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**FEEDBACK NASH EQUILIBRIA FOR LINEAR QUADRATIC
DESCRIPTOR DIFFERENTIAL GAMES**

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Feedback Nash Equilibria for Linear Quadratic Descriptor Differential Games

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Abstract

In this note we consider the non-cooperative linear feedback Nash quadratic differential game with an infinite planning horizon for descriptor systems of index one. The performance function is assumed to be indefinite. We derive both necessary and sufficient conditions under which this game has a Nash equilibrium.

Keywords: linear-quadratic games, linear feedback Nash equilibrium, affine systems, solvability conditions, Riccati equations.

Jel-codes: C61, C72, C73.

1 Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games. In particular in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in Dockner et al. [8], Jørgensen et al. [22], Plasmans et al. [32] and Grass et al. [19]). Moreover, in optimal control theory it is well-known that, e.g., the issue to obtain robust control strategies can be approached as a dynamic game problem (see e.g. [2], [26] and [5]).

In this paper we consider the linear quadratic differential game under the assumption that the dynamics of the underlying process are described by a descriptor system. That is, by both a set of differential equations and linear equations. Problems of this kind appear in studying systems which operate under different timescales like e.g. in mechanical engineering where an electrically

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driven robot manipulator typically has slow mechanical dynamics and fast electrical dynamics, or in environmental economics where global warming is assumed to be a system which has slow dynamics that is affected by various processes that have fast dynamics. They also sometimes naturally appear in modeling systems like e.g. the Leontieff model in economics describing the relationship between the levels of production of a number of interrelated production sectors. See e.g. [27], [21] or [20] and the references therein.

The "one-player" regulator problem for descriptor systems that have index one has been considered by many authors. One of the first who considered this control problem was Pandolfi [30]. His results were later on generalized by e.g. Cobb [7] who gave both necessary and sufficient conditions under which the regular definite control problem has a solution in terms of a transformed system. In the seminal paper [4] Bender and Laub show, amongst other things, by considering the stable eigenspace of a generalized eigenvalue problem that the (generally non-unique) optimal feedback gains are constrained to lie in a linear variety, all of which yield the same minimal cost of (1). Unfortunately, however, this approach does not render itself to be generalized to a multi-player context with a feedback information structure. In [12] an explicit characterization was obtained for this set of optimal feedback gains in terms of a transformed system, which make it possible to extend results for a multi-player context. The reason to consider index one descriptor systems is that, see e.g. [17] (or references cited there), for all initial states of the system there exists a smooth control that generates a smooth state trajectory if and only if the system is controllable at infinity (or, impulse controllable). Further, all impulsive modes of the system can be transformed then into finite dynamic modes using a static state feedback control. After this transformation the system has then the property of being of index one (which will be formally introduced in section 2). Since we do not want to consider impulsive control actions in this paper we will therefore restrict our attention to impulse controllable systems, or without loss of generality, to systems that are of index one.

In this note we consider the linear quadratic differential game under a feedback information structure. The reason to consider this information structure is that the corresponding linear feedback Nash equilibria (FBNE) have the nice property of strong time consistency. A property which, e.g., does not hold under an open-loop information structure.

The case that the system is just described by a set of differential equations has been considered by many authors and dates back to the seminal work of Starr and Ho in [33]. For the fixed finite planning horizon, one can show that there exists at most one FBNE (see e.g. [28], [31]). For the infinite planning horizon it is well-known that the problem may have up to an infinite number of FBNE. References and related issues can be found in, e.g., [3] and [9].

For descriptor systems having an open-loop information structure in [10] the general multi-person game problem for index one systems was solved. Further, in [11] some first results were obtained for higher order index systems.

Probably the first reference where in fact descriptor systems were considered with a feedback information structure within the context of singularly perturbed systems is [16] (see also [15] for the zero-sum case). For descriptor systems with a closed-loop perfect state information structure Xu and Mizukami [34] derived for a finite planning horizon some first results for zero-sum games. Further, these authors considered the leader-follower information structure in [29] and [35]. Glizer [18] considered the asymptotic behavior of the zero-sum game solution from a cheap control perspective. As already mentioned above Engwerda et al. [12] solved the general multi-person game problem where the solvability conditions are posed in terms of a transformed system. However, the results

were obtained under the simplifying assumption that the state cost are nonnegative. In particular this implies that the results obtained there are not directly applicable to a zero-sum game setting. In this paper we generalize these feedback results for performance criteria that also include "cross-terms", i.e. products of the state and control variables and with no definiteness assumptions concerning the involved state cost. We provide a complete parametrization of all Nash solutions. As a special case we derive existence results for the zero-sum game. Performance criteria of this type often naturally appear in economic policy making.

The outline of this note is as follows. Section two introduces the problem and contains some preliminary results. The main results of this paper are stated in Section three, whereas Section four illustrates the presented theory with an example. The final section contains some concluding remarks.

2 Preliminaries

In this paper we assume that player $i = 1, 2$ likes to minimize:

$$\lim_{t_f \rightarrow \infty} J_i(t_f, x_0, u_1, u_2), \text{ where } J_i(t_f, x_0, u_1, u_2) := \int_0^{t_f} [x^T(t), u_1^T(t), u_2^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (1)$$

$$M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}, R_{ii} > 0, i = 1, 2, \text{ and } x(t) \text{ satisfies the linear differential equation}$$

$$E\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t), x(0) = x_0. \quad (2)$$

Notice that we make no definiteness assumptions w.r.t. matrix Q_i .

Next, we recall some standard definitions and results concerning descriptor systems. Consider the system

$$E\dot{x}(t) = Ax(t), x(0) = x_0, \quad (3)$$

where $\text{rank}(E) = r < n$. An initial state x_0 in (3) is called consistent if with this choice of the initial state the system has a solution. Let λ be any complex number. Then, system (3) (or the matrix pair (E, A)) is called regular if $\det(\lambda E - A) \neq 0$. System (3) has a unique solution, for any consistent initial state, if and only if (E, A) is regular.

The solutions of $|\lambda E - A| = 0$ are called the finite eigenvalues of (E, A) , and the corresponding (generalized) eigenvectors are exponential modes of the system. From [14] we recall the so-called Weierstrass canonical form.

Theorem 2.1 *If (3) is regular there exist nonsingular matrices X and Y such that*

$$Y^T E X = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T A X = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \quad (4)$$

where A_1 is a matrix in Jordan form, N is a nilpotent matrix also in Jordan form and I is the identity matrix. A_1 and N are unique up to permutation of Jordan blocks. \square

From (4) we have that $\lambda E - A = Y^{-T} \text{diag}\{\lambda I - A_1, \lambda N - I\} X^{-1}$. Since $\lambda N - I = -\lambda(\mu I - N)$, $\mu = \lambda^{-1}$ and N is nilpotent, one says that $\lambda N - I$ describes the eigenstructure at ∞ (or the infinite eigenvalues) of the pencil $\lambda E - A$. Notice that in case we consider the system

$$\begin{bmatrix} I & 0 \\ 0 & N + \epsilon I \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (5)$$

where $\epsilon > 0$ and $N \neq 0$, there exist initial states that grow arbitrarily fast in time if $\epsilon \rightarrow 0$. Moreover, if we consider in (5) $\epsilon = 1$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $x_2(0) = [1 \ 0]^T$, we observe that the solution of the second part of the system has to jump to $x_2(t) = [0 \ 0]^T$, for all $t > 0$. For these reasons system (3) is said to create impulsive modes if there exist generalized eigenvectors, x_k , satisfying the relations $E x_1 = 0$ and $E x_k = x_{k-1}$ ($k \geq 2$). System (3) has no impulsive modes if and only if $N = 0$ or, equivalently, $\text{rank } E = \deg |\lambda E - A|$. Further, system (3) is said to be of index one¹ if $N = 0$ (its degree of nilpotency is one). Next recall from, e.g., [17] that for all initial states in (2) there exists a smooth control that generates a smooth state trajectory if and only if (2) is controllable at infinity (or, impulse controllable²). Further, all impulsive modes of (2) can be transformed then into finite dynamic modes using static state feedback control. Since we do not want to consider impulsive control actions in this paper we restrict our attention to impulse controllable systems. As we consider the state feedback control problem this motivates why we may assume for our problem, without loss of generality, that the matrix pair (E, A) in (2) is index one.

Under this index one assumption, with

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x, \text{ where } x_1 \in \mathbb{R}^r \text{ and } x_2 \in \mathbb{R}^{n-r} \quad (6)$$

and $B_{i1} := [I_r \ 0] Y^T B_i$ and $B_{i2} := [0 \ I_{n-r}] Y^T B_i$, the performance function J_i in (1) can be rewritten as follows:

$$J_i = \int_0^\infty \{ [x_1^T(t) \ x_2^T(t)] X^T [I \ F_1^T \ F_2^T] \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix} X \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \} dt \quad (7)$$

and the system as

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \left(\begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} F_1 X + \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} F_2 X \right) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X^{-1}x_0. \quad (8)$$

Of course problem (7,8) is not completely specified, as we did not specify the set of admissible feedback strategies yet and neither paid attention to the fact that not for every initial state a solution exists of (8). The set of all consistent initial states of (3) is given by $\{x_0 \mid x_0 = X[\bar{x}_1^T \ 0]^T, \bar{x}_1 \in \mathbb{R}^r\}$. As shown in [24] this set does not depend on the choice of matrix X that is chosen in (4). In the sequel we will assume that if $x_0 = X[\bar{x}_1^T \ \bar{x}_2^T]^T$ is an inconsistent initial state of (4), the system's state will jump to $x_0^+ := X[\bar{x}_1^T \ 0]^T$. This assumption is motivated by the idea that the part of the state of the system that is governed by the fast dynamics of the system can, compared to the part

¹This is equivalent (see e.g. [25]) to the assumption that $\text{rank}([E \ AW]) = n$, where the image of matrix W equals the null space of E . Notice furthermore that in that case $|\lambda E - A|$ is not a constant.

