

Dynamic Adjustment of Supply Constrained Disequilibria to Walrasian Equilibrium

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Abstract

It is well-known that the classical Walrasian tatonnement process of adjusting prices does not need to converge. In several papers Weddepohl has shown cyclic or even chaotic behavior in discrete time tatonnement processes. Starting from a trivial non-trade disequilibrium with complete rationing on the supplies at a given vector of initial prices, a quantity and price adjustment process is proposed, converging always to a Walrasian equilibrium. At any price and rationing system generated by the process all markets clear, only supply rationing occurs, in the short run only rationing schemes are adjusted and in the long run both prices and rations are adjusted. Furthermore, prices can only take values equal or above their initial values, while supply rationing only may occur when the corresponding price is on its initial value.

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1 Introduction

It is well-known that the classical Walrasian tatonnement process of adjusting prices in an economy has a number of drawbacks. First, it is not guaranteed that the price adjustment process converges to Walrasian equilibrium prices. In Scarf (1960) examples of economies have been given for which the process fails to converge. Other studies have shown that the discrete time process does not need to converge, even if the economy satisfies conditions that guarantee convergency of the continuous time adjustment, see e.g. Bela and Majumdar (1992) and Weddepohl (1994, 1995). In fact, Saari (1985) has shown that any process based on a finite amount of local information may fail to converge. This lack of convergence has been solved by Smale (1976), van der Laan and Talman (1987a,b) and Kamiya (1990), where price adjustment processes have been presented in which the adjustment is based on local information about the excess demands combined with global information on the starting price system.

Economists since Walras have been focussing on the convergency properties of price adjustment processes. However, stimulated by the computer making simulations easily executable, more recently a shift of interests to the economic interpretation of the adjustment process has taken place. This brings us to the second drawback of the tatonnement process, namely that demand and supply are not in equilibrium as long as the process has not achieved Walrasian equilibrium prices. So, trade must be excluded until an equilibrium is reached. Although the process can be seen as taking place in a sequence economy, where in each period the excesses or shortages determine prices for the next period and trade occurs at disequilibrium prices, see e.g. Weddepohl (1997), also such an interpretation is unsatisfactory because it leaves the way in which the trade is determined unspecified. A possible way out has been provided by Veendorp (1975), who noticed that the relevant market signals for an adjustment process are based on the effective demands associated with an equilibrium under rationing constraints, and who specified a process generating a sequence of rationed equilibria in which prices are adjusted according to effective excess demands and supplies. However, also such a process does not necessarily converge to a Walrasian equilibrium price system and even chaotic behavior may be expected, see Day and Pianigiani (1991). Using computer simulations the possibility of chaotic behavior has been confirmed in several studies, see e.g. Böhm (1993) and Weddepohl (1997).

Weddepohl (1999) argues that both the classical Walrasian tatonnement process and price adjustments based on effective demands with in each period trade under some rationing scheme at disequilibrium prices are not very satisfactory; the Walrasian tatonnement because of the drawbacks already mentioned above, and the latter one because the determination of the rations is at least as complicated as solving for an equilibrium price. Therefore he takes a more realistic approach by assuming that markets are open

consecutively, not simultaneously. Applying such a process to a simple model with two goods, labor and a produced good, and money as numeraire commodity, shows that again cyclical movements and even chaos may appear, albeit that for moderate step sizes long cyclic movements similar to business cycles in reality can occur. This illustrates nicely that not only the convergency properties, but also the non-convergent behavior of adjustment processes is a very interesting topic of research, because it may help us to obtain a better understanding in real phenomena.

In this contribution in honor to Weddepohl's work on 'Equilibrium, Disequilibrium and Dynamics', we focus on convergency and consider an alternative approach to solve the disequilibrium problem as long as the price adjustment process has not reached yet a Walrasian equilibrium price system. This approach has been introduced in Herings et al. (1997), in which an adjustment process is proposed satisfying that at any price system generated by the process all markets clear endogenously by rationing constraints. In this process the problem of finding the market clearing rations does not appear. Instead, any state generated by the process gives a feasible price system and a rationing scheme, i.e. at these prices and rations the corresponding total constrained excess demand is equal to zero on each market. Moreover, convergency of the process to an unconstrained Walrasian equilibrium state is guaranteed under standard assumptions with respect to the economy.

