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Structural Restrictions in Cooperation

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Özer Selçuk
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CONTENTS

CHAPTER 1

INTRODUCTION

Game theory is the study of mathematical models of conflict and cooperation between intelligent rational decision makers (Myerson (1991)). The book "Theory of Games and Economics Behavior" written by von Neumann and Morgenstern (1947) is considered as the starting point of game theory literature. In this book, von Neumann and Morgenstern distinguish between two main approaches in game theory. The possibility of signing a binding contract among the players is the main distinction between these two approaches. A cooperative game corresponds to a game where commitments are fully binding and enforceable while for non-cooperative games the commitments have no binding force, see, e.g., Harsanyi (1966). When it is assumed that all players choose to cooperate, the fundamental question in cooperative game theory deals with the problem of how much payoff every player should receive. A solution concept assigns a set of suitable payoff vectors to each cooperative game.

Cooperative games with transferable utilities, or simply TU-games, refers to the case where the revenues created by a coalition of players through cooperation can be freely distributed to the members of the coalition. Formally, a TU-game consists of a set of players and a characteristic function which assigns to each coalition of players its worth being the highest revenue that the coalition can earn without cooperating with the rest of the players. As a general solution concept for TU-games, Gillies (1959) introduces the core which is the set of payoff vectors that are efficient and stable. A payoff vector is efficient if it distributes the worth of the grand coalition of all players and in order to be stable each coalition of players should receive at least its worth as the total joint payoff.

The best known single-valued solution concept for TU-games is the Shapley value. For the Shapley value, all permutations on the player set are considered and the average of the marginal contribution vectors corresponding to those permutations is calculated. At such a payoff vector, a player receives what he contributes in worth to the set of his predecessors in the permutation. The Shapley value is characterized by efficiency, additivity, the null-player property, and symmetry, see Shapley (1953). In the literature there are various other characterizations of the Shapley value. Together with efficiency and symmetry Young (1985) uses strong monotonicity and in van den Brink (2002) a fairness axiom is used together with efficiency and the null player property.

The classical assumption for TU-games states that every coalition is able to form and earn the worth created by cooperation. However, in many practical situations the collection of coalitions that can be formed is restricted by some social, economical, hierarchical, or technical structure. In the literature there are several different modifications of TU-games in order to cover the cases where cooperation among the players is restricted.

Aumann and Dréze (1974) and Owen (1977) consider cooperative games with coalition structure which is called a priori unions in Owen (1977). The coalition structure in these models is an exogenously given partition of the set of all players. Aumann and Dréze (1974) assumes that cooperation is not restricted within each member of the partition, but on the other hand, for the players that belong to different elements of the partition it is impossible to cooperate. For such situations, Aumann and Dréze (1974) studies well-known solution concepts including the Shapley value. According to Owen (1977), the players that are in the same a priori union are more likely to cooperate compared to the players of different a priori unions. For games with a priori unions, Owen (1977) introduces Owen's value and provides a characterization for it. Different than the Shapley value which is the average of the marginal contribution vectors corresponding to all permutations, Owen's value considers those permutations in which the players in each element of the partition appear successively. Instead of a partition of the set of players, there are various papers that consider other specific set systems. For example, Algaba et al. (2001) considers union stable cooperation structures, Bilbao and Edelman (2000a) considers convex geometries, Bilbao et al. (2001) considers matroids, Algaba et al. (2003) considers antimatroids, Bilbao and Ordónez

(2009a) considers augmenting systems, Ui et al. (2011a) considers complete coalition structures, and Koshevoy and Talman (2014) considers building sets. For more models of games with restricted cooperation represented by other combinatorial structures we refer to Bilbao (2000).

Another way to include restricted cooperation into TU-games is by assuming a permission structure. A permission structure is modeled by means of a directed graph and players need the permission of their superiors in the directed graph in order to cooperate. There are two main approaches for TU-games with permission structure. Gilles et al. (1992), Derks and Gilles (1995) and van den Brink and Gilles (1996) consider situations where each player needs the permission of all his superiors to cooperate (conjunctive approach). On the other hand, in Gilles and Owen (1999), and van den Brink (1997) another assumption is employed which states that the permission of one direct superior is sufficient to cooperate (disjunctive approach). In the two approaches, by taking both the underlying game and the permission structure into account, a new TU-game is defined and the Shapley value of this game is taken as solution.

