A.J.J. TALMAN and H.C. TIJMS, Amsterdam

A SIMPLE APPROACH TO SOME PRODUCTION-INVENTORY PROBLEMS

1. *Introduction.*

Consider a production-inventory problem in which the production of a single commodity can be controlled by varying the production rate. The system is supposed to have a finite storage capacity K. We assume that the system is either in phase 1 or phase 2 where at any moment the system can be switched from one phase to another without loss of time. If the system is in phase i, then at epochs generated by a Poisson process with rate λ_i orders for the product are placed where the size of any order is distributed as the positive random variable S_i with $F_i(x) = \Pr\{S_i \le x\}$ and ES_i is finite. If the order size exceeds the inventory, the excess demand is lost. When the inventory reaches the capacity level K, then production is stopped until the next order arrives. In phase 1 we have that between demand epochs the inventory increases linearly at rate $\sigma_1 > 0$. For phase 2 we distinguish between

Case A. If the inventory is less than K, then between demand epochs the inventory *increases* linearly at rate $\sigma_2 > 0$.

Case B. If the inventory is positive, then between demand epochs the inventory *decreases* linearly at rate $\sigma_2 > 0$ (e.g. by deterioration).

The following control rule specified by two critical levels y_1 and y_2 with $0 \le y_2 < y_1 \le K$ is considered. If in phase 1 the inventory reaches the level y_1 the system is switched to phase 2 and if in phase 2 the inventory assumes a value less than or equal to y_2 the system is switched to phase 1.

We consider the following cost structure. In phase i there is incurred a holding-production cost at rate $h_i(x)$ when the inventory is x where $h_i(x)$ is a bounded function having only a countable number of discontinuities. If in phase i there is an excess demand y then lost sales costs $p_i(y)$ are incurred where $p_i(y)$ is a non-negative Baire function. Finally, there are fixed costs R for switching from phase 1 to phase 2.

We shall present for the cases A and B a simple and unified derivation for the long-run average costs per unit time for the (y_1,y_2) rule. By an appropriate choice of the cost parameters, we obtain from this formula various operating characteristics like the joint stationary distribution of the inventory and the phase of the system and the average number of switch-overs per unit time.

The results of this paper are new for case B and were already obtained for case A in [2] by a closely related but somewhat more complex derivation, cf. also [5]. We note that the problem with backlogging of excess demand can be analyzed in the same way.

In section 2 we discuss an integral-differential equation which is essential in the analysis and in section 3 we give the derivation of the average costs.

2. *Preliminaries.*

Let a(x) be a given bounded function defined on (a,b) and let α be a given nonzero constant. Further, let F be a given probability distribution function with F(0) = 0 and finite first moment. Suppose that the function u(x) is continuous on (a,b) and, for all except countably many $x \in (a,b)$ satisfies the integral-differential equation

$$(1) \quad \frac{du(x)}{dx} = a(x) + \alpha\{u(x) - \int_0^{x-a} u(x-y)\, dF(y)\}.$$

Also, let some boundary condition for u(x) be given. We shall now demonstrate that (1) can be reduced to a renewal equation. Therefore we first note that, by partial integration (cf. p. 77 in [1])

$$(2) \quad \frac{d}{dx} \int_0^{x-a} u(x-y)\, \{1-F(y)\}dy = u(x) - \int_0^{x-a} u(x-y)\, dF(y).$$

Hence, by (1) and (2),

$$(3) \quad u(x) = A(x) + \alpha \int_0^{x-a} u(x-y)\{1-F(y)\}dy \qquad \text{for all } a < x < b$$

where $A(x) = \gamma + \int_0^x a(y)\, dy$ for some constant γ which follows from the given boundary condition. Define now the functions H and G and the number δ by

$$(4) \quad H(x) = 0 \text{ for } x < 0,\ H(x) = |\alpha| \int_0^x\{1-F(y)\}dy \qquad \text{for } x \geq 0$$

$$(5) \quad \int_0^\infty e^{-\delta y}\, dH(y) = 1 \text{ and } G(x) = \int_0^x e^{-\delta y}\, dH(y).$$

Since H is non-negative and non-decreasing, it follows that δ is uniquely determined by (5) and that G is a probability distribution function. Letting $H^{(n)}(F^{(n)})$ denote the n-fold convolution of H(G) with itself, we readily verify

$$(6) \quad G^{(2)}(x) = \int_0^x e^{-\delta y}\, dH^{(2)}(y) \text{ and } \int_0^\infty e^{-\delta y}\, dH^{(2)}(y) = 1.$$

Finally, define the renewal functions M and $\bar{M}$ by

$$(7)\qquad M(x) = \sum_{n=1}^{\infty} G^{(n)}(x) \text{ and } \bar{M}(x) = \sum_{n=1}^{\infty} G^{(2n)}(x)$$

Now, to give the solution to (1) we distinguish between two cases

(i) $\alpha > 0$. Then we can write (3) as

$$(8)\qquad u(x) = A(x) + \int_0^{x-a} u(x-y)\, dH(y) \quad \text{for } a < x < b.$$

