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FOUR PERSON SYMMETRIC INFORMATION GAMES

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TOHOKU MANAGEMENT & ACCO

Discussion Paper No.4

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March 1987

Abstract:

Description is given of a stable set of a four-person game generated from an information market. Implications of the stable set on traders' behavior are also examined.

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1. Introduction

Since J. von Neumann and O. Morgenstern defined the stable set as a solution concept of n-person cooperative games in their voluminous book (von Neumann and Morgenstern [1]), a great number of works have been done on this concept. Though its general existence was denied by Lucas [2] and Lucas and Rabie [3], it has been shown, in many games, that stable sets exist and further that they give insights into players' behavior concerning coalition formations.

For three-person games, all stable sets are known. For constant-sum three-person games, stable sets were already given by von Neumann and Morgenstern [1]. For four-person games, though the existence of a stable set was proved by Bondareva, Kulakovskaya, and Naumova [4], its exact forms were determined only for special classes; symmetric games, simple games, extreme games, etc. See Owen [5], Mills [6], Nering [7], Shapley [8], Hebert [9], etc.

The aim of this paper is to present a stable set of a class of four-person games derived from the analysis of an information market recently done by the authors (Muto, Potters, and Tijs [10]), and to examine the traders' behavior which it reveals.

The remainder of the paper is organized as follows. In the next section, the definition of the stable set is given as well as the definition of the information game; the cooperative game generated from an information market. In Section 3, the main theorem, describing a stable set of a class of four-person information games, is presented, and the implications of this stable set on the traders' behavior are studied. The proof of the main theorem is given in Section 4. The paper closes in Section 5 with a short conclusion.

2. Stable sets and information games

Let (N,v) be an n-person cooperative game; $N=\{1,2,\ldots,n\}$ is the set of <u>players</u> and v is a real-valued function on 2^N (the set of all subsets of N) with $v(\emptyset) = 0$, called the <u>characteristic function</u>. A subset S of N is called a <u>coalition</u>. For each coalition S, v(S) denotes the value that S can gain by itself. (N,v) is called <u>symmetric w.r.t. (with respect to) a coalition S</u> if the values of v are unchanged by any permutation of the players in S. If (N,v) is symmetric w.r.t. the grand coalition N, it is simply called <u>symmetric</u>. Or, (N,v) is symmetric if $v(S) = v(T)$ whenever $|S|=|T|$ where $|S|$ and $|T|$ are the numbers of players in S and T, respectively.

Let A(v) be the set of <u>imputations</u> of (N,v), i.e.,

$A(v) = \{x \in R^n : \sum_{i\in N} = v(N),\ x_i \geqq v(\{i\}) \text{ for all } i \in N\}$

where R^n is the n-dimensional Euclidean space. For two imputations $x, y \in A(v)$ and a nonempty coalition $S \subseteqq N$, we say <u>x dominates y via S</u>, denoted by x dom_S y, if (i) $x_i > y_i$ for all $i \in S$ and (ii) $\sum_{i\in S} \leqq v(S)$; the latter condition is called the <u>effectiveness of S for x</u>. We write x dom y if there is some S such that x dom_S y. For a set $B \subseteqq A(v)$, let Dom B = $\{x \in A(v)$: there is some $y \in B$ such that y dom x$\}$.

A set $K \subseteqq A(v)$ is called a stable set if it satisfies (i)

$K \cap \text{Dom } K = \emptyset$ (internal stability) and (ii) $K \cup \text{Dom } K = A(v)$ (external stability). A set $C(v)$ is called the core if $C(v) = A(v) - \text{Dom } A(v)$. A set $B \subseteqq A(v)$ is called <u>symmetric w.r.t. a coalition S</u>, if it is unchanged by any permutation of the players in S. If B is symmetric w.r.t. N, it is simply called <u>symmetric</u>. For each S, if the game is symmetric w.r.t. S, the core is also symmetric w.r.t. S. But this is not necessarily true for stable sets: In fact, the three-person simple majority game, $v(S)=1$ if $|S| \geqq 2$ and $v(S) = 0$ if $|S| \leqq 1$, has a non-symmetric stable set $\{x \in A(v) : x_1 = c\}$ $(0 \leqq c < 1/2)$, though the game is symmetric.[1)]