Faigle and Kern (1992) considers TU-games with precedence constraints which are modeled by some partially ordered set of players. In case of precedence constraints, only the coalitions that respect the precedence structure on the set of players are able to form. Faigle and Kern (1992) defines the Shapley value for TU-games with precedence constraints and provides a characterization.

Yet, another way to represent restricted cooperation in a TU-game is to assume that cooperation depends on a communication structure which is generally represented by an undirected graph on the set of players. The study of TU-games with communication structure represented by undirected graphs is initiated by Myerson (1977), see also Owen (1986) and Borm et al. (1994). In an undirected graph on the set of players, an edge between two players is interpreted as the players' ability to communicate bilaterally with each other. Given a communication structure represented by an undirected graph, Myerson (1977) assumes that only connected sets of players in the graph are able to form a coalition. For games with such communication structure, Myerson (1977) introduces a value, called the Myerson value, which is the Shapley value of the so called Myerson restricted game. The Myerson value is characterized by component efficiency and fairness. Borm et al. (1992) also considers cooperative games with communication structure and studies the position value as a solution concept for such games, see Meessen (1988). A characterization of the position value for TU-games with cycle-free communication structure is provided by Borm et al. (1992). In case of an arbitrary undirected graph representing the communication structure, a characterization of the position value is provided by Slikker (2005). Together with introducing the position value, Borm et al. (1992) also provides a new axiomatic characterization of the Myerson value by using component efficiency, additivity, the superfluous arc property, and the communication ability property. As a more general structure, Myerson (1980) introduces conference structures as a way to model communication in TU-games. In contrast to graphs, in a conference structure a communication link can also be formed among the members of a coalition with more than two players. In van den Nouweland et al. (1992), the communication structure in TU-games is modeled with hypergraphs and characterizations of the Myerson value and the position value are provided.

For games with communication structure which are represented by a cycle-free undirected graph, Herings et al. (2008) introduces as solution concept the average tree solution and Herings et al. (2010) defines the average tree solution for games with communication structure represented by an arbitrary undirected graph. For TU-games with cycle-free communication structure, the average tree solution is the average of the marginal contribution vectors corresponding to all spanning trees of the graph. For this class of games, Herings et al. (2008) characterizes the average tree solution with component efficiency and component fairness. Other characterizations of the average tree solution for TU-games with cycle-free communication structure are provided by Mishra and Talman (2010) and van den Brink (2009). In Mishra and Talman (2010) efficiency, linearity, strong symmetry, the dummy property, and independence in unanimity games are used for a characterization. On the other hand, van den Brink (2009) uses component efficiency, collusion neutrality, additivity, the communication ability property, the equal gain/loss property, and component independence.

Chapters 2, 3 and 4 of this monograph deal with TU-games with restricted cooperation. Chapter 2 considers TU-games with communication structure. The communication structure in that chapter, which restricts cooperation among the players, is represented by a cycle-free graph on the set of players. A new characterization of the average tree solution for TU-games with cycle-free communication structure is provided. The axioms used for this characterization are in the same spirit of the ones that are used to characterize the Shapley value for TU-games. For this characterization, we use efficiency, linearity, the restricted null player property, strong symmetry, and restricted marginality.

Chapter 3 considers TU-games with dominance structure. Different than a communication structure, the dominance structure is represented by a directed graph on the set of players and similar to a communication structure it restricts the cooperation among the players. For TU-games with dominance structure, we introduce two solution concepts, the average covering tree solution and the dominance value. The average covering tree solution is the average of the marginal contribution vectors corresponding to all covering trees of the digraph and the dominance value is the average of the marginal contribution vectors corresponding to all consistent permutations. Given a TU-game with dominance structure, each node in the directed graph may be considered as a task that needs to be completed and different assumptions about the ordering of the tasks results in different solution concepts. For the average covering tree solution it is assumed that at each time several tasks can be completed as long as they belong to independent groups of tasks and the subordination of tasks in the digraph is not violated. On the other hand, for the dominance value it is assumed that at each time only one task can be completed as long as the subordination of tasks in the digraph is not violated. In case the dominance structure is represented by a cycle-free directed graph, the Shapley value introduced in Faigle and Kern (1992) and the dominance value coincide. Both the average covering tree solution and the dominance value are efficient, linear and independent of inessential arcs. Moreover, the average covering tree solution satisfies the superfluous player property and hierarchical efficiency, while the dominance value satisfies the restricted null player property and the restricted equal treatment property. Additionally, for each of these solution concepts we provide a convexity type of condition that guaranties the core stability of the solution and we provide characterizations on the class of TU-games with special types of dominance structures.

