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Maximal type test statistics based on conditional processes

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Aims and Scope: This is a broad based journal covering all branches of statistics, with special encouragement to w the field of statistical planning and related combinatorial mathematics and probability theory. We look upon Plan Inference as the two twin branches of statistics, the former being concerned with how to collect data (or info appropriately, and the latter with how to analyze or summarise the information after it has been collected. The dat collected in one or more attempts, or one or more variables, and may be subject to relatively simple or more stochastic processes. Thus, the major areas, such as experimental design (single- or multi-stage or sequential), s certain branches of information theory, multivariate analysis, decision theory, distribution free methods, data probabilistic modelling, reliability, etc. are all included. A large variety of statistical problems, particularly statistical planning, necessarily involve combinatorial or discrete mathematics. One main feature of this journal particularly encourages papers in all branches of combinatorial mathematics which have some bearing on s problems. The journal encourages papers on the application of Statistics to scientific problems. It welcomes papers, survey articles, book reviews, and material for the Statistical Discussion Forum.

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Abstract

A general methodology is presented for non-parametric testing of independence, location and dispersion in multiple regression. The proposed testing procedures are based on the concepts of conditional distribution function, conditional quantile, and conditional shortest t-fraction. Techniques involved come from empirical process and extreme-value theory. The asymptotic distributions are standard Gumbel.

AMS classifications: 62G07; 62G10; 62G20

Keywords: Non-parametric regression; Empirical processes; Extreme-value theory

1. Introduction and main results

Let (X,Y), $(X_1,Y_1),\ldots,(X_n,Y_n)$ be i.i.d. random vectors from a distribution $\tilde{\mu}$ on $\mathbb{R}^{d+1}$, $X_i \in \mathbb{R}^d$, $Y_i \in \mathbb{R}$ $(i=1,\ldots,n)$. The marginal distribution of the X's is denoted by μ; let S be the support of μ. In this paper we are concerned with the conditional distribution of Y given $X=x$, determined by (a version of) the conditional distribution function (df) F_x. The corresponding conditional quantiles

$$Q_x(p)=\inf\{y:F_x(y)\geqslant p\},\quad p\in(0,1)$$

can be used to describe the location of Y given $X=x$, as employed in median regression. Dispersion characteristics will be measured by means of lengths of shortest t-fractions (shortt); see e.g. Rousseeuw and Leroy (1988), Grübel (1988), and Einmahl and Mason (1992). For any df G and any interval $[c,d]\subset\mathbb{R}$ we use the notation $G([c,d])$ for $G(d)-G(c-)$. The conditional length of a shortt is now defined by

$$U_x(t)=\inf\{b-a:F_x([a,b])\geqslant t\},\quad t\in(0,1).$$

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It is our aim to provide new tests for independence, constant location, and homoscedasticity through F_x, $Q_x(p)$ and $U_x(t)$, respectively. More precisely, the following hypotheses will be considered for $0 < p,t < 1$ fixed:

- $\mathrm{H}_0^{(1)}$: F_x is independent of $x \in S$ (μ a.e.);
- $\mathrm{H}_0^{(2)}$: $Q_x(p)$ is independent of $x \in S$ (μ a.e.);
- $\mathrm{H}_0^{(3)}$: $U_x(t)$ is independent of $x \in S$ (μ a.e.).

Our statistical test procedures will be based on an appropriately chosen partition $\{A_{j,n} : j = 1,\ldots,m_n\}$ of S, with for convenience,

$$\mu_j := \mu(A_{j,n}) \geqslant \mu(A_{j+1,n}) =: \mu_{j+1}, \quad \text{for all} \quad 1 \leqslant j \leqslant m_n - 1.$$

Empirical estimates of

$$F_j(y) := P(Y \leqslant y \mid X \in A_{j,n}),$$

$$Q_j(p) := \inf\{y : F_j(y) \geqslant p\},$$

and

$$U_j(t) := \inf\{b - a : F_j([a,b]) \geqslant t\}$$

are given by

$$F_{j,n}(y) := \frac{\sum_{i=1}^{n} I_{A_{j,n}\times(-\infty,y]}(X_i, Y_i)}{\sum_{i=1}^{n} I_{A_{j,n}}(X_i)},$$

$$Q_{j,n}(p) := \inf\{y : F_{j,n}(y) \geqslant p\},$$

and

$$U_{j,n}(t) := \inf\{b - a : F_{j,n}([a,b]) \geqslant t\}.$$

Throughout we assume F_j ($j = 1,\ldots,m_n$) to be continuous on $\mathbb{R}$. Let μ_n denote the empirical measure based on $X_1, X_2, \ldots, X_n$, and set

$$\mu_{j,n} = \mu_n(A_{j,n}), \quad 1 \leqslant j \leqslant m_n.$$

Note that the common values of $F_x, Q_x(p)$ under $\mathrm{H}_0^{(1)}, \mathrm{H}_0^{(2)}$, respectively, are equal to $F, Q(p)$, the marginal df and pth quantile of the Y-distribution. Hence they are appropriately estimated by F_n and $Q_n(p)$, with

$$F_n(y) = n^{-1}\sum_{i=1}^{n} I_{(-\infty,y]}(Y_i), \quad y \in \mathbb{R},$$

$$Q_n(p) = \inf\{y : F_n(y) \geqslant p\}.$$

Concerning the hypothesis $\mathrm{H}_0^{(3)}$, observe that the common value of $U_x(t)$, denoted by $U.(t)$, is not necessarily equal to the length of the marginal shortt of the Y-distribution (e.g., consider a degenerate bivariate (normal) distribution on the line $y = x$). We will estimate $U.(t)$ by

$$U_{\cdot n}(t) = \sum_{j=1}^{m_n} \mu_{j,n} U_{j,n}(t).$$

Now we are ready to state our main results. Let

$$\Lambda(x) = \exp(-\mathrm{e}^{-x}), \quad x \in \mathbb{R},$$

be the standard Gumbel df, Γ a rv with df Λ, and write

$$I_n = \sup_{y\in\mathbb{R}} \max_{1\leqslant j\leqslant m_n} \sqrt{n\mu_{j,n}}|F_{j,n}(y) - F_n(y)|.$$

Theorem 1. *If* $n\mu_{m_n}/((\log n)^2 \log m_n) \to \infty$ *and* $\mu_1 \log m_n \to 0$ *as* $n \to \infty$, *then we have under* $\mathrm{H}_0^{(1)}$ *that*

$$\sqrt{8\log m_n}\left(I_n - \sqrt{\tfrac{1}{2}\log(2m_n)}\right) \xrightarrow{\mathrm{d}} \Gamma.$$

Let c_α be such that $1 - \Lambda(c_\alpha) = \alpha$, $\alpha \in (0,1)$. Our asymptotic test for independence can now be specified.

Corollary 1. *The test which rejects* $\mathrm{H}_0^{(1)}$ *when*

$$I_n \geqslant \sqrt{\tfrac{1}{2}\log(2m_n)} + c_\alpha \Big/ \sqrt{8\log m_n}$$

has asymptotic significance level α *if the assumptions of Theorem* 1 *are satisfied.*

The following corollary can be applied when the $\boldsymbol{X}$-distribution is known and continuous.

