

Weighted constrained egalitarianism in TU-games

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Abstract

The constrained egalitarian solution of Dutta and Ray (1989) for TU-games is extended to asymmetric cases, using the notion of weight systems as in Kalai and Samet (1987,1988). This *weighted constrained egalitarian solution* is based on the *weighted Lorenz-criterion* as an inequality measure. It is shown that in general there is at most one such weighted egalitarian solution for TU-games. Existence is proved for the class of convex games. Furthermore, the core of a positive valued convex game is covered by weighted constrained egalitarian solutions.

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1 Introduction

The constrained egalitarian solution of Dutta and Ray (1989) is a solution concept for cooperative games with transferable utility which combines commitment to egalitarianism and promotion of individual interests in a consistent way. This solution is developed in a framework where, on one hand, each member of the society believes in egalitarianism as a social value, and on the other hand, private preferences dictate selfish behaviour. The constrained egalitarian solution, however, deals with completely symmetric players. In many situations this seems an overly strong assumption. For a discussion on examples where lack of symmetry is present, the reader is referred to Kalai and Samet (1987,1988) and Shapley (1981). It is assumed that the asymmetries between the players are reflected by an exogenously given vector of positive weights, which is based on considerations not captured by the parameters of the game itself.

In this paper we concentrate on the *weighted constrained egalitarian solution* that promotes the minimization of differences in weighted allocations subject to those constraints imposed by the selfishness of the players. So if the vector of shares proportional to the individual weights is *credible* with respect to these restrictions, then it is the weighted constrained egalitarian solution. In case all the players have equal weights, this approach leads again to the Dutta and Ray solution. We will extend the result of Dutta and Ray (1989) by showing that in general the weighted constrained egalitarian solution consists of at most one element. Where the constrained egalitarian solution is based on the Lorenz-ordering as an inequality measure, we use a weighted version like in Ebert (1997,1999) and Jaffray and Mongin (1998).

Existence for the class of convex games follows essentially by replication of the proof in Dutta and Ray (1989); a weighted version of the algorithm for calculation of the constrained egalitarian solution works in our case. The notion of weighted constrained egalitarian solutions extends to fully asymmetric situations through the concept of hierarchical systems (see the *weight systems* in Kalai and Samet (1987,1988)). Again, this notion corresponds to at most one egalitarian allocation. For convex games, we propose an algorithm for determining the constrained egalitarian solution incorporating a hierarchical system. It returns the concept of

weighted constrained egalitarianism of Hokari (1998). Where this work generalizes the results in Dutta and Ray (1989), Hokari (1998) generalizes the results in Dutta (1990). By variation of the hierarchical systems, the related constrained egalitarian solutions fill up the core of a positive valued TU game. This result is analogous to that for the class of weighted Shapley values (Monderer *et al.* (1992)).

2 Preliminaries

The cardinality of a finite set N is denoted $|N|$ and its power set by $\mathcal{P}(N)$. The set of real numbers is denoted \mathbb{R} . For a finite set N we denote by \mathbb{R}^N the set of all functions from N to \mathbb{R} . An element of \mathbb{R}^N will be identified with and $|N|$ -dimensional vector whose coordinates are indexed by the elements of N . Let $x \in \mathbb{R}^N$. For $i \in N$ we will use x_i to denote $x(i)$. Furthermore, for nonempty $S \subseteq N$ we write x_S for the restriction of x to S , i.e. $x_S = (x_i)_{i \in S}$. In addition $x(S) := \sum_{i \in S} x_i$. On \mathbb{R}^N we define the following relations. For $x, y \in \mathbb{R}^N$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. In addition $x > y$ if $x \geq y$ and $x \neq y$. If $x_i > y_i$ for all $i \in N$ then we write $x \gg y$. In addition we define $\mathbb{R}_+^N := \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}_{++}^N := \{x \in \mathbb{R}^N \mid x \gg 0\}$.

A *cooperative game with transferable utility* or *TU-game* (von Neumann and Morgenstern (1944)) is an ordered pair (N, v) , where N is a finite set and v is a function from the power set of N to the real numbers, i.e. $v : \mathcal{P}(N) \rightarrow \mathbb{R}$. The elements of N are called *players* and the *characteristic function* v associates with each *coalition* of players $S \subseteq N$ the number $v(S)$, interpreted as the benefits from cooperation of the coalition S . In particular, the benefits associated with the empty coalition are defined to be 0, i.e. $v(\emptyset) = 0$. The class of all TU-games is denoted \mathcal{G} ; the class of all TU-games with player set N is denoted \mathcal{G}^N . The vector $x \in \mathbb{R}^N$ is a *pre-imputation* for $(N, v) \in \mathcal{G}$ if $x(N) = v(N)$. Moreover, it is an *imputation* if it is an individual rational allocation as well, i.e. for all $i \in N$ it holds that $x_i \geq v(\{i\})$. The set of all pre-imputations for (N, v) is denoted by $I^*(N, v)$, and the set of all imputations is denoted $I(N, v)$. The *core* of the game (N, v) consists of all elements in $I^*(N, v)$ that satisfy collective rationality constraints as well, i.e. the set $\{x \in I^*(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. It will subsequently be denoted by $core(N, v)$. The elements of $core(N, v)$ are called *core elements*. A

game (N, v) is *convex* if for all $i \in N$ and all $S \subseteq T \subset N \setminus \{i\}$ it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

Throughout this paper we will assume a fixed and finite set of players $N = \{1, 2, \dots, n\}$ for some natural number $n \geq 1$. Since no confusion will arise, it will be sometimes convenient to abbreviate $(N, v) \in \mathcal{G}$ by v .

3 Positive weights

Suppose that an exogeneous asymmetry between the players of N is described by a vector $\omega \in \mathbb{R}_{++}^N$. The constrained egalitarian solution is based on the Lorenz-ordering as a measure of economic inequality¹. It is a measure that corresponds to the idea that a vector of payoffs x is more equal than another vector y , if according to x each fraction of the population consisting of the players with lowest payoffs gets (weakly) more than it would get in case of y . This happens if only if the *Lorenz-curve for x* gives (weakly) higher values everywhere on its domain than that corresponding to y . For the asymmetric case with weight ω we will use the following natural extension of this inequality measure (see Ebert (1997,1998)) . For $y \in \mathbb{R}^N$ take a permutation $\sigma_y : N \rightarrow N$ such that for all $i \in \{1, 2, \dots, n-1\}$

$$\frac{y_{\sigma_y(i)}}{\omega_{\sigma_y(i)}} \leq \frac{y_{\sigma_y(i+1)}}{\omega_{\sigma_y(i+1)}}.$$

Then the ω -Lorenz curve for y is the piecewise linear function $L_y^\omega : [0, \omega(N)] \rightarrow \mathbb{R}$ with $L_y^\omega(0) = 0$ and $L_y^\omega(\omega_{\sigma_y(i)}) = \sum_{j=1}^i y_{\sigma_y(j)}$ for all $i \in \{1, 2, \dots, n\}$, such that it is linear on each of the intervals of type $[0, \omega_{\sigma_y(1)}]$ and $(\omega_{\sigma_y(i)}, \omega_{\sigma_y(i+1)})$. More formally, for $p \in [0, \omega_{\sigma_y(1)}]$,

$$L_y^\omega(p) = p \cdot \frac{y_{\sigma_y(1)}}{\omega_{\sigma_y(1)}},$$

and for $p \in (\omega_{\sigma_y(i)}, \omega_{\sigma_y(i+1)})$, $i \in \{1, 2, \dots, n-1\}$,

$$L_y^\omega(p) = \sum_{j=1}^i y_{\sigma_y(j)} + \left(p - \sum_{j=1}^i \omega_{\sigma_y(j)} \right) \frac{y_{\sigma_y(i+1)}}{\omega_{\sigma_y(i+1)}}.$$

¹For an overview of inequality measures we refer the reader to Sen (1997).