It has been shown on pp. 362-363 in [3] (cf. also p. 77 in [1]) that this equation is in fact a renewal equation and has the unique solution

$$(9)\qquad u(x) = A(x) + \int_0^{x-a} e^{\delta y} A(x-y)dM(y) \quad \text{for } a < x < b.$$

(ii) $\alpha < 0$. Then we can write (3) as

$$(10)\qquad u(x) = A(x) - \int_0^{x-a} u(x-y)\, dH(y) \quad \text{for } a < x < b$$

so, by substitution,

$$(11)\qquad u(x) = B(x) + \int_0^{x-a} u(x-y)\, dH^{(2)}(y) \quad \text{for } a < x < b$$

where $B(x) = A(x) - \int_0^{x-a} A(x-y)\, dH(y)$. Hence, using (6)

$$(12)\qquad u(x) = B(x) + \int_0^{x-a} e^{\delta y} B(x-y) d\bar{M}(y).$$

Finally, for the special case $F(x) = 1 - e^{-\eta x}$ we have

$$(13)\qquad \delta = |\alpha| - \eta,\ \frac{dM(y)}{dy} = |\alpha|,\ \frac{d\bar{M}(y)}{dy} = |\alpha|\, e^{-|\alpha|y}\{e^{|\alpha|y} - e^{-|\alpha|y}\}/2.$$

3. *The derivation for the average costs.*

We first observe that under the (y_1,y_2) control rule the stochastic process describing the inventory and the phase of the system is regenerative. Define now a cycle as the period between two successive epochs at which the inventory reaches the level y_1 while in phase 1. Denote by Z(t) the total costs incurred up to time t, then (e.g. [4])

$$(14)\qquad \lim_{t\to\infty} \frac{EZ(t)}{t} = \frac{\text{total expected costs in one cycle}}{\text{expected length of one cycle}}$$

From the average costs we may obtain various operating characteristics for the system, cf. [2] and [5].

For example fix i and let $h_i(x) = 1$ for $x \le z$, $h_i(x) = 0$ for $x > z$ and the other costs equal to zero, then the left side of (12) equals the long-run expected fraction of time that the system is in phase 1 and the inventory is less than or equal to z. This quantity gives in fact the joint stationary distribution of the inventory and the phase of the system. To determine (14), we define for the (y_1,y_2) rule (cf. [2] and [5]),

$t_1(x)[k_1(x)]$ = expected time (expected holding and lost-sales costs incurred) up to the first epoch at which the inventory assumes the value y_1 starting in phase 1 with an inventory x, $0 \le x < y_1$.

$t_2(x)[k_2(x)]$ = expected time (expected holding and lost-sales costs incurred) up to the first epoch at which the inventory assumes a value less than or equal to y_2 starting in phase 2 with an inventory x, $y_2 < x \le K$.

$p(x,v)$ = probability that the inventory assumes a value in $[0,v]$ at the first epoch at which the inventory assumes a value less than or equal to y_2 starting in phase 2 with an inventory x, $y_2 < x \le K$ and $0 \le v \le y_2$.

Then, by (14),

$$(15)\qquad \lim_{t\to\infty} \frac{EZ(t)}{t} = \frac{R + k_2(y_1) + \int_0^{y_2} k_1(v)\, p(y_1,dv)}{t_2(y_1) + \int_0^{y_2} t_1(v)\, p(y_1,dv)}.$$

It is immediate that the formulae for $t_i(x)$ follow from those for $k_i(x)$ by putting $h_i(x) \equiv 1$ and the other costs equal to zero. We readily verify that the functions $k_i(x)$ for $i = 1,2$ and $p(x,v)$ for any v are continuous in x. Clearly the function $k_1(x)$ is the same for both cases A and B. Considering $k_1(x-\Delta x)$ for Δx small and using standard arguments, we find for all except countably many $x \in (0,y_1)$ that (cf. [2] and [5])

$$(16)\qquad \frac{dk_1(x)}{dx} = \frac{-h_1(x)}{\sigma_1} - \frac{\lambda_1}{\sigma_1}\int_x^\infty \{k_1(0)+p_1(y-x)\}dF_1(y) + \frac{\lambda_1}{\sigma_1}\{k_1(x) + - \int_0^x k_1(x-y)\, dF_1(y)\}$$

with the boundary condition $\lim_{x\to y_1} k_1(x) = 0$. Using the relations (3), (9), (16) and the continuity of $k_1(x)$ at $x = 0$, we obtain a formula for $k_1(x)$. Similarly, we find the following integral-differential equations. For all except countably many $x \in (y_2,K)$,

$$(17)\qquad \frac{dk_2(x)}{dx} = \pm\left[\frac{h_2(x)}{\sigma_2} + \frac{\lambda_2}{\sigma_2}\int_x^\infty p_2(y-x)dF_2(y) - \frac{\lambda_2}{\sigma_2}\{k_2(x) - \int_0^{x-y_2} k_2(x-y)dF_2(y)\}\right]$$

and for any $v \in [0,y_2]$,

$$(18)\qquad \frac{\partial p(x,v)}{\partial x} = \pm\left[\frac{\lambda_2}{\sigma_2}F_2(x-v) - \frac{\lambda_2}{\sigma_2}\{p(x,v) - \int_0^{x-y_2} p(x-y,v)\,dF_2(y)\}\right]$$

with the - sign in (17) - (18) for case A and the + sign for case B. We have the following boundary conditions.