²System (2) is impulse controllable (see e.g. [25]) if and only if $\text{rank}[E \ AW \ B] = n$.

of the state governed by the slow dynamics, easily adapt. We will restrict the analysis to the set of linear state feedback controls that stabilize the system for all consistent initial states. A necessary and sufficient condition for the existence of such a stabilizing feedback matrix is that system (2) (or (E, A, B) , where $B = [B_1 \ B_2]$) is finite dynamics stabilizable³. This property holds if and only if $\text{rank} [\lambda E - A \ B] = n$ for all $\lambda \in \mathbb{C}_0^+$ ⁴. These requirements lead then to the assumption that $F \in \mathcal{F}$, where

$$\mathcal{F} := \{F = [F_1^T \ F_2^T]^T \mid \text{all finite eigenvalues of } (E, A + BF) \text{ are stable and } (E, A + BF) \text{ has index one}\}.$$

We assume that the matrix pairs (A_i, B_{i1}) , $i = 1, 2$, are stabilizable. So, in principle, each player is capable to stabilize the first part of the transformed system on his own. Notice furthermore from the above discussion (and the fact that for any fixed matrix $F \in \mathcal{F}$, matrix $G := I + [B_{12} \ B_{22}]FX_2$ is invertible (see Lemma 2.3 below)) that for a fixed $F \in \mathcal{F}$ the set of consistent initial states for (8) equals $\{(x_1(0), x_2(0)) \mid x_2(0) = G^{-1}[B_{12} \ B_{22}]FX_1x_1(0), x_1(0) \in \mathbb{R}^r\}$.

Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of both players to stabilize the system (see e.g. [9] for a discussion).

Then, $u^* := (u_1^*, u_2^*) \in \mathcal{F}$ is called a feedback Nash equilibrium if the usual inequalities apply, i.e., no player can improve his performance by a unilateral deviation from this set of equilibrium actions. Introducing the notation $u_{-i}^*(\alpha) := u^*$ where u_i^* has been replaced by the arbitrary input function α the formal definition reads as follows

Definition 2.2 (F_1^*, F_2^*) or $(u_1^*, u_2^*) \in \mathcal{F}$ is called a *feedback Nash equilibrium* (FBN) if for $i = 1, 2$, $J_i(x_0, u^*) \leq J_i(x_0, u_{-i}^*(\alpha))$ for every x_0 and input α such that $u_{-i}^*(\alpha) \in \mathcal{F}$. \square

So, summarizing, the main problem addressed in this paper reads as follows.

Problem 1: Consider the performance criterion (1) and system (2), where $\text{rank}(E) = r < n = \dim(x)$. Assume (E, A) is regular and has index one; and (E, A, B) is finite dynamics stabilizable. Let X be as defined in (4). Decompose $X =: [X_1 \ X_2]$, with $X_1 \in \mathbb{R}^{n \times r}$ and $X_2 \in \mathbb{R}^{n \times (n-r)}$. Find conditions under which (1,2) has a FBN solution $u_i = F_i x$, with $F \in \mathcal{F}$. \square

To solve Problem 1 we first state some preliminary results. The following Lemma 2.3 can be found in [12].

Lemma 2.3 *Assume $(E, A + BF)$ is regular and has index one. Then for all $F \in \mathcal{F}$, $G := I + [B_{12} \ B_{22}]FX_2$ is invertible.*

Furthermore we recall from, e.g., [6, p.97] the following property.

Lemma 2.4 *Assume $C \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{m \times n}$. Then the following holds:*

1. $I_n + CD$ is invertible if and only if $I_m + DC$ is invertible.

2. If $I_n + CD$ is invertible then: $C(I_m + DC)^{-1} = (I_n + CD)^{-1}C$. \square

³(2) is finite dynamics stabilizable if there exists a feedback $u(t) = Fx(t)$ such that all finite eigenvalues of the system $E\dot{x}(t) = (A + BF)x(t)$ are stable.

⁴ \mathbb{C}_0^+ is the set of complex numbers with non-negative real part.

The following lemma can be proved directly from the definition of Nash equilibrium.

Lemma 2.5 (F_1^*, F_2^*) is a FBN for the game defined by the cost $J_i(F_1H, F_2H)$ and the system $\dot{x}(t) = (A + B_1F_1H + B_2F_2H)x(t)$, $x(0) = x_0$ if and only if (G_1^*, G_2^*) is a FBN for the game defined by the cost $J_i(G_1, G_2)$ and the system $\dot{x}(t) = (A + B_1G_1 + B_2G_2)x(t)$, $x(0) = x_0$ and (F_1^*, F_2^*) solve the set of equations $F_1^*H = G_1^*$ and $F_2^*H = G_2^*$. \square

Finally, we recall from [13] the following result.

Theorem 2.6 Assume that matrix $G = \begin{bmatrix} R_{11} & N_1 \\ N_2^T & R_{22} \end{bmatrix}$ is invertible. Then the differential game (1,2), with $E = I$, has a feedback Nash equilibrium (F_1, F_2) for every initial state if and only if

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -G^{-1} \begin{bmatrix} B_1^T K_1 + V_1^T \\ B_2^T K_2 + W_2^T \end{bmatrix}, \quad (9)$$

Here (K_1, K_2) are a symmetric stabilizing solution of the coupled algebraic Riccati equations

$$0 = (A + B_2F_2)^T K_1 + K_1(A + B_2F_2) - F_1^T R_{11} F_1 + F_2^T R_{21} F_2 + F_2^T W_1^T + W_1 F_2 + Q_1 \quad (10)$$

$$0 = (A + B_1F_1)^T K_2 + K_2(A + B_1F_1) - F_2^T R_{22} F_2 + F_1^T R_{12} F_1 + F_1^T V_2^T + V_2 F_1 + Q_2. \quad (11)$$

\square

3 Main results

To solve Problem 1 we first notice from (8), using Lemma 2.3, and our jump assumption on inconsistent initial states, that for all $t > 0$

$$x_2 = Hx_1, \text{ where } H := -(I + [B_{12} \ B_{22}]FX_2)^{-1}[B_{12} \ B_{22}]FX_1. \quad (12)$$

Substitution of this into (7,8) shows that Problem 1 has a FBNE if and only if (F_1, F_2) is a FBNE for the problem with

$$J_i = \int_0^\infty \{x_1^T(t)[I \ H^T]X^T[I \ F_1^T \ F_2^T] \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ F_2 \end{bmatrix} X \begin{bmatrix} I \\ H \end{bmatrix} x_1(t)\} dt \quad (13)$$

and

$$\dot{x}_1(t) = \left(A_1 + [B_{11} \ B_{21}] \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} X \begin{bmatrix} I \\ H \end{bmatrix} \right) x_1(t), \quad x_1(0) = [I \ 0]X^{-1}x_0. \quad (14)$$

Introducing $\tilde{F}_i := F_i X \begin{bmatrix} I \\ H \end{bmatrix}$, we can rewrite (13,14) above as

$$J_i = \int_0^\infty \{x_1^T(t) [[I \ H^T] X^T \ \tilde{F}_1^T \ \tilde{F}_2^T] \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix} \begin{bmatrix} X \begin{bmatrix} I \\ H \end{bmatrix} \\ \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} x_1(t)\} dt \quad (15)$$

subject to

$$\dot{x}_1(t) = \left(A_1 + [B_{11} \ B_{21}] \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} \right) x_1(t), \quad x_1(0) = [I \ 0] X^{-1} x_0. \quad (16)$$

Next notice that, by Lemma 2.4,

$$\begin{aligned} H &= -(I + [B_{12} \ B_{22}] F X_2)^{-1} [B_{12} \ B_{22}] F X_1 = -[B_{12} \ B_{22}] F (I + X_2 [B_{12} \ B_{22}] F)^{-1} X_1 \\ &= -(B_{12} \tilde{F}_1 + B_{22} \tilde{F}_2). \end{aligned} \quad (17)$$

Using (17) we can rewrite (15) as follows

$$J_i = \int_0^\infty \{x_1^T(t) [I \ \tilde{F}_1^T \ \tilde{F}_2^T] \tilde{M}_i \begin{bmatrix} I \\ \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} x_1(t)\} dt, \quad (18)$$

where

$$\tilde{M}_i := \begin{bmatrix} X_1^T & 0 & 0 \\ -B_{12}^T X_2^T & I & 0 \\ -B_{22}^T X_2^T & 0 & I \end{bmatrix} \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix} \begin{bmatrix} X_1 & -X_2 B_{12} & -X_2 B_{22} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} =: \begin{bmatrix} \tilde{Q}_i & \tilde{V}_i & \tilde{W}_i \\ \tilde{V}_i^T & \tilde{R}_{1i} & \tilde{N}_i \\ \tilde{W}_i^T & \tilde{N}_i^T & \tilde{R}_{2i} \end{bmatrix}$$

We now have the following result.