Corollary 2. *If* $m_n \to \infty$, $\mu_1 = \mu_{m_n}$, *and* $n\mu_1/(\log n)^3 \to \infty$, *then*

$$\sqrt{8\log m_n}\left(I_n - \sqrt{\tfrac{1}{2}\log(2m_n)}\right) \xrightarrow{\mathrm{d}} \Gamma.$$

In the statement of our next result we make use of the following conditions:
(C.1) for some constant $c_1 > 0$,

$$\limsup_{n\to\infty} \max_{1\leqslant j\leqslant m_n} \sup_{y\in\mathbb{R}} f_j(y) < c_1$$

where f_j denotes the derivative of F_j;
(C.2) the derivative f of F exists at $Q(p)$ and satisfies $f(Q(p)) > 0$.
Furthermore, let

$$c_{\alpha,n} = \sqrt{2\log m_n} + \left(c_\alpha - \tfrac{1}{2}(\log\log m_n + \log\pi)\right) / \sqrt{2\log m_n}.$$

Theorem 2. *Let $p \in (0,1)$ be fixed. The test which rejects $H_0^{(2)}$ when for some $j \in \{1,2,\ldots,m_n\}$*

$$Q_n(p) \notin \left[Q_{j,n}\left(p - c_{\alpha,n}\sqrt{\frac{p(1-p)}{n\mu_{j,n}}}\right),\ Q_{j,n}\left(p + c_{\alpha,n}\sqrt{\frac{p(1-p)}{n\mu_{j,n}}}\right)\right)$$

has asymptotic significance level α if (C.1) *and* (C.2) *are satisfied and if $n\mu_{m_n}/((\log n)^2 \log m_n) \to \infty$ and $\mu_1 \log m_n \to 0$.*

In order to establish our last result some additional regularity conditions are required. The first one reads as follows:

(C.3) for large n, every F_j $(1 \leqslant j \leqslant m_n)$ has a density f_j which is continuous on $\mathbb{R}$ and has support $(\beta_j, \gamma_j), -\infty \leqslant \beta_j < \gamma_j \leqslant \infty$, is strictly increasing on $(\beta_j, y_{0,j}]$ and strictly decreasing on $[y_{0,j}, \gamma_j)$ for some $y_{0,j} \in (\beta_j, \gamma_j)$. Moreover, every $f_x, x \in S$, satisfies this unimodality assumption.

Let $t \in (0,1)$ be fixed. Under (C3) we have for large n that there exists a unique interval $[a_{j,t}, b_{j,t}]$ (the shortt) such that $F_j([a_{j,t}, b_{j,t}]) = t$, $f_j(a_{j,t}) = f_j(b_{j,t})$, and $f_j(y) > f_j(a_{j,t})$ for every $y \in (a_{j,t}, b_{j,t})$ $(1 \leqslant j \leqslant m_n)$.
We also need that

(C.4) there exist constants $c_2, \delta_2 > 0$ such that the derivatives f_j' of f_j satisfy

$$\liminf_{n\to\infty} \min_{1\leqslant j\leqslant m_n} \inf_{y\in[a_{j,t},b_{j,t}]\setminus[a_{j,t}+\delta_2,\ b_{j,t}-\delta_2]} |f_j'(y)| > c_2.$$

Introducing the derivative u_j of U_j $(1 \leqslant j \leqslant m_n)$ we assume

(C.5) there exist constants $c_3, c_4 > 0$ such that for every $s \in (0,1)$

$$\limsup_{n\to\infty} \max_{1\leqslant j\leqslant m_n} |u_j(s) - u_j(t)| \leqslant c_3|s-t|,$$

$$\limsup_{n\to\infty} \max_{1\leqslant j\leqslant m_n} u_j(t) < c_4.$$

Finally, we will assume

(C.6) $$\sqrt{n\mu_1 \log m_n} \max_{1\leqslant j\leqslant m_n} \left(U_j(t) - \sup_{x\in A_{j,n}} U_x(t)\right)^+ \to 0.$$

Theorem 3. *Let $t \in (0,1)$ be fixed. The test which rejects $\mathrm{H}_0^{(3)}$ when for some $j \in \{1,2,\ldots,m_n\}$*

$$U_{\cdot n}(t) \notin \left[U_{j,n}\left(t - c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right),\ U_{j,n}\left(t + c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right)\right)$$

has asymptotic significance level α if (C.1), (C.3)–(C.6) *are satisfied and if $\mu_1 \log m_n \to 0$, $\mu_1^4(\sum_{j=1}^{m_n} \mu_j^{1/2})^8 (\log n)^4 (\log m_n)^5/(n\mu_{m_n}) \to 0$.*

For any $x \in S$, let $m_t(x)$ be defined as the midpoint of the interval pertaining to $U_x(t)$. This robust regression curve is strongly related to the least median of squares regression estimator introduced in Rousseeuw (1984) (see also Rousseeuw and Leroy, 1988). The following smoothness conditions on m_t and F_x ($x \in S$) can be used instead of assumption (C.6), as shown by the following corollaries.

(C.7) for some constant $c_5 > 0$,

$$|m_t(x_1) - m_t(x_2)| \leqslant c_5 \|x_1 - x_2\|$$

for any $x_1, x_2 \in S$;

(C.8) the second-order derivatives f_x' of F_x exist, and for some $c_6 > 0$,

$$\sup_{x \in S} \sup_{y \in \mathbb{R}} |f_x'(y)| < c_6.$$

Let $\operatorname{diam}(A) := \sup\{\|x_1 - x_2\| : x_1, x_2 \in A\}$, where $\|x_1 - x_2\|$ denotes the Euclidean distance between x_1 and x_2.

Corollary 3. *The test which rejects* $\mathrm{H}_0^{(3)}$ *when for some* $j \in \{1,2,\ldots,m_n\}$

$$U_{\cdot n}(t) \notin \left[U_{j,n}\left(t - c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right),\ U_{j,n}\left(t + c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right)\right)$$

has asymptotic significance level α *if* (C.1), (C.3)–(C.5), (C.7) *and* (C.8) *are satisfied, and if*

$$n\mu_1 \log m_n \left(\max_{1 \leqslant j \leqslant m_n} \operatorname{diam}(A_{j,n})\right)^4 \to 0,\ \mu_1 \log m_n \to 0 \quad \text{and}$$

$$\mu_1^4 \left(\sum_{j=1}^{m_n} \mu_j^{1/2}\right)^8 (\log n)^4 (\log m_n)^5 / (n\mu_{m_n}) \to 0.$$

Corollary 4. *If* $m_n \to \infty$, $\mu_1 = \mu_{m_n}$, $n\mu_1/(\log n)^9 \to \infty$, *and* $n\mu_1 \log m_n$ $(\max_{1 \leqslant j \leqslant m_n} \operatorname{diam}(A_{j,n}))^4 \to 0$, *then it follows under* (C.1), (C.3)–(C.5), (C.7) *and* (C.8) *that the test which rejects* $\mathrm{H}_0^{(3)}$ *when for some* $j \in \{1,2,\ldots,m_n\}$

$$U_{\cdot n}(t) \notin \left[U_{j,n}\left(t - c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right),\ U_{j,n}\left(t + c_{\alpha,n}\sqrt{\frac{t(1-t)}{n\mu_{j,n}}}\right)\right)$$

has asymptotic significance level α.

Remark. (1) The choice of $Q_x(p)$, resp. $U_x(t)$, rather than $m_p(x)$, resp. the interquartile range $Q_x(\frac{1}{2}(1+t)) - Q_x(\frac{1}{2}(1-t))$, to produce tests for $\mathrm{H}_0^{(2)}$, resp. $\mathrm{H}_0^{(3)}$, was motivated in part by considerations of statistical relevance. Indeed, $m_p(x)$ ($x \in A_{j,n}$) can only be estimated at a rate of $(n\mu_j)^{-1/3}$ (see e.g. Kim and Pollard, 1990), whereas interquartile ranges have a lower breakdown point than the corresponding shortt measures when $t > \frac{1}{3}$ (see Rousseeuw and Leroy (1988) for the case $t = \frac{1}{2}$). The techniques we use to derive our results however, can also be applied to other testing procedures, e.g. those based on $m_p(x)$ and interquartile ranges.

(2) In the cases considered in Theorems 2 and 3, similar results on sup-norm statistics where t, p vary over non-degenerate intervals can be obtained with the technique of proof introduced in the next section.