For an economy with restrictions on the set of feasible prices, an equilibrium concept has been introduced by Drèze (1975). A Drèze equilibrium is given by a feasible price vector and a rationing scheme on net demands and net supplies, such that (i) the constrained demands of the agents are feasible, i.e. all markets clear, (ii) on each market rationing may affect either supply or demand, but not both sides of the market simultaneously and (iii) rationing of a commodity only occurs when its price restriction is binding. In the price and quantity adjustment process considered in Herings et al. (1997) the existence of a connected set of Drèze equilibria leading to a Walrasian equilibrium is shown, such that at any point in time rationing on the net demands may occur, while the net supplies are never rationed. That process was defined to incorporate features of Eastern European economies, where rationing on demand used to be the rule rather than the exception. On the other hand, in van der Laan (1980) it is argued that 'constraints on the supply side can often more easily be realized than constraints on the demand side', while Kurz (1982) argues that 'demand rations rarely occur in market economies while resource unemployments are very common'. This motivated these authors to consider Drèze equilibria with only rationing on the net supplies. In van der Laan (1980, 1982) and Kurz (1982) the existence of a Drèze equilibrium, satisfying that only rationing on the net supplies occurs and at

least one commodity is not rationed at all, has been proven. Such an equilibrium is also called an unemployment equilibrium. In Dehez and Drèze (1984) and van der Laan (1984) it has been shown that in economies with money as numeraire commodity, there exists a supply constrained equilibrium without rationing on money, when the set of feasible prices allows for enough flexibility of the price level of the non-numeraire commodities. Weddepohl (1987) discusses extensively linkages between prices through index functions and shows that a supply constrained equilibrium in which at least one commodity is not rationed exists if the system of indexes is non-circular, i.e. no commodity is directly or indirectly indexed by itself. In these papers it is also argued that the equilibrium condition that rationing of a commodity is not allowed unless its price restriction is binding, can not be maintained when prices are tied to each other through index functions.

In this paper we propose an adjustment process satisfying that at any price system generated by the process all markets clear endogenously by rationing on the net supplies only. The starting point of the process is a historically given price system. At this system of fixed prices, the process starts with a trivial non-trade disequilibrium in which all markets clear by imposing complete rationing on the supplies of all commodities, and then subsequently adjusts rations and prices in such a way that at any moment in time all markets clear by rationing on the net supplies only, while there is no rationing on the demand side of the markets. In the beginning of the process only the rationing schemes are simultaneously adjusted until one of the commodities is no longer rationed and thus an unemployment equilibrium is reached. This part of the process in which the rations are reduced, can be seen as short term adjustment of the rations at a given fixed initial price system. Continuing from this unemployment equilibrium, the supply rations on the other commodities can not be reduced further without either imposing demand rationing on the unrationed commodity or allowing that its price adjusts upwards. So, from this short term market clearing disequilibrium at given fixed prices, the prices of the unrationed commodities are allowed to adjust upwards. Keeping all markets in equilibrium, the process therefore continues with simultaneously adjusting the prices of the unrationed commodities above their initial level and adjusting the rations on the supplies of the other commodities. This long term process continues until none of the commodities is rationed anymore and thus a Walrasian equilibrium has been obtained. Summarizing, a process is proposed satisfying that at any price and rationing system generated by the process (i) all markets clear, so that trade can take always place, (ii) only supply rationing occurs, (iii) in the short run only rationing schemes are adjusted, (iv) in the long run both prices and rations are adjusted, (v) prices are never below their initial value, (vi) supply rationing only occurs when the corresponding price is on its initial value, and (vii) eventually a Walrasian equilibrium is reached.

In Herings et al. (1997) the existence of a path of approximate market clearing states with only demand rationing connecting the initial state with a Walrasian equilibrium has been shown by using simplicial techniques. Using the existence of such a path, they show that there is a connected set of market clearing disequilibria with demand rations containing both the trivial starting state and the Walrasian equilibrium state. In this paper we will show the existence of a connected set of supply constrained market clearing disequilibria directly by using Browder's fixed point theorem. Again simplicial path following techniques can be used to obtain an implementable algorithm for finding a path of approximate market clearing states.

The paper is organized as follows. In Section 2 we introduce the model and define the concept of a supply constrained disequilibrium. In Section 3 we discuss the equilibrating mechanism by constructing a reduced total excess demand function satisfying that any zero point of this function yields a supply constrained disequilibrium. In Section 4 we prove the existence of a connected set of supply constrained disequilibria. The dynamics of the adjustment process is discussed in Section 5.