Theorem 3.1 *Assume that matrix $\tilde{G} := \begin{bmatrix} \tilde{R}_{11} & \tilde{N}_1 \\ \tilde{N}_2^T & \tilde{R}_{22} \end{bmatrix}$ is invertible and the matrices $\tilde{R}_{ii} > 0$, $i = 1, 2$.*

Then (F_1, F_2) is a FBN for (1,2) for every initial state if and only if

$$F_i = \tilde{F}_i P^+ + Z_i (I - P P^+), \quad \text{where } Z_i \in \mathbb{R}^{m_i \times n}, \quad P = X \begin{bmatrix} I \\ -B_{12} \tilde{F}_1 - B_{22} \tilde{F}_2 \end{bmatrix} \quad (19)$$

and $(\tilde{F}_1, \tilde{F}_2)$ are given by

$$\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = -\tilde{G}^{-1} \begin{bmatrix} B_{11}^T K_1 + \tilde{V}_1^T \\ B_{21}^T K_2 + \tilde{W}_2^T \end{bmatrix}, \quad (20)$$

where (K_1, K_2) are a symmetric stabilizing solution of the coupled algebraic Riccati equations

$$0 = (A_1 + B_{21} \tilde{F}_2)^T K_1 + K_1 (A_1 + B_{21} \tilde{F}_2) - \tilde{F}_1^T \tilde{R}_{11} \tilde{F}_1 + \tilde{F}_2^T \tilde{R}_{21} \tilde{F}_2 + \tilde{F}_2^T \tilde{W}_1^T + \tilde{W}_1 \tilde{F}_2 + \tilde{Q}_1 \quad (21)$$

$$0 = (A_1 + B_{11} \tilde{F}_1)^T K_2 + K_2 (A_1 + B_{11} \tilde{F}_1) - \tilde{F}_2^T \tilde{R}_{22} \tilde{F}_2 + \tilde{F}_1^T \tilde{R}_{12} \tilde{F}_1 + \tilde{F}_1^T \tilde{V}_2^T + \tilde{V}_2 \tilde{F}_1 + \tilde{Q}_2. \quad (22)$$

Moreover, $J_i = x_0^T X^{-T} [I \ 0]^T K_i [I \ 0] X^{-1} x_0$. \square

Proof: From the above reformulation it follows directly from Lemma 2.5 that (F_1, F_2) is a FBN for (1,2) for every initial state if and only if $(\tilde{F}_1, \tilde{F}_2)$ is a FBN for the game defined by the cost (18) and system (16) and (F_1, F_2) solve the set of equations $\tilde{F}_i := F_i X \begin{bmatrix} I \\ H \end{bmatrix}$. From, e.g. [1, p.295], we have that this equation always has a solution and that all solutions are parameterized by (19). Further, by Theorem 2.6 we have that $(\tilde{F}_1, \tilde{F}_2)$ is a FBN for every initial state for the game defined by the cost (18) and system (16) if and only if (20) holds where (K_1, K_2) are a stabilizing solution of (21,22). \square

Remark 3.2 Notice that since matrix X is invertible and $\begin{bmatrix} I \\ -B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2 \end{bmatrix}$ is full column rank, the Moore-Penrose inverse P^+ of P equals

$$\begin{aligned} P^+ &= \begin{bmatrix} I \\ -B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2 \end{bmatrix}^+ X^{-1} \\ &= \left([I \ (-B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2)^T] \begin{bmatrix} I \\ -B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2 \end{bmatrix} \right)^{-1} [I \ (-B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2)^T] X^{-1}. \end{aligned}$$

\square

Remark 3.3 For the zero-sum game, i.e. $J_1 = -J_2$, we obtain by addition of (21) and (22) (followed by some elementary rewriting) that K_i satisfy the equation

$$A_{cl}^T(K_1 + K_2) + (K_1 + K_2)A_{cl} = 0,$$

where A_{cl} is the stable closed-loop matrix $A_1 + B_{11}\tilde{F}_1 + B_{21}\tilde{F}_2$. Since A_{cl} is a stable matrix it follows from this linear matrix equation that necessarily $K_1 + K_2 = 0$. So we have that $K_2 = -K_1$. Substitution of this into (21) and (22) shows that these equations have a stabilizing solution if and only if the equation

$$A_1^T K_1 + K_1 A_1 + \tilde{Q}_1 - [\tilde{V}_1 + K_1 B_{11} \ \tilde{W}_1 + K_1 B_{21}] \tilde{G}^{-1} [\tilde{V}_1 + K_1 B_{11} \ -(\tilde{W}_1 + K_1 B_{21})]^T = 0 \quad (23)$$

has a solution K_1 such that $\sigma(A_1 - [B_{11} \ B_{21}] \tilde{G}^{-1} \begin{bmatrix} \tilde{V}_1^T + B_{11}^T K_1 \\ -\tilde{W}_1^T - B_{21}^T K_1 \end{bmatrix}) \subset \mathbb{C}^-$. Notice that $\tilde{G} = \begin{bmatrix} \tilde{R}_{11} & \tilde{N}_1 \\ -\tilde{N}_1^T & \tilde{R}_{22} \end{bmatrix}$. Since \tilde{R}_{11} is invertible one can use, e.g., the expression for the inverse of a block matrix (see e.g. [23, p.656]) to verify that for the inverse matrix $\tilde{G}^{-1} =: \tilde{G}^{inv}$, $G_{12}^{inv} = -G_{21}^{invT}$. Consequently,

$$\begin{aligned} &[\tilde{V}_1 + K_1 B_{11} \ \tilde{W}_1 + K_1 B_{21}] \tilde{G}^{-1} \begin{bmatrix} \tilde{V}_1^T + B_{11}^T K_1 \\ -\tilde{W}_1^T - B_{21}^T K_1 \end{bmatrix} = \\ &= [\tilde{V}_1 + K_1 B_{11} \ \tilde{W}_1 + K_1 B_{21}] \begin{bmatrix} \tilde{G}_{11}^{inv} & -\tilde{G}_{21}^{invT} \\ \tilde{G}_{21}^{inv} & \tilde{G}_{22}^{inv} \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T + B_{11}^T K_1 \\ -\tilde{W}_1^T - B_{21}^T K_1 \end{bmatrix} \\ &= [\tilde{V}_1 + K_1 B_{11} \ \tilde{W}_1 + K_1 B_{21}] \begin{bmatrix} \tilde{G}_{11}^{inv} & \tilde{G}_{21}^{invT} \\ \tilde{G}_{21}^{inv} & -\tilde{G}_{22}^{inv} \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T + B_{11}^T K_1 \\ \tilde{W}_1^T + B_{21}^T K_1 \end{bmatrix} \end{aligned}$$

is clearly symmetric too.

So (23) is an ordinary Riccati equation, which we know has at most one stabilizing solution. Therefore, if the zero-sum game has a solution, then the equilibrium actions $(\tilde{F}_1, \tilde{F}_2)$ are unique and given by

$$\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = -\tilde{G}^{-1} \begin{bmatrix} B_{11}^T K_1 + \tilde{V}_1^T \\ -(B_{21}^T K_1 + \tilde{W}_1^T) \end{bmatrix}.$$

So from (19) we have that the set of equilibrium strategies for the zero-sum game (1,2) is

$$(\tilde{F}_1 P^+ + Z_1(I - PP^+), \tilde{F}_2 P^+ + Z_2(I - PP^+)),$$

where $Z_i \in \mathbb{R}^{m_i \times n}$ and $P = X \begin{bmatrix} I \\ -B_{12}\tilde{F}_1 - B_{22}\tilde{F}_2 \end{bmatrix}$. □

Remark 3.4 [16] considered convergence of the feedback Nash solution of the singularly perturbed system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + 2x_2(t) + u_1(t) + u_2(t); & x_1(0) &= x_1(0); \\ \epsilon \dot{x}_2(t) &= -x_1(t) - 2x_2(t) + 2u_1(t) + 2u_2(t); & x_2(0) &= x_2(0); \end{aligned}$$

with performance criteria

$$J_i = \int_0^\infty \{[x_1^T(t) \ x_2^T(t)] Q_i [x_1^T(t) \ x_2^T(t)]^T + u_i^T(t) R_{ii} u_i(t) + u_j^T(t) R_{ij} u_j(t)\} dt, \quad i, j = 1, 2, \quad i \neq j,$$

where $Q_i = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $R_{ii} = 1$, $R_{ij} = 2$. In this example the converged (symmetric) feedback Nash equilibrium strategies, i.e. $F_i^* = \lim_{\epsilon \downarrow 0} F_i(\epsilon) = -[1.5908 \ 0.7321]$, $i = 1, 2$.

With $Y^T = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & -1 \end{bmatrix}$ the transformed system parameters if $\epsilon = 0$ (cf. (8,18)) are $A_1 = 0$, $B_{i1} = 3$, $B_{i2} = 1$, $\tilde{Q}_i = \frac{3}{2}$, $\tilde{V}_i = 0$, $\tilde{W}_i = 0$, $\tilde{R}_{ii} = 3$, $\tilde{R}_{ij} = 4$, $j \neq i$, and $\tilde{N}_i = 2$, $i = 1, 2$. Straightforward calculations show that the set of FBN for this game are (see (19)):

$$F_i = \frac{\sqrt{\frac{1}{6}}}{1 + (\frac{1}{2} + \sqrt{\frac{2}{3}})^2} [-1 \ \frac{1}{2} + \sqrt{\frac{2}{3}}] + \lambda_i [\frac{1}{2} + \sqrt{\frac{2}{3}} \ 1], \quad i = 1, 2; \quad -0.4906 < \lambda_1 + \lambda_2 < 1.1966.$$

It is easily verified that F_i^* does not belong to this class of FBN solutions.

So, this example demonstrates that in general the FBN strategies of the singularly perturbed system do not converge to a FBN strategy of the corresponding reduced order system. This, though the fast dynamics in the singularly perturbed system are stable (i.e. $A_{22} = -2 < 0$).

