the next round, and each of these changes will in turn affect income and confidence in yet further rounds*" (p. 16).

4.4 Testable moments restrictions

In this section we focus on the testability of the equilibrium conditions of the model simple moment conditions that can be derived from the model with heterogeneous expectations. Let M be the set of assets. From (4.7), the time t price of an asset $m \in M$ must satisfy the following Euler for agent i :

$$1 = \beta \mathbb{E}_t^i \left[\left(\frac{C_t^i}{C_{t+1}^i} \right)^{-\alpha} R_{t+1}^m \right], m \in M, i \in \{o, p\} \text{ and } t \in T, \quad (4.15)$$

where $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$ is the gross return of asset m .

Substituting the share of optimistic agent, eq. (4.15) becomes:

$$\begin{aligned} 1 &= \beta \mathbb{E}_t^o \left[\left(\frac{(\eta_t)^{\frac{1}{\alpha}} + 1}{(\eta_{t+1})^{\frac{1}{\alpha}} + 1} \right)^{-\alpha} \left(\frac{C_{t+1}}{C_t} \right)^{-\alpha} R_{t+1}^m \right] \\ &= \beta \mathbb{E}_t^o f(\eta_{t+1}, y_{t+1}, R_{t+1}^m), \end{aligned}$$

where $y_{t+1} = \log \frac{C_{t+1}}{C_t}$. It follows that

$$\begin{aligned} 1 &= \beta \mathbb{E}_t^o f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \\ &= \beta \sum_{y_{t+1}} f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \text{prob}^o(y_{t+1}|y^t). \end{aligned} \quad (4.16)$$

It is easy to show that

$$\text{prob}^o(y_{t+1}|y^t) = (1 - \omega^o) \text{prob}(y_{t+1}|y^t) + \omega^o \text{prob}(y_{t+1}|S_{t+1} = h), \quad (4.17)$$

where $\text{prob}(y_{t+1}|y^t)$ is the full-Bayesian probability⁵. Substituting this definition in (4.16) gives the condition:

$$1 = \sum_{y_{t+1}} f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \{ (1 - \omega^o) \text{prob}(y_{t+1}|y^t) + \omega^o \text{prob}(y_{t+1}|S_{t+1} = h) \}, \quad (4.18)$$

⁵See Appendix 3 for details.

in which we need to work out more explicitly the term $\text{prob}(y_{t+1}|S_{t+1} = h)$. By the properties of the conditional probabilities the following equalities can be derived:

$$\begin{aligned}
\text{prob}(y_{t+1}|S_{t+1} = h) &= \frac{\text{prob}(y_{t+1}, S_{t+1} = h)}{\text{prob}(S_{t+1} = h)} \\
&= \frac{\text{prob}(S_{t+1} = h|y_{t+1}) \text{prob}(y_{t+1})}{\text{prob}(S_{t+1} = h)} \\
&= \frac{\pi_{t+1,t+1}(h) \text{prob}(y_{t+1})}{\pi(h)},
\end{aligned} \tag{4.19}$$

so that substituting the term back into (4.18) we have:

$$\begin{aligned}
1 &= \sum_{y_{t+1}} f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \left\{ (1 - \omega^o) \text{prob}(y_{t+1}|y^t) + \omega^o \frac{\pi_{t+1,t+1}(h) \text{prob}(y_{t+1})}{\pi(h)} \right\} \\
&= (1 - \omega^o) \sum_{y_{t+1}} \text{prob}(y_{t+1}|y^t) f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \\
&\quad + \omega^o \sum_{y_{t+1}} \text{prob}(y_{t+1}) f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \frac{\pi_{t+1,t+1}(h)}{\pi(h)} \\
&= (1 - \omega^o) E_t f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) + \frac{\omega^o}{\pi(h)} \mathbb{E} f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \pi_{t+1,t+1}(h).
\end{aligned}$$