Chapter 4 is concerned with the fact that in some real life cases, graphs and specific combinatorial structures are not the appropriate tools to represent restricted cooperation. In that chapter, a set system is taken to represent the restricted cooperation in a TU-game. We assume that the grand coalition is a member of the set system or can be partitioned into coalitions such that each feasible coalition is a subset of a partition member. For TU-games with coalitional structures, where the restricted cooperation is represented by a set system on the set of players, we introduce the average coalitional tree solution. As the Shapley value for TU-games with arbitrary coalitional structure, Aguilera et al. (2010) considers the average of the marginal contribution vectors corresponding to all maximal chains of the set system. However, the average coalitional tree solution is defined as the average of the marginal contribution vectors corresponding to all maximal nested sets of the set system. A nested set of a set system is a more general concept than a chain. We study the properties that are satisfied by the average coalitional tree solution and consider the special cases where the coalitional structure is a building set and forms a partition of the set of players.

The remaining two chapters of this monograph belong to the area of social choice theory which deals with collective decision making by aggregating individual preferences to obtain a social preference.

In a voting situation, if voters are asked to compare all candidates pairwise, the result of this procedure will be a tournament if the number of voters is odd. Formally, a tournament is a complete and asymmetric binary relation on a set of alternatives and it can also be considered as a complete asymmetric directed graph. Given a tournament, a Condorcet winner is the alternative that has an arc in the directed graph to every other alternative. For a tournament, a Condorcet winner may not exist and if a solution picks the Condorcet winner whenever it exists, then this solution is said to be Condorcet consistent. In the literature, several different methods, called tournament solutions, are proposed to choose the winner of a tournament. Zermelo (1929) employs a probabilistic approach and as a self consistent choice rule, Grivko and Levchenkov (1994) proposes the Markovian solution, and Slikker et al. (2012) employs an iterative approach to rank the alternatives in a tournament. Additionally, Fishburn (1977) and Miller (1980) proposes the uncovered set, and Dutta (1988) introduces the minimal covering set. For a tournament, the Copeland solution consists of the alternatives that have an arc to a maximum number of alternatives, see Copeland (1951). Chapter 5 provides a new characterization of the Copeland solution that is defined for tournaments. It is shown that the Copeland winner is not only a score maximizer but also is minimizing the number of steps required to reach every other alternative.

Chapter 6 deals with preference aggregation in case the social preferences are sophisticated. Arrow (1951) defines a social welfare function as a rule that assigns a social preference to each possible profile of individual preferences on the set of alternatives. A preference on a set of alternatives is a complete, reflexive, and transitive binary relation. The celebrated impossibility result in Arrow (1951) shows that the dictatorial social welfare function, that cares about only the preference of a single individual, is the only social welfare function that satisfies a set of desirable properties, including Pareto optimality and independence of irrelevant alternatives. A sophisticated preference, which is more general than a standard preference, adds ambiguity into the preferences. A sophisticated social welfare function is defined as a mapping from the profiles of individual standard preferences to the set of sophisticated preferences. In this chapter, we characterize sophisticated social welfare functions, that are Pareto optimal and independent of irrelevant alternatives, in terms of oligarchies that are induced by some power distribution in the society. This class is quite large which contains both dictatoriality and equal power distribution as two the extreme cases. When the range of the sophisticated social welfare function is restricted to the set of standard preferences, it can be shown that the induced oligarchy contains only one individual. Hence, the results in this chapter generalize the impossibility theorem of Arrow (1951).