We now briefly review the definition of the information game studied by the authors (Muto, Potters, and Tijs [10]). Let $N=\{1,2,\ldots,n\}$ be the set of traders; 1 is the owner (patent holder) of an information indispensable for manufacturing a new product, and $2,\ldots,n$ are firms demanding this information for producing and selling the product. A perfect patent protection is supposed to be provided. We assume that the (consumer) market of the new product is divided into parts (submarkets) according to the group of firms which have the right and possibility to enter. For each $\emptyset \neq T \subseteqq N-\{1\}$, let M_T denote the submarket into which only the firms in T can enter.[2)] We further assume that for each submarket M_T, a maximal profit gained by selling the product in M_T, denoted by r_T, is known.

The cooperative game derived from this information market, called the <u>information game</u>, is given by (N,v) where

$$v(S) = \begin{cases} \sum_{T:T\cap S\neq\emptyset} r_T & \text{if } \{1\} \subsetneq S \\ 0 & \text{otherwise.} \end{cases}$$

Namely, if a coalition S contains the patent holder 1, all of the members of S - {1} can produce and sell the product cooperatively, and thus they can gain profits from all the submarkets into which they can enter. Here we note that since the perfect patent protection is present, there is no possibility of relicensing by the members of S. On the other hand, coalitions not containing the patent holder cannot produce the product and thus cannot gain any profit. In paticular, if $r_T = r_{T'}$ whenever $|T| = |T'|$, i.e., r_T depends only on the number of firms in T, the information game is symmetric w.r.t. {2,...,n}. We simply call such a game the symmetric information game. The readers may refer to Muto, Potters, and Tijs [10] for the details of the information game.

In the following, we describe a stable set of four-person symmetric information games which is symmetric w.r.t. {2,...,n}, i.e., reflects the symmetry of the players 2,...,n, and we further examine the implications of this stable set on the traders' behavior.

3. A stable set of four-person symmetric information games

Leting $r(1) = r_{\{2\}} = r_{\{3\}} = r_{\{4\}}$, $r(2) = r_{\{2,3\}} = r_{\{2,4\}} = r_{\{3,4\}}$, and $r(3) = r_{\{2,3,4\}}$, the four-person symmetric information game is given by (N,v) where N = {1,2,3,4}, and

$$v(S) = \begin{cases} 3r(1) + 3r(2) + r(3) & \text{if } S = N \\ 2r(1) + 3r(2) + r(3) & \text{if } 1 \in S \text{ and } |S| = 3 \\ r(1) + 2r(2) + r(3) & \text{if } 1 \in S \text{ and } |S| = 2 \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

The imputation set A(v) is given by

$$A(v) = \{x \in R^4 : \textstyle\sum_{i=1}^4 x_i = 3r(1) + 3r(2) + r(3),$$
$$x_i \geqq 0 \text{ for } i = 1,2,3,4\}.$$

It was further shown in Muto, Potters, and Tijs [10] that the core C(v) is given by

$$C(v) = \{x \in A(v) : x_i \leqq r(1) \text{ for } i = 2,3,4\}. \qquad (2)$$

We now define the sets K1, K2, and K3 by

$$K1 = K1_{23} \cup K1_{24} \cup K1_{34}$$
where $K1_{ij} = \{x \in A(v) : r(2)/2 + r(1) \geqq x_i = x_j \geqq r(1) \geqq x_k\}$,
$\{i,j,k\} = \{2,3,4\}$,

$$K2 = K2_{23} \cup K2_{24} \cup K2_{34}$$
where $K2_{ij} = \{x \in A(v) : r(2)/2 + r(1) = x_i = x_j \geqq x_k \geqq r(1)\}$,
$\{i,j,k\} = \{2,3,4\}$,

and $K3 = \{x \in A(v) : x_2 = x_3 = x_4 \geqq r(2)/2 + r(1)\}$.

Then we have the following theorem. The proof will be given in the next section.