2 The model

We consider a pure exchange economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$. In this economy there are m consumers, indexed $i = 1, \dots, m$, and n commodities, indexed $j = 1, \dots, n$. For k a positive integer, we denote $I_k = \{1, \dots, k\}$. Each consumer $i \in I_m$ is characterized by a consumption set X^i , a preference preordering \succeq^i on X^i , and a vector of initial endowments w^i . The vector w is defined by $w = \sum_{i \in I_m} w^i$. We consider the situation that the economy \mathcal{E} is faced with a fixed price \underline{p} in the short run. In the long run prices are allowed to increase and are therefore restricted to the set $P = \{p \in \mathbb{R}^n \mid p \geq \underline{p}\}$. Since prices are regulated and restricted, quantity constraints have to accompany the price mechanism to equilibrate the markets. The following standard assumptions X, U and W with respect to the economy \mathcal{E} are made.

Assumption X

For every consumer $i \in I_m$, the consumption set X^i is a closed and convex subset of \mathbb{R}_+^n and $X^i + \mathbb{R}_+^n \subset X^i$.

Assumption U

For every consumer $i \in I_m$, the preference preordering \succeq^i on X^i is complete, transitive,

reflexive, continuous, strongly monotonic, and strictly convex.³

Assumption W

For every consumer $i \in I_m$, the vector of initial endowments w^i belongs to the interior of X^i .

Notice that the assumption of strict convexity allows us to work with demand functions instead of demand correspondences. We further assume that the short run price vector p is a strictly positive vector. Following van der Laan (1980) and Kurz (1982) we only allow for quantity rationing on the supply side of the market. Given an arbitrary price system $p \in \mathbb{R}_{++}^n$ and rationing scheme on supply $\ell \in \mathbb{R}_+^n$, the constrained budget set of consumer $i \in I_m$ is given by

$$B^i(p, \ell) = \{x^i \in X^i \mid p \cdot x^i \leq p \cdot w^i \text{ and } w_k^i - x_k^i \leq \ell_k w_k, \forall k \in I_n\}.$$

Observe that the number ℓ_k is the fraction of the total endowment of commodity k each consumer is allowed to supply maximally. The corresponding constrained demand $d^i(p, \ell)$ of consumer i is defined as the best element for \succeq^i in $B^i(p, \ell)$. Because of the strict convexity and strong monotonicity assumptions, this element is unique and lies on the budget hyperplane, i.e. $p \cdot d^i(p, \ell) = p \cdot w^i$. A supply constrained disequilibrium is defined as follows.

Definition 2.1 (Supply Constrained Disequilibrium)

A Supply Constrained Disequilibrium for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ is a price system $p^* \in P$, a rationing scheme $\ell^* \in \mathbb{R}_+^n$, and, for every consumer $i \in I_m$, a consumption bundle $x^{*i} \in X^i$ such that

- (i) for all $i \in I_m$, $x^{*i} = d^i(p^*, \ell^*)$;
- (ii) $\sum_{i=1}^m x^{*i} = w$;
- (iii) for all $k \in I_n$, if $\exists i \in I_m$ such that $w_k^i - x_k^{*i} = \ell_k^* w_k$, then $p_k^* = \underline{p}_k$.

In the definition, the rationing schemes on supply are assumed to be uniform, i.e. the same for each consumer. This assumption can be easily relaxed. Condition (i) requires that the consumption of each consumer equals his constrained demand, while condition (ii) is the market clearing condition. Condition (iii) precludes supply rationing on the market of some commodity as long as its price is not on its lower bound. Observe that in a supply

³A preference preordering \succeq^i is said to be strongly monotonic if $\bar{x}^i, \hat{x}^i \in X^i$, $\bar{x}^i \leq \hat{x}^i$, and $\bar{x}^i \neq \hat{x}^i$ implies $\hat{x}^i \succ^i \bar{x}^i$. A preference preordering \succeq^i is said to be strictly convex when for any pair $\bar{x}^i, \hat{x}^i \in X^i$, such that $\bar{x}^i \neq \hat{x}^i$, $\bar{x}^i \sim^i \hat{x}^i$, holds $\lambda \bar{x}^i + (1 - \lambda)\hat{x}^i \succ^i \bar{x}^i$ for $\lambda \in (0, 1)$.

constrained disequilibrium all markets clear and so trade is possible. However, trade takes place against disequilibrium prices as long as at least one of the rations is binding. In case for every commodity $k \in I_n$ it holds that $w_k^i - x_k^{*i} < \ell_k^* w_k$ for all $i \in I_m$, no supply constraint is binding and p^* and x^{*i} , $i \in I_m$, form a Walrasian equilibrium. Further, the definition implies the existence of a trivial disequilibrium with allocation $x^{*i} = w^i$, for all $i \in I_m$, by setting $p = \underline{p}$ and $\ell = 0^n$. Notice that $B^i(\underline{p}, 0^n) = \{w^i\}$ for all $i \in I_m$. We will show that there exists a connected set of supply constrained disequilibria containing both the trivial disequilibrium with complete supply rationing and a Walrasian equilibrium without any rationing.