CHAPTER 2

THE AVERAGE TREE SOLUTION FOR TU-GAMES WITH CYCLE-FREE COMMUNICATION STRUCTURE

2.1 Introduction

In this chapter, we consider TU-games with communication structure which is represented by an undirected graph on the set of players, see Myerson (1977). For TU-games with communication structure, only the players that form a connected set in the undirected graph are able to cooperate. To illustrate this point, consider a situation where several cities are located along a river. Suppose that using the river is vital for trade because it is the only way to transport goods between the cities. Therefore, in order to cooperate and trade with each other, any subset of these cities must form a connected set along the river. In this setting, cities can be considered as the set of players and the river corresponds to the set of bilateral communication links connecting players to each other. Once a subset of cities is able to trade, there will emerge a revenue which is the worth corresponding to that coalition of cities. Given that all cities are connected by the river, a solution for such a situation deals with the problem of distributing the total revenue resulting from the cooperation of all cities.

For TU-games, where every subset of players is able to cooperate, the Shapley value is the most well known solution concept. The Shapley value is defined as the average of the marginal contribution vectors corresponding to all permutations on the set of players, see Shapley (1953). The original characterization of the Shapley value, as in Shapley (1953), uses efficiency, additivity, the null-player property, and symmetry. In the literature, there exist several other characterizations of the Shapley value where Young (1985) uses strong monotonicity together with efficiency and symmetry. As a way to include restricted cooperation, Myerson (1977) considers TU-games with communication structure which is represented by an arbitrary undirected graph on the set of players. In a TU-game with communication structure, Myerson (1977) assumes only connected sets of players in the undirected graph are able to cooperate. For TU-games with communication structure, the Myerson value is the Shapley value of the so-called Myerson restricted game. On the class of TU-games with communication structure, Myerson (1977) provides a characterization of the Myerson value by using component efficiency and fairness.

As a solution concept, Herings et al. (2008) introduces the average tree solution for TU-games with communication structure which is represented by a cycle-free graph. Herings et al. (2010) studies the average tree solution for TU-games with communication structure represented by an arbitrary undirected graph. Given a TU-game with cycle-free communication structure, for the average tree solution all spanning trees of the undirected graph are considered and the average of the corresponding marginal contribution vectors is calculated. The marginal contribution vector corresponding to a tree is first defined by Demange (2004) as the hierarchical outcome. For a cycle-free undirected graph, a spanning tree is a cycle-free directed graph where each arc between any pair of nodes corresponds to an edge between the same pair of nodes in the undirected graph. For this class of games, Herings et al. (2008) characterizes the average tree solution with component efficiency and component fairness. Other characterizations of the average tree solution for TU-games with cycle-free communication structure are provided by Mishra and Talman (2010) and van den Brink (2009). In Mishra and Talman (2010) efficiency, linearity, strong symmetry, the dummy property, and independence in unanimity games are used for this characterization, while van den Brink (2009) uses component efficiency, collusion neutrality, additivity, the communication ability property, the equal gain/loss property, and component independence as axioms. Additionally, for TU-games with communication structure which is represented by a cycle on the set of players, Selçuk et al. (2013) provides a characterization of the average tree solution. For this characterization, together with efficiency, linearity, and the restricted dummy property some modified symmetry axioms are used.

In this chapter, we provide a new characterization of the average tree solution for TU-games with cycle-free communication structure by using axioms that are in the same spirit of the ones that characterize the Shapley value as in Shapley (1953) and Young (1985). For this characterization, we use efficiency, linearity, the restricted null player property, strong symmetry, and restricted marginality. Among those axioms, efficiency and linearity are well-known in the literature and are also satisfied by the Myerson value. Given a TU-game with cycle-free communication structure, if a player's marginal contributions to any collection of his satellites, which are the components arising when this player is erased from the communication structure, are zero, then the restricted null player property requires this player to receive zero payoff. The restricted null player property is not satisfied by the Myerson value and may be considered as a strong form of the null player property used by Shapley (1953). Strong symmetry is also satisfied by the Myerson value and it requires equal payoff for all players if any proper subset of the grand coalition has zero worth, see Mishra and Talman (2010). On the class of TU-games with connected cycle-free communication structure, strong symmetry coincides with the weak communication ability property introduced by van den Brink et al. (2011). Restricted marginality requires a player to receive the same payoff in two different TU-games with the same cycle-free communication structure, if this player's marginal contributions to some specific coalitions are the same in both games. This property is not satisfied by the Myerson value.

This chapter is organized as follows. Section 2 contains the preliminaries. Section 3 introduces the Shapley value, the Myerson value and the average tree solution. Section 4 provides the new characterization of the average tree solution.