<u>Theorem 1</u>: Let $K = C(v) \cup K1 \cup K2 \cup K3$. Then K is a stable set of the four-person symmetric information game (N,v). Further this K is symmetric w.r.t. {2,3,4}.

Figure 1 depicts this stable set K. We now examine its implications on the traders' behavior. Recalling (2), we see that, in the core C(v), each player i (i = 2,3,4) can never gain a payoff exceeding the profit of the submarket which he governs alone. The parts K1, K2, K3 show that these players can gain more by forming coalitions among them. In K1, there are imputations corresponding to situations where two of them form a coalition. For instance, in $K1_{23}$, players 2 and 3 gain a portion of the profit from the overlapped submarket $M_{\{2,3\}}$ by forming a coalition; they divide it equally. K2 indicates the situation in which three-person coalition {2,3,4} is formed, but within it a subcoalition with two players is formed. For instance, in $K2_{23}$, players 2 and 3 form a subcoalition and they can gain the whole profit arising from the overlapped submarket $M_{\{2,3\}}$; but a weak player (player 4) can gain only a portion of the profit from the overlapped submarket $M_{\{2,4\}}$ or $M_{\{3,4\}}$. The profit arising from $M_{\{2,3,4\}}$ goes to the patent holder (player 1) in K1 and K2. K3 shows the possibility of a tight three-person coalition {2,3,4} being formed; they equally share the profits. If their negotiation power would be extremely strong, they could gain all the profits arising from the overlapped submarkets.

We hereupon mention briefly how the shape of K is changed as r(1), r(2), r(3) vary. As r(2) decreases, K1 (the three parallelograms FLMK, JPQK, HNOK in Figure 1) and K2 (the three lines MR, QR, OR) diminish; eventually when r(2) becomes 0, both of them vanish, and K consists of C(v) and K3. See Figure 2. This shows that in case there is no submarket shared by two players, players 2, 3, and 4 do not have the incentives to form two-person coalitions. Further if r(3) is also 0, then K consists only of the core C(v).[3] Now if r(1) decreases, then the core C(v) and K1 diminish; eventually when r(1) becomes 0, the core consists of only one point (the vertex A) and K1 consists of the three lines on the faces of the tetrahedron. See Figure 3. Further if r(2) is also 0, K1 and K2 vanish, and K consists of C(v) (the vertex A) and K3, i.e., the line segment connecting the vertex A and the middle point of the face BCD. Namely, in case there exists only the submarket shared by the three players, only the tight three-person coalition is formed.