3 The equilibrating mechanism

To prove the existence of a connected set of SCDs, we first focus on the equilibrating mechanism to find price systems and rationing schemes satisfying the equilibrium conditions of Definition 2.1. To formalize the equilibrating mechanism, we introduce a set $Q \subset \mathbb{R}^n$ containing the set P in its interior and define for every $q \in Q$ a price $p(q) \in P$ and a rationing scheme $\ell(q) \in \mathbb{R}_+^n$. The set Q is taken to be equal to

$$Q = \{q \in \mathbb{R}^n \mid q \geq \underline{p} - e\},$$

where e is the n -vector of ones. For $q \in Q$, the price $p(q) \in P$ is defined to be the orthogonal projection of q on P , i.e.

$$p_k(q) = \max\{q_k, \underline{p}_k\}, \quad k \in I_n,$$

and the rationing scheme $\ell(q) \in \mathbb{R}_+^n$ is defined by

$$\ell_k(q) = 1 - \underline{p}_k + q_k, \quad k \in I_n.$$

Observe that $\ell_k(q) = 0$ and $p_k(q) = \underline{p}_k$ if $q_k = \underline{p}_k - 1$ and that $\ell_k(q) \geq 1$ and $p_k(q) = q_k$ if $q_k \geq \underline{p}_k$. Clearly, both $p(q)$ and $\ell(q)$ are continuous in q . For any $q \in Q$ we now define for every consumer $i \in I_m$ his reduced budget set $B^i(q)$ by

$$B^i(q) = B^i(p(q), \ell(q)).$$

Finally, for any consumer $i \in I_m$ we define his reduced excess demand correspondence $d^i: Q \rightarrow \mathbb{R}^n$ by

$$d^i(q) = \{x^i \in B^i(q) \mid x^i \succeq^i y^i, \text{ for all } y^i \in B^i(q)\}.$$

Because $\underline{p}_k > 0$ for all k , we have that $p(q)$ is strictly positive for all $q \in Q$. According to Herings (1996) it follows that B^i is a continuous correspondence at any $q \in Q$. With

Assumption U it then follows that d^i is a continuous function and so is the reduced excess demand function $z: Q \rightarrow \mathbb{R}^n$ defined by

$$z(q) = \sum_{i \in I_m} d^i(q) - w.$$

Since for any $i \in I_m$ the budget constraint $p(q) \cdot d^i(q) \leq p(q) \cdot w^i$ is always satisfied with equality, Walras' law holds, i.e. $p(q) \cdot z(q) = 0$ for all $q \in Q$. Moreover, for any $j \in I_n$, we have that $q_j = \underline{p}_j - 1$ implies $\ell_j(q) = 0$ and therefore $z_j(q) \geq 0$. So, we have the following lemma.

Lemma 3.1 *Under Assumptions X, U and W, the reduced excess demand function $z: Q \rightarrow \mathbb{R}^n$ satisfies the following properties:*

- i) z is continuous on Q ;
- ii) for all $q \in Q$ it holds that $p(q) \cdot z(q) = 0$ (Walras' law);
- iii) $q_j = \underline{p}_j - 1$ implies $z_j(q) \geq 0$.

The next theorem shows that any zero point of z induces an SCD.

Theorem 3.2 *Let q^* be a zero point of z in Q , i.e. $z(q^*) = 0^n$. Then the price system $p^* = p(q^*) \in P$, the rationing scheme $\ell^* = \ell(q^*) \in \mathbb{R}_+^n$, and for each $i \in I_m$, the bundle $x^{*i} \in X^i$ given by $x^{*i} = d^i(q^*)$, form a supply constrained disequilibrium.*

Proof.

We have to show that the three conditions of Definition 2.1 are satisfied. Clearly the conditions (i) and (ii) hold by construction of the reduced demand functions. For any $i \in I_m$ and $k \in I_n$ it holds that $x_k^{*i} \geq 0$ and $w_k^i < w_k$ and thus $w_k^i - x_k^{*i} < w_k$. So, if for some $i \in I_n$ it holds that $w_k^i - x_k^{*i} = \ell_k(q^*)w_k$, we must have $\ell_k(q^*) < 1$, and therefore $q_k^* < p_k(q^*)$, which implies $p_k(q^*) = \underline{p}_k$. This proves condition (iii) of Definition 2.1. Q.E.D.