(3) Our statistic I_n discussed in Theorem 1 is somewhat similar to the V-quantities in Kiefer (1959) to test equality of distributions in a one-way layout of several populations (see also the references in that paper). The situation considered here provides a generalization of Kiefer's result to the case where the number of groups increases with the sample size.

(4) Somewhat related papers are Bhattacharya and Gangopadhyay (1990), Stute (1986) and Ruymgaart (1994).

(5) In a non-regression setting an analogue of our type of test statistics is the goodness-of-fit test statistic in Dijkstra et al. (1984). In case S is compact, these authors propose to reject uniformity on S when $P_n = \max_{1 \leqslant j \leqslant m_n} \mu_{j,n}$ becomes too large, where the partition is taken to be such that under the null hypothesis the μ_j are all equal. Their simulation study shows that the power of this test is at least comparable to the power of the classical χ^2-test for uniformity against peaked alternatives.

A 'continuous' version of this 'peak-test' is given by the scan statistic (see e.g. Naus, 1966, 1982; Cressie, 1980, 1987) which uses a maximal type statistic obtained from continuous scanning of S with a fixed window. In case $d = 1$, Deheuvels and Révész (1987) derived asymptotics for the scan statistic using a similar condition as in Corollary 2; i.e. $(na_n)/(\log n)^3 \to \infty$, where a_n is the window length.

When $\mu_1 = \mu_{m_n}$ one can also derive the following result for P_n: if $(n\mu_1)/(\log n)^3 \to \infty$ and $\mu_1 \to 0$, we have under the hypothesis of uniformity that

$$\sqrt{2 \log m_n} \left\{ \sqrt{\frac{n}{\mu_1}}(P_n - \mu_1) - \sqrt{2 \log m_n} + \frac{1}{2}(\log \log m_n + \log 4\pi)/(2 \log m_n)^{1/2} \right\} \xrightarrow{\text{d}} \Gamma.$$

(6) The condition $n\mu_1(\log m_n)(\max_{1 \leqslant j \leqslant m_n} \operatorname{diam}(A_{j,n}))^4 \to 0$ specifies to $nh_n^{4+d} \log 1/h_n \to 0$ in case X possesses a uniform distribution on $[0, 1]^d$, say, and the partition is taken to be cubic with $\operatorname{diam}(A_{j,n}) \sim h_n$ $(j = 1, \ldots, m_n)$. This rate condition of h_n lies close to the optimal rate of the window size in kernel density estimation when minimizing the mean squared error.

(7) If one wants to restrict attention to a subset of the support S of X, all of our results can still be used by translating them in terms of conditional distributions given X belongs to that subset.

2. Proofs

The proofs of our main results rely on the following important proposition which states that jointly over all elements $A_{j,n}$ of the partition of S, we can approximate the different empirical processes

$$\alpha_{j,n} = \sqrt{n\mu_{j,n}}(F_{j,n} - F_j), \ j = 1, \ldots, m_n,$$

by independent Gaussian processes, and this, per j, at a rate which is comparable to the one attained by the Komlós–Major–Tusnády (1975) approximation of the (one-dimensional) uniform empirical process.

Denoting the joint distribution of (X, Y) by $\tilde{\mu}$, and the empirical measure based on $(X_1, Y_1), \ldots, (X_n, Y_n)$ by $\tilde{\mu}_n$, we will use the following quantities:

$$\tilde{\mu}_j(y) = P(X \in A_{j,n} \text{ and } Y \leqslant y) = \tilde{\mu}(A_{j,n} \times (-\infty, y]),$$

$$\tilde{\mu}_{j,n}(y) = \tilde{\mu}_n(A_{j,n} \times (-\infty, y]),$$

so that

$$F_{j,n}(y) = \frac{\tilde{\mu}_{j,n}(y)}{\mu_{j,n}}.$$

Proposition. *If* $m_n \to \infty$ *and* $(n\mu_{m_n})/(\log n)^2 \to \infty$, *then there exists a triangular scheme of rowwise independent Brownian bridges* $\{B_{j,n}(t), 0 \leqslant t \leqslant 1\}$ $(1 \leqslant j \leqslant m_n,\ n \geqslant 1)$ *such that*

$$\sup_{y \in \mathbb{R}} \max_{1 \leqslant j \leqslant m_n} |\alpha_{j,n}(y) - B_{j,n}(F_j(y))| = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n\mu_{m_n}}}\right).$$

Proof. We consider the transformation from $S \times \mathbb{R}$ to $[0, 1]$

$$(x, y) \to T(x, y) = \sum_{j=1}^{m_n} 1_{A_{j,n}}(x) \left[\sum_{k=1}^{j-1} \mu_k + \mu_j F_j(y)\right],$$

and the transformed rv's

$$Z_i = T(X_i, Y_i), \quad i = 1, 2, \ldots, n.$$

One easily checks that $Z_1, Z_2, \ldots, Z_n$ are independent uniformly (0,1) distributed rv's. Let $\{e_n(t), 0 \leqslant t \leqslant 1\}$ denote the empirical process based on $Z_1, Z_2, \ldots, Z_n$. The approximation theorem of Komlós et al. (1975) entails then the existence of a sequence of Brownian bridges $\{\tilde{B}_n(t), 0 \leqslant t \leqslant 1\}$ such that as $n \to \infty$

$$\sup_{0 \leqslant t \leqslant 1} |e_n(t) - \tilde{B}_n(t)| = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right).$$

Setting $T(A_{j,n} \times (-\infty, y]) := \{T(x, v) : x \in A_{j,n}, v \leqslant y\}$, $y \in \mathbb{R} \cup \{\infty\}$, we have $\lambda(T(A_{j,n} \times (-\infty, y])) = \mu_j F_j(y)$, with λ denoting Lebesgue measure; also for a closed, half-open, or open interval A with endpoints $a \leqslant b$ we set $\tilde{B}_n(A) := \tilde{B}_n(b) - \tilde{B}_n(a)$.

It now follows that

$$\max_{1 \leqslant j \leqslant m_n} |\sqrt{n}(\mu_{j,n} - \mu_j) - \tilde{B}_n(T(A_{j,n} \times (-\infty, \infty]))| = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right)$$

and that

$$\max_{1 \leqslant j \leqslant m_n} \sup_{y \in \mathbb{R}} |\sqrt{n}(\tilde{\mu}_{j,n}(y) - \tilde{\mu}_j(y)) - \tilde{B}_n(T(A_{j,n} \times (-\infty, y]))| = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right).$$

Now uniformly in $j \in \{1,\ldots,m_n\}$ and $y \in \mathbb{R}$ we have

$$\begin{aligned}\alpha_{j,n}(y) &= \sqrt{n\mu_{j,n}}\left(\frac{\tilde{\mu}_{j,n}(y)}{\mu_{j,n}} - \frac{\tilde{\mu}_j(y)}{\mu_j}\right)\\ &= \sqrt{n}(\tilde{\mu}_{j,n}(y) - \tilde{\mu}_j(y))/\sqrt{\mu_{j,n}} - \sqrt{n}(\mu_{j,n} - \mu_j)(\tilde{\mu}_j(y)/(\mu_j\sqrt{\mu}_{j,n}))\\ &= \left\{\left\{\tilde{B}_n(T(A_{j,n} \times (-\infty, y])) + \mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right)\right\}\mu_j^{-1/2}\right.\\ &\quad \left. - \left\{\tilde{B}_n(T(A_{j,n} \times (-\infty, \infty])) + \mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right)\right\} F_j(y)\mu_j^{-1/2}\right\}\tau_{j,n}^{-1/2},\end{aligned}$$

where

$$\tau_{j,n} = \mu_{j,n}/\mu_j = 1 + n^{-1/2}\mu_j^{-1}\tilde{B}_n(T(A_{j,n} \times (-\infty,\infty])) + \mu_j^{-1}\mathrm{O_P}\left(\frac{\log n}{n}\right). \quad (2.1)$$

We can define a sequence of Wiener processes $\{W_n(t), 0 \leqslant t \leqslant 1\}$ such that $\tilde{B}_n = W_n - IW_n(1)$, where I denotes the identity function. Hence, we find that

$$\begin{aligned}\alpha_{j,n}(y) = &\left\{\{W_n(T(A_{j,n} \times (-\infty, y])) - F_j(y)W_n(T(A_{j,n} \times (-\infty,\infty]))\}\mu_j^{-1/2}\right.\\ &\left. + \mu_j^{-1/2}\mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right)\right\}\tau_{j,n}^{-1/2} \quad (n \to \infty).\end{aligned}$$

We now set

$$B_{j,n}(F_j(y)) := \mu_j^{-1/2}\{W_n(T(A_{j,n} \times (-\infty, y])) - F_j(y)W_n(T(A_{j,n} \times (-\infty,\infty]))\}.$$

One easily checks that the $B_{j,n}$ are indeed independent in $j \in \{1,2,\ldots,m_n\}$ and distributed as Brownian bridges.