The example presented below in Section 4 shows that there are also cases where, independent of the stability properties of the fast dynamics, always convergence occurs. □

4 An Example

In this section we illustrate the obtained results in a small example. We consider the following dynamic equations

$$\dot{f}(t) = \beta f(t) - p(t) - u_1(t); f(0) = f_0 \quad (24)$$

$$\epsilon \dot{p}(t) = \alpha p(t) + u_2(t) \quad (25)$$

with revenue functions

$$\tilde{J}_1 = \int_0^\infty \{\tau_f f^2(t) - u_1^2(t)\} dt; \quad (26)$$

$$\tilde{J}_2 = \int_0^\infty \{\tau_p p^2(t) - u_2^2(t)\} dt. \quad (27)$$

The example might be interpreted along the following lines. Consider a ranger who has to take care about the fish stock in a lake. The lake is fed by a river. Upstream the river there is a firm who wants to dump some of its waste into the river. The waste, however, has the property that it contains a product which spreads quickly in the water and from which cause the fish to die. Both the ranger and the firm know this. The firm has a license to dump some of its waste into the river provided the pollution is not getting too excessive in the sense that the fish stock will get below its natural equilibrium value (which is set to 0 here for ease of calculations). In the above model the fish stock (above its natural equilibrium value) is represented by the variable f , the fishing by the ranger by u_1 (a negative value can be interpreted as planting fish into the lake), the waste dropped into the river by u_2 (a negative value can be interpreted as efforts undertaken by the firm to clean the river again) and p the amount of pollution. It is assumed that the negative effect of pollution on the growth of the fish stock is proportional to p . The ranger is paid for the number of hours he spends on keeping the fish stock of the lake near its equilibrium value. The quadratic structure of the revenues for the ranger could be motivated by the fact that if the fish stock is far from its equilibrium it will cost him much more time to fix everything again.

Introducing $x^T(t) := [f(t) \ p(t)]$, we can rewrite the above model (24,25) as

$$\tilde{E}\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) \text{ with cost functions } J_i = \int_0^\infty [x^T \ u_1(t) \ u_2(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (28)$$

where

$$\tilde{E} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}; \quad A = \begin{bmatrix} \beta & -1 \\ 0 & \alpha \end{bmatrix}; \quad B_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$M_1 = \begin{bmatrix} -\tau_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \text{and } M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\tau_p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Elementary calculations show that with $Y^T := \frac{-1}{\alpha^2} \begin{bmatrix} \alpha & 1 \\ 0 & -\alpha \end{bmatrix}$ and $X := \begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix}$, $Y^T E X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Y^T A X = \begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix}$, $Y^T B_1 = \begin{bmatrix} \frac{1}{\alpha} \\ 0 \end{bmatrix}$ and $Y^T B_2 = \begin{bmatrix} \frac{-1}{\alpha^2} \\ \frac{1}{\alpha} \end{bmatrix}$.

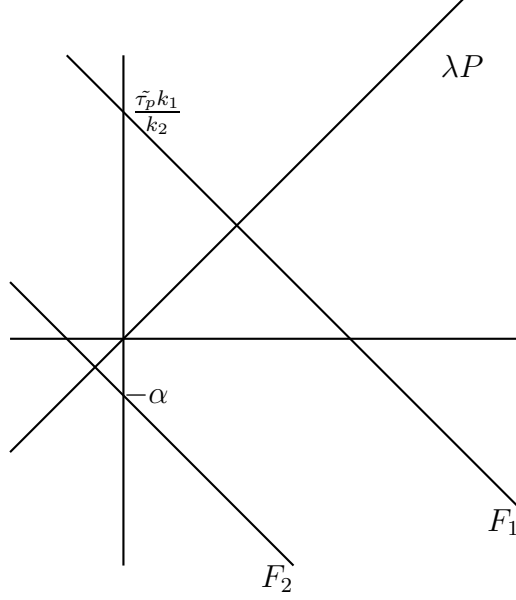


Figure 1: The set of feedback matrices (19) for the Example.

It is easily verified that $(E, A + BF)$ has index one iff., with $F_2 := [f_{21} f_{22}]$, $f_{22} \neq -\alpha$. Furthermore straightforward calculations show that $\tilde{M}_1 = \begin{bmatrix} -\alpha^2 \tau_f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\tilde{M}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\tilde{\tau}_p}{\alpha^2} \end{bmatrix}$. Then, with $\tilde{G} := \begin{bmatrix} 1 & 0 \\ 0 & \frac{\tilde{\tau}_p}{\alpha^2} \end{bmatrix}$ and assuming that $\tilde{\tau}_p := \alpha^2 - \tau_p > 0$ and $\beta - \frac{k_1}{\alpha^2} - \frac{k_2}{\alpha^2 \tilde{\tau}_p} < 0$, we get that $\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & \frac{\alpha^2}{\tilde{\tau}_p} \end{bmatrix} \begin{bmatrix} \frac{k_1}{\alpha^2} \\ \frac{-k_2}{\alpha^2} \end{bmatrix} = \begin{bmatrix} \frac{-k_1}{\alpha^2} \\ \frac{k_2}{\tilde{\tau}_p} \end{bmatrix}$, where k_i solve the equations

$$0 = 2(\beta - \frac{k_2}{\alpha^2 \tilde{\tau}_p})k_1 - \frac{1}{\alpha^2}k_1^2 - \alpha^2 \tau_f \text{ and } 0 = k_2(2\beta - \frac{2}{\alpha^2}k_1 - \frac{1}{\alpha^2 \tilde{\tau}_p}k_2). \quad (29)$$

Notice if matrix $P = [-\alpha \frac{k_2}{\alpha \tilde{\tau}_p}]^T$ in (19) is denoted by $[p_1 \ p_2]^T$, matrix $P^+ = \frac{1}{p_1^2 + p_2^2}[p_1 \ p_2]$. So, matrix F_i , $i = 1, 2$, in (19) is given by (with $z_i, s_i \in \mathbb{R}$):

$$F_1 = \frac{1}{p_1^2 + p_2^2} \tilde{F}_1 [p_1 \ p_2] + \frac{z_1 p_2 - z_2 p_1}{p_1^2 + p_2^2} [p_2 \ -p_1] \text{ and } F_2 = \frac{1}{p_1^2 + p_2^2} \tilde{F}_2 [p_1 \ p_2] + \frac{s_1 p_2 - s_2 p_1}{p_1^2 + p_2^2} [p_2 \ -p_1].$$

In case $k_2 \neq 0$ this can be simplified as

$$F_1 = [0 \ \frac{\tilde{\tau}_p k_1}{k_2}] + \lambda_1 [\frac{-k_2}{\alpha \tilde{\tau}_p} \ \alpha] \text{ and } F_2 = [0 \ -\alpha] + \lambda_2 [\frac{-k_2}{\alpha \tilde{\tau}_p} \ \alpha]. \quad (30)$$

That is, geometrically, all equilibrium feedback matrices F_i are located on a line through $\tilde{F}_i P^+$ which is perpendicular to the line λP , $\lambda \in \mathbb{R}$. We illustrate this in Figure 1. In the Appendix it is shown that depending on the choice of parameters either zero up to three different equilibria can occur.

In the rest of this example we will concentrate on the case that $\tau_f = \beta^2$, where just one equilibrium occurs. From the Appendix we have that under this assumption $(k_1, k_2) = (\frac{-\alpha^2 \beta}{3}, \frac{8\alpha^2 \beta \tilde{\tau}_p}{3})$ solve (29) and the equilibrium actions are $[\tilde{F}_1 \ \tilde{F}_2] = [\frac{\alpha \beta}{3}, \frac{8\alpha^2 \beta}{3}]$ and the cost are $J_1 = -\frac{\beta}{3} f_0^2$ and $J_2 = \frac{8}{3} \beta \tilde{\tau}_p f_0^2$.

From (30) it follows then that

$$F_1 = [0 \quad \frac{-1}{8}] + \lambda_1[\frac{-8}{3}\beta \quad 1] \text{ and } F_2 = [0 \quad -\alpha] + \lambda_2[\frac{-8}{3}\beta \quad 1], \text{ with } \lambda_2 \neq 0. \quad (31)$$

So the closed-loop dynamics, using this choice of F_i , are determined by an initial amount of pollution of $p_0 = -\frac{\frac{-8}{3}\beta\lambda_2}{\alpha+\lambda_2-\alpha}f_0 = \frac{8}{3}\beta f_0$ followed by a decline of pollution and a return to its natural equilibrium of the fish stock along the trajectories $p(t) = \frac{8}{3}\beta f(t)$, where $f(t) = e^{-\frac{4}{3}t}f_0$.

In the Appendix also the feedback Nash equilibria are determined in case $\epsilon \neq 0$. In particular it is shown there that the corresponding equilibrium costs coincide with the above formulae in the limiting case ($\epsilon \downarrow 0$). Furthermore the limiting equilibrium actions are obtained by choosing $\lambda_1 = \frac{1}{8}$ and $\lambda_2 = -\sqrt{\tilde{\tau}_p}$ in (31) above. So, the ranger will be just planting fish in this equilibrium and the firm's dumping of waste depends positively on the fish stock and negatively on the amount of pollution.

To analyze the question which choice of λ_i in (31) might provide some additional robustness properties, we next consider the cost if one uses the equilibrium actions F_i from (31) to control system (24,25) if $\epsilon \neq 0$. Then the system dynamics are described by

$$\begin{bmatrix} \dot{f}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \beta(1 + \frac{8}{3}\lambda_1) & -(\lambda_1 + \frac{7}{8}) \\ \frac{-8}{3}\frac{\beta\lambda_2}{\epsilon} & \frac{\lambda_2}{\epsilon} \end{bmatrix} \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} =: A_\epsilon \begin{bmatrix} f(t) \\ p(t) \end{bmatrix}, \quad \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} f_0 \\ p_0 \end{bmatrix}.$$

Notice that this system is stable iff $\lambda_2 < 0$ and $\frac{\lambda_2}{\epsilon} + \beta(1 + \frac{8}{3}\lambda_1) < 0$.