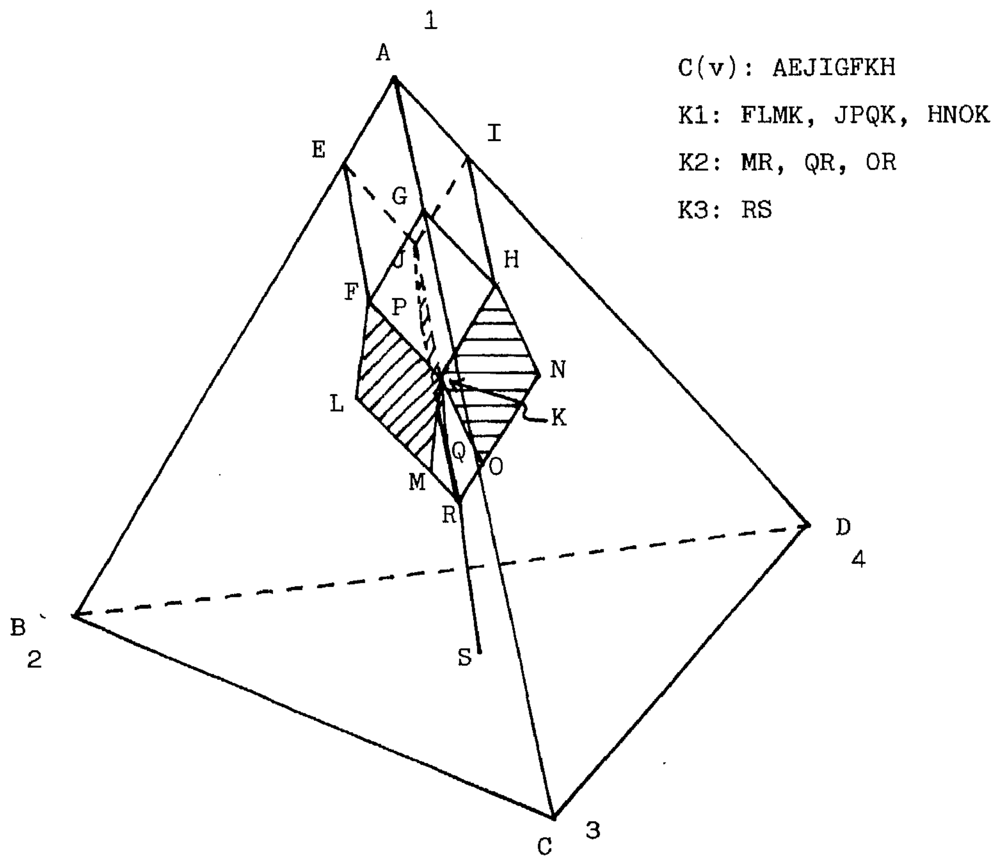

Figure 1. The stable set given in Theorem 1

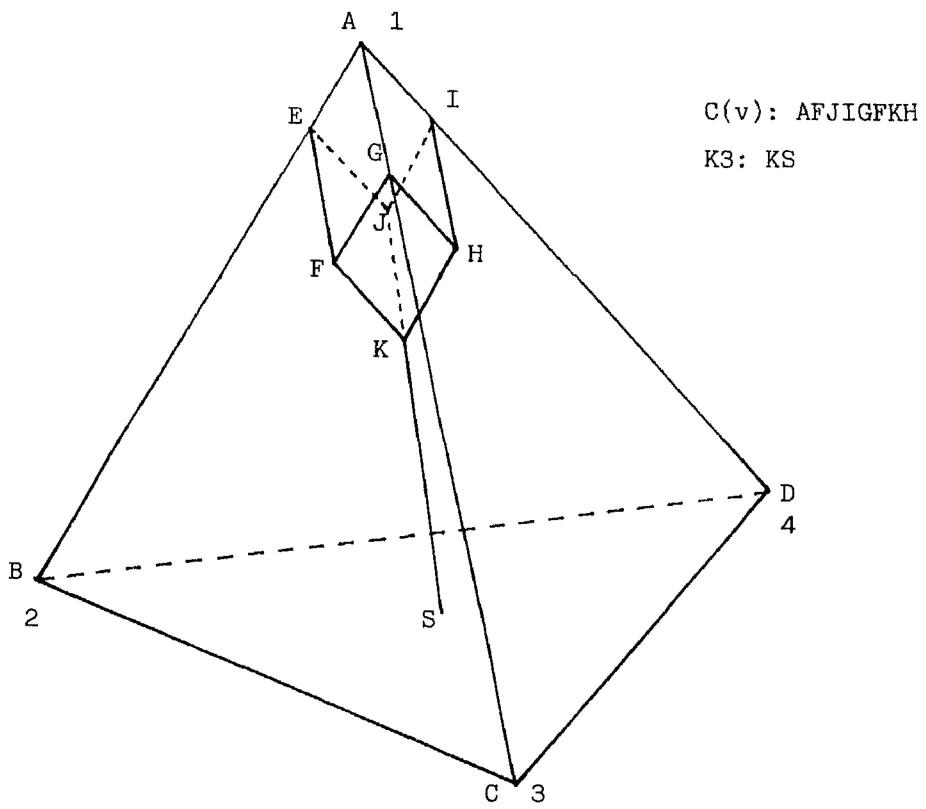

Figure 2. The stable set given in Theorem 1 when r(2) = 0

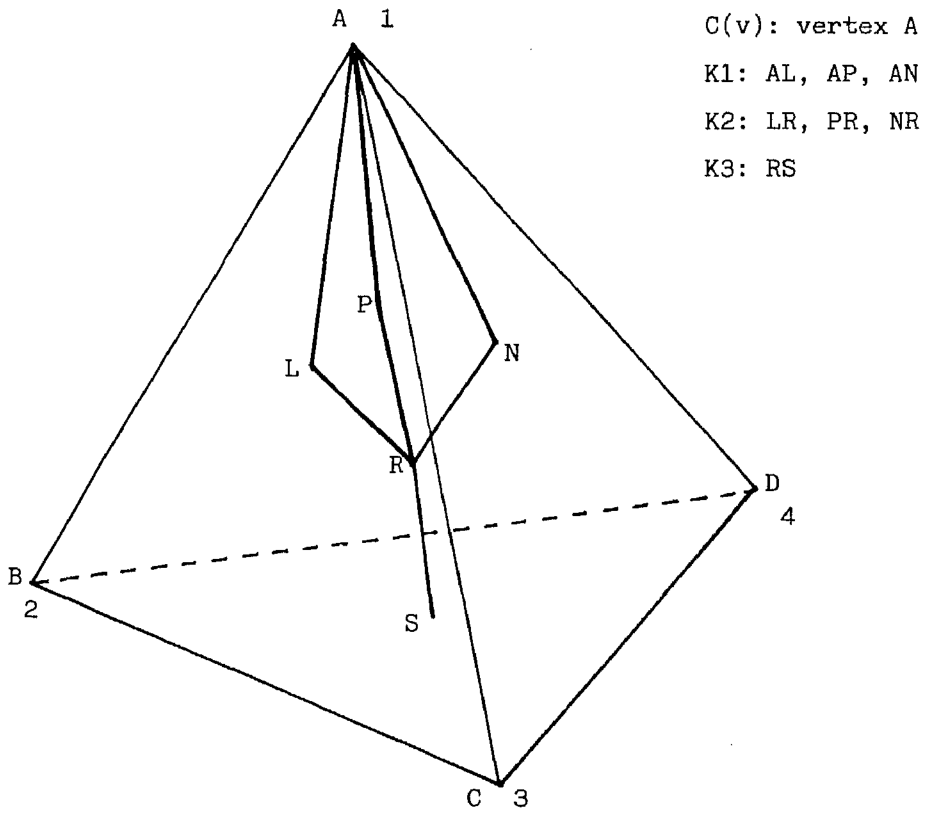

Figure 3. The stable set given in Theorem 1 when r(1) = 0

4. Proof of the theorem

From the definition of v (see (1)), we see that dominations can be done only via coalitions {1,2}, {1,3}, {1,4}, {1,2,3}, {1,2,4}, and {1,3,4}.

Internal stability of K: Take two imputations x and y, and suppose x dom_S y. Note that $y \notin C(v)$. Case (1): $x \in C(v)$: Recalling (2), we have $x_i \leqq r(1)$ for i = 2,3,4. Therefore if $y \in K2 \cup K3$, then we never have x dom_S y since for all $y \in K2 \cup K3$, $y_i \geqq r(1)$ for i = 2,3,4. Suppose $y \in K1_{ij}$ where $\{i,j\} \subsetneqq \{2,3,4\}$. Then S must be {1,k} ($k \in \{2,3,4\}$, $k \neq i$, $k \neq j$) since $y_i = y_j \geqq r(1)$. Therefore we obtain $x_1 > y_1$, $x_k > y_k$, and $x_1 + x_k \leqq v(\{1,k\}) = r(1) + 2r(2) + r(3)$; the last inequality implies that $x_i + x_j \geqq v(N) - v(\{1,k\}) = 2r(1) + r(2)$. Meanwhile, from the definition of $K1_{ij}$, we have $y_i + y_j \leqq 2r(1) + r(2)$. Hence we obtain a contradiction