Now as $n \to \infty$

$$|\alpha_{j,n}(y) - B_{j,n}(F_j(y))| \leqslant |B_{j,n}(F_j(y))|\ (\tau_{j,n}^{-1/2} - 1) + \mu_j^{-1/2}\tau_{j,n}^{-1/2}\mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right).$$

For a function φ on [0,1], write $\|\varphi\| = \sup_{0 \leqslant t \leqslant 1}|\varphi(t)|$. First remark that as the F_j are assumed to be continuous:

$$\max_{1 \leqslant j \leqslant m_n}\sup_{y \in \mathbb{R}}|B_{j,n}(F_j(y))| = \max_{1 \leqslant j \leqslant m_n}\|B_{j,n}\| = \mathrm{O_P}(\sqrt{\log m_n}),$$

as, because of the independence of the rv's $\|B_{j,n}\|$ $(1 \leqslant j \leqslant m_n)$, we have for any $M > 0$, that

$$P\left(\max_{1 \leqslant j \leqslant m_n}\|B_{j,n}\| > M\sqrt{\log m_n}\right) \leqslant 2m_n\,\mathrm{e}^{-2M^2\log m_n} = 2m_n^{1-2M^2},$$

which tends to zero as $m_n \to \infty$ when $M > 2^{-1/2}$. (Here we also used the fact that for a Brownian bridge B we have $P(\|B\| > u) \leqslant 2\mathrm{e}^{-2u^2}$.)

Furthermore,

$$
\begin{aligned}
&n^{-1/2} \max_{1\leqslant j\leqslant m_n} \mu_j^{-1} \left| \tilde{B}_n\left(\left(\sum_{k=1}^{j-1}\mu_k, \sum_{k=1}^{j}\mu_k\right]\right)\right| \\
&\leqslant n^{-1/2}\mu_{m_n}^{-1/2} \max_{1\leqslant j\leqslant m_n} \left(\mu_j^{-1/2}\left| W_n\left(\left(\sum_{k=1}^{j-1}\mu_k, \sum_{k=1}^{j}\mu_k\right]\right)\right| + \mu_{m_n}^{1/2}\, |W_n(1)|\right) \\
&= (n\mu_{m_n})^{-1/2}\mathrm{O_P}(\sqrt{\log m_n}),
\end{aligned}
$$

since $W_n((\sum_{k=1}^{j-1}\mu_k,\ \sum_{k=1}^{j}\mu_k])/\sqrt{\mu_j}$ $(1\leqslant j\leqslant m_n)$ are m_n independent standard normal rv's whose maximum is well known to be of order $\mathrm{O_P}(\sqrt{\log m_n})$ as $n\to\infty$.

Hence,

$$
\max_{1\leqslant j\leqslant m_n} |\tau_{j,n}-1| = \mathrm{O_P}\left(\sqrt{\frac{\log n}{n\mu_{m_n}}} + \frac{\log n}{n\mu_{m_n}}\right) \tag{2.2}
$$

and

$$
\max_{1\leqslant j\leqslant m_n} \sup_{y\in\mathbb{R}} |B_{j,n}(F_j(y))|\ |\tau_{j,n}^{-1/2}-1| = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n\mu_{m_n}}} + \frac{(\log n)^{3/2}}{n\mu_{m_n}}\right)\ (n\to\infty).
$$

Finally, with

$$
\left(\max_{1\leqslant j\leqslant m_n} (\mu_j\tau_{j,n})^{-1/2}\right)\mathrm{O_P}\left(\frac{\log n}{\sqrt{n}}\right) = \mathrm{O_P}\left(\frac{\log n}{\sqrt{n\mu_{m_n}}}\right),
$$

the result follows. □

Proof of Theorem 1. First remark that by the well-known fact that

$$
\sqrt{n}\ \sup_{y\in\mathbb{R}} |F_n(y)-F(y)| = \mathrm{O_P}(1) \quad (n\to\infty),
$$

we have

$$
\sqrt{\log m_n}\ \sup_{y\in\mathbb{R}} \max_{1\leqslant j\leqslant m_n} \sqrt{n\mu_{j,n}}\ |F_n(y)-F(y)| = (\mu_1\log m_n)^{1/2}\mathrm{O_P}(1) \quad (n\to\infty)
$$

since, as in the proof of the proposition we find that uniformly in $j\in\{1,\ldots,m_n\}$

$$
\mu_{j,n}^{1/2} = \mu_j^{1/2}\left(1+\mathrm{O_P}\left(\sqrt{\frac{\log m_n}{n\mu_{m_n}}}\right)\right).
$$

Hence, since $\mu_1\log m_n\to 0$, it suffices to show that, under $\mathrm{H}_0^{(1)}$,

$$
\sqrt{8\log m_n}\left(\sup_{y\in\mathbb{R}} \max_{1\leqslant j\leqslant m_n} \sqrt{n\mu_{j,n}}\ |F_{j,n}(y)-F(y)| - \sqrt{\tfrac{1}{2}\log(2m_n)}\right) \xrightarrow{\mathrm{d}} \Gamma
$$

$(n\to\infty)$.

Under $\mathrm{H}_0^{(1)}$ it now follows from the proposition that

$$\sqrt{8\log m_n}\,\left|\sup_{y\in\mathbb{R}}\max_{1\leqslant j\leqslant m_n}\sqrt{n\mu_{j,n}}\,\left|F_{j,n}(y)-F(y)\right| - \sup_{y\in\mathbb{R}}\max_{1\leqslant j\leqslant m_n}|B_{j,n}(F_j(y))|\,\right|$$
$$= \mathrm{O_P}\left(\frac{\log n\sqrt{\log m_n}}{\sqrt{n\mu_{m_n}}}\right) = \mathrm{o_P}(1)$$

if $n\to\infty$ and $n\mu_{m_n}/((\log n)^2\log m_n)\to\infty$.

Finally, remark that by the independence of the $\|B_{j,n}\|$ $(1\leqslant j\leqslant m_n)$ we can apply standard extreme value theory to show that

$$\sqrt{8\log m_n}\,\left\{\max_{1\leqslant j\leqslant m_n}\|B_{j,n}\| - \sqrt{\tfrac{1}{2}\log(2m_n)}\right\} \xrightarrow{\mathrm{d}} \Gamma \tag{2.3}$$

since $P(\|B_{j,n}\|>u)\sim 2\mathrm{e}^{-2u^2}$ (see Resnick, 1987, Proposition 1.19). □

Proof of Corollary 2. If $\mu_1=\mu_2=\cdots=\mu_{m_n}$, then $m_n=\mu_1^{-1}$.