Example 3.1 Let $y = (9, 3, 6)^T$ and $\omega = (1, 2, 3)^T$. Then $\sigma_y(1) = 2, \sigma_y(2) = 3$ and $\sigma_y(3) = 1$. Furthermore, $L_y^\omega : [0, 6] \rightarrow \mathbb{R}$ is defined through

$$L_y^\omega(p) = \begin{cases} \frac{3}{2}p & \text{if } p \in [0, 2], \\ 2p - 1 & \text{if } p \in (2, 5], \\ 9p - 36 & \text{if } p \in (5, 6]. \end{cases} \quad \triangleleft$$

Definition 3.2 Let $x, y \in \mathbb{R}^N$. Then y *weakly ω -Lorenz dominates* x , notation $yL^\omega x$, if $L_y^\omega(p) \geq L_x^\omega(p)$ for all $p \in [0, \omega(N)]$. If $yL^\omega x$ and not $xL^\omega y$, then y *L^ω -dominates* x . Accordingly, $x, y \in \mathbb{R}^N$ are *L^ω -comparable* if $xL^\omega y$ or $yL^\omega x$.

Example 3.3 Consider $z = (4, 4, 4)^T$ and y as in the above example. Then for $\omega = (1, 1, 1)^T$ we have $\sigma_z = (1 \ 2 \ 3)$ and $L_z^\omega(p) = 6p$ for all $p \in [0, 3]$. It is easy to show that $L_z^\omega(p) > L_y^\omega(p)$ for all $p \in (0, 3)$, and thus z L^ω -dominates y . \triangleleft

Example 3.4 The (weighted) Lorenz-ordering is not complete. Just consider $x = (4, 5)^T, y = (1, 9)^T$ and $\omega = (1, 2)^T$. Then $\sigma_y = (1 \ 2), \sigma_x = (2 \ 1)$, and thus

$$L_y^\omega(p) = \begin{cases} p & \text{if } p \in [0, 1], \\ 4\frac{1}{2}p - 3\frac{1}{2} & \text{if } p \in (1, 3] \end{cases}, \quad L_x^\omega(p) = \begin{cases} 2\frac{1}{2}p & \text{if } p \in [0, 2], \\ 4p - 3 & \text{if } p \in (2, 3]. \end{cases}$$

Then for $p \in (0, 1\frac{3}{4})$ it holds that $L_x^\omega(p) > L_y^\omega(p)$, while $L_x^\omega(p) < L_y^\omega(p)$ for $p \in (1\frac{3}{4}, 3)$. Therefore x and y are not L^ω -comparable. \triangleleft

So by definition we have that y (weakly) ω -Lorenz dominates x if it holds that all fractions of the grand coalition N associated with the lowest weighted payoffs are (weakly) better of with the allocation y than with x . Note that we do not restrict ourselves to comparison of only allocations x, y with $x(N) = y(N)$. In Example 3.4 we have $x(N) = 9 < 10 = y(N)$. In our sense, x ω -Lorenz dominates the allocation y if, accordingly, all configurations of players with the lowest weighted payoffs are better of with x than y . Only in this section we will only be concerned with the more usual and restricted concept since we will compare only allocations x and y if they are *feasible* payoff vectors in the game (N, v) for some coalition $S \subset N$, i.e. $x(S) = y(S) = v(S)$.

According to Dasgupta *et al.* (1973) and Rothschild and Stiglitz (1973) it holds for two allocations $x, y \in \mathbb{R}^N$ such that $x(N) = y(N)$ and $\omega = e_N$ that x Lorenz-dominates y if and only if x can be obtained from y by a finite sequence of progressive transfers and permutations, i.e. if payoff is redistributed from a rich player to a poor one this means a strict decrease of inequality. Furthermore, Atkinson (1970) shows that x Lorenz-dominates y if and only if x is preferred to y by any increasing and concave social welfare function. Ebert (1997) generalizes the ideas and shows the following equivalence. Denote by $\Omega(\omega)$ for $\omega \in \mathbb{R}_{++}^N$ the set of all functions $W : \mathbb{R}^N \rightarrow \mathbb{R}$ such that there is a continuous, increasing and concave function $U : \mathbb{R} \rightarrow \mathbb{R}$ with

$$W(z) = \sum_{i \in N} \omega_i U\left(\frac{z_i}{\omega_i}\right).$$

The characteristic U is seen as a normalized utility function, and W a corresponding social welfare function taking into account the impacts of the different participants. Ebert (1999) characterizes the weighted Lorenz ordering as follows:

Theorem 3.5 (Ebert (1999)) *Let $x, y \in \mathbb{R}^N$ such that $x(N) = y(N)$. Let $\omega \in \mathbb{R}_{++}^N$ be a vector of weights. The following statements are equivalent*

- (i) $xL^\omega y$,
- (ii) $W(x) \geq W(y)$ for all $W \in \Omega(\omega)$.

Remark 3.6 Consider $(N, v) \in \mathcal{G}$. In case $\omega_i = \omega_j$ for all $i, j \in N$ then L^ω is no more than the well known Lorenz-ordering. The element $(\frac{\omega_i}{\omega(N)}v(N))_{i \in N}$ is the L^ω -dominant payoff vector in the hyperplane $\{x \in \mathbb{R}^N \mid x(N) = v(N)\}$.

Example 3.7 Let $(N, v) \in \mathcal{G}$ such that $N = \{1, 2, 3\}$, $v(N) = 12$, and let $\omega = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})^T$ be the vector of weights. Then the corresponding vectors of weighted payoffs for $x = (2, 4, 6)^T$ and $y = (6, 4, 2)^T$ respectively, together with the corresponding ordering permutations are $(4, 16, 24)^T$, $\sigma_x = (1 \ 2 \ 3)$ and $(12, 16, 8)^T$, $\sigma_y = (3 \ 1 \ 2)$ respectively. The reader may verify that y L^ω -dominates x , and that the L^ω -maximal pre-imputation for (N, v) is $(6, 3, 3)^T$. \triangleleft

For $A \subseteq \mathbb{R}^K$ with K being a finite subset in \mathbb{N} , we define $E^\omega(A)$ as the set of all payoff vectors in A that are ω -Lorenz undominated within A . More formally,

E^ω is the map on the domain $\{A \mid A \subseteq \mathbb{R}^K \text{ for } K \subset \mathbb{N}, |K| < \infty\}$ defined by

$$E^\omega(A) = \{x \in A \mid \text{there is no } y \in A, y \neq x \text{ such that } y L^\omega x\}.$$