When q^* is a zero point of z satisfying $q^* \geq \underline{p}$, i.e. $q^* \in P$, then $p(q^*) = q^*$ and $\ell(q^*) \geq e$ and hence for every $k \in I_n$ it holds that $w_k^i - x_k^{*i} < \ell_k(q^*)w_k$ for all $i \in I_m$, which implies that q^* induces a Walrasian equilibrium. On the other hand, the point $q^0 = \underline{p} - e$ induces the trivial non-trade disequilibrium with complete rationing on all supplies.

In the next section we show that there exists a connected set of zero points of the reduced excess demand function z containing both q^0 , inducing the trivial disequilibrium, and some $q^* \in P$, inducing a Walrasian equilibrium. This implies that there exists a connected set of supply constrained disequilibria leading from the trivial disequilibrium with complete supply rationing to a Walrasian equilibrium without rationing.

4 Connected set of disequilibria

To show the existence of a connected set of zeros of the reduced excess demand function z containing both $q^0 = \underline{p} - e$ and a vector q^* in P , let $s_0 = \sum_{k=1}^n q_k^0$ and define for $s \geq s_0$ the set $Q(s)$ by

$$Q(s) = \{q \in Q \mid \sum_{k=1}^n q_k \leq s\}.$$

For any $s > s_0$, let $\overline{Q}(s) = \{q \in Q(s) \mid \sum_{k=1}^n q_k = s\}$ be the upper boundary of $Q(s)$ and let $\underline{Q}(s) = \{q \in Q(s) \mid q_k = \underline{p}_k - 1 \text{ for some } k \in I_n\}$ be the lower boundary of $Q(s)$. The next theorem says that for any $s > s_0$ there exists a connected set of zero points of z in $Q(s)$ containing both q^0 and a point in $\overline{Q}(s)$.

Theorem 4.1 *Let $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ be an economy satisfying Assumptions X , U and W . Then for any $s > s_0$ there exists a connected set $C(s)$ of zero points of the reduced excess demand function z in $Q(s)$ such that $q^0 \in C(s)$ and $C(s) \cap \overline{Q}(s) \neq \emptyset$.*

Proof.

Take any $s > s_0$. For some $M > s_0$, define X_0 by

$$X_0 = \{y \in \mathbb{R}^n \mid \sum_{k=1}^n y_k = s_0, \max_{k \in I_n} y_k \leq M\}$$

and, for $0 < \alpha \leq 1$, define X_α by

$$X_\alpha = \{x \in \mathbb{R}^n \mid x = y + \frac{\alpha}{n}(s - s_0)e, y \in X_0\}.$$

Clearly, for $x \in X_\alpha$, it holds that $\sum_{k=1}^n x_k = \sum_{k=1}^n y_k + \alpha(s - s_0) = s_0 + \alpha(s - s_0) = (1 - \alpha)s_0 + \alpha s$. Further, define $X = \cup_{\alpha \in [0,1]} X_\alpha$ and take M so large that $Q(s) \subset X$ and X_1 contains $\overline{Q}(s)$ in its relative interior. Notice that $Q(s) \cap X_0 = \{q^0\}$ and $Q(s) \cap X_1 = \overline{Q}(s)$. Let $Y = \text{aff}(X_0)$ be the affine hull of X_0 , i.e. $Y = \{y \in \mathbb{R}^n \mid \sum_{k=1}^n y_k = s_0\}$, and for $x \in \mathbb{R}^n$ let $\tau(x)$ be the projection of x on Y , i.e. $\tau(x)$ is the unique point in Y satisfying $\tau(x) = x - \lambda e$ for some $\lambda \in \mathbb{R}$. Next, let the set Y_0 be defined by

$$Y_0 = \{y \in Y \mid y = \tau(q + z(q)), q \in Q(s)\} \cup \{y \in Y \mid y = \tau(p(q)), q \in \underline{Q}(s)\}.$$

Since the set $Q(s)$ is compact and the functions z , p and τ are continuous, it follows that Y_0 is a bounded subset of Y , and thus M can be taken so large that X_0 contains Y_0 in its relative interior.