The condition $\mu_1\log m_n\to 0$ is then automatically satisfied when $m_n\to\infty$. □

Proof of Theorem 2. Observe that, under $\mathrm{H}_0^{(2)}$,

$$P\Bigg(Q_n(p)\notin\left[Q_{j,n}\left(p-c_{\alpha,n}\sqrt{\frac{p(1-p)}{n\mu_{j,n}}}\right),\ Q_{j,n}\left(p+c_{\alpha,n}\sqrt{\frac{p(1-p)}{n\mu_{j,n}}}\right)\right),$$
$$\text{for some } j\in\{1,\ldots,m_n\}\Bigg)\to\alpha \quad \text{as } n\to\infty$$

if

$$\sqrt{2\log m_n}\Bigg\{\max_{1\leqslant j\leqslant m_n}\left(\sqrt{n\mu_{j,n}}\,|F_{j,n}(Q_n(p))-p|\,/\sqrt{p(1-p)}\right)-\sqrt{2\log m_n}$$
$$+\tfrac{1}{2}(\log\log m_n+\log\pi)(2\log m_n)^{-1/2}\Bigg\}\xrightarrow{\mathrm{d}}\Gamma. \tag{2.4}$$

Indeed, for any df G on the real line and any $p\in(0,1)$ we have

$$G(x)\geqslant p \quad \text{if and only if} \quad G^{-1}(p)\leqslant x$$

and hence

$$G(x)<p \quad \text{if and only if} \quad G^{-1}(p)>x\,.$$

We first show that under (C.1), (C.2), $n\mu_{m_n}/((\log n)^2\log m_n)\to\infty$ and $\mu_1\log m_n\to 0$

$$\sqrt{2\log m_n}\left\{\max_{1\leqslant j\leqslant m_n}\sqrt{n\mu_{j,n}}|F_{j,n}(Q_n(p))-p| - \max_{1\leqslant j\leqslant m_n}|B_{j,n}(F_j(Q_n(p)))|\right\}\xrightarrow{\mathrm{P}}0 \tag{2.5}$$

where $\{B_{j,n}\}$ $(1 \leqslant j \leqslant m_n, n \geqslant 1)$ is the sequence of Brownian bridges described in the Proposition. Now (2.5) follows from the Proposition if we can show that under our assumptions

$$\sqrt{\log m_n} \max_{1 \leqslant j \leqslant m_n} \sqrt{n\mu_{j,n}}\ |F_j(Q_n(p)) - p| \xrightarrow{\mathrm{P}} 0. \tag{2.6}$$

The well-known central limit theorem for quantiles yields that under $\mathrm{H}_0^{(2)}$ and (C.2)

$$Q_n(p) = Q(p) + \mathrm{O_P}(n^{-1/2}) = Q_j(p) + \mathrm{O_p}(n^{-1/2})$$

when $n \to \infty$. Hence, by the mean value theorem we have under $\mathrm{H}_0^{(2)}$ that

$$F_j(Q_n(p)) = F_j(Q_j(p) + \mathrm{O_P}(n^{-1/2})) = p + \mathrm{O_P}(n^{-1/2}) f_j(\tilde{Q}_{j,n}(p))$$

with $\tilde{Q}_{j,n}(p) \in (Q_n(p) \wedge Q_j(p),\ Q_n(p) \vee Q_j(p))$ $(1 \leqslant j \leqslant m_n)$. Hence, by (C.1) and under $\mathrm{H}_0^{(2)}$,

$$\sqrt{n} \max_{1 \leqslant j \leqslant m_n} |F_j(Q_n(p)) - p| = \mathrm{O_P}(1) \quad (n \to \infty), \tag{2.7}$$

so that it remains to check that $(\log m_n)(\max_{1 \leqslant j \leqslant m_n} \mu_{j,n}) \xrightarrow{\mathrm{P}} 0$ $(n \to \infty)$ for (2.6) (and hence (2.5)) to hold.

However, using $\tau_{j,n}$ in (2.1) again, we get that

$$\log m_n \left(\max_{1 \leqslant j \leqslant m_n} \mu_{j,n} \right) \leqslant \mu_1 (\log m_n) \left(\max_{1 \leqslant j \leqslant m_n} \tau_{j,n} \right),$$

which tends to zero in probability as $n \to \infty$ and $\mu_1 \log m_n \to 0$ because of (2.2).

Next, it follows from (2.7), and the modulus of continuity behaviour of Brownian bridges (see e.g. Csörgő and Révész, 1981, Lemma 1.1.1) that

$$\sqrt{\log m_n} \max_{1 \leqslant j \leqslant m_n} \Big|\ |B_{j,n}(F_j(Q_n(p)))| - |B_{j,n}(p)|\ \Big| = \mathrm{O_P}(n^{-1/4}((\log n)(\log m_n))^{1/2})$$

$$= \mathrm{o_P}(1) \quad (n \to \infty). \tag{2.8}$$

As $B_{j,n}(p)$ $(1 \leqslant j \leqslant m_n)$ are independent $\mathcal{N}(0, p(1-p))$ rv's, standard techniques form extreme value theory yield that

$$\sqrt{2 \log m_n}\ \Big\{ (p(1-p))^{-1/2} \max_{1 \leqslant j \leqslant m_n} |B_{j,n}(p)| - \sqrt{2 \log m_n}$$

$$+ \tfrac{1}{2} (\log\log m_n + \log \pi) \cdot (2 \log m_n)^{-1/2} \Big\} \xrightarrow{\mathrm{d}} \Gamma \quad (m_n \to \infty). \tag{2.9}$$

Limit statement (2.4) now follows from (2.5), (2.8) and (2.9). □

Proof of Theorem 3. We introduce the functions

$$H_j(z) = \sup\{F_j([a,b]) : b - a \leqslant z\}.$$

Note that H_j is the inverse of U_j (for n large enough). The derivative of H_j is denoted by h_j. Remark that condition (C.1) implies

$$\limsup_{n\to\infty} \max_{1\leqslant j\leqslant m_n} \sup_{z\geqslant 0} h_j(z) < \infty, \tag{2.10}$$

as for each $j \in \{1,\ldots,m_n\}$ we find that h_j is non-increasing and $h_j(0) = \max_{y\in\mathbb{R}} f_j(y)$.

Analogously, we define the inverse function $H_{j,n}$ of $U_{j,n}$ by

$$H_{j,n}(z) = \inf\{t : U_{j,n}(t)\geqslant z\}$$

and note that

$$H_{j,n}(z) = \sup\{F_{j,n}([a,b]) : b-a\leqslant z\},\ 1\leqslant j\leqslant m_n.$$

To prove Theorem 3 it now suffices to show that under $\mathrm{H}_0^{(3)}$

$$\sup_{t\in(0,1)} \max_{1\leqslant j\leqslant m_n} |\sqrt{n\mu_{j,n}}(H_{j,n}(U_{\cdot n}(t)) - t) - \tilde{B}_{j,n}(t)|$$
$$= O_{\mathrm{P}}((n\mu_{m_n})^{-1/8}(\log m_n)^{1/8}(\log n)^{1/2}) \tag{2.11}$$

for some triangular scheme of rowwise independent Brownian bridges $\{\tilde{B}_{j,n}\}(1\leqslant j \leqslant m_n, n\geqslant 1)$; cf. the proof of Theorem 2. We derive (2.11) in three steps by showing that under the given conditions

$$\sup_{t\in(0,1)} \max_{1\leqslant j\leqslant m_n} |\sqrt{n\mu_{j,n}}\ (H_{j,n}(U_j(t)) - t) - \tilde{B}_{j,n}(t)|$$
$$= O_{\mathrm{P}}((n\mu_{m_n})^{-1/8}(\log m_n)^{1/8}(\log n)^{1/2}), \tag{2.12}$$

$$\sqrt{\log m_n} \max_{1\leqslant j\leqslant m_n} \sqrt{n\mu_{j,n}}\ |H_j(U_{\cdot n}(t)) - t| \xrightarrow{\mathrm{P}} 0 \tag{2.13}$$

and

$$\sqrt{\log m_n} \max_{1\leqslant j\leqslant m_n} |\tilde{B}_{j,n}\ (H_j(U_{\cdot n}(t))) - \tilde{B}_{j,n}(t)| \xrightarrow{\mathrm{P}} 0 \quad (n\to\infty). \tag{2.14}$$