In the Appendix it is shown that $\tilde{J}_i = [f_0 \quad p_0]H_i \begin{bmatrix} f_0 \\ p_0 \end{bmatrix}$, where

$$H_1 = \frac{1}{\epsilon\beta(8\lambda_1 + 3) + 3\lambda_2} \begin{bmatrix} \beta(\lambda_2 + \frac{\epsilon\beta}{6}(8\lambda_1 + 3)(8\lambda_1 - 3)) & \frac{-1}{16}\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 5) \\ \frac{-1}{16}\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 5) & \frac{-1}{128}\frac{\epsilon}{\lambda_2}h_1 \end{bmatrix}$$

and

$$H_2 = \frac{1}{\epsilon\beta(8\lambda_1 + 3) + 3\lambda_2} \begin{bmatrix} \frac{-8}{3}\beta\lambda_2(-4\epsilon\beta\lambda_2 + 3\tilde{\tau}_p) & -\epsilon\beta(4\lambda_2^2 + (8\lambda_1 + 3)\tilde{\tau}_p) \\ -\epsilon\beta(4\lambda_2^2 + (8\lambda_1 + 3)\tilde{\tau}_p) & \frac{-1}{8}\frac{\epsilon}{\lambda_2}h_2 \end{bmatrix}$$

with $h_1 = 4\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 9) - 3\lambda_2(8\lambda_1 - 1)^2$ and $h_2 = \epsilon\beta\{(8\lambda_1 + 3)^2\tilde{\tau}_p + 8\lambda_2(2\lambda_2 + \alpha(8\lambda_1 + 3))\} + 12\lambda_2(\tau_p - (\lambda_2 - \alpha)^2)$.

Notice that if $\epsilon \downarrow 0$, $H_1 \rightarrow \begin{bmatrix} \frac{\beta}{3} & 0 \\ 0 & 0 \end{bmatrix}$ and $H_2 \rightarrow \begin{bmatrix} -\frac{8}{3}\beta\tilde{\tau}_p & 0 \\ 0 & 0 \end{bmatrix}$.

The question which control strategy will be chosen by the players, depends on the additional objectives of the players. If they don't have any additional objectives it seems reasonable that ultimately the λ_i parameters are such that a Nash equilibrium occurs. That is, each player i tries to maximize \tilde{J}_i w.r.t. λ_i , assuming a fixed value for the other player. If, e.g., $p_0 = 0$, this would result in solving for the ranger the equation $\frac{\partial H_1(1,1)}{\partial \lambda_1} = 0$ and for the firm $\frac{\partial H_2(1,1)}{\partial \lambda_2} = 0$. This yields, with $\tilde{\lambda}_1 := 8\lambda_1 - 1$, the set of simultaneous equations

$$\begin{aligned} \epsilon\beta(\tilde{\lambda}_1 + 4)^2 + 6\lambda_2\tilde{\lambda}_1 &= 0 \\ 8\epsilon\beta\lambda_2(\tilde{\lambda}_1 + 4) + 12\lambda_2^2 - 3\tilde{\tau}_p(\tilde{\lambda}_1 + 4) &= 0, \end{aligned}$$

under the (stability) constraint that $\tilde{\lambda}_1 > 0$.

The solution of the above set of equations is given by $\lambda_2 = -\frac{\epsilon\beta(\tilde{\lambda}_1+4)^2}{6\tilde{\lambda}_1}$ where $\tilde{\lambda}_1$ is the (unique) positive solution of the equation

$$-3\tilde{\lambda}_1^3 - (20 + \frac{9}{\epsilon^2\beta^2}\tilde{\tau}_p)\tilde{\lambda}_1^2 - 16\tilde{\lambda}_1 + 64 = 0.$$

Notice that in the limiting case, $\epsilon \downarrow 0$, this leads again to the solution $\lambda_1 = \frac{1}{8}$, $\lambda_2 = -\sqrt{\tilde{\tau}_p}$.

From a robustness point of view another option could be that players like to choose the λ_i in such a way that the closed-loop system becomes as stable as possible. Or, stated differently, to determine λ_i in such a way that the real part of the largest eigenvalue of matrix A_ϵ ,

$$\lambda(\lambda_1, \lambda_2) := \frac{1}{2} \left\{ \frac{\lambda_2}{\epsilon} + \beta(1 + \frac{8}{3}\lambda_1) + \sqrt{(\frac{\lambda_2}{\epsilon} + \beta(1 + \frac{8}{3}\lambda_1))^2 + \frac{16}{3}\frac{\lambda_2\beta}{\epsilon}} \right\},$$

is minimal.

Fist notice that for fixed λ_i , $\lim_{\epsilon \downarrow 0} \lambda(\lambda_1, \lambda_2) = -\frac{4}{3}\beta$. Choosing for a fixed negative λ_2 , λ_1 such that $1 + \frac{8}{3}\lambda_1 = -\frac{\lambda_2}{\epsilon\beta} - \sqrt{-\frac{16}{3}\frac{\lambda_2}{\epsilon\beta}}$, we have that $\lambda(\lambda_1, \lambda_2) = -\sqrt{-\frac{16}{3}\frac{\beta\lambda_2}{\epsilon}}$. So, in case players would agree to coordinate their actions, as far as it concerns the choice of the λ_i parameters, we see that this largest eigenvalue can be made arbitrarily small by an appropriate choice of λ_i . A clear disadvantage of this choice is that its implementation requires knowledge of ϵ . Furthermore, it easily results in high gain feedbacks.

On the other hand we see that if the ranger chooses $\lambda_1 = -\frac{7}{8}$, $\lambda(-\frac{7}{8}, \lambda_2) = -\frac{\beta}{4}$, whatever player's 2 choice of $\lambda_2 (< 0)$ is. Furthermore, $\lambda(\lambda_1, \lambda_2) > -\frac{\beta}{4}$ if $\lambda_1 < -\frac{7}{8}$ and $\lambda(\lambda_1, \lambda_2) < -\frac{\beta}{4}$ if $-\frac{7}{8} < \lambda_1 < \frac{-\lambda_2}{\epsilon\beta(1+\frac{8}{3})}$. So, in case it is unclear what the value of ϵ will be and, moreover, the ranger doesn't know the firm's choice of λ_2 , the ranger can enforce always a certain stability of the closed-loop system. In this case the ranger considers both the fish stock and the level of pollution in his decision how much he should fish. This at the expense that he might have earned more profits (for small values of ϵ) or that, in retrospect, a more stabilizing control could have been implemented if he would have known the firm's action.

5 Concluding Remarks

In this note we considered the regular indefinite infinite-planning horizon linear-quadratic differential game for index one descriptor systems. Both necessary conditions and sufficient conditions were derived for the existence of a feedback Nash equilibrium. These conditions were stated in terms of a transformed system. A complete parametrization was derived for the set of FBN equilibria. The transformation used here was based on the Weierstrass canonical form. Notice, however, that for numerical purposes it suffices to make a singular value decomposition of matrix E . Using this, one can calculate then easily matrices Y and X yielding the same structural form (18), with matrix A_1 a square matrix instead of a Jordan matrix (see [12, Remark 3.2]).

In case there exists a FBN equilibrium, usually, there exists an infinite number of feedback Nash equilibria which all give rise to the same closed-loop behavior of the system due to the informational non-uniqueness of the problem. This makes it possible to look for equilibria within this set that satisfy some additional properties, like e.g. robustness. We illustrated in an example how different criteria

may yield different choices for the feedback design. In particular we showed that in this example one of these equilibria can be interpreted as the converged FBN of the corresponding singular perturbed system. By means of a counterexample (based on [16]) we showed that this property does not hold for arbitrary systems. Furthermore we showed that in this example one of the players can enforce a certain stability of the closed-loop system for the corresponding singularly perturbed system by an appropriate choice of his strategy, this independent of the FBN strategy choice of the other player and independent of the time-scale of the fast system. Clearly these observations ask for a more detailed study in a more general setting.

The obtained theoretical results can be generalized straightforwardly to the N -player case. Furthermore, since Q_i are allowed to be indefinite, the obtained results were used directly to derive properties for the zero-sum game. We showed that in that case all FBN equilibria yield the same closed-loop system.

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Appendix

Feedback Nash equilibria from game considered in (28) if $\epsilon \neq 0$.

First we rewrite this game into its standard form as

$$\dot{x}(t) = \tilde{E}^{-1}Ax(t) + \tilde{E}^{-1}B_1u_1(t) + \tilde{E}^{-1}B_2u_2(t) =: \tilde{A}x(t) + \tilde{B}_1u_1(t) + \tilde{B}_2u_2(t)$$

with cost functions

$$J_i = \int_0^\infty x^T(t)Q_ix(t) + u_i^T R_{ii}u_i(t)dt,$$

where

$$\tilde{A} = \begin{bmatrix} \beta & -1 \\ 0 & \frac{\alpha}{\epsilon} \end{bmatrix}; \tilde{B}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; \tilde{B}_2 = \begin{bmatrix} 0 \\ \frac{1}{\epsilon} \end{bmatrix}; Q_1 = \begin{bmatrix} -\tau_f & 0 \\ 0 & 0 \end{bmatrix}; Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\tau_p \end{bmatrix}; \text{ and } R_{ii} = 1.$$

From, e.g. [9][Theorem 8.5], we recall that the feedback Nash equilibria $u_i(t) = F_i x(t)$ for this game are given by $F_i^* := -R_{ii}^{-1}\tilde{B}_i^T K_i$, $i = 1, 2$, where (K_1, K_2) are a set of symmetric solutions of the next coupled algebraic Riccati equations

$$0 = -(\tilde{A} - S_2 K_2)^T K_1 - K_1(\tilde{A} - S_2 K_2) + K_1 S_1 K_1 - Q_1, \quad (32)$$

$$0 = -(\tilde{A} - S_1 K_1)^T K_2 - K_2(\tilde{A} - S_1 K_1) + K_2 S_2 K_2 - Q_2, \quad (33)$$

that satisfy the additional requirement that the closed-loop system matrix $A_{cl} := \tilde{A} - S_1 K_1 - S_2 K_2$ is stable. Here $S_i := \tilde{B}_i R_{ii}^{-1} \tilde{B}_i^T$.