2.2 Preliminaries

A *cooperative game with transferable utility* (TU-game) is represented by a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of n players with $n \geq 2$, and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \in 2^N$ is a *coalition* and the associated real number $v(S)$ stands for the *worth* of coalition S, being the total joint revenue that is achievable by S without cooperating with the rest of the players and can be freely distributed among the players in S. We denote the set of TU-games with fixed player set N by $\mathcal{G}_N$. Shapley (1953) introduces the class of *unanimity games* which forms a linear basis for $\mathcal{G}_N$. For a coalition $S \in 2^N \setminus \{\emptyset\}$, the unanimity game $u_S \in \mathcal{G}_N$ is defined by

$$u_S(Q) = \begin{cases} 1 & \text{if } Q \supseteq S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $Q \in 2^N$.

A payoff vector $x \in \mathbb{R}^n$ is an n-dimensional vector at which x_i is the payoff available for player $i \in N$. A single-valued *solution* on $\mathcal{G}_N$ is a function $\xi\colon \mathcal{G}_N \to \mathbb{R}^n$ that assigns to every TU-game $(N, v) \in \mathcal{G}_N$ a payoff vector $\xi(N, v) \in \mathbb{R}^n$.

A *graph* on N is a pair (N, L), consisting of a set of nodes N and a collection of unordered pairs of nodes $L \subseteq L^c_N$, where $L^c_N = \{\, \{i, j\} \mid i, j \in N,\ i \neq j\}$ is the complete (undirected) graph without loops on N and an unordered pair $\{i, j\} \in L$ is called an *edge*. For $S \in 2^N$, $(S, L|_S)$ stands for the subgraph of (N, L) on S, where $L|_S = \{\{i, j\} \in L \mid i, j \in S\}$. In a graph (N, L), a sequence of different nodes $(i_1, \ldots, i_k)$, $k \geq 2$, is a *path* between node i_1 and node i_k if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. A path $(i_1, \ldots, i_k)$, $k \geq 3$, is a *cycle* in (N, L) if $\{i_k, i_1\} \in L$. A graph (N, L) is *cycle-free* if it does not contain any cycle. Two nodes $i, j \in N$ are *connected* in (N, L) if there exists a path in (N, L) between these nodes. In a graph (N, L), a subset S of N is *connected* if for any two distinct nodes of S there exists a path between these nodes in the subgraph $(S, L|_S)$. For a graph (N, L), a subset S of N is a *component* of (N, L) if S is maximally connected, i.e., S is connected and for any $j \in N \setminus S$ the set $S \cup \{j\}$ is not connected. The collection of all connected subsets of S in the graph (N, L) is denoted by $C^L(S)$ and the collection of all components of $(S, L|_S)$ is denoted by $\widehat{C}^L(S)$. For $i \in N$, $\widehat{C}^L_i$ denotes the component of (N, L) that contains player i, i.e., $S \in \widehat{C}^L(N)$ and $i \in S$ implies $\widehat{C}^L_i = S$.

The combination of a TU-game and an (undirected) graph results in a *TU-game with communication structure* which is denoted by a triple (N, v, L) where N is the set of players, (N, v) is a TU-game, and (N, L) is a graph on N. We denote the set of TU-games with communication structure and fixed player set N by $\mathcal{G}^{cs}_N$. The set of TU-games with cycle-free communication structure and fixed player set N is denoted by $\mathcal{G}^{cf}_N$. The set of TU-games with connected cycle-free communication structure and fixed player set N is denoted by $\mathcal{G}^{ccf}_N$. Note that $\mathcal{G}^{ccf}_N \subseteq \mathcal{G}^{cf}_N \subseteq \mathcal{G}^{cs}_N$. A single valued solution on a subset $\mathcal{G} \subseteq \mathcal{G}^{cs}_N$ is a function $\xi : \mathcal{G} \to \mathbb{R}^n$ such that $\xi(N, v, L) \in \mathbb{R}^n$ is the payoff vector assigned to the TU-game with communication structure $(N, v, L) \in \mathcal{G}$.