For any $y \in X_0$ and $\alpha \in [0, 1]$, let $y^\alpha \in X_\alpha$ be given by $y^\alpha = y + \frac{\alpha}{n}(s - s_0)e$. Then the point-to-set mapping $\varphi: X_0 \times [0, 1] \rightarrow X_0$ is defined by

$$\varphi(y, \alpha) = \begin{cases} \{\tau(y^\alpha + z(y^\alpha))\} & \text{if } y^\alpha \in Q(s) \setminus \underline{Q}(s), \\ \text{Conv}(\{\tau(y^\alpha + z(y^\alpha))\} \cup \{\tau(p(y^\alpha))\}) & \text{if } y^\alpha \in \underline{Q}(s), \\ \{\tau(p(q(y^\alpha)))\} & \text{if } y^\alpha \in X \setminus Q(s), \end{cases}$$

where $\text{Conv}(\cdot)$ denotes the convex hull of a set and $q(x)$ is the orthogonal projection of a point $x \in \mathbb{R}^n$ on $Q(s)$. Clearly, X_0 is nonempty, convex and compact, φ is upper semi-continuous, and for every $(y, \alpha) \in X_0 \times [0, 1]$ it holds that $\varphi(y, \alpha)$ is compact, convex and nonempty. According to Browder's theorem, see Browder (1960), there exists a connected set $\overline{C}(s)$ of fixed points of φ in $X_0 \times [0, 1]$ such that $\overline{C}(s) \cap (X_0 \times \{0\}) \neq \emptyset$ and $\overline{C}(s) \cap (X_0 \times \{1\}) \neq \emptyset$, i.e. there exists a connected set $\overline{C}(s)$ in $X_0 \times [0, 1]$ satisfying

$$y \in \varphi(y, \alpha), \text{ for all } (y, \alpha) \in \overline{C}(s)$$

and containing a point $(\tilde{y}, 0) \in X_0 \times \{0\}$ and a point $(\hat{y}, 1) \in X_0 \times \{1\}$. Hence, the set $C(s)$ in X defined by

$$C(s) = \{x \in X | x = y^\alpha, (y, \alpha) \in \overline{C}(s)\}$$

is a connected set in X satisfying $C(s) \cap X_0 \neq \emptyset$ and $C(s) \cap X_1 \neq \emptyset$. It remains to show that every element of $C(s)$ lies in $Q(s)$ and is a zero point of z . Since $Q(s) \cap X_0 = \{q^0\}$ and $Q(s) \cap X_1 = \overline{Q}(s)$, it then follows that $\tilde{y}^0 = q^0$ and $\hat{y}^1 \in \overline{Q}(s)$, so that $C(s)$ is a connected set in $Q(s)$ of zero points of z satisfying $q^0 \in C(s)$ and $C(s) \cap \overline{Q}(s) \neq \emptyset$.

Take any $y^\alpha \in C(s)$. First, suppose $y^\alpha \in X \setminus Q(s)$. By definition of φ it then follows that

$$y = \tau(p(q(y^\alpha))) = p(q(y^\alpha)) - \lambda e$$

for some $\lambda \in \mathbb{R}$. On the other hand, $y = y^\alpha - \frac{\alpha}{n}(s - s_0)e$, implying that

$$y^\alpha - \frac{\alpha}{n}(s - s_0)e = p(q(y^\alpha)) - \lambda e$$

so that $y^\alpha - p(q(y^\alpha)) = \beta e$ with $\beta = \frac{\alpha}{n}(s - s_0)e - \lambda$. However, since $y^\alpha \notin Q(s)$, there must be a component k with $y_k^\alpha < \underline{p}_k - 1$ and a component h with $y_h^\alpha \geq \underline{p}_h - 1$. Taking first the projection $q(y^\alpha)$ of y^α on Q and then the projection $p(q(y^\alpha))$ of $q(y^\alpha)$ on P implies that $y_k^\alpha - p_k(q(y^\alpha)) < -1 \leq y_h^\alpha - p_h(q(y^\alpha))$, yielding a contradiction. Consequently, $y^\alpha \in Q(s)$ for every $y^\alpha \in C(s)$.

Now, suppose $y^\alpha \in Q(s) \setminus \underline{Q}(s)$. Then

$$y^\alpha - \frac{\alpha}{n}(s - s_0)e = y = \tau(y^\alpha + z(y^\alpha)).$$

Since $\tau(y^\alpha + z(y^\alpha)) = y^\alpha + z(y^\alpha) - \lambda e$ for some $\lambda \in \mathbb{R}$ it follows that

$$z(y^\alpha) = (\lambda - \frac{\alpha}{n}(s - s_0))e.$$

Because of Walras' law, we have

$$p(y^\alpha) \cdot z(y^\alpha) = (\lambda - \frac{\alpha}{n}(s - s_0))p(y^\alpha) \cdot e = 0.$$

Hence, since $p(y^\alpha) \cdot e > 0$, we obtain that $\lambda = \frac{\alpha}{n}(s - s_0)$ and thus $z(y^\alpha) = 0^n$, showing that y^α is a zero point of z .