$$v(N) = \sum_{i=1}^{4} x_i > \sum_{i=1}^{4} y_i = v(N). \tag{3}$$

Case (2) $x \in K1$: Suppose $x \in K1_{ij}$. (2.1) $y \in K1$: Suppose first $y \in K1_{ij}$. If S = {1,i,k} or {1,j,k}, then we have the contradiction (3) since $x_i = x_j$ and $y_i = y_j$. If S = {1,i,j}, then we must have $x_i > y_i$, $x_j > y_j$, and $x_1 + x_i + x_j \leqq v(\{1,i,j\})$; the last inequality implies $x_k \geqq r(1)$. Since if $y \in K1_{ij}$, then $y_k \leqq r(1)$, we obtain the contradiction (3). The same contradiction follows in the cases of S = {1,i}, {1,j}, and {1,k}. Now suppose $y \in K1_{ik}$. Since $x_k \leqq r(1) \leqq y_k$, S must be {1,i,j}, {1,i}, or {1,j}. If S = {1,i,j}, then we have $x_i > y_i$, $x_j > y_j$, and $x_k = r(1)$. It thus follows $x_i = x_j > y_i = y_k$ and $x_k = r(1) \geqq y_j$ which lead to the contradiction (3). If S = {1,i}, then we have $x_i > y_i$ and $x_1 + x_i \leqq v(\{1,i\})$, i.e., $x_j + x_k \geqq 2r(1) + r(2)$. Since $y_j + y_k \leqq 2r(1) + r(2)$ for all $y \in K1_{ik}$, the contradiction (3) follows. Similar arguments lead to the same contradiction in case $y \in K1_{jk}$.

<u>(2.2)</u> $y \in$ <u>K2</u>: If $y \in K2_{ij}$, then we never have x dom_S y since $x_i = x_j \leqq (r(2)/2) + r(1)$ and $x_k \leqq r(1) \leqq y_k$. Suppose $y \in K2_{ik}$. Since $x_k \leqq r(1) \leqq y_k$ and $x_i \leqq (r(2)/2) + r(1) = y_i$, S must be $\{1,j\}$. It thus follows $x_j > y_j$ and $x_1 + x_j \leqq v(\{1,j\})$, i.e., $x_i + x_k \geqq 2r(1) + r(2)$. Since $y_i + y_k = 2r(1) + r(2)$ for all $y \in K2_{ik}$, the contradiction (3) follows. In case $y \in K2_{jk}$, we get (3) in a similar manner.

<u>(2.3)</u> $y \in$ <u>K3</u>: We never have x dom_S y since $x_2, x_3, x_4 \leqq (r(2)/2) + r(1) \leqq y_2 = y_3 = y_4$.

<u>Case (3)</u> $x \in$ <u>K2</u>: Suppose $x \in K2_{ij}$. <u>(3.1)</u> $y \in$ <u>K1</u>: We never have x dom_S y since $x_1 = v(N) - (x_2 + x_3 + x_4) \leqq 2r(2) + r(3)$ for all $x \in K2$ and $y_1 \geqq 2r(2) + r(3)$ for all $y \in K1$. <u>(3.2)</u> $y \in$ <u>K2</u>: If $y \in K2_{ij}$, then we never have x dom_S y since $x_i = x_j = (r(2)/2) + r(1) = y_i = y_j$. Suppose $y \in K2_{ik}$. Since $x_k \leqq x_i = x_j = (r(2)/2) + r(1) = y_i = y_k$, S must be $\{1,j\}$. Hence we have $x_j > y_j$ and $x_1 + x_j \leqq v(\{1,j\})$, i.e., $x_i + x_k \geqq 2r(1) + r(2)$ which leads to the contradiction (3) since $y_i + y_k = 2r(1) + r(2)$ for all $y \in K2_{ik}$. Similar arguments lead to the same contradiction in case $y \in K2_{jk}$. <u>(3.3)</u> $y \in$ <u>K3</u>: We never have x dom_S y since $x_2, x_3, x_4 \leqq (r(2)/2) + r(1) \leqq y_2 = y_3 = y_4$.

<u>Case (4)</u> $x \in$ <u>K3</u>: If $y \in K1 \cup K2$, then we never have x dom_S y since $x_1 = v(N) - (x_2 + x_3 + x_4) \leqq (3/2)r(2) + r(3)$ for all $x \in K3$ and $y_1 \geqq (3/2)r(2) + r(3)$ for all $y \in K1 \cup K2$. If $y \in K3$, then

$x \text{ dom}_S\, y$ leads to the contradiction (3) since $x_2 = x_3 = x_4$ and $y_2 = y_3 = y_4$.