First, we prove the existence of a sequence $\{\tilde{B}_{j,n}\}$ of Brownian bridges for which (2.12) holds. Remark that from the Proposition it follows that

$$\sup_{[a,b]} \max_{1\leqslant j\leqslant m_n} |\alpha_{j,n}([a,b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))| = O_{\mathrm{P}}\left(\frac{\log n}{\sqrt{n\mu_{m_n}}}\right) \tag{2.15}$$

as $B_{j,n}([F_j(a),F_j(b)]) = B_{j,n}(F_{j,n}(b)) - B_{j,n}(F_{j,n}(a))$. To derive (2.12) from (2.15) we apply and refine the method of proof of Proposition 3.1 in Einmahl and Mason (1992). We define

$$\tilde{B}_{j,n}(t) = B_{j,n}(F_j(b_{j,t})) - B_{j,n}(F_j(a_{j,t})),\quad 1\leqslant j\leqslant m_n.$$

As the intervals $[a_{j,t}, b_{j,t}]$ are nested for different values of t, one easily checks that the $\tilde{B}_{j,n}$ are distributed as Brownian bridges for every $j\in\{1,2,\ldots,m_n\}$ and large n;

moreover, $\tilde{B}_{1,n},\ldots,\tilde{B}_{m_n,n}$ are clearly independent. Notice that for any $j \in \{1,\ldots,m_n\}$ and $0 < t < 1$

$$\tilde{B}_{j,n}(t) - \sqrt{n\mu_{j,n}}(H_{j,n}(U_j(t)) - t) \leqslant (B_{j,n}(F_j(b_{j,t})) - B_{j,n}(F_j(a_{j,t}))) \\ -\alpha_{j,n}([a_{j,t}, b_{j,t}]) \tag{2.16}$$

which, by (2.15), is seen to be $O_P(\log n/\sqrt{n\mu_{m_n}})$, uniformly in $j \in \{1,\ldots,m_n\}$.

Next, we also have for any $j \in \{1,\ldots,m_n\}$ and any sequence $\varepsilon_n \downarrow 0$

$$\sqrt{n\mu_{j,n}}(H_{j,n}(U_j(t)) - t) - \tilde{B}_{j,n}(t)$$

$$\leqslant \left\{ \sqrt{n\mu_{j,n}} \sup_{\substack{b-a \leqslant U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leqslant t}} (F_{j,n}([a,b]) - t) - \tilde{B}_{j,n}(t) \right\}$$

$$\vee \left\{ \sqrt{n\mu_{j,n}} \sup_{F_j([a,b]) \leqslant t-\varepsilon_n} (F_{j,n}([a,b]) - t) - \tilde{B}_{j,n}(t) \right\}. \tag{2.17}$$

The second term on the right-hand side of (2.17) is

$$\leqslant \sqrt{n\mu_{j,n}} \sup_{F_j([a,b]) \leqslant t} (F_{j,n}([a,b]) - F_j([a,b])) + |\tilde{B}_{j,n}(t)| - \varepsilon_n\sqrt{n\mu_{j,n}}$$
$$\leqslant 2 \max_{1 \leqslant j \leqslant m_n} \sup_{[c,d]} |B_{j,n}([c,d])|$$
$$+ \max_{1 \leqslant j \leqslant m_n} \sup_{[a,b]} |\alpha_{j,n}([a,b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))| - \varepsilon_n \min_{1 \leqslant j \leqslant m_n} \sqrt{n\mu_{j,n}}.$$

From (2.3), (2.15) and (2.2) it now follows that the second term on the right-hand side of (2.17) can be asymptotically bounded from above by 0 in probability, by making the appropriate choice

$$\varepsilon_n = M(\log m_n/(n\mu_{m_n}))^{1/2}$$

with M a large enough positive constant.

The first term on the right-hand side of (2.17) is

$$\leqslant \sqrt{n\mu_{j,n}} \sup_{\substack{b-a \leqslant U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leqslant t}} ((F_{j,n}(b) - F_{j,n}(a)) - (F_j(b) - F_j(a))) - \tilde{B}_{j,n}(t)$$

$$\leqslant \sup_{\substack{b-a \leqslant U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leqslant t}} |\alpha_{j,n}([a,b]) - (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a)))|$$

$$+ \sup_{\substack{b-a \leqslant U_j(t) \\ t-\varepsilon_n < F_j([a,b]) \leqslant t}} (B_{j,n}(F_j(b)) - B_{j,n}(F_j(a))) - \tilde{B}_{j,n}(t). \tag{2.18}$$

The first term on the right-hand side of (2.18) is of order $O_P(\log n/\sqrt{n\mu_{m_n}})$, uniformly in $j \in \{1,2,\ldots,m_n\}$, by (2.15).

Finally, observe that for any $j \in \{1,\dots,m_n\}$

$$\sup_{\substack{b-a\leqslant U_j(t)\\ t-\varepsilon_n<F_j([a,b])\leqslant t}} \{B_{j,n}(F_j(b)) - B_{j,n}(F_j(a))\} - \tilde{B}_{j,n}(t)$$

$$\leqslant \max_{1\leqslant j\leqslant m_n} \sup_{\substack{b-a\leqslant U_j(t)\\ t-\varepsilon_n<F_j([a,b])\leqslant t}} \{|B_{j,n}(F_j(b)) - B_{j,n}(F_j(b_{j,t}))|$$

$$+ |B_{j,n}(F_j(a)) - B_{j,n}(F_j(a_{j,t}))|\}. \tag{2.19}$$

For any interval $[a,b]$ with $b-a = U_j(t)$ and $t-\varepsilon_n < F_j([a,b])\leqslant t$, we find by (C.4) that (uniformly in j) $|a-a_{j,t}|$ and $|b-b_{j,t}|$ become arbitrarily small as $n\to\infty$. We then find, with δ_2 as in (C.4), that eventually as $n\to\infty$ whether $a\in[a_{j,t}, a_{j,t}+\delta_2)$ or $b\in[b_{j,t}-\delta_2, b_{j,t}]$. In case $a < a_{j,t} < y_{0,j} < b < b_{j,t}$, we have that

$$F_j([b,b_{j,t}])\leqslant |b_{j,t}-b|\ f_j(y_{0,j})\leqslant c_1(b_{j,t}-b).$$

On the other hand, if $\varepsilon_n\geqslant F_j([a_{j,t},b_{j,t}]) - F_j([a,b])\geqslant 0$, then

$$\varepsilon_n\geqslant F_j([b,b_{j,t}]) - (b_{j,t}-b)f_j(b_{j,t}) = -((b_{j,t}-b)^2/2)f_j'(\tilde{b}_{j,t})$$

with $\tilde{b}_{j,t}\in(b_{j,t}\wedge b, b_{j,t}\vee b)$, so that (C.4) implies that for n large enough $F_j(b_{j,t})-F_j(b)$ $\leqslant C\varepsilon_n^{1/2}$, for some $C>0$. Also in the other possible cases we can obtain this same bound for $|F_j(b_{j,t})-F_j(b)|\vee|F_j(a_{j,t})-F_j(a)|$. Hence the expression on the right-hand side of (2.19) can be bounded by

$$\omega(n,\varepsilon_n) := 2\max_{1\leqslant j\leqslant m_n}\ \sup_{0\leqslant s\leqslant 1-C\sqrt{\varepsilon_n}}\ \sup_{0\leqslant t\leqslant C\sqrt{\varepsilon_n}} |B_{j,n}(s+t) - B_{j,n}(s)|. \tag{2.20}$$

By Lemma 1.1.1 in Csörgő and Révész (1981), the representation of Brownian bridges in terms of Wiener processes, and the independence of the Brownian bridges $B_{j,n}$ $(1\leqslant j\leqslant m_n)$, we obtain that for any $K>0$ there exist constants $K_1, K_2>0$ such that

$$P(\omega(n,\varepsilon_n) > K\gamma_n)\leqslant K_1 m_n\varepsilon_n^{-1/2}\exp(-K_2K^2\gamma_n^2\varepsilon_n^{-1/2}).$$

Choosing

$$\gamma_n = \varepsilon_n^{1/4}(\log n)^{1/2} = M^{1/4}(\log m_n)^{1/8}(n\mu_{m_n})^{-1/8}(\log n)^{1/2},$$

one easily checks that

$$P(\omega(n,\varepsilon_n) > K\gamma_n)\to 0 \quad (n\to\infty)$$

choosing $K>0$ large enough. This together with (2.16)–(2.20) implies (2.12).