It is easy to construct examples of sets A for which $E^\omega(A)$ is empty. However, when A is non-empty and closed, then $E^\omega(A) \neq \emptyset$. Secondly, for $A \subset \mathbb{R}$, we have $|A| = 1$ and $|E^\omega(A)| = 1$. In general $E^\omega(A)$ will not be a singleton set by the partial nature of the ω -Lorenz ordering. Fix $(N, v) \in \mathcal{G}$. Analogous to what has been done by Dutta and Ray (1989), we define the ω -Lorenz cores of coalitions, in a recursive way. The ω -Lorenz core of a singleton coalition is $L^\omega(\{i\}) := \{v(i)\}$. Now suppose that the ω -Lorenz cores for all coalitions of cardinality k or less have been defined, where $1 < k < n$. The ω -Lorenz core of $S \subset N$ with $|S| = k + 1$ is defined by

$$L^\omega(S) := \{x \in \mathbb{R}^S \mid x(S) = v(S), \nexists T \subset S, y \in E^\omega(L^\omega(T)), y > x_T\}.$$

For $(N, v) \in \mathcal{G}$, the *subgame for* $T \subset N$ is game $(T, v|_T) \in \mathcal{G}$ such that $v|_T(S) = v(S)$ for all $S \subseteq T$. The Lorenz core captures the idea of consistent egalitarianism, in the sense that a coalition T may successfully reject a proposed allocation, only if it is able to show an egalitarian solution for its own subsociety, independent from other players, in the subgame $(T, v|_T)$, implying a weak improvement for all, and a strict improvement for at least one of the players in T . In the sequel, as no confusion arises, we will write $\text{EL}^\omega(S)$ instead of $E^\omega(L^\omega(S))$. If $x \in \mathbb{R}^T$, and there is $S \subset T$, $y \in \text{EL}^\omega(S)$ with $y > x_S$, we say that y ω -Lorenz blocks x (L^ω -blocks x). In this fashion we will also say that S L^ω -blocks T . A coalition T will be called *viable* if $\text{EL}^\omega(T)$ is nonempty. A ω -constrained egalitarian allocation exists if the grand coalition is viable for ω . Then $\text{EL}^\omega(N)$ is the set of ω -constrained egalitarian allocations.

The approach is different from Arin and Iñarra (1997) or Arin *et al.* (1998) who focus on the core as the basic stability concept. In their terminology the *egalitarian set* becomes the set of all Lorenz-undominated elements in the core. In this respect we stress that we do not restrict ourselves to balanced games. Furthermore, for 2-person games the solution concepts coincide, since then the ω -Lorenz core equals the core of the game for all weight vectors $\omega \in \mathbb{R}_{++}^N$. We will give an example to illustrate this.

Example 3.8 Consider the 2-person game (N, v) , that is specified through $N = \{1, 2\}$, $v(\{1\}) = 4$, $v(\{2\}) = 3$ and $v(N) = 8$. Then $\text{EL}^\omega(\{1\}) = v(\{1\}) = 4$, $\text{EL}^\omega(\{2\}) = v(\{2\}) = 3$. Next we have

$$L^\omega(N) = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 8, x_1 \geq 4 \text{ and } x_2 \geq 3\} = \text{core}(N, v).$$

The L^ω -dominant element in this set is $(5, 3)^T$, and therefore $\text{EL}^\omega(N) = \{(5, 3)^T\}$.

◁

There is no tight relationship between the existence of the egalitarian solution and the balancedness of a game. Dutta and Ray (1989) shows that the egalitarian solution exists in case (N, v) is convex. However, they provide an example of a *totally balanced game*² for which the egalitarian solution does not exist. As we will show, restricting ourselves to balanced games which are *average convex*³ will not help.

Example 3.9 Consider $(N, v) \in \mathcal{G}$ be the three person game defined by $N = \{1, 2, 3\}$, $v(N) = 1$, $v(\{1, 2\}) = v(\{1, 3\}) = \frac{7}{9}$, $v(\{2, 3\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1\}) = \frac{1}{3}$. Straightforward calculations yield $\text{EL}(\{1\}) = \{v(1)\} = \frac{1}{3}$, $\text{EL}(\{2\}) = \{0\}$, $\text{EL}(\{3\}) = \{0\}$, $\text{EL}(\{1, 3\}) = \text{EL}(\{1, 2\}) = \{(\frac{7}{18}, \frac{7}{18})^T\}$. But then $L(N) = \{x \in \mathbb{R}_+^N \mid x(N) = 1, x_1 > \frac{7}{18}\}$ and therefore $\text{EL}(N) = \emptyset$, while $(1, 0, 0)^T \in \text{core}(N, v)$. Moreover, it is easy to check that (N, v) is average convex.

◁

Essentially, in the above example the constrained egalitarian solution does not exist for the reason that the Lorenz-dominant pre-imputation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is blocked. Shifting some power from the players 2 and 3 to player 1 restores nonemptiness, as is shown in the next example.

Example 3.10 Notice that for the vector of relative weights $(18, 3, 1)^T$ in the game in the former example 3.9 we calculate, consecutively, $\text{EL}^\omega(\{i\}) = v(\{i\})$ for $i = 1, 2, 3$, $\text{EL}^\omega(\{1, 2\}) = \{(\frac{2}{3}, \frac{1}{9})^T\}$, $\text{EL}^\omega(\{1, 3\}) = \{(\frac{126}{171}, \frac{8}{171})^T\}$, and $\text{EL}^\omega(\{2, 3\}) = \{(0, 0)^T\}$. The ω -Lorenz dominant pre-imputation for (N, v) is the vector $y = (\frac{18}{22}, \frac{3}{22}, \frac{1}{22})^T$. It is easily verified that it is not blockable by any subcoalition in N .

²A game is *totally balanced* if all corresponding subgames are balanced.

³A game (N, v) is *average convex* (see Iñarra and Usategui (1995)) if for all $i \in S \subset T \subseteq N$ it holds that $\sum_{i \in S} \{g(S) - g(S \setminus \{i\})\} \leq \sum_{i \in S} \{g(T) - g(T \setminus \{i\})\}$.

Therefore it must hold $\text{EL}^\omega(N) = \{y\}$. So we obtain $\text{EL}^\omega(N) \neq \emptyset$ while $\text{EL}(N) = \emptyset$. Especially we see that by changing the relative weights, (non)emptiness of the weighted constrained egalitarian solution is *not* preserved. \triangleleft

The main result in Dutta and Ray (1989) applies to the generalized class of weighted constrained egalitarian solutions.

Theorem 3.11 *There is at most one ω -constrained egalitarian allocation for $(N, v) \in \mathcal{G}$ and $\omega \in \mathbb{R}_{++}^N$.*

Proof. The theorem is obtained as corollary to Theorem 4.6. \square

In the sequel we will write $\text{CES}^\omega(N, v)$ for the ω -constrained egalitarian solution associated with (N, v) . When the weights are all equal, we will write $\text{CES}(N, v)$. The next achievement of Dutta and Ray (1989) is showing nonemptiness of the constrained egalitarian solution for the class of convex games. The result is constructive; Dutta and Ray (1989) show an algorithm, though of exponential complexity, for calculating $\text{CES}(N, v)$ for convex (N, v) . Essentially, the same approach gives the weighted constrained egalitarian solution (see also Hokari (1998)). The procedure differs from that of Dutta and Ray (1989) only by the notion of average value of coalitions. For standard fixed tree games the procedure can be adapted in order to be able to calculate the weighted constrained egalitarian solution in polynomial time (see Koster *et al.* (1998)).

Definition 3.12 The *average value* of a coalition S in a game $(N, v) \in \mathcal{G}$ for a given vector $x \in \mathbb{R}_+^N$ is defined by

$$\alpha_x(v, S) = \begin{cases} \frac{v(S)}{x(S)} & \text{if } x(S) > 0, \\ \infty & \text{if } x(S) = 0. \end{cases}$$

In this chapter we consider only $\alpha_\omega(v, S)$ for $\omega \in \mathbb{R}_{++}^N$, so in this definition only the first line applies. For convex games the set of coalitions that maximize the average weighted value is closed under union, as we will show in the next lemma.