Case A.

$$(19)\qquad k_2(K) = \frac{h_2(K)}{\lambda_2} + \int_K^\infty p_2(y-K)dF_2(y) + \int_{(0,K-y_2)} k_2(K-y)dF_2(y)$$

$$(20)\qquad p(K,v) = \int_{K-v}^\infty dF_2(y) + \int_{(0,K-y_2)} p(K-y,v)dF_2(y)$$

Case B. $\lim_{x\to y_2} k_2(x) = 0$ and $\lim_{x\to y_2} p(x,v) = 1$ for $v = y_2$ and 0 otherwise.

The equations (17) and (18) are of the type (1) and can be solved using (9), (12), (19) and (20) and the continuity of the functions $k_2(x)$ and $p(x,v)$.

Finally, consider the special case where $F_i(x) = 1 - e^{-\eta_i x}$ for $x > 0$ and $i = 1,2$. Then, by (13), the solutions to (16)-(18) can be further simplified. We give only the resulting expressions for case B, cf. [2] for case A. For any $y_2 < x \le K$, we have

$$p(x,v) = \begin{cases} \dfrac{\lambda_2}{\sigma_2\beta_2} e^{-\eta_2(y_2-v)}\{1-e^{-\lambda_2(x-y_2)/\sigma_2}\} & \text{for } 0 \le v < y_2 \\ 1 & \text{for } v = y_2, \end{cases}$$

where $\beta_2 = \eta_2 + \lambda_2/\sigma_2$.

Taking $h_i(x) = 1$ for $x \le z$ and $h_i(x) = 0$ for $x > z$ and assuming $\gamma_1 = \eta_1 - \lambda_1/\sigma_1 \ne 0$, we find

$$k_1(x) = \begin{cases} \dfrac{(z-x)\eta_1}{\sigma_1\gamma_1} + \dfrac{\lambda_1}{\sigma_1^2\gamma_1^2}[1 + e^{-y_1\gamma_1} - e^{-(y_1-z)\gamma_1} - e^{-x\gamma_1}], & 0 \le x \le z \\ \dfrac{\lambda_1}{\sigma_1^2\gamma_1^2}[e^{-y_1\gamma_1} - e^{-(y_1-z)\gamma_1} - e^{-x\gamma_1} + e^{-(x-z)\gamma_1}], & z \le x < y_1 \end{cases}$$

and

$$k_2(x)=\begin{cases}\dfrac{(x-y_2)\eta_2}{\sigma_2\beta_2}-\dfrac{\lambda_2}{\sigma_2^2\beta_2^2}[e^{-(x-y_2)\beta_2}-1], & y_2 < x \le z\\[2ex] \dfrac{(z-y_2)\eta_2}{\sigma_2\beta_2}-\dfrac{\lambda_2}{\sigma_2^2\beta_2^2}[e^{-(x-y_2)\beta_2}-e^{-(x-z)\beta_2}], & z \le x \le K.\end{cases}$$

Remark. Consider the case where in phase i between demand epochs the inventory changes at a general rate of $\sigma_i(x)$ when the inventory is x where $\sigma_i(x)$ is either nonnegative or nonpositive and satisfies some regularity conditions. Then, in the analysis we get integral-differential equations of the type

$$(21)\qquad \sigma(x)\,\frac{du(x)}{dx} = a(x) + \alpha\{u(x) - \int_0^{x-a} u(x-y)\,dF(y)\}$$

instead of (1). In general (21) can be only solved numerically. In case $F(x)= =1-e^{-\eta x}$ then, by using $\int_0^{x-a} u(x-y)\eta e^{-\eta y}\,dy = \eta e^{-\eta x}\int_a^x u(t)e^{\eta t}dt$, we can reduce (21) by differentiation to a first-order linear differential equation in $du(x)/dx$.

References.

1. J.W. Cohen (1976), On Regenerative Processes in Queueing Theory, Lecture Notes in Economics and Mathematical Systems 121, Springer, Berlin.
2. B.T. Doshi, F. van der Duyn Schouten and A.J.J. Talman (1978), A production-inventory control model with a mixture of backorders and lost-sales (to appear in Manag. Sci.).
3. W. Feller (1966), An Introduction to Probability Theory and its Applications, Vol. II, Wiley, New York.
4. S.M. Ross (1970), Applied Probability Models with Optimization Applications, Holden-Day, San Francisco.
5. H.C. Tijms and F. van der Duyn Schouten (1978), Inventory control with two switch-over levels for a class of M/G/1 queueing systems with variable arrival and service rate, Stoch. Proc. and Appl. 6, 213-223.