Let z_t be a q -dimensional vector of instrumental variables that are in the agents' information set. The equilibrium condition in terms of unconditional expectations can be defined as follows:

$$\begin{aligned}
z_t &= (1 - \omega^o) \mathbb{E}_t \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \otimes z_t \} + \\
&\quad + \frac{\omega^o}{\pi(h)} \mathbb{E} \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \pi_{t+1,t+1}(h) \} \otimes z_t,
\end{aligned}$$

where \otimes denotes the Kronecker product and $\pi_{t+1,t+1}(h)$ is the full-Bayesian posterior probability of the high-mean state, at time $t + 1$, after observing y_{t+1} , that is, $\pi_{t+1,t+1}(h) = \text{prob}[S_{t+1} = h | y^{t+1}]$. Taking the unconditional expectation of both sides, we have:

$$\begin{aligned}
0 &= \mathbb{E} \left\{ \begin{aligned} &(1 - \omega^o) \mathbb{E}_t \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \otimes z_t \} + \\ &+ \frac{\omega^o}{\pi(h)} \mathbb{E} \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \pi_{t+1,t+1}(h) \} \otimes z_t - z_t \end{aligned} \right\} \\
&= (1 - \omega^o) \mathbb{E} \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \otimes z_t \} + \\
&\quad + \frac{\omega^o}{\pi(h)} \mathbb{E} \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \pi_{t+1,t+1}(h) \} \otimes \mathbb{E} z_t - \mathbb{E} z_t
\end{aligned}$$

and the sample equivalent of the equilibrium condition can be defined as:

$$0 = \frac{1}{T} \sum_{t=1}^T \left\{ \begin{aligned} &(1 - \omega^o) \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \otimes z_t \} + \\ &+ \frac{\omega^o}{\pi(h)} \{ f(\eta_{t+1}, y_{t+1}, R_{t+1}^m) \pi_{t+1,t+1}(h) \} \otimes \frac{1}{T} \sum_{t=1}^T z_t - z_t \end{aligned} \right\}.$$

In our model the ratio of beliefs, η_{t+1} , is a constant function of y_{t+1} . Indeed,

$$\eta_{t+1} = \frac{\prod_{j=1}^{t+1} prob^p(y_j|y_{j-1})}{\prod_{j=1}^{t+1} prob^o(y_j|y_{j-1})}$$

so that, at each time t , η_t is known and a function of y_t given the distortion parameters.

Therefore, the Generalized Method of Moments testable condition can be written as:

$$0 = \frac{1}{T} \sum_{t=1}^T \left\{ \begin{aligned} &(1 - \omega^o) f(y_{t+1}, R_{t+1}^m; \omega^o) \otimes z_t + \\ &\frac{\omega^o}{\pi(h)} f(y_{t+1}, R_{t+1}^m; \omega^o) \pi_{t+1,t+1}(h) \otimes \frac{1}{T} \sum_{t=1}^T z_t - z_t \end{aligned} \right\}.$$

In order to estimate the parameters α, ω^o and β , we use the nondurable plus services series of personal consumption expenditure, available at the Bureau of Economic Analysis (*BEA*) website. Nominal market returns and risk free rates are available in the CRSP website. We used quarterly data from 1947:2 to 2009:2. Nominal values are converted into real quantities by dividing by the implicit price deflator associated with the consumption series⁶.

We consider different combinations of instrumental variables, made of lagged values of con-

⁶ A detailed description of the data is in the Appendix 2.

sumption and lagged returns. A first round of consistent but inefficient estimates is obtained using the identity matrix as the weighting matrix. The consistent estimates are used to construct the efficient weighting matrix. Table 4.2 reports the estimates, the column DF denotes the degrees of freedom or the number of overidentifying restrictions, and $prob$ represents the probability that a $\chi^2(DF)$ random variate is less than the computed value of the test statistic under the hypothesis that the model equilibrium conditions are satisfied.

<i>INST</i>	α	ω^o	β	<i>DF</i>	<i>Prob</i>
<i>const; y₋₁; y₋₂; R₋₁</i>	3.0991 (5.0869)	0.1000 (0.0505)	0.9925 (0.0329)	1	0.9268
<i>const; y₋₁; y₋₂; R₋₁ * y₋₁; R₋₁ * y₋₂</i>	3.1055 (13.1846)	0.1000 (0.2269)	0.9900 (0.0761)	2	0.1729
<i>const; y₋₁²; R₋₁²; R₋₁² * y₋₁²</i>	3.0986 (17.1676)	0.1000 (0.2652)	0.9902 (0.1069)	1	0.1981
<i>const; y₋₁²; R₋₁²; y₋₁² * R₋₁²; y₋₂²</i>	3.0995 (17.0850)	0.1000 (0.2651)	0.9901 (0.1064)	2	0.1497

Table 4.2: Generalized Methods of Moments estimates with overidentified conditions. Standard errors are in parentheses.