Lemma 3.13 *If (N, v) is convex, then the set of coalitions that maximize the weighted average value is closed under union.*

Proof. Let $\mu := \max \{\alpha_\omega(v, S) \mid S \subseteq N\}$. Suppose there are two maximally weighted coalitions S and T with $\alpha_\omega(v, S) = \alpha_\omega(v, T) = \mu$. Then $\alpha_\omega(v, S \cup T) \leq \mu$, since S and T have maximally weighted average value. Suppose that $\alpha_\omega(v, S \cup T) < \mu$. Then

$$\begin{aligned}
v(S \cup T) &< \frac{v(S)}{\omega(S)} \omega(S \cup T) \\
&= \frac{v(S)}{\omega(S)} \{\omega(S) + \omega(T) - \omega(S \cap T)\} \\
&= v(S) + \frac{v(T)}{\omega(T)} \omega(T) - \frac{v(S)}{\omega(S)} \omega(S \cap T) \\
&= v(S) - v(T) - \frac{v(S)}{\omega(S)} \frac{\omega(S \cap T)}{v(S \cap T)} v(S \cap T) \\
&\leq v(S) + v(T) - v(S \cap T),
\end{aligned}$$

since $\frac{v(S)}{\omega(S)} \geq \frac{v(S \cap T)}{\omega(S \cap T)}$. This contradicts with the fact that (N, v) is convex. \square

Lemma 3.14 *If (N, v) is convex, then there is a unique maximally weighted coalition that maximizes the average value.*

Proof. This is a direct consequence of Lemma 3.13. \square

For a convex game (N, v) the weighted constrained egalitarian solution is calculated as follows.

Algorithm 3.15

1. *Input:* $(N, v) \in \mathcal{G}$, $\omega \in \mathbb{R}_{++}^N$.
2. *Set* $v_1 = v$, $N_1 := N$.
3. *Repeat, as long as* $N_i \neq \emptyset$, *the following step.*
If possible determine the unique maximally weighted coalition $S_i \subseteq N_i$, *that maximizes the weighted average value in* v_i . *In presence of such a set* S_i , *define for* $j \in S_i$, $\text{ALG}_j(v, \omega) := \omega_j \alpha_\omega(v_i, S_i)$. *Let* $N_{i+1} := N_i \setminus S_i$, *and*

$(N_{i+1}, v_{i+1}) \in \mathcal{G}^{N_{i+1}}$ by

$$v_{i+1}(S) = v_i(S \cup S_i) - v_i(S_i) \text{ for all } S \subseteq N_{i+1}.$$

Stop if there is no such set S_i , and put $\text{ALG}(N, v, \omega) := \emptyset$.

4. Output is $\text{ALG}(N, v, \omega)$.

Dutta and Ray (1989) noticed that in case of a convex game (N, v) and with equally weighted players, $\text{ALG}(N, v, \omega)$ is the constrained egalitarian solution, the Lorenz-dominant element in $\text{core}(N, v)$. This result is easily extended in the next theorem.

Theorem 3.16 *The allocation $\text{ALG}(N, v, \omega)$ equals $\text{CES}^\omega(N, v)$. Moreover it is the L^ω -dominant element in $\text{core}(N, v)$.*

Hokari (1998) provides alternative expressions for calculating the weighted constrained egalitarian solution.

Remark 3.17 The weighted constrained egalitarian solution never gives coalitions of players less than their average values. It is this property that Klijn *et al.* (1998) call *equal division stability* in case of all weights being one. It also plays a role in Selten (1972).

Dutta and Ray (1989) show that convexity is not a necessary condition in order to ensure existence of the constrained egalitarian solution. It is shown that whenever the above allocation procedure determines an element in the Lorenz core for the grand coalition N , then the resulting allocation is in fact the constrained egalitarian solution. This result carries over to our model, using exact copies of the proofs in Dutta and Ray (1989),

Theorem 3.18 *Suppose $\text{ALG}(N, v, \omega) \neq \emptyset$ determines an element in $L^\omega(N)$. Then $\text{EL}^\omega(N) = \{\text{ALG}(N, v, \omega)\}$, so $\text{CES}^\omega(N, v) = \text{ALG}(N, v, \omega)$.*

An application to cooperative production situations

The notion of weighted constrained egalitarianism is generalized for the class of all

cooperative *cost* games, where the values of the characteristic function are interpreted in terms of costs instead of benefits. This is done in the obvious way, by simply reversing the strict inequality sign in the definition of the ω -Lorenz core. Then correspondingly the weighted constrained egalitarian solution is the set of all ω -Lorenz undominated elements in the ω -Lorenz core for N . Then the same proofs can be used to show that the solution consists of one element at the most, and that nonemptiness holds for the class of concave cost games, i.e., those games (N, k) such that for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$ it holds

$$k(S \cup \{i\}) - k(S) \geq k(T \cup \{i\}) - k(T).$$

The weighted constrained egalitarian solution is calculated using an adaptation of Algorithm 3.15 where the focus is on the maximally weighted coalition that *minimizes* the weighted average cost.

Consider the situation that a collective of agents N share the technology for the production of a certain perfectly divisible good. Assume that the technology is described by a cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that for each output level t , $c(t)$ stands for the level of input/costs. Moreover we take c nondecreasing and assume the absence of fixed cost, which amounts to $c(0) = 0$. The agents in N have demands for output as is summarized by the vector $q \in \mathbb{R}_+^N$, such that agent i demands q_i . Then the aggregate demand $q(N)$ is produced and the corresponding total costs $c(q(N))$ have to be shared. The ordered pair (q, c) is called a cooperative production situation. The *stand alone cost game* for the situation (q, c) is the cost game (N, c^{SA}) defined by $c^{\text{SA}}(S) := c(q(S))$ for all $S \subseteq N$. It is easily shown that if c is a concave function, then the corresponding stand alone cost game is concave. But then by Theorem 3.16 we obtain an easy way to determine the corresponding weighted constrained egalitarian solution for a specified vector of weights $\omega \in \mathbb{R}_{++}^N$. Suppose that each agent $i \in N$ demands a positive amount $q_i > 0$. Then with weight vector $\omega = q$ we show equality of the vector of average cost shares and the corresponding weighted egalitarian solution for the stand alone game (N, c^{SA}) .

Theorem 3.19 *Let c be a concave cost function and let $q \in \mathbb{R}_{++}^N$ be the profile of individual demands. Then for all $i \in N$ we have*

$$\text{CES}_i^q(N, c^{\text{SA}}) = \frac{q_i}{q(N)} c(q(N)). \quad (1)$$

Proof. The concavity of c implies increasing returns to scale, i.e. the minimal average cost in the stand alone cost game is attained by the grand coalition. Then the combination of Theorem 3.16 and Algorithm 3.15 implies

$$\text{CES}_i^q(N, c^{\text{SA}}) = \frac{q_i}{q(N)} c^{\text{SA}}(N) = \frac{q_i}{q(N)} c(q(N)). \quad \square$$

Tijs and Koster (1998) model the cooperative production situation by the *pessimistic cost game*. For absolutely continuous cost functions c , define for $y \in [0, q(N)]$

$$c^*(y) := \sup \left\{ \int_U c'(t) dt \mid U \in \mathcal{B}([0, q(N)]), \lambda(U) = y \right\}. \quad (2)$$

Here $\mathcal{B}([0, q(N)])$ and λ denote the Borel- σ -algebra on $[0, q(N)]$ and the Lebesgue measure respectively. Then c^* is trivially extended to a (concave) *pessimistic* cost function \bar{c}^* . Now the *pessimistic cost game* (N, c^{P}) corresponding to the cooperative production situation (q, c) is defined by considering the stand alone game corresponding to (q, \bar{c}^*) , i.e.

$$c^{\text{P}}(S) := \bar{c}^*(q(S)) \quad \text{for all } S \subseteq N.$$

But then by Theorem 3.19 and the fact that $\bar{c}^*(q(N)) = c(q(N))$ we immediately obtain that average cost shares equal the weighted constrained egalitarian allocations where the demand vector is taken as the vector of weights. A precise statement is given below.