To derive (2.13), note that under $\mathrm{H}_0^{(3)}$ we have for any $j \in \{1,\ldots,m_n\}$

$$\begin{aligned} &|H_j(U_{\cdot n}(t)) - t| \\ &\leqslant |H_j(U_{\cdot n}(t)) - H_j(U_{\cdot}(t))| + |H_j(U_{\cdot}(t)) - H_j(U_j(t))| \\ &\leqslant |U_{\cdot n}(t)) - U_{\cdot}(t)|\ h_j(\tilde{U}_n(t)) + \ |U_{\cdot}(t) - U_j(t)|\ h_j(\tilde{U}_j(t), \end{aligned} \tag{2.21}$$

where $\tilde{U}_n(t) \in (U_{\cdot n}(t) \wedge U_{\cdot}(t),\ U_{\cdot n}(t) \vee U_{\cdot}(t))$ and $\tilde{U}_j(t) \in (U_j(t) \wedge U_{\cdot}(t),\ U_j(t) \vee U_{\cdot}(t))$, $1 \leqslant j \leqslant m_n$.

Now using (2.2) and (2.10), and the fact that under $\mathrm{H}_0^{(3)}$ from (C.6) and $U_j(t) \geqslant U_{\cdot}(t)$ it follows that

$$\sqrt{n\mu_1 \log m_n} \max_{1 \leqslant j \leqslant m_n} |U_j(t) - U_{\cdot}(t)| \to 0 \quad (n \to \infty), \tag{2.22}$$

we now find that as $n \to \infty$

$$\begin{aligned} &\sqrt{\log m_n} \max_{1 \leqslant j \leqslant m_n} (\sqrt{n\mu_{j,n}}\ |U_{\cdot}(t) - U_j(t)|\ ) h_j(\tilde{U}_j(t)) \\ &= \mathrm{O_P}\left(\sqrt{n\mu_1 \log m_n} \max_{1 \leqslant j \leqslant m_n} |U_j(t) - U_{\cdot}(t)| \right) = \mathrm{o_P}(1). \end{aligned} \tag{2.23}$$

On the other hand, by (2.2) and (2.10), as $n \to \infty$

$$\begin{aligned} &\sqrt{\log m_n} \max_{1 \leqslant j \leqslant m_n} (\sqrt{n\mu_{j,n}} h_j(\tilde{U}_n(t)))\ |U_{\cdot n}(t) - U_{\cdot}(t)| \\ &\quad = \mathrm{O_P}(1)\sqrt{n\mu_1 \log m_n}\ |U_{\cdot n}(t) - U_{\cdot}(t)|. \end{aligned} \tag{2.24}$$

Furthermore,

$$|U_{\cdot n}(t) - U_{\cdot}(t)| \leqslant \sum_{j=1}^{m_n} \mu_{j,n} |\ U_j(t) - U_{\cdot}(t)|\ + \left| \sum_{j=1}^{m_n} \mu_{j,n}(U_{j,n}(t) - U_j(t)) \right|. \tag{2.25}$$

Now

$$\sqrt{n\mu_1 \log m_n} \sum_{j=1}^{m_n} \mu_{j,n}\ |U_j(t) - U_{\cdot}(t)|\ \leqslant \sqrt{n\mu_1 \log m_n} \max_{1 \leqslant j \leqslant m_n} |U_j(t) - U_{\cdot}(t)| \tag{2.26}$$

which tends to zero by (2.22).

The mean value theorem yields that for some $\tilde{t}_{j,n} \in (H_j(U_{j,n}(t)) \wedge t,\ H_j(U_{j,n}(t)) \vee t)$

$$\begin{aligned} \sum_{j=1}^{m_n} \mu_{j,n}\ (U_{j,n}(t) - U_j(t)) &= \sum_{j=1}^{m_n} \mu_{j,n}\ (U_j(H_j(U_{j,n}(t))) - U_j(t)) \\ &= \sum_{j=1}^{m_n} \mu_{j,n}\ u_j(\tilde{t}_{j,n})(H_j(U_{j,n}(t)) - t). \end{aligned} \tag{2.27}$$

We now show that in this last expression we can replace $\mu_{j,n}u_j(\tilde{t}_{j,n})$ by $\sqrt{\mu_j}u_j(t)\sqrt{\mu_{j,n}}$. To this end we first remark that using (2.2) and (2.12) we have as $n \to \infty$ that

$$\max_{1 \leqslant j \leqslant m_n} \sup_{t \in (0,1)} |H_j(U_{j,n}(t)) - t| = \max_{1 \leqslant j \leqslant m_n} \sup_{t \in (0,1)} |H_{j,n}(U_j(t)) - t|$$

$$= (n\mu_{m_n})^{-1/2} \max_{1 \leqslant j \leqslant m_n} \|\tilde{B}_{j,n}\| + \mathrm{O_P}((n\mu_{m_n})^{-5/8}(\log m_n)^{1/8}(\log n)^{1/2})$$

$$= \mathrm{O_P}((n\mu_{m_n})^{-1/2} (\log m_n)^{1/2} + (n\mu_{m_n})^{-5/8} (\log m_n)^{1/8} (\log n)^{1/2})$$

$$= \mathrm{O_P}((n\mu_{m_n})^{-1/2}(\log m_n)^{1/2}). \tag{2.28}$$

A similar argument yields that

$$\max_{1 \leqslant j \leqslant m_n} \sqrt{n\mu_{j,n}} \sup_{t \in (0,1)} |H_j(U_{j,n}(t)) - t| = \mathrm{O_P}((\log m_n)^{1/2}) \quad (n \to \infty). \tag{2.29}$$

Using (C.5) we obtain that

$$|u_j(\tilde{t}_{j,n}) - u_j(t)| \leqslant c_3 |\tilde{t}_{j,n} - t| \leqslant c_3 \max_{1 \leqslant j \leqslant m_n} |H_j(U_{j,n}(t)) - t|$$
$$= \mathrm{O_P}((n\mu_{m_n})^{-1/2}(\log m_n)^{1/2}) \; (n \to \infty).$$