Thus we have shown the internal stability of K.

<u>External stability</u>: Take an imputation $x \in A(v) - K$, and suppose without loss of generality $x_2 \geqq x_3 \geqq x_4$. Then $x_2 > r(1)$ since $x \notin C(v)$. If $x_3 < r(1)$, then $x \in \text{Dom}\, C(v)$. In fact, define an imputation y by

$$y = (x_1 + e_1,\ r(1),\ x_3 + e_3,\ x_4 + e_4)$$

where e_1, e_3, e_4 are positive numbers such that $e_1 + e_3 + e_4 = x_2 - r(1)$ and $x_4 + e_4$, $x_3 + e_3 \leqq r(1)$. Then $y \in C(v)$ and $y \text{ dom}_{\{1,3,4\}}\, x$ since $y_1 > x_1$, $y_3 > x_3$, $y_4 > x_4$, and $y_1 + y_3 + y_4 = v(N) - r(1) = v(\{1,3,4\})$. Suppose $x_3 \geqq r(1)$ in the following. The following proof is divided into six cases according to the values of x_4 and $x_2 + x_3$.

<u>Case (1) $x_4 \leq r(1)$, $x_2 + x_3 \leqq 2r(1) + r(2)$</u>: Since $x \notin K$, we must have $x_2 > x_3$, and thus $x_3 < (r(2)/2) + r(1)$. Define $y \in A(v)$ by

$$y = (x_1 + e_1,\ x_3 + e_3,\ x_3 + e_3,\ x_4 + e_4)$$

where e_1, e_3, and e_4 are positive numbers such that $e_1 + 2e_3 + e_4 = x_2 - x_3$, $x_3 + e_3 \leqq (r(2)/2 + r(1))$, and $x_4 + e_4 \leqq r(1)$. (Since $x_3 \geqq r(1)$, we must have $x_3 + e_3 > r(1)$.) Thus $y \in Kl_{23}$ and $y \text{ dom}_{\{1,3,4\}}\, x$.

<u>Case (2) $x_4 \leq r(1)$, $x_2 + x_3 > 2r(1) + r(2)$</u>: Define $y \in A(v)$ by

$$y = (x_1 + e_1,\ r(2)/2 + r(1),\ r(2)/2 + r(1),\ x_4 + e_4)$$

where e_1 and e_4 are positive numbers such that $e_1 + e_4 = x_2 + x_3 - (2r(1) + r(2))$ and $x_4 + e_4 \leqq r(1)$. Then $y \in Kl_{23}$ and $y \text{ dom}_{\{1,4\}}\, x$.

Case (3) $x_4 = r(1)$, $x_2 + x_3 \leqq 2r(1) + r(2)$: Since $x \notin K$, we must have $x_2 > x_3$, and thus $x_3 < r(2)/2 + r(1)$. Define $y \in A(v)$ by

$$y = (x_1 + e_1,\ r(1),\ x_3 + e_3,\ x_3 + e_3)$$

where e_1 and e_3 are positive numbers such that $e_1 + 2e_3 = x_2 - x_3$ and $x_3 + e_3 \leqq (r(2)/2) + r(1)$. (Since $x_3 \geqq x_4 = r(1)$, $x_3 + e_3 > r(1)$.) Then $y \in K1_{34}$ and $y\ \mathrm{dom}_{\{1,3,4\}}\ x$.