Theorem 3.20 *Consider the cooperative production problem (q, c) such that $q \in \mathbb{R}_{++}^N$ and c is absolutely continuous. Then it holds for all $i \in N$ that*

$$\text{CES}_i^q(N, c^{\text{P}}) = \frac{q_i}{q(N)} c(q(N)).$$

4 Hierarchical systems

Clearly, the above framework still lacks the possibility of dealing with completely asymmetric situations, where it is socially desirable to enforce the largest transfer

of all the economic opportunities from a certain group of players S to those in $N \setminus S$, such that it is not in the interest of any subcoalition of S to separate. S should enjoy as less as possible from the benefits of the grand coalition, apart from what they are able to generate themselves through internal cooperation. Also within S there may be a set S' of players that should have *zero impact* compared to those in $S \setminus S'$, expressing that society demands the highest possible transfer of the economic prosperity of this group to the higher rewarded players in $N \setminus S$ and $S \setminus S'$. We will focus on the situation where the society can enforce these transfers, without needing the consent of players of S' . Still we will allow a proposed allocation to be attacked by S' if some of its members are able to do better without the support of other players. So society may enforce cooperation on a large scale, but it has to be sensible to the possible disagreements raised by the selfishness of the subgroups of players. A way to model fully asymmetric situations is by *hierarchical systems*. The concept is, in mathematical terms, equivalent to the weight systems in Kalai and Samet (1987,1988), Monderer *et al.* (1992)), however the interpretation is different. Using hierarchical systems, we will be able to generalize weighted constrained egalitarian ideas.

Definition 4.1 A *hierarchical system* for N consists of an ordered pair $\Sigma = (\mathcal{S}, \mathcal{W})$, where $\mathcal{S} = (S_1, S_2, \dots, S_k)$ is an ordered partition of N , and $\mathcal{W} = (\omega_1, \omega_2, \dots, \omega_k)$ is the k -tuple with corresponding weights, $\omega_i \in \mathbb{R}_{++}^{S_i}$ for all $i \in \{1, 2, \dots, k\}$.

The partition \mathcal{S} stands for the hierarchical structure of the relative impacts. More specifically, the players in S_j are said to have zero impact compared to those in S_i whenever $i < j$. Within each such layer S_i , we take ω_{S_i} to represent the relative weights. Note that whenever \mathcal{S} is just the trivial partition such that $k = 1$ and $S_k = \{1, 2, \dots, n\}$, then we are back in a situation as we studied already in the former section. A hierarchical system $\Sigma = (\mathcal{S}, \mathcal{W})$ for N defines for each $T \subset N$ a hierarchical system $\Sigma_T = (\mathcal{S}_T, \mathcal{W}_T)$, where $\mathcal{S}_T = (T_1, T_2, \dots, T_r)$ is the ordered partition of T induced by \mathcal{S} , such that $T_k = T \cap S_k$ for $k \in \{1, 2, \dots, r\}$, and $\mathcal{W}_T = (\omega_{T_1}, \omega_{T_2}, \dots, \omega_{T_k})$ is the corresponding tuple of weight vectors from \mathcal{W} induced by \mathcal{S} . Let $T(0) = \emptyset$ and $T(i) = \cup_{j=1}^i T_j$, and for $x \in \mathbb{R}^T$ and $i \in \{1, 2, \dots, r\}$, denote the Lorenz mapping corresponding to x_{T_i} and the weight vector ω_{T_i} by $L_x^{T_i}$. On \mathbb{R}^T we define the Lorenz-ordering with respect to Σ as follows:

Definition 4.2 Let $x, y \in \mathbb{R}^T$. Then y is said to *weakly Σ -Lorenz dominate* x , notation $yL^\Sigma x$, if for all $i \in \{1, 2, \dots, r\}$ it holds

$$y(T(i-1)) + L_y^{T_i}(p) \geq x(T(i-1)) + L_x^{T_i}(p) \text{ for all } p \in [0, 1].$$

In case the above inequalities are strict for some i and p , then y L^Σ -dominates x .

Basically, this ordering has the property that any transfer of profits to players in top layers of the hierarchy is positively rewarded. Suppose that we have two payoff vectors, x and y , that prescribe the same payoff per layer. This means that for all $i \in \{1, 2, \dots, r\}$ we have $x(T(i)) = y(T(i))$. Then $yL^\Sigma x$ if and only if the restriction of y to each set T_i is considered to be (weakly) more egalitarian than the restriction of x to T_i , if we use the weighted Lorenz-criterion as the measure of inequality and take ω_{T_i} as the corresponding weight vector.

Remark 4.3 The above extension of the usual Lorenz-ordering depends only on the *relative* weights; for all $\alpha \in \mathbb{R}_{++}^k$ and $\bar{\omega}_{S_i} = \alpha_i \omega_{S_i}$ we have

$$xL^\Sigma y \iff xL^{\bar{\Sigma}} y,$$

where $\bar{\Sigma}$ is the hierarchical system out of Σ with weights $(\bar{\omega}_{S_i})_{i \in \{1, 2, \dots, k\}}$.

Example 4.4 Let $\Sigma = ((\{1\}, \{2\}), (1, 1))$ be a weight system for $N = \{1, 2\}$. Define $x, y \in \mathbb{R}^N$ by $x = (1, 1)^T$ and $y = (2, 0)^T$. For $p \in (0, 1]$ we have $L_y^{\{1\}}(p) > L_x^{\{1\}}(p)$, and

$$y_1 + L_y^{\{2\}}(p) = 2 + 0 \geq 1 + p = x_1 + L_x^{\{2\}}(p).$$

Hence y L^Σ -dominates x . ◁

Analogous to the former section, we define the set of undominated allocations and Lorenz cores with respect to a hierarchical system Σ as follows. For $A \subset \mathbb{R}^K$ with $K \subseteq N$, we define the set $E^\Sigma(A)$ as the set of all elements that are undominated within A with respect to L^Σ . The Σ -Lorenz core of a single player $i \in N$ is defined as $\{v(i)\}$. Given the Σ -Lorenz core for all coalitions of size k or smaller, we define for $S \subseteq N$ with $|S| = k + 1$ the Σ -Lorenz core by

$$L^\Sigma(S) := \{x \in \mathbb{R}^S \mid x(S) = v(S), \nexists T \subset S \{y \in E^\Sigma(L^\Sigma((T)), y > x_T)\}\}.$$

Instead of $E^\Sigma(L^\Sigma(S))$ we will use a more convenient notation, i.e. $\text{EL}^\Sigma(S)$. In the line of the former section a coalition S is considered *viable for Σ* if $\text{EL}^\Sigma(S)$ is nonempty. We say that a Σ -egalitarian solution exists if the grand coalition is viable for Σ .