<i>INST</i>	α	ω^o	β
<i>const; y₋₁; R₋₁</i>	3.0995 (5.1169)	0.1000 (0.0581)	0.9923 (0.0330)
<i>const; y₋₁; y₋₂</i>	3.1070 (13.3396)	0.1000 (0.2599)	0.9898 (0.0766)
<i>const; y₋₁²; y₋₂²</i>	3.0797 (61.9299)	0.1000 (2.7636)	0.9901 (0.5304)
<i>const; y₋₁²; R₋₁²</i>	3.0885 (17.4155)	0.1640 (0.4155)	0.9862 (0.1208)

Table 4.3: Generalized Methods of Moments estimates in the case of exactly identification. Standard errors are in parentheses.

The estimates of α , ω^o and β appear similar in the different cases. The J -test suggests that the model is correctly specified. However, the standard errors of the risk aversion parameter are usually very high. We should recall, indeed, that we are considering a standard utility specification, where the risk aversion parameter is constrained to be the inverse of the elasticity of substitution. Such tension has been emphasized in the literature and, starting from Epstein and Zin (1989), more flexible frameworks have been provided. However, we do not address this issue here: we want to concentrate exclusively on the role of distortion of beliefs and we keep the model as simple as possible in order to make the interpretation easier. The estimates of the distortion parameter are usually positive, but not extremely high. Requiring the discount factor to be less than unity, is, in most cases, a binding constraint. This feature is common among expected utility models suggesting that this framework have some problems in fitting the levels of asset returns. Similar results are obtained in case of exact identification, which is summarized in Table 4.3.

4.5 Conclusions

We propose a simple asset pricing model with (risk averse) heterogeneous agents. Heterogeneity is modelled in a very simple way, captured by one single parameter. In this way, we want to propose a practical alternative to complex heterogeneous agents models, which, starting from Brock and Hommes (1997), have been able to imply interesting and complex price dynamics at the expenses of the robustness of the validation: such models, indeed, often rely on numerous parameters which are usually calibrated. Furthermore, we are not only interested in the ability of the model to produce realistic dynamics (like waves of optimism and pessimism, volatility clustering, etc.), but we would also like to build a model which allows to reproduce the observed price-dividend ratios during the postwar sample.

Agents have access to the same information, but they interpret it with different attitudes: the optimistic group believes that good times are more likely than full-Bayesian learners would, while the pessimists increase the probability of bad times happening. In order to control the effect of such types of heterogeneity, we consider a simple complete market: in our economy, there are two possible states and two Arrow-Debreu assets which pay the aggregate dividend if a

specific predefined state of nature realizes. A key assumption of our model is that heterogeneity is not going to be resolved: pessimists and optimists are equally wrong about the probability of future states of nature, they both survive in the long run so that prices reflect the temporary deviation from the long-run persistence of heterogeneity.

The model is able to reproduce the waves of optimism and pessimism observed in the dynamics of asset prices. The amplitude of such waves is influenced by the risk aversion parameter and the distortion in beliefs formation imposed on the agents. Low levels of risk aversion determine a noticeable amplification effect which is peculiar to the case of an economy with heterogeneous agents: risk neutral investors tend to make more extreme investment decisions, which, if based on wrong beliefs, lead them to be driven out of the market. Furthermore, the more agents are (equally) wrong about the probabilities of future realizations, the more price fluctuations are volatile. This suggests that heterogeneity in the form of persistent different attitudes is relevant in understanding the main forces driving asset prices and can constitute a useful alternative to the existing heterogeneity models in terms of robustness and validation.

We also present an attempt to estimate the model with GMM methods and we show that it is possible to derive testable moment conditions.

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4.6 Appendix

4.6.1 Appendix 1: The price-dividend ratio

We consider the price function (4.7) in terms of the optimistic agent and we substitute on it the utility function form and eq. (4.12) for C_t^o :