Finally, suppose $y^\alpha \in \underline{Q}(s)$. Then there exists $\beta_1 \geq 0$ and $\beta_2 \geq 0$ with $\beta_1 + \beta_2 = 1$ such that

$$\beta_1 \tau(y^\alpha + z(y^\alpha)) + \beta_2 \tau(p(y^\alpha)) = y = y^\alpha - \frac{\alpha}{n}(s - s_0)e.$$

Using that $\tau(x)$ is the projection of x on Y , it follows that this can be rewritten as

$$\beta_1 y^\alpha + \beta_1 z(y^\alpha) + \beta_2 p(y^\alpha) = y^\alpha - \mu e$$

for some $\mu \in \mathbb{R}$. Since $\beta_1 + \beta_2 = 1$ we obtain

$$\beta_1 z(y^\alpha) = \beta_2 (y^\alpha - p(y^\alpha)) - \mu e.$$

When $\beta_1 = 0$, we have $p(y^\alpha) = y^\alpha - \mu e$, which implies that $y^\alpha = q^0$, yielding the trivial equilibrium. When $\beta_1 > 0$, we obtain

$$z(y^\alpha) = \frac{\beta_2}{\beta_1} (y^\alpha - p(y^\alpha)) - \frac{\mu}{\beta_1} e.$$

Since $y^\alpha \in \underline{Q}(s)$, we have that $y_k^\alpha - p_k(y^\alpha) = -1$ for some $k \in I_n$ and $y_h^\alpha - p_h(y^\alpha) \geq -1$ for all $h \neq k$. Hence,

$$z_k(y^\alpha) \leq z_h(y^\alpha) \text{ for all } h \in I_n.$$

On the other hand, $\ell_k(y^\alpha) = 0$ and thus $z_k(y^\alpha) \geq 0$. Therefore, $z_h(y^\alpha) \geq 0$ for all $h \in I_n$, and so $z(y^\alpha) = 0^n$ due to Walras' law. Q.E.D.

The theorem guarantees that for any $s > s_0$ there is a connected set of supply constrained disequilibria connecting the trivial disequilibrium with a supply constrained disequilibrium induced by a point $\bar{q}(s)$ in $\bar{Q}(s)$. The next theorem shows that when s is large enough $\bar{q}(s)$ induces a Walrasian equilibrium.

Theorem 4.2 *Let $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ be an economy satisfying Assumptions X, U, W. Then there exists an $\bar{s} > s_0$ such that for all $q \in Q$ with $\sum_{k=1}^n q_k \geq \bar{s}$ it holds that $z(q) \neq 0^n$ when $q \notin P$.*

Proof.

Suppose, without loss of generality, that there exists a commodity $j \in I_n$ and a sequence $(q^r)_{r \in \mathbb{N}}$ in Q , satisfying $\sum_{h=1}^n q_h^r \geq r$, $z(q^r) = 0^n$ and $q_j^r < \underline{p}_j$ for all $r \in \mathbb{N}$. For any $r \in \mathbb{N}$, consider the vector $p^r = p(q^r) / \sum_h p_h(q^r)$. Clearly, we have

$$d^i(p^r, \ell(q^r)) = d^i(q^r), \quad i \in I_m.$$

Since $z(q^r) = 0^n$ and thus $0^n \leq d^i(q^r) \leq w$, we may restrict the budget set of any consumer i to the set of consumption bundles with upper bound $w^i + w$. Now, denote $\ell^r = \ell(q^r)$ and $d^{i,r} = d^i(q^r)$ for all $i \in I_m$, $r \in \mathbb{N}$. Then $(p^r, \ell^r, d^{1,r}, \dots, d^{m,r})_{r \in \mathbb{N}}$ is a sequence in a compact set. Hence there is a subsequence converging to some $(\bar{p}, \bar{\ell}, \bar{d}^1, \dots, \bar{d}^m)$ satisfying $\sum_{i=1}^m \bar{d}^i = w$. Moreover, since for all $r \in \mathbb{N}$, $\sum_{k=1}^n p_k^r = 1$ and $p_j(q^r) = \underline{p}_j$, we also have that $\bar{p}_j = 0$ and there exists some $h \neq j$ with $\bar{p}_h > 0$. This latter property implies that $q_h^r > \underline{p}_h$ for sufficiently large r and hence $\bar{\ell}_h \geq 1$. Following Drèze (1975), this implies that the demand functions are continuous at $(\bar{p}, \bar{\ell})$. Consequently, $\bar{d}^i = d^i(\bar{p}, \bar{\ell})$ for all $i \in I_m$. Since $\bar{p}_j = 0$ and $\bar{\ell}_k = 1$ for some $k \neq j$, it follows from the monotonicity of the preferences that $\bar{d}_j^i = w_j^i + w_j$ for all $i \in I_m$, and thus $\sum_{i=1}^m \bar{d}_j^i > w_j$, which contradicts $z_j(q^r) = 0$ for all $r \in \mathbb{N}$. Q.E.D.