Case (4) $x_4 = r(1)$, $x_2 + x_3 > 2r(1) + r(2)$: In case $r(2) > 0$, define $y \in A(v)$ by

$$y = (x_1 + e_1,\ (r(2)/2) + r(1),\ (r(2)/2) + r(1),\ x_4 + e_4)$$

where e_1 and e_4 are positive numbers such that $e_1 + e_4 = x_2 + x_3 - (2r(1) + r(2))$ and $x_4 + e_4 \leqq (r(2)/2) + r(1)$. (Note that $x_4 < (r(2)/2) + r(1)$ since $r(2) > 0$.) Then $y \in K2_{23}$ and $y\ \mathrm{dom}_{\{1,4\}}\ x$. In case $r(2) = 0$, noting the fact that $x_2 + x_3 > 2r(1) = 2x_4$, define $y \in A(v)$ by

$$y = (x_1 + e_1,\ x_4 + e_4,\ x_4 + e_4,\ x_4 + e_4)$$

where e_1 and e_4 are positive numbers such that $e_1 + e_4 = x_2 + x_3 - 2x_4$. Then $y \in K3$ and $y\ \mathrm{dom}_{\{1,4\}}\ x$.

Case (5) $x_4 > r(1)$, $x_2 + x_3 \leqq 2r(1) + r(2)$: If $x_3 = (r(2)/2) + r(1)$, then $x_2 = x_3 = (r(2)/2) + r(1)$, and thus $x \in K2_{23}$. Hence we suppose $x_3 < (r(2)/2) + r(1)$. Define $y \in A(v)$ by

$$y = (x_1 + e_1,\ r(1),\ x_3 + e_3,\ x_3 + e_3)$$

where e_1 and e_3 are positive numbers such that $e_1 + 2e_3 = (x_2 - x_3) + (x_4 - r(1))$ and $x_3 + e_3 \leqq (r(2)/2) + r(1)$. Then $y \in K2_{34}$ and $y\ \mathrm{dom}_{\{1,3,4\}}\ x$.

Case (6) $x_4 > r(1)$, $x_2 + x_3 > 2r(1) + r(2)$: If $x_4 <$

$(r(2)/2) + r(1)$, then we can show $x \in \text{Dom } K2_{23}$ in a similar manner as in the first half of Case (4). Suppose $x_4 \geqq (r(2)/2) + r(1)$. Since $x \notin K$, we must have $x_2 > x_4$. Define $y \in A(v)$ by

$$y = (x_1 + e_1, x_4 + e_4, x_4 + e_4, x_4 + e_4)$$

where e_1 and e_4 are positive numbers such that $e_1 + 3e_4 = x_2 + x_3 - 2x_4$. Then $y \in K3$ and $y \text{ dom}_{\{1,4\}} x$.

Thus we have shown the external stability of K.

Finally we easily see that K1, K2, and K3 are symmetric w.r.t. {2,3,4}. Further the core C(v) is symmetric w.r.t. {2,3,4} since (N,v) is symmetric information games. Therefore K is symmetric w.r.t. {2,3,4}. Q.E.D.

5. Concluding remarks

We have obtained a stable set of four-person symmetric information games which reflects the symmetry of the players 2, 3, and 4 (demanders of the information), and further we have examined its implications on traders' behavior in the information market.

In concluding this paper, we observe that the stable set K obtained in this paper is closely related with the stable set of four-person symmetric games presented in Muto [11]. This observation suggests the following two possibilities of further research: (i) One may construct other stable sets (reflecting the symmetry of the players 2, 3, and 4) of four-person symmetric information games from the symmetric stable sets of four-person symmetric games due to Nering [7]. (Refer to Muto [11].);

(ii) one may find a stable set (reflecting the symmetry of the palyers 2, 3, 4, and 5) of five-person symmetric information games, based on the symmetric stable set of five-person symmetric games presented in Muto [11].

Acknowledgements:

This work was begun while Muto was visiting Catholic University in May 1986. He was staying at CORE in Belgium in the academic year 1985. The financial supports from CORE and Catholic University are gratefully acknowledged.

Footnotes

1. This game has also a symmetric stable set, {(1/2, 1/2, 0), (1/2, 0 ,1/2), (0, 1/2, 1/2)}.
2. Unlike the original model studied in Muto, Potters, and Tijs [10], we here assume that trader 1 (patent holder) has no facilities of manufacturing the product. The cooperative games derived from these different models are, however, mathematically equivalent. Precisely speaking, the information game developed in this paper is the (0,1)-normalization of the original information game.
3. Refer to Muto, Potters, and Tijs [10] concerning the condition for the core being the stable set in general information games.

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