Moreover, the cost incurred by player i by playing this equilibrium action is $x_0^T K_i x_0$, $i = 1, 2$.

With $K_1 = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$ and $K_2 = \begin{bmatrix} \bar{k}_1 & \bar{k}_2 \\ \bar{k}_2 & \bar{k}_3 \end{bmatrix}$ an elementary spelling of equations (32,33) yields the next equations

$$\begin{aligned}
(k_1 - \beta)^2 + 2(k_2 + 1)\frac{\bar{k}_2}{\epsilon^2} - 2\frac{\bar{k}_2}{\epsilon^2} + \tau_f - \beta^2 &= 0 \\
\frac{\bar{k}_2}{\epsilon^2}k_3 - \frac{1}{\epsilon}\left(\alpha - \frac{\bar{k}_3}{\epsilon}\right)(k_2 + 1) + \frac{1}{\epsilon}\left(\alpha - \frac{\bar{k}_3}{\epsilon}\right) + (k_2 + 1)(k_1 - \beta) + \beta &= 0 \\
(k_2 + 1)^2 - 1 - \frac{2}{\epsilon}\left(\alpha - \frac{\bar{k}_3}{\epsilon}\right)k_3 &= 0 \\
2(k_1 - \beta)\bar{k}_1 + \frac{\bar{k}_2^2}{\epsilon^2} &= 0 \\
(k_1 - \beta)\bar{k}_2 + (k_2 + 1)\bar{k}_1 - \frac{1}{\epsilon}\left(\alpha - \frac{\bar{k}_3}{\epsilon}\right)\bar{k}_2 &= 0 \\
2(k_2 + 1)\bar{k}_2 + \left(\alpha - \frac{\bar{k}_3}{\epsilon}\right)^2 + \tau_p - \alpha^2 &= 0.
\end{aligned}$$

Introducing next $r := k_2 + 1$, $s := \alpha - \frac{\bar{k}_3}{\epsilon}$, $v := k_1 - \beta$, $\tilde{\tau}_p := \alpha^2 - \tau_p$ and $\tilde{\tau}_f := \beta^2 - \tau_f$ we can rewrite the above set of equations as

$$v^2 + 2(r - 1)\frac{\bar{k}_2}{\epsilon^2} - \tilde{\tau}_f = 0 \quad (34)$$

$$\frac{\bar{k}_2}{\epsilon^2}k_3 - \frac{s}{\epsilon}(r - 1) + rv + \beta = 0 \quad (35)$$

$$r^2 - 1 - \frac{2s}{\epsilon}k_3 = 0 \quad (36)$$

$$2v\bar{k}_1 + \frac{\bar{k}_2^2}{\epsilon^2} = 0 \quad (37)$$

$$v\bar{k}_2 + r\bar{k}_1 - \frac{s}{\epsilon}\bar{k}_2 = 0 \quad (38)$$

$$2r\bar{k}_2 + s^2 - \tilde{\tau}_p = 0. \quad (39)$$

Furthermore, $A_{cl} = \begin{bmatrix} -v & -r \\ -\frac{\bar{k}_2}{\epsilon^2} & \frac{s}{\epsilon} \end{bmatrix}$. Notice that both eigenvalues of A_{cl} are stable iff.

$$i) \frac{s}{\epsilon} - v < 0 \text{ and } ii) \frac{sv}{\epsilon} + \frac{r\bar{k}_2}{\epsilon^2} < 0. \quad (40)$$

Now, first consider the case $v = 0$. Then from (37) it follows that $\bar{k}_2 = 0$ and therefore the eigenvalues of A_{cl} are not stable. So this case does not provide an appropriate solution.

Next consider the case $v \neq 0$. Then from (37) it follows that $\bar{k}_1 = -\frac{\bar{k}_2^2}{2v\epsilon^2}$. From (38) it follows next that either $\bar{k}_2 = 0$ or $s = \epsilon v - \frac{r\bar{k}_2}{2v\epsilon}$.

In case $\bar{k}_2 = 0$, it follows that $\bar{k}_1 = 0$ too, and (34-39) reduce to

$$v^2 - \tilde{\tau}_f = 0 \quad (41)$$

$$-\frac{s}{\epsilon}(r - 1) + rv + \beta = 0 \quad (42)$$

$$r^2 - 1 - \frac{2s}{\epsilon}k_3 = 0 \quad (43)$$

$$s^2 - \tilde{\tau}_p = 0. \quad (44)$$

Since the eigenvalues of A_{cl} must be stable it follows that $v = \sqrt{\tilde{\tau}_f}$, $s = -\sqrt{\tilde{\tau}_p}$, $r = -\frac{\beta + \frac{s}{\epsilon}}{v - \frac{s}{\epsilon}}$ and $k_3 = \frac{\epsilon}{2s}(r^2 - 1)$ is an appropriate solution provided the conditions $\tilde{\tau}_f > 0$ and $\tilde{\tau}_p > 0$ hold.

Next, consider the case that $\bar{k}_1 = -\frac{\bar{k}_2^2}{2v\epsilon^2}$ and $s = \epsilon v - \frac{r\bar{k}_2}{2\epsilon v}$. Then, (34-39) reduce to

$$v^2 + 2(r-1)\frac{\bar{k}_2}{\epsilon^2} - \tilde{\tau}_f = 0 \quad (45)$$

$$\frac{\bar{k}_2}{\epsilon^2}k_3 - \frac{s}{\epsilon}(r-1) + rv + \beta = 0 \quad (46)$$

$$r^2 - 1 - \frac{2s}{\epsilon}k_3 = 0 \quad (47)$$

$$2r\bar{k}_2 + s^2 - \tilde{\tau}_p = 0. \quad (48)$$

In case $s = 0$ it follows from (40) that $v > 0$ and $r\bar{k}_2 < 0$, which violates the condition that $0 = s = \epsilon v - \frac{r\bar{k}_2}{2\epsilon v}$. So, we may assume in the following $s \neq 0$.

In case $r = 1$, by (47), necessarily $k_3 = 0$. However, it is easily verified from (45) and (46) that the case $k_3 = 0$ does not provide an appropriate solution too.

So, w.l.o.g., we may assume that both $r \neq 1$ and $s \neq 0$. Now, introduce $w := \epsilon v$. By (47), $k_3 = \frac{\epsilon(r^2-1)}{2s}$. Substitution into (46) gives:

$$\begin{aligned} 0 &= \frac{\bar{k}_2(r^2-1)}{2s} - s(r-1) + rw + \epsilon\beta \\ &= \frac{1}{2s}[\bar{k}_2(r^2-1) - 2s^2(r-1) + 2s(rw + \epsilon\beta)]. \end{aligned}$$

Substitution of, first, s^2 from (48) and, next, $s = w - \frac{r\bar{k}_2}{2w}$ into the above expression yields then

$$\bar{k}_2(4r^2 - (4 + \frac{\epsilon\beta}{w})r - 1) - 2\tilde{\tau}_p(r-1) + 2w(rw + \epsilon\beta) = 0.$$

From (45) it follows that $\bar{k}_2 = \frac{\tilde{\tau}_f\epsilon^2 - w^2}{2(r-1)}$. Substitution of \bar{k}_2 into the above equality and some elementary calculations show then that the next equality must hold

$$(\tilde{\tau}_f\epsilon^2 - w^2)(4wr^2 - (4w + \epsilon\beta)r - w) - 4w(r-1)^2\tilde{\tau}_p + 4(r-1)w^2(rw + \epsilon\beta) = 0.$$

Which can be rewritten as

$$w^3 + \epsilon\beta(5r-4)w^2 + (\tilde{\tau}_f\epsilon^2(4r^2 - 4r - 1) - 4\tilde{\tau}_p(r-1)^2)w - \tilde{\tau}_f\epsilon^3\beta r = 0. \quad (49)$$

Furthermore, substitution of \bar{k}_2 into (48) and some elementary rewriting shows that (48) can be rewritten as

$$\frac{(3r-4)^2}{16(r-1)^2}w^2 + \frac{\tilde{\tau}_f^2\epsilon^4r^2}{16(r-1)^2} \frac{1}{w^2} + \frac{2r(3r-4)\tilde{\tau}_f\epsilon^2}{16(r-1)^2} - \tilde{\tau}_p = 0. \quad (50)$$

So (45-48) has an appropriate solution iff. (49,50) has a solution (r, w) satisfying the conditions i) $\frac{r\bar{k}_2}{2w} > 0$ and ii) $w^2 + \frac{1}{2}r\bar{k}_2 < 0$, where $\bar{k}_2 = \frac{\tilde{\tau}_f\epsilon^2 - w^2}{2(r-1)}$.

Case $\tau_f = \beta^2$

From the above reasoning it follows that the game has an equilibrium iff. (49,50) has a solution (r, w) satisfying the conditions i) and ii). Substitution of $\tau_f = \beta^2$ into this shows that the next conditions should hold

$$w^3 + \epsilon\beta(5r - 4)w^2 - 4\tilde{\tau}_p(r - 1)^2w = 0, \quad (51)$$

$$\frac{(3r - 4)^2}{16(r - 1)^2}w^2 - \tilde{\tau}_p = 0, \quad (52)$$

where (r, w) has the properties that i) $\frac{r\bar{k}_2}{2w} > 0$ and ii) $w^2 + \frac{1}{2}r\bar{k}_2 < 0$, with $\bar{k}_2 = \frac{-w^2}{2(r-1)}$.