Remark 4.5 It is immediately seen that $\text{core}(S, v|_S) \subseteq L^\Sigma(S)$ for all $S \subseteq N$, where $(S, v|_S)$ denotes the subgame from (N, v) induced by S . Especially, $\text{core}(N, v) \subseteq L^\Sigma(N)$.

We will prove that there can be at most one constrained egalitarian solution with respect to a particular weight system, and for trivial weight systems in particular (see the statement in Theorem 3.11).

Theorem 4.6 *The set $\text{CES}^\Sigma(N, v)$ consists of one element at most.*

Proof. Essentially the proof hinges on ideas laid down in Dutta and Ray (1989). Fix a TU-game (N, v) . We claim that $|\text{EL}^\Sigma(S)| \leq 1$ for all $S \subseteq N$.

Whenever $|S| \leq 2$, the above claim is clearly true. Assume that for all coalitions S with $|S| \leq k < n$ the claim holds. We will show that it must hold for coalitions of size $k + 1$.

Suppose on the contrary, that there is a coalition S of size $k + 1$ and two distinct allocations $y, y' \in \text{EL}^\Sigma(S)$. Let $r(1), r(2), \dots, r(q)$ be an increasing row of indices such that S is the union of the sets $S^t := S \cap S_{r(t)} \neq \emptyset$ for $t = 1, 2, \dots, q$. Let $m \in \{1, 2, \dots, q\}$ be the highest number such that $y_{S^m} \neq y'_{S^m}$. Without loss of generality we assume that the players in S^m are numbered 1 to $|S^m|$ such that for all $i, j \in \{1, 2, \dots, |S^m|\}$ with $j \geq i$ it holds $\frac{y_i}{\omega_i} \geq \frac{y_j}{\omega_j}$.

Define i as the smallest integer in S^m for which $y_i \neq y'_i$. Then either (i) $y_i < y'_i$ or (ii) $y_i > y'_i$.

Case (i). Suppose $y_i < y'_i$. Then

$$P(i, y') := \bigcup_{t < m} S^t \cup \left\{ j \in S^m \mid \frac{y'_j}{\omega_j} < \frac{y'_i}{\omega_i} \right\} \neq \emptyset.$$

Define $M(i) := \{T \subset S \mid i \in T, T \text{ is viable}\}$. Choose any $T \in M(i)$. For every such viable coalition T of size smaller than k , the existence of a vector y^T such that

$\text{EL}^\Sigma(T) = \{y^T\}$ is guaranteed by our induction hypothesis. Because $y' \in \text{EL}^\Sigma(S)$ it can not be L^Σ -blocked by T . As a result one of the following statements must be true:

- a) $y'_i > y_i^T$,
- b) $y'_j > y_j^T$ for some $j \in T, j \neq i$,
- c) $y'_j = y_j^T$ for all $j \in T$.

Claim: If c) holds, then $T \cap P(i, y') \neq \emptyset$. We prove this by contradiction. So suppose it is not the case that $T \cap P(i, y') \neq \emptyset$. Then first of all $T \cap S^t = \emptyset$ for all $t = 1, 2, \dots, m-1$, and consequently $T \subset \cup_{t=m}^q S^t$. Then $\frac{y'_j}{\omega_j} \geq \frac{y'_i}{\omega_i}$ for all $j \in T \cap S^m$. But then we have for $j \in \cup_{t=m+1}^q S^t \cap T$ as well as for $j \in \{1, 2, \dots, i-1\}$, by definition of i ,

$$y_j^T = y'_j = y_j, \quad y_i^T = y'_i > y_i.$$

Moreover, for $j \in T \cap S^m$ such that $j > i$ it holds

$$\frac{y_j^T}{\omega_j} = \frac{y'_j}{\omega_j} \geq \frac{y'_i}{\omega_i} > \frac{y_i}{\omega_i} \geq \frac{y_j}{\omega_j}.$$

So $y_j^T > y_j$. But then y is L^Σ -blocked by coalition T , thereby contradicting the assumption that $y \in \text{EL}^\Sigma(S)$. So the claim is established.

Now let $M'(i)$ be the subset of coalitions of $M(i)$ for which a) is true. Then $M'(i) \neq \emptyset$, because $y'_i > y_i \geq v(\{i\})$, so $\{i\} \in M'(i)$. Choose $\delta > 0$ so that the following two inequalities are satisfied,

$$\delta < \min_{T \in M'(i)} (y'_i - y_i^T), \quad (3)$$

$$\frac{1}{\omega_j} (y'_j + \frac{\delta}{z}) < \frac{1}{\omega_i} (y_i - \delta) \text{ for all } j \in P(i, y'). \quad (4)$$

In the last inequality z is defined as $|P(i, y')|$. We construct a feasible allocation for S, y'' , in the following way,

$$\begin{aligned} y''_j &= y'_j && \text{for all } j \notin P(i, y') \cup \{i\}, \\ y''_i &= y'_i - \delta, \\ y''_j &= y'_j + \frac{\delta}{z} && \text{for } j \in P(i, y'). \end{aligned}$$

Clearly, y'' L^Σ -dominates y' , since a transfer is made only to players that are located higher in the hierarchy, or players that have lower weighted payoffs.

Notice that by the second condition on δ , per layer in the hierarchy, the same ordering of the set of weighted allocations is established. Then it is immediately seen that $y''L^\Sigma y'$. So we have arrived at a contradiction if we can show that no coalition $T \subset S$ can L^Σ -block y'' .

Discern three cases:

- (I) Let $T \subseteq S$ and suppose that $T \notin M(i)$. Then, either $i \in T$ and T is not viable, or T is viable but $i \notin T$. In the latter case, all the players but i get more in y'' . Since T cannot block y' by assumption, it certainly cannot do better than y'' , i.e. T cannot L^Σ -block y'' .
- (II) By our choice of δ , it is also clear that if $T \in M'(i)$, T cannot L^Σ -block y'' .
- (III) Finally, if $T \in M(i)$, and satisfies b) or c), then we use the above claim and the definition of y'' to argue that there is $j \in T$ with $y''_j > y'_j$. But then, also in this case, T cannot L^Σ -block y'' .

Since (I), (II) and (III) together exhaust all possibilities, we must have $y'' \in L^\Sigma(S)$ and $y''L^\Sigma y$, which gives the desired contradiction.