The theorem implies that for s sufficiently large any point $\bar{q}(s)$ of the connected set $C(s)$ in $\bar{Q}(s)$ lies in P and hence induces a Walrasian equilibrium price system $p^* = \bar{q}(s)$. From this it follows that there exists a connected set C of zero points of z in Q containing both q^0 and a q^* inducing a Walrasian equilibrium. Notice that due to Walras' law $z(q) = 0^n$ is generically a system of $n - 1$ independent equations in n unknowns, so that C is typically a 1-dimensional set, e.g. see Herings (1996). Hence, there exists a connected set of market clearing supply constrained disequilibria connecting the trivial non-trade disequilibrium to a Walrasian equilibrium.

5 The dynamics of the adjustment process

In this section we consider the adjustment process induced by the connected set of supply constrained disequilibria. Under suitable differentiability conditions it can be shown that the connected set C contains a piecewise smooth path of zero points of z in Q connecting $q^0 = \underline{p} - e$ inducing the trivial non-trade market clearing disequilibrium and a $q^* \in P$ inducing a Walrasian equilibrium. The path starts in q^0 with supply rationing and price vector \underline{p} and in the short run adjusts rations keeping all markets in equilibrium and all prices fixed until a point q^u is reached with $q_j^u = \underline{p}_j$ for at least one $j \in I_n$. At this point an SCD is reached with commodity j not being rationed, inducing a so-called unemployment equilibrium. Then the process continues with long run adjustment of prices and rations. Keeping all markets in equilibrium, prices of commodities are allowed to increase above their initial value when they are unrationed, while simultaneously the supply rations of the other commodities are adjusted with their prices still on the initial values. This means that as soon as a commodity becomes unrationed its price is allowed to increase. However, it may happen that in the long run the price of an unrationed commodity becomes equal

again to its initial value. In that case the price of that commodity is kept equal to its initial value and the commodity is allowed to become rationed in its supply again. In case in the long run all prices return to their initial values, all commodities are being rationed in their supply and the process switches back to the short run. As soon as one of the commodities becomes unrationed, the process switches back to the long run. Eventually, the process reaches a point q^* in P and thus no commodity is rationed anymore. This q^* induces a Walrasian equilibrium without rationing.

To follow a path of zero points several mathematical path following techniques are available, for instance the well-known predictor-corrector continuation methods and techniques based on piecewise linear approximation, see Allgower and Georg (1980). Predictor-corrector methods can not be applied directly because they require differentiability of the reduced excess demand function z , which is not the case due to the switches in regimes between price and quantity adjustments and between rationing and nonrationing. Therefore, to use such a method a transformation of all variables as proposed in Garcia and Zangwill (1981) is needed. The piecewise linear approximation technique is based on taking a simplicial subdivision of the underlying set Q and linearizing the function z on each subsimplex, resulting in a piecewise linear function Z from Q to \mathbb{R}^n . Then there exists a piecewise linear path in Q from q^0 to a point q^* in P satisfying that for any point q on the path it holds that

$$Z(q) = \beta(q)e$$

for some $\beta(q) \in \mathbb{R}$. For every $\epsilon > 0$, it can be guaranteed that $|\beta(q)| < \epsilon$ for any q on the path by taking the mesh size of the simplicial subdivision small enough. Hence, any point on the piecewise linear path gives an approximating supply constrained disequilibrium, i.e. it induces a price system and rationing scheme with the absolute values of the reduced excess demands less than ϵ . Each linear piece of the path lies in one of the subsimplices of the subdivision and can be followed by a linear programming pivot step in a system of $n+1$ equations and variables. The pivot step determines endogenously which vector q has to be considered in the next step. This makes that the path can be seen as an adjustment process in discrete time. At any point in time the linear programming step determines a new point based on the vectors of excess demands in at most $n+1$ of the points generated before. This new point induces a new system of prices and rations yielding an approximating SCD. This solves the problem of finding the next price and rationing scheme, as was raised by Weddepohl (1999) as an argument against adjustment of prices through a sequence of market clearing disequilibria.

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