From (52) it is clear that this set of equations has a solution only if $\tilde{\tau}_p \geq 0$. Since the conditions i) and ii) imply in particular that $w < 0$ it follows, moreover, that $w = -\sqrt{\tilde{\tau}_p} \frac{|4(r-1)|}{|3r-4|}$, unless $r = \frac{4}{3}$ and $\tilde{\tau}_p = 0$.

In case $r = \frac{4}{3}$, $w^2 + \frac{1}{2}r\bar{k}_2 = 0$, which violates condition ii). So, if $r = \frac{4}{3}$ there is no equilibrium. Recall that $v = 0$ did not provide a solution. Therefore, $w \neq 0$, and it follows from (51) that

$$w^2 + \epsilon\beta(5r - 4)w - 4\tilde{\tau}_p(r - 1)^2 = 0, \text{ with } w = -\sqrt{\tilde{\tau}_p} \frac{|4(r - 1)|}{|3r - 4|}. \quad (53)$$

Moreover, i) and ii) hold iff. $1 < r < \frac{4}{3}$.

Substitution of w^2 into (53) gives after some elementary calculations that w also satisfies

$$w = 12 \frac{\tilde{\tau}_p(r - 1)^2(r - 2)(3r - 2)}{\epsilon\beta(5r - 4)(3r - 4)^2}. \quad (54)$$

Combining (53) and (54) yields then after some elementary manipulations that r solves the equation

$$f(r) := 12\sqrt{\tilde{\tau}_p}(r - 1)(r - 2)(3r - 2) - 4\epsilon\beta(5r - 4)(3r - 4) = 0. \quad (55)$$

Notice that $f(0) < 0$, $f(1) > 0$ and $f(\frac{4}{3}) < 0$. So there is precisely one solution of $f(r) = 0$ in the interval $1 < r < \frac{4}{3}$.

So, summarizing, the game has no equilibrium if $\tilde{\tau}_p \leq 0$ and precisely one equilibrium iff $\tilde{\tau}_p > 0$. In this lastmentioned case the parameters characterizing this equilibrium are (with r as defined above):

- $k_1 = v + \beta = \frac{w}{\epsilon} + \beta = 4\sqrt{\tilde{\tau}_p} \frac{r-1}{\epsilon(3r-4)} + \beta$
- $k_2 = r - 1$
- $k_3 = \frac{\epsilon(r^2-1)}{2s} = \frac{1}{2\sqrt{\tilde{\tau}_p}} \frac{\epsilon(r^2-1)(3r-4)}{5r-4}$
- $\bar{k}_1 = -\frac{\bar{k}_2^2}{2\epsilon^2v} = -8\tilde{\tau}_p^{3/2} \frac{r-1}{\epsilon(3r-4)^3}$
- $\bar{k}_2 = -8\tilde{\tau}_p \frac{r-1}{(3r-4)^2}$
- $\bar{k}_3 = \epsilon(\alpha - s) = \epsilon(\alpha - \sqrt{\tilde{\tau}_p} \frac{5r-4}{3r-4})$.

The equilibrium actions are in this case $u_1 = [k_1 \ k_2]x(t) = [4\sqrt{\tilde{\tau}_p} \frac{r-1}{\epsilon(3r-4)} + \beta \ r - 1]x(t)$ and $u_2 = \frac{-1}{\epsilon}[k_2 \ \bar{k}_3]x(t) = [\frac{8}{\epsilon}\tilde{\tau}_p \frac{r-1}{(3r-4)^2} \ \sqrt{\tilde{\tau}_p} \frac{5r-4}{3r-4} - \alpha]x(t)$, where $x(t)$ satisfies the differential equation $\dot{x}(t) = (\tilde{A} + \tilde{B}_1F_1 + \tilde{B}_2F_2)x(t)$, $x(0) = x_0$.

Notice that if $\epsilon \downarrow 0$, the appropriate solution of $f(r) = 0$ in (55) converges to $r = 1$. The corresponding equilibrium actions converge then to $u_1 = [\frac{-\beta}{3} \ 0]x(t)$ and $u_2 = [\frac{8}{3}\beta\sqrt{\tilde{\tau}_p} - (\sqrt{\tilde{\tau}_p} + \alpha)]x(t)$. The corresponding cost $J_i = x_0^T K_i x_0$ converge to $(J_1, J_2) = (\frac{-\beta}{3}x_{01}^2, \frac{8}{3}\beta\tilde{\tau}_p x_{01}^2)$. Therefore, the revenues for the ranger are $\frac{\beta}{3}f_0^2$ and the cost for the firm $\frac{8}{3}\beta\tilde{\tau}_p f_0^2$.

Feedback Nash equilibria from game considered in (28) if $\epsilon = 0$.

From (29) we have that either $k_2 = 0$ or $k_2 = 2\tilde{\tau}_p(\beta\alpha^2 - k_1)$. This gives rise to potentially three different equilibria.

Equilibrium 1: If $k_2 = 0$ it follows from (29) that $k_1 = \alpha^2(\beta + \sqrt{\tilde{\tau}_f})$, provided $\tilde{\tau}_f > 0$. This yields $[\tilde{F}_1 \ \tilde{F}_2] = [-\alpha(\beta + \sqrt{\tilde{\tau}_f}) \ 0]$; $F_1 = (\beta + \sqrt{\tilde{\tau}_f})[1 \ 0] + \lambda_1[0 \ 1]$ and $F_2 = \lambda_2[0 \ 1]$, $\lambda_2 \neq -\alpha$. The corresponding cost for the ranger is $J_1 = (\sqrt{\tilde{\tau}_f} + \beta)f_0^2$ and for the firm, $J_2 = 0$.

If $k_2 = 2\tilde{\tau}_p(\beta\alpha^2 - k_1)$ we obtain from (29) two potential solutions for k_1 if $\beta^2 + 3\tau_f \geq 0$. That is, $k_1 = \frac{\alpha^2}{3}(\beta - \sqrt{\beta^2 + 3\tau_f})$ and $k_1 = \frac{\alpha^2}{3}(\beta + \sqrt{\beta^2 + 3\tau_f})$.

Equilibrium 2: First consider the case $(k_1, k_2) = (\frac{\alpha^2}{3}(\beta - \sqrt{\beta^2 + 3\tau_f}), \frac{2\alpha^2\tilde{\tau}_p}{3}(2\beta + \sqrt{\beta^2 + 3\tau_f}))$. This yields $[\tilde{F}_1 \ \tilde{F}_2] = [\frac{-\alpha}{3}(\beta - \sqrt{\beta^2 + 3\tau_f}), \frac{2\alpha^2}{3}(2\beta + \sqrt{\beta^2 + 3\tau_f})]$; $F_1 = [0 \ \frac{\beta - \sqrt{\beta^2 + 3\tau_f}}{2(2\beta + \sqrt{\beta^2 + 3\tau_f})}] + \lambda_1[\frac{-2}{3}(2\beta + \sqrt{\beta^2 + 3\tau_f}) \ 1]$ and $F_2 = [0 \ -\alpha] + \lambda_2[\frac{-2}{3}(2\beta + \sqrt{\beta^2 + 3\tau_f}) \ 1]$. The cost for the ranger is $J_1 = \frac{1}{3}(\beta - \sqrt{\beta^2 + 3\tau_f})f_0^2$ and for the firm, $J_2 = \frac{2\tilde{\tau}_p}{3}(2\beta + \sqrt{\beta^2 + 3\tau_f})f_0^2$.

Equilibrium 3: Next consider the case $(k_1, k_2) = (\frac{\alpha^2}{3}(\beta + \sqrt{\beta^2 + 3\tau_f}), \frac{2\alpha^2\tilde{\tau}_p}{3}(2\beta - \sqrt{\beta^2 + 3\tau_f}))$. This yields the equilibrium actions $[\tilde{F}_1 \ \tilde{F}_2] = [\frac{-\alpha}{3}(\beta + \sqrt{\beta^2 + 3\tau_f}), \frac{2\alpha^2}{3}(2\beta - \sqrt{\beta^2 + 3\tau_f})]$ provided $-2\beta + \sqrt{\beta^2 + 3\tau_f} < 0$. The corresponding cost is $J_1 = \frac{1}{3}(\beta + \sqrt{\beta^2 + 3\tau_f})f_0^2$ and for the firm, $J_2 = \frac{2\tilde{\tau}_p}{3}(2\beta - \sqrt{\beta^2 + 3\tau_f})f_0^2$.

Case $\tau_f = \beta^2$

From the above reasoning it follows directly that for this case only equilibrium 2 applies. Substitution of τ_f by β^2 shows that the equilibrium actions become $[\tilde{F}_1 \ \tilde{F}_2] = [\frac{\alpha\beta}{3}, \frac{8\alpha^2\beta}{3}]$ and the cost are $J_1 = -\frac{\beta}{3}f_0^2$ and $J_2 = \frac{8}{3}\beta\tilde{\tau}_p f_0^2$.

Since $(k_1, k_2) = (\frac{-\alpha^2\beta}{3}, \frac{8\alpha^2\beta\tilde{\tau}_p}{3})$ it follows from (30) that

$$F_1 = [0 \ \frac{-1}{8}] + \lambda_1[\frac{-8}{3}\beta \ 1] \text{ and } F_2 = [0 \ -\alpha] + \lambda_2[\frac{-8}{3}\beta \ 1], \text{ where } \lambda_i \in \mathbb{R}. \quad (56)$$

Notice that this equilibrium coincides with the limiting case ($\epsilon \downarrow 0$) we studied before. Furthermore the limiting equilibrium actions are obtained by choosing $\lambda_1 = \frac{1}{8}$ and $\lambda_2 = -\sqrt{\tilde{\tau}_p}$.