Case (ii). Suppose $y_i > y'_i$. We can then assume, without loss of generality, that

$$P(i, y) := \bigcup_{j=1}^{m-1} S^j \cup \left\{ j \in S^m \mid \frac{y_j}{w_j} < \frac{y_i}{w_i} \right\}$$

is not empty. Then, for if this is not true, $m = 1$ and $\frac{y_j}{w_j} = \frac{y_i}{w_i}$ for all $j \in S^m$ with $j > i$. Since y, y' are both feasible, there must be some $j > i$ such that $\frac{y'_j}{w_j} > \frac{y_i}{w_i}$. Then one can verify that there is a $p^* \in (0, \omega(S^m)]$ such that for $\bar{\omega} := \omega_{S^m}$,

$$L_{y'}^{\bar{\omega}}(p) = L_y^{\bar{\omega}}(p) \quad \text{if } p \in [p^*, \omega(S^m)],$$

$$L_{y'}^{\bar{\omega}}(p) < L_y^{\bar{\omega}}(p) \quad \text{if } p \in (0, p^*).$$

But this means that $yL^\Sigma y'$, since on layers S_t with $t > 1$ the payoffs according to y and y' are the same. This provides a contradiction with the assumption that $y' \in \text{EL}^\Sigma(S)$. So we may assume that $P(i, y) \neq \emptyset$. Again, for $T \in M(i)$ one of the following statements will be true

$$\text{a')} \quad y_i > y_i^T,$$

$$\text{b')} \quad y_j > y_j^T \text{ for some } j \in T, j \neq i,$$

$$\text{c')} \quad y_j = y_j^T \text{ for all } j \in T.$$

Define $M''(i) := \{T \in M(i) \mid y_i > y_i^T\}$. Then $M''(i) \neq \emptyset$, because $y_i > y'_i \geq v(\{i\})$, so $\{i\} \in M''(i)$. Again, choose $\delta > 0$ small enough such that the following conditions are satisfied,

$$\delta < \min_{T \in M''(i)} (y_i - y_i^T) \quad (5)$$

$$\frac{1}{\omega_j} \left(y_j + \frac{\delta}{z} \right) < \frac{1}{\omega_i} (y_i - \delta) \text{ for all } j \in P(i, y). \quad (6)$$

Here $z := |P(i, y)|$. Just in the same way as before we define with δ, z a new allocation y^* ,

$$y_j^* = y_j \quad \text{for all } j \notin P(i, y) \cup \{i\},$$

$$y_i^* = y_i - \delta,$$

$$y_j^* = y_j + \frac{\delta}{z} \quad \text{for } j \in P(i, y).$$

Clearly $y^* L^\Sigma y$. So if $y^* \in \text{EL}^\Sigma(S)$, we have a contradiction and the theorem is proved. Check that given $y \in \text{EL}^\Sigma(S)$, y^* cannot be blocked by *any* viable coalition T satisfying any one of the following conditions,

- 1) $T \notin M(i)$,
- 2) T satisfies a'),
- 3) T satisfies b'),
- 4) T satisfies c') and $T \cap P(i, y) \neq \emptyset$.

So T can only L^Σ -block y^* , when $T \in M(i)$, T satisfies c') and $T \cap P(i, y) = \emptyset$. In such a case, T cannot be a subset of $\bigcup_{t=m+1}^q S^t \cup \{1, 2, \dots, i\}$. Suppose not. The combination of condition c'), $y_i > y'_i$, and $y_j = y'_j$ for $j < i$ or $j \in \bigcup_{t=m+1}^q S^t$, would imply $y_j^T = y_j \geq y'_j$ for all $j \in T$ with strict inequality for $j = i$. Then T would L^Σ -block y' , a contradiction.

So T is not a subset of $\bigcup_{t=m+1}^q S^t \cup \{1, 2, \dots, i\}$ and $T \cap P(i, y) = \emptyset$. This means that $\frac{y_j}{\omega_j} \geq \frac{y_i}{\omega_i}$ for all $j \in T \cap S^m$. Then combining

$$T \cap \left\{ j \in S^m \mid \frac{y_j}{\omega_j} < \frac{y_i}{\omega_i} \right\} = \emptyset$$

together with the fact that the weighted payoffs of the players in S are decreasing in the corresponding index numbers, there is an $r > i$ with $\frac{y_i}{\omega_i} = \frac{y_{i+1}}{\omega_{i+1}} = \dots = \frac{y_r}{\omega_r}$ and with $\frac{y_{r+1}}{\omega_{r+1}} < \frac{y_r}{\omega_r}$ if $r < k + 1$. On the other hand, by c') it can not be the case that $y_j \geq y'_j$ for all $j \in T$ while $y_i > y'_i$. Since then T would L^Σ -block y' because $i \in T$. So there exists $j \in T$, with $i + 1 \leq j \leq r$ such that $y'_j > y_j$. If we take the smallest j with this property, we simply permute i and j now, and leave all other indices unchanged, and notice that we are back in Case (i).

Therefore, no $T \subset S$ can L^Σ -block y^* , which contradicts $y \in \text{EL}^\Sigma(S)$. This completes the proof of the theorem. \square

In order to determine the set of weighted constrained egalitarian solutions, the following algorithm may be useful. It is based on the former algorithm for games (N, v) with trivial hierarchical systems.

First we apply ALG to the subgame for the players in S_k with the corresponding vector of weights ω_{S_k} . If it returns the empty solution we stop: the weighted egalitarian solution is empty. Else we proceed with determining the allocation for each of the players in S_{k-1} , by applying ALG on the reduced game v_{k-1} with corresponding weight vector $\omega_{S_{k-1}}$. If $\text{ALG}(S_{k-1}, v_{k-1}, \omega_{S_{k-1}}) = \emptyset$, then so is $\text{ALG}^*(N, v, \Sigma)$ and we stop. Otherwise we continue with the players S_{k-2} and the game v_{k-2} , etc....

Algorithm 4.7

1. *Input: game (N, v) , hierarchical system Σ .*
2. *For $i = k$ to 1*
 - do* $\text{ALG}(S_i, v_i, \omega_{S_i})$
 - if* $\text{ALG}(S_i, v_i, \omega_{S_i}) = \emptyset$, *then stop and put* $\text{ALG}^*(N, v, \Sigma) = \emptyset$.
 - else* $\text{ALG}_{S_i}^*(N, v, \Sigma) := \text{ALG}(S_i, v_i, \omega_{S_i})$.

3. *Output:* $\text{ALG}^*(N, v, \Sigma)$.

Theorem 4.8 *If $\emptyset \neq \text{ALG}^*(N, v, \Sigma) \in L^\Sigma(N)$, then $\text{ALG}^*(N, v, \Sigma) = \text{EL}^\Sigma(N)$.*

Proof. Let $z = \text{ALG}^*(N, v, \Sigma) \in L^\Sigma(N)$. Take an increasing row of numbers $t(1), t(2), \dots, t(k) \in \mathbb{N}$ such that $\{S'_{t(i)+1}, S'_{t(i)+2}, \dots, S'_{t(i+1)}\}$ is the collection of sets of players forming the partition of S_{i+1} , induced by the above algorithm in step $k - i$, for the calculation of $\text{ALG}(S_i, v_i, \omega_{S_i})$. Then by Theorem 3.16 we have $z_{S_k} = \text{EL}^{\omega_{S_k}}(S_k)$ in the game v_k , and so $z_{S_k} = \text{EL}^\Sigma(S_k)$. We will prove the theorem by induction on the S'_i 's. More precisely, we will show that for all $j \in \{1, 2, \dots, t(k) - 1\}$ it holds that, for $S_\ell^+ := \bigcup_{t=\ell}^{t(k)} S'_t$, for all $\ell \in \{1, 2, \dots, t(k)\}$,

$$z_{S_{j+1}^+} = \text{EL}^\Sigma(S_{j+1}^+) \implies z_{S_j^+} = \text{EL}^\Sigma(S_j^+).$$

Suppose that this implication does not hold for some j . If it would be the case that $z(S_j^+) \notin L^\Sigma(S_j^+)$ then $z \notin L^\Sigma(N)$, a contradiction. So there must be an $y \in \text{EL}^\Sigma(S_j^+)$ that L^Σ -dominates $z_{S_j^+}$. Let $q := \max\{\ell \geq j \mid z_{S_\ell} \neq y_{S_\ell}\}$. Then there is $p \in S_q$ such that (i) $y_p < z_p$ and (ii) for all $\ell \in T = \{s \in S_q \mid \frac{z_s}{\omega_s} \geq \frac{z_p}{\omega_p}\}$, we have $y_\ell \leq z_\ell$. By the construction of z , $T = S_q^1 \cup S_q^2 \cup \dots \cup S_q^t$ for some $t \leq t(q)$. By hypothesis $z_{T \cup S_{q+1}^+} = \text{EL}^\Sigma(T \cup S_{q+1}^+)$. But then $T \cup S_{q+1}^+$ L^Σ -blocks y , and so $y \notin L^\Sigma(S_j^+)$, a contradiction. So, we proved the necessary induction step. \square