$$\begin{aligned}
P_t^m(y^t) &= \beta \sum_{y^{t+1}} \text{prob}^o(y_{t+1}|y_t) \frac{u'(C_{t+1}^o(y^{t+1}))}{u'(C_t^o(y^t))} (P_{t+1}^m(y^{t+1}) + D_{t+1}^m(y^{t+1})), \quad (4.20) \\
&= \beta \sum_{y^{t+1}} \text{prob}^o(y_{t+1}|y_t) \frac{(C_{t+1}^o(y^{t+1}))^{-\alpha}}{(C_t^o(y^t))^{-\alpha}} (P_{t+1}^m(y^{t+1}) + D_{t+1}^m(y^{t+1})), \\
&= \beta \sum_{y^{t+1}} \text{prob}^o(y_{t+1}|y_t) \frac{\left(\frac{1}{(\eta_{t+1}(y^{t+1}))^{\frac{1}{\alpha}+1}} D_{t+1}(y^{t+1}) \right)^{-\alpha}}{\left(\frac{1}{(\eta_t(y^t))^{\frac{1}{\alpha}+1}} D_t(y^t) \right)^{-\alpha}} (P_{t+1}^m(y^{t+1}) + D_{t+1}^m(y^{t+1})), \\
&= \beta \sum_{y^{t+1}} \text{prob}^o(y_{t+1}|y_t) \left(\frac{\eta_t^{\frac{1}{\alpha}}(y^t) + 1}{\eta_{t+1}^{\frac{1}{\alpha}}(y^{t+1}) + 1} \right)^{-\alpha} \times \\
&\quad \times \left(\frac{D_{t+1}(y^{t+1})}{D_t(y^t)} \right)^{-\alpha} (P_{t+1}^m(y^{t+1}) + D_{t+1}^m(y^{t+1})),
\end{aligned}$$

where $D_{t+1}^m(j) = D_{t+1}$ if $m = 1(2)$, $\wedge j = h(l)$, zero otherwise.

Dividing both sides by D_t and denoting with $\varphi_t^k = P_t^k/D_t$ the price-dividend ratio we get

$$\begin{aligned}
\varphi_t^m &= \beta \left(\eta_t^{\frac{1}{\alpha}}(y^t) + 1 \right)^{-\alpha} \times \quad (4.21) \\
&\quad \times \left[\begin{array}{l} \pi_t^o(j) \left(\left[\frac{\pi_t^p(j)}{\pi_t^o(j)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}}(y^t) + 1 \right)^\alpha \exp((1-\alpha)y_j) [1 + \varphi_{t+1}^m] + \\ \pi_t^o(k) \left(\left[\frac{\pi_t^p(k)}{\pi_t^o(k)} \right]^{\frac{1}{\alpha}} \eta_t^{\frac{1}{\alpha}}(y^t) + 1 \right)^\alpha \exp((1-\alpha)y_k) \varphi_{t+1}^m \end{array} \right]
\end{aligned}$$

for $j \neq k, \forall m$.

4.6.2 Appendix 2: Data description

We have used quarterly real observations. Consumption is derived using the quarterly series of Personal Consumption Expenditure from the Bureau of Economic Analysis (*BEA*) website. Value weighted portfolio returns are from the *CRSP* dataset. Nominal values are deflated using the Implicit Price Deflator relative to the consumption series from the *BEA* website. Data span the period 1947:2-2010:2.

4.6.3 Appendix 3: Derivation of moment restrictions

We present here the algebraic steps which prove the equivalence stated in eq. (4.17):

$$\begin{aligned}
\text{prob}^o(y_{t+1}|y^t) &= \text{prob}(y_{t+1}|S_{t+1} = h) \pi_t^o(h) + \text{prob}(y_{t+1}|S_{t+1} = l) \pi_t^o(l) \\
&= \text{prob}(y_{t+1}|S_{t+1} = h) [(1 - \omega^o) \pi_t(h) + \omega^o] + \\
&\quad \text{prob}(y_{t+1}|S_{t+1} = l) [(1 - \omega^o) \pi_t(l)] \\
&= \text{prob}(y_{t+1}|S_{t+1} = h) (1 - \omega^o) \pi_t(h) + \\
&\quad \text{prob}(y_{t+1}|S_{t+1} = l) [(1 - \omega^o) \pi_t(l)] + \\
&\quad \omega^o \text{prob}(y_{t+1}|S_{t+1} = h) \\
&= (1 - \omega^o) [\text{prob}(y_{t+1}|S_{t+1} = h) \pi_t(h) + \text{prob}(y_{t+1}|S_{t+1} = l) \pi_t(l)] + \\
&\quad \omega^o \text{prob}(y_{t+1}|S_{t+1} = h) \\
&= (1 - \omega^o) \text{prob}(y_{t+1}|y^t) + \omega^o \text{prob}(y_{t+1}|S_{t+1} = h).
\end{aligned}$$