Next consider the cost if one uses the equilibrium actions F_i from (56) to control system (24,25). Then the system dynamics are described by

$$\begin{bmatrix} \dot{f}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \beta(1 + \frac{8}{3}\lambda_1) & -(\lambda_1 + \frac{7}{8}) \\ \frac{-8}{3}\frac{\beta\lambda_2}{\epsilon} & \frac{\lambda_2}{\epsilon} \end{bmatrix} \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} =: A_\epsilon \begin{bmatrix} f(t) \\ p(t) \end{bmatrix}, \quad \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} f_0 \\ p_0 \end{bmatrix}.$$

Notice that this system is stable iff $\lambda_2 < 0$ and $\frac{\lambda_2}{\epsilon} + \beta(1 + \frac{8}{3}\lambda_1) < 0$.
Furthermore,

$$\tilde{J}_i = \int_0^\infty [f(t) \ p(t)] W_i \begin{bmatrix} f(t) \\ p(t) \end{bmatrix} dt$$

where $W_1 := \begin{bmatrix} \beta^2(1 - \frac{64}{9}\lambda_1^2) & \frac{\beta\lambda_1}{3}(8\lambda_1 - 1) \\ \frac{\beta\lambda_1}{3}(8\lambda_1 - 1) & -(\lambda_1 - \frac{1}{8})^2 \end{bmatrix}$ and $W_2 := \begin{bmatrix} -\frac{64}{9}\beta^2\lambda_2^2 & \frac{8}{3}\beta\lambda_2(\lambda_2 - \alpha) \\ \frac{8}{3}\beta\lambda_2(\lambda_2 - \alpha) & \tau_p - (\lambda_2 - \alpha)^2 \end{bmatrix}$.

Since A_ϵ is stable, $\tilde{J}_i = [f_0 \ p_0] H_i \begin{bmatrix} f_0 \\ p_0 \end{bmatrix}$, where H_i is the unique solution of the linear matrix equation

$$H_i A_\epsilon + A_\epsilon^T H_i = -W_i.$$

Straightforward calculations show that

$$H_1 = \frac{1}{\epsilon\beta(8\lambda_1 + 3) + 3\lambda_2} \begin{bmatrix} \beta(\lambda_2 + \frac{\epsilon\beta}{6}(8\lambda_1 + 3)(8\lambda_1 - 3)) & \frac{-1}{16}\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 5) \\ \frac{-1}{16}\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 5) & \frac{-1}{128}\frac{\epsilon}{\lambda_2}h_1 \end{bmatrix}$$

and

$$H_2 = \frac{1}{\epsilon\beta(8\lambda_1 + 3) + 3\lambda_2} \begin{bmatrix} \frac{-8}{3}\beta\lambda_2(-4\epsilon\beta\lambda_2 + 3\tilde{\tau}_p) & -\epsilon\beta(4\lambda_2^2 + (8\lambda_1 + 3)\tilde{\tau}_p) \\ -\epsilon\beta(4\lambda_2^2 + (8\lambda_1 + 3)\tilde{\tau}_p) & \frac{-1}{8}\frac{\epsilon}{\lambda_2}h_2 \end{bmatrix}$$

where $h_1 = 4\epsilon\beta(8\lambda_1 + 3)(8\lambda_1 - 9) - 3\lambda_2(8\lambda_1 - 1)^2$ and $h_2 = \epsilon\beta\{(8\lambda_1 + 3)^2\tilde{\tau}_p + 8\lambda_2(2\lambda_2 + \alpha(8\lambda_1 + 3))\} + 12\lambda_2(\tau_p - (\lambda_2 - \alpha)^2)$.

Notice that if $\epsilon \downarrow 0$, $H_1 \rightarrow \begin{bmatrix} \frac{\beta}{3} & 0 \\ 0 & 0 \end{bmatrix}$ and $H_2 \rightarrow \begin{bmatrix} -\frac{8}{3}\beta\tilde{\tau}_p & 0 \\ 0 & 0 \end{bmatrix}$.

References

- [1] Abadir, K.M., & Magnus, J.R., (2005). *Matrix Algebra*, Cambridge University Press, Cambridge.
- [2] Başar T. and Bernhard P., 1995, *H_∞-Optimal Control and Related Minimax Design Problems*, Birkhäuser, Boston.
- [3] Başar T. and Olsder G.J., 1999. *Dynamic Noncooperative Game Theory*, SIAM, Philadelphia.
- [4] Bender, D.J., & Laub, A.J., (1987). The linear-quadratic optimal regulator for descriptor systems. *IEEE TAC*, 32, 672-688.
- [5] Broek W.A. van den, Engwerda J.C. and Schumacher J.M., 2003. Robust equilibria in indefinite linear-quadratic differential games, *Journal of Optimization Theory and Applications*, 119, pp.565-595.
- [6] De Carlo, R.A., & Saeks, R., (1981). *Interconnected Dynamical Systems*, Marcel Dekker, New York.
- [7] Cobb D., (1983). Descriptor variable systems and optimal state regulation. *IEEE TAC*, 28, 601-611.

- [8] Dockner E., Jørgensen S., Long N. van and Sorger G., 2000. *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge.
- [9] Engwerda J.C., 2005. *LQ Dynamic Optimization and Differential Games*, John Wiley & Sons.
- [10] Engwerda, J.C., & Salmah, (2009). The open-loop linear quadratic differential game for index one descriptor systems. *Automatica*, 45, 585-592.
- [11] Engwerda, J.C., Salmah, & Wijayanti, I.E., (2009). The open-loop discounted linear quadratic differential game for regular higher order index descriptor systems. *International Journal of Control*, 82, 2365-2374.
- [12] Engwerda, J.C., Salmah and Wijayanti, I.E., 2009. The (multi-player) optimal linear quadratic feedback state regulator problem for index one descriptor systems. *Proc. European Control Conference*, Budapest.
- [13] Engwerda, J.C. and Salmah, 2009. Necessary and Sufficient Conditions for Feedback Nash Equilibria for the Affine-Quadratic Differential Game, Internal paper, Tilburg University.
- [14] Gantmacher, F., (1959). *Theory of Matrices*, vol. I,II. Chelsea: New York.
- [15] Gardner, B.F., 1977. Zero-sum Nash strategy for systems with fast and slow modes. *Proceedings 15th Allerton conference on Communication, Computers and Control*, Univerisity of Illinois, Urbana, 96-103.
- [16] Gardner, B.F., and Cruz, J.B., 1978. Well-posedness of singularly perturbed Nash games. *Journal of The Franklin Institute*, 306, 355-374.
- [17] Geerts, T., (1993). Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case. *Linear Algebra and its Applications*, 181, 111-130.
- [18] Glizer V.Y., 2000. Asymptotic solution of zero-sum linear-quadratic differential game with cheap control for minimizer. *Nonlinear Differential Equations and Applications*, Vol.7, pp.231-258.
- [19] Grass, D., Caulkins, J.P., Feichtinger, G., Tragler, G., and Behrens D.A., 2008. *Optimal Control of Nonlinear Processes: with applications in drugs*. Springer-Verlag, Berlin.
- [20] Grimsrud, K.M., & Huffaker R.,(2006). Solving multidimensional bioeconomic problems with singular-perturbation reduction methods: Application to managing pest resistance to pesticidal crops. *Journal of Environmental Economics and Management*, 51, 336-353.
- [21] Hodge, A.M., & Newcomb, R.W., (2002). Semistate theory and analogue VLSI design. *IEEE Circuit and System Magazine*, 2, 30-51.
- [22] Jørgensen S. and Zaccour G., 2003, *Differential Games in Marketing*, Kluwer, Deventer.
- [23] Kailath T., 1980. *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J.

- [24] Kalogeropoulos, G., & Arvanitis, K.G., (1998). A matrix-pencil-based interpretation of inconsistent initial conditions and system properties of generalized state-space. *IMA Journal of Mathematical Control & Information*, 15, 73-91.
- [25] Kautsky, J., Nichols, N.K., & Chu, E.K.-W, (1989). Robust pole assignment in singular control systems, *Linear Algebra and its Applications*, 121, 9-37.
- [26] Kun G., 2001. *Stabilizability, Controllability, and Optimal Strategies of Linear and Nonlinear Dynamical Games*, PhD. Thesis, RWTH-Aachen, Germany.
- [27] Lewis, F.L., (1986). A survey of linear singular systems. *Circuits, Systems & Signal Processing*, 5, 3-36.
- [28] Lukes D.L., 1971. Equilibrium feedback control in linear games with quadratic costs, *SIAM J. Contr. Optim.*, Vol.9, no.2, pp.234-252.
- [29] Mizukami, K. and Xu H. (1992). Closed-loop Stackelberg strategies for linear-quadratic descriptor systems. *JOTA*, 74, 151-170.
- [30] Pandolfi, L., (1981). On the regulator problem for linear degenerate control systems. *JOTA*, 33, 241-254.
- [31] Papavassilopoulos G.P., Medanic J. and Cruz J., 1979. On the existence of Nash strategies and solutions to coupled Riccati equations in linear-quadratic games, *Journal of Optimization Theory and Applications*, Vol. 28, pp.49-75.
- [32] Plasmans J., Engwerda J., van Aarle B., Di Bartolomeo B. and Michalak T., 2006. Dynamic Modeling of Monetary and Fiscal Cooperation Among Nations, Series: Dynamic Modeling and Econometrics in Economics and Finance, Vol. 8., Springer-Verlag, Berlin.
- [33] Starr A.W. and Ho Y.C., 1969. Nonzero-sum differential games, *Journal of Optimization Theory and Applications*, Vol.3, pp.184-206.
- [34] Xu, H., & Mizukami, K., (1994). On the Isaacs equation of differential games for descriptor systems. *JOTA*, 83, 405-419.
- [35] Xu, H., & Mizukami, K., (1996). Linear feedback closed-loop Stackelberg strategies for descriptor systems with multilevel hierarchy. *JOTA*, 88, 209-231.