Hence, with (2.29) and the rate condition in the statement of the theorem we have that

$$\sqrt{n\mu_1 \log m_n} \sum_{j=1}^{m_n} \mu_{j,n} |u_j(\tilde{t}_{j,n}) - u_j(t)| \; |H_j(U_{j,n}(t)) - t|$$

$$= \mathrm{O_P}(\mu_1^{1/2}(n\mu_{m_n})^{-1/2} \log m_n) \sum_{j=1}^{m_n} \mu_{j,n}^{1/2} (\sqrt{n\mu_{j,n}} \; |H_j(U_{j,n}(t)) - t|)$$

$$= \mathrm{O_P}\left((\log m_n)^{3/2}(n\mu_{m_n})^{-1/2}\mu_1^{1/2} \left(\sum_{j=1}^{m_n} \mu_{j,n}^{1/2} \right) \right)$$

$$= \mathrm{o_P}(1) \quad (n \to \infty). \tag{2.30}$$

Next, using (C.5), (2.29), and $\max_{1 \leqslant j \leqslant m_n} |\mu_{j,n}^{1/2} - \mu_j^{1/2}| = \mathrm{O_P}((\log n)^{1/2} n^{-1/2})$ $(n \to \infty)$ we find

$$\sqrt{n\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_{j,n}^{1/2}(\mu_{j,n}^{1/2} - \mu_j^{1/2}) u_j(t)(H_j(U_{j,n}(t)) - t) \right|$$

$$= \mathrm{O_P}\left(\sqrt{\frac{\mu_1 \log m_n \log n}{n}} \right) \sum_{j=1}^{m_n} u_j(t) \; |\sqrt{n\mu_{j,n}}(H_j(U_{j,n}(t)) - t)|$$

$$= \mathrm{O_P}\left(\log m_n \sqrt{\frac{\mu_1 \log n}{n}} \; m_n \right)$$

$$= \mathrm{O_P}\left(\log m_n \sqrt{\frac{\mu_1 \log n}{n\mu_{m_n}}} \sum_{j=1}^{m_n} \mu_j^{1/2} \right) \quad (n \to \infty), \tag{2.31}$$

which is $o_P(1)$ as $n \to \infty$ because of the rate conditions in the statement of the theorem.

From (2.27), (2.30) and (2.31) it now remains to show that

$$\sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - t) \right| \xrightarrow{P} 0 \tag{2.32}$$

as $n \to \infty$ in order to verify (2.13).

To this end, as $|H_{j,n}(U_{j,n}(t)) - t| \leqslant (n\mu_{j,n})^{-1}$ a.s., the expression on the left-hand side of (2.32) is equal to

$$\sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - H_{j,n}(U_{j,n}(t)) \right|$$
$$+ O_P\left(\left(\sum_{j=1}^{m_n} \mu_j^{1/2} \right) \sqrt{\frac{\mu_1 \log m_n}{n\mu_{m_n}}} \right) \quad (n \to \infty). \tag{2.33}$$

Now, by (2.11),

$$\sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \sqrt{n\mu_{j,n}} (H_j(U_{j,n}(t)) - H_{j,n}(U_{j,n}(t))) \right|$$
$$= \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(H_j(U_{j,n}(t))) \right|$$
$$+ O_P\left(\mu_1^{1/2} \left(\sum_{j=1}^{m_n} \mu_j^{1/2} \right) (n\mu_{m_n})^{-1/8} (\log m_n)^{5/8} (\log n)^{1/2} \right). \tag{2.34}$$

Using the modulus of continuity behaviour of Brownian bridges together with (2.27), we get

$$\sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(H_j(U_{j,n}(t))) \right|$$
$$= \sqrt{\mu_1 \log m_n} \left| \sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(t) \right|$$
$$+ O_P\left(\mu_1^{1/2} \left(\sum_{j=1}^{m_n} \mu_j^{1/2} \right) (n\mu_{m_n})^{-1/4} (\log n)^{1/2} (\log m_n)^{3/4} \right). \tag{2.35}$$

Observe that because of the independence of the $\tilde{B}_{j,n}$ we have that

$$\sum_{j=1}^{m_n} \mu_j^{1/2} u_j(t) \tilde{B}_{j,n}(t) \sim \mathcal{N}\left(0, t(1-t) \sum_{j=1}^{m_n} \mu_j u_j^2(t) \right).$$

With (C.5)

$$t(1-t)\sum_{j=1}^{m_n}\mu_j u_j^2(t) = \mathrm{O}(1) \quad (n \to \infty)$$

and hence

$$\sqrt{\mu_1 \log m_n}\sum_{j=1}^{m_n}\mu_j^{1/2}u_j(t)\tilde{B}_{j,n}(t) \xrightarrow{\mathrm{P}} 0 \quad (n \to \infty). \tag{2.36}$$

Statements (2.33)–(2.36) yield (2.32), and (2.13) follows from (2.21)–(2.27) and (2.30)–(2.32).

Finally, statement (2.14) follows by (2.13), the behaviour of the modulus of continuity of Brownian bridges, and the independence of the $\tilde{B}_{j,n}$ ($j = 1,\ldots,m_n$). This concludes the proof of Theorem 3. □

Proof of Corollary 3. It suffices to show that, under $\mathrm{H}_0^{(3)}$, $\sqrt{n\mu_1 \log m_n}\max_{1\leqslant j\leqslant m_n}(U_j(t)-U_{\cdot}(t)) \to 0$ ($n \to \infty$) is implied by (C.7), (C.8) and the rate $n\mu_1 \log m_n(\max_{1\leqslant j\leqslant m_n} \mathrm{diam}\,(A_{j,n}))^4 \to 0$ ($n \to \infty$).

Let $K_x = [a_x, b_x]$ denote the shortt pertaining to F_x, let $\alpha_j = \inf_{x\in A_{j,n}} a_x$, $\tilde{\beta}_j = \sup_{x\in A_{j,n}} b_x$, and set

$$\beta_j = \alpha_j + U_{\cdot}(t),\ \tilde{\alpha}_j = \tilde{\beta}_j - U_{\cdot}(t).$$

Let a be such that $\alpha_j \leqslant a < a+U_{\cdot}(t) \leqslant \tilde{\beta}_j$. A Taylor expansion, using $f_x(a_x) = f_x(b_x)$ and (C.8), yields that for some $\tilde{a}_x \in (a_x \wedge a,\ a_x \vee a)$ and $\tilde{b}_x \in (b_x \wedge (a+U_{\cdot}(t)),\ b_x \vee (a+U_{\cdot}(t)))$ we have

$$\begin{aligned} t - F_x([a, a+U_{\cdot}(t)]) &= (F_x(a) - F_x(a_x)) - (F_x(a+U_{\cdot}(t)) - F_x(b_x)) \\ &= \tfrac{1}{2}(a-a_x)^2 f_x'(\tilde{a}_x) - \tfrac{1}{2}(a+U_{\cdot}(t)-b_x)^2 f_x'(\tilde{b}_x) \\ &\leqslant c_6(\alpha_j - \tilde{\alpha}_j)^2, \end{aligned}$$

and hence,

$$t - F_j([a, a+U_{\cdot}(t)]) \leqslant \max_{1\leqslant j\leqslant m_n}(\alpha_j - \tilde{\alpha}_j)^2 c_6 =: v_n.$$

Set $\eta = (U_j(t) - U_{\cdot}(t))/v_n$. Since $U_j(t) \geqslant U_{\cdot}(t)$, we have $\eta \geqslant 0$. Observe that for $y_1 \in [\tilde{\alpha}_j, \beta_j]$ and $y_2 \leqslant \alpha_j$ or $y_2 \geqslant \tilde{\beta}_j$ we have $f_j(y_1) \geqslant f_j(y_2)$. Hence, it readily follows that $[\tilde{\alpha}_j, \beta_j] \subset K_j$. This means that we can find an a as above such that $K_j = [a - \eta v_n, a + U_{\cdot}(t)]$ or such that $K_j = [a, a + U_{\cdot}(t) + \eta v_n]$.

Without loss of generality assume the first equality holds. Observe that the second condition in (C.5) implies that

$$\liminf_{n\to\infty}\ \min_{1\leqslant j\leqslant m_n}\ \inf_{y\in[a_{j,t}, b_{j,t}]} f_j(y) > 1/c_4.$$

Hence,

$$0 = t - F_j([a - \eta v_n,\ a + U_{\cdot}(t)]) \leqslant v_n - F_j([a - \eta v_n, a]) \leqslant v_n(1 - \eta/c_4),$$

which (when $v_n > 0$) implies $\eta \leqslant c_4$. This, in combination with $v_n\sqrt{n\mu_1 \log m_n} \to 0 \quad (n \to \infty)$, completes the proof. □

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