Especially, this means that whenever $\text{ALG}^*(N, v, \Sigma)$ determines a core element of (N, v) then it is the weighted egalitarian solution. A sufficient condition for this to happen is convexity of (N, v) . In that case $\text{ALG}^*(N, v, \Sigma)$ determines the *sequential weighted Dutta-Ray solution* of Hokari (1998):

Theorem 4.9 *If (N, v) is convex, then $\text{ALG}^*(N, v, \Sigma) = \text{CES}^\Sigma(N, v)$.*

Proof. By Theorem 4.8 it suffices to prove that $\emptyset \neq \text{ALG}^*(N, v, \Sigma) \in \text{core}(N, v)$, since $\text{core}(N, v) \subseteq L^\Sigma(v)$ (see Remark 4.5).

Let $z = \text{ALG}^*(N, v, \Sigma)$. First of all, we have $z \neq \emptyset$ since $z_{S_i} = \text{CES}^{\omega_{S_i}}(S_i, v_i) \neq \emptyset$ by convexity of v_i . Let $T \subset N$. By convexity of (N, v) it holds for all $i \in \{1, 2, \dots, k\}$ that

$$v\left(\bigcup_{j=i}^k (S_j \cap T)\right) - v\left(\bigcup_{j=i+1}^k (S_j \cap T)\right) \leq v((S_i \cap T) \cup S_i^*) - v(S_i^*) = v_i(S_i \cap T).$$

Together with the fact that $\text{CES}^{\omega_{S_i}}(S_i, v_i) \in \text{core}(S_i, v_i)$ and hence $z(S_i \cap T) \geq v_i(S_i \cap T)$ for all $i \in \{1, 2, \dots, k\}$, we thus have

$$v(T) \leq \sum_{i=1}^k v_i(S_i \cap T) \leq \sum_{i=1}^k z(S_i \cap T) = z(T).$$

This means that $z \in \text{core}(N, v)$. □

Remark 4.10 Consider a convex game (N, v) and a weight system Σ . Let A_1, A_2, \dots, A_p be the coalitions that are consecutively calculated by Algorithm 4.7. Then observe that for all i and $j \in A_i$ we have

$$\text{CES}_j^\Sigma(v) = \frac{\omega_{ij}}{\omega_i(A_i)} \left\{ v \left(\bigcup_{t \leq i} A_t \right) - v \left(\bigcup_{t \leq i-1} A_t \right) \right\}. \quad (7)$$

Monderer *et al.* (1992) show that by varying the weight systems, the weighted Shapley values cover the core of a convex game. We will show, in a constructive way, that for nonnegative convex games the set of all weighted constrained egalitarian solutions coincides with its core.

Theorem 4.11 *Let (N, v) be convex and $v(S) \geq 0$ for all $S \subseteq N$. Then for each $x \in \text{core}(N, v)$ there is a hierarchical system Σ such that $\text{CES}^\Sigma(N, v) = x$.*

Proof. Let (N, v) be a nonnegative convex game and let $x \in \text{core}(N, v)$. By the fact that $v(S) \geq 0$ for all $S \subseteq N$ it follows $x \geq 0$. Define $S_1 = \{i \in N \mid x_i \neq 0\}$ and $S_2 = \{i \in N \mid x_i = 0\}$. We distinguish between the following cases.

Case (i). Suppose $S_2 = \emptyset$. Take $\omega = x$ as the weight vector for $S_1 = N$ and let $\Sigma = (\{N\}, \omega)$ be the corresponding hierarchical system. Since $x \in \text{core}(N, v)$ we have $\omega(S) = x(S) \geq v(S)$ for all $S \subseteq N$ with equality for $S = N$. So we may take N as the largest coalition for which the weighted value is maximal, i.e. 1. Then $\text{CES}_i^\Sigma(N, v) = \text{CES}_i^\omega(N, v) = x_i \frac{v(N)}{\omega(N)} = x_i$ for all $i \in N$.

Case (ii). Suppose $S_2 \neq \emptyset$. Then we discern two cases:

Case (ii'). If $S_1 = \emptyset$ then define $\omega \in \mathbb{R}_{++}^N$ by $\omega_i := \frac{1}{|N|}$ and define correspondingly a hierarchical system $\Sigma = (S_2, \omega)$. It is clear that $\text{CES}^\Sigma(N, v) = 0_N = x$.

Case (ii''). In case $S_1 \neq \emptyset$, define weight vectors ω_1 and ω_2 for S_1 and S_2 respectively, by $\omega_1 := x_{S_1}$ and $\omega_2 := 1$ for all $i \in S_2$. Then, according to the above Theorem 4.9 the weighted constrained egalitarian solution corresponding to the hierarchical system $\Sigma = ((S_1, S_2), (\omega_1, \omega_2))$ is calculated using the following relations: $\text{CES}_{S_1}^\Sigma(N, v) = \text{CES}^{\omega_1}(S_1, v_1)$ and $\text{CES}_{S_2}^\Sigma(N, v) = \text{CES}^{\omega_2}(S_2, v_2)$, where $v_1(S) := v(S \cup S_2) - v(S_2)$ for all $S \subseteq S_1$ and $v_2(S) = v(S)$ for all $S \subseteq S_2$. For $S \subseteq S_2$ it holds $0 = x(S) \geq v(S) \geq 0$, and thus $v(S) = 0$. Hence it follows easily that $\text{CES}_{S_2}^\Sigma(N, v) = 0$. But $v(S_2) = 0$ implies $v_1(S) = v(S \cup S_2)$ for all S . According to the first algorithm we obtain the first weighted constrained egalitarian values of players in S_1 by considering the maximal coalitions attaining the maximal weighted value

$$\frac{v_1(S)}{\omega_1(S)} = \frac{v(S \cup S_2)}{x(S)}.$$

Since $x \in \text{core}(N, v)$ it holds that $x(S) = x(S \cup S_2) \geq v(S \cup S_2)$ and thus

$$\frac{v_1(S)}{\omega_1(S)} \leq 1 \quad \text{for all } S \subset S_1.$$

In Lemma 3.14 we proved that the set $\arg \max \left\{ \frac{v(S)}{\omega(S)} \mid \emptyset \neq S \subseteq N \right\}$ is closed under union. So the maximal coalition in S_1 that maximizes the weighted value is S_1 , since by $x \in \text{core}(N, v)$ it holds $\omega_1(S_1) = x(S_1) = x(N) = v(N) = v(S_1 \cup S_2)$ and thus $\frac{v_1(S_1)}{\omega_1(S_1)} = 1$. Then for the players i in this maximal set S_i , the weighted constrained egalitarian values with respect to Σ are thus

$$\text{CES}_i^\Sigma(N, v) = \omega_{1i} \frac{v_1(S_1)}{\omega_1(S_1)} = x_i \frac{v(N)}{v(N)} = x_i.$$

Hence, $\text{CES}^\Sigma(N, v) = x$. □

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