A *directed graph* or *digraph* is a pair (N, D) with D being a collection of ordered pairs of different nodes, i.e., $D \subseteq D^c_N$, where $D^c_N = \{(i,j) \mid i,j \in N,\ i \neq j\}$ is the complete directed graph without loops on N and an ordered pair $(i,j) \in D$ is called an *arc* from i to j. For a digraph (N,D), a sequence of different nodes $(i_1,\ldots,i_k)$, $k \geq 2$, is a *path* in (N,D) between node i_1 and node i_k if $\{(i_h,i_{h+1}),(i_{h+1},i_h)\} \cap D \neq \emptyset$ for $h = 1,\ldots,k-1$. A sequence of different nodes $(i_1,\ldots,i_k)$, $k \geq 2$, is a *directed path* in (N,D) from i_1 to i_k if $(i_h,i_{h+1}) \in D$ for $h = 1,\ldots,k-1$. If there exists a directed path in (N,D) from node $i \in N$ to node $j \in N$, then j is a *successor* of i and i is a *predecessor* of j in (N,D). If $(i,j) \in D$, then node j is an *immediate successor* of node i and player i is an *immediate predecessor* of j in (N,D). For $i \in N$, $S_D(i)$ is the set of successors of node i in (N,D) and $\bar{S}_D(i) = S_D(i) \cup \{i\}$. A path $(i_1,\ldots,i_k)$, $k \geq 3$, in (N,D) is a *cycle* if $\{(i_k,i_1),(i_1,i_k)\} \cap D \neq \emptyset$, and a directed path $(i_1,\ldots,i_k)$, $k \geq 2$, in (N,D) is a *directed cycle* if $(i_k,i_1) \in D$. A digraph (N,D) is *cycle-free* if it contains no directed cycles, i.e., no node is a successor of itself. A digraph (N,D) is *strongly cycle-free* if it is cycle-free and contains no cycles. A directed graph (N,T) is a *tree* if it has a unique node without any predecessors, called the *root* of the tree, and for every other node in N there is a unique directed path in (N,T) from the root to that node. A tree (N,T) is a *spanning tree* of an undirected graph (N,L) if every arc of T induces an edge of L, i.e., $(i,j) \in T$ implies $\{i,j\} \in L$. A tree (N,T) is a *line tree* if each node, different than the root, has exactly one immediate predecessor.

2.3 The Shapley value, the Myerson value, and the average tree solution: Existing characterizations

Shapley (1953) introduces one of the most well-known single-valued solution concepts, the Shapley value, for TU-games. The Shapley value is the average of the marginal contribution vectors corresponding to all permutations on the set of players. For a permutation $\pi\colon N \to N$, $\pi(i)$ denotes the (unique) position of player $i \in N$ in π, $P_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of predecessors of i in π, and $\bar{P}_\pi(i) = P_\pi(i) \cup \{i\}$. For a TU-game $(N,v) \in \mathcal{G}_N$, the *marginal contribution vector* corresponding to permutation π on N is given by the payoff vector $m^\pi(N,v) \in \mathbb{R}^n$, defined by

$$m^\pi_i(N,v) = v(\bar{P}_\pi(i)) - v(P_\pi(i)) \text{ for all } i \in N.$$

Let Π_N stand for the collection of all permutations on N. Note that $|\Pi_N| = n!$.

Definition 2.3.1 The *Shapley value* of a TU-game $(N,v) \in \mathcal{G}_N$ is given by the

payoff vector
$$Sh(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi_N} m^{\pi}(N,v).$$

In the literature the Shapley value is characterized in many different ways. For the characterization, Shapley (1953) uses efficiency, additivity, the null player property and symmetry.

Definition 2.3.2 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies *efficiency* if for any $(N,v) \in \mathcal{G}_N$ it holds that $\sum_{i \in N} \xi_i(N,v) = v(N)$.

For a solution in order to satisfy efficiency it should distribute the worth of the grand coalition to the players.

Definition 2.3.3 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies *additivity* if for any (N,v), $(N,w) \in \mathcal{G}_N$ it holds that $\xi(N,v+w) = \xi(N,v) + \xi(N,w)$.

According to the additivity axiom, given any two TU-games with the same set of players, if an additive solution assigns a payoff vector to each of these two TU-games, then the sum of these two payoff vectors should be assigned to the TU-game, for which the worth of every coalition is obtained by summing up the worths of that coalition in both TU-games.

A player $i \in N$ is called a *null player* in TU-game $(N,v) \in \mathcal{G}_N$ if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.

Definition 2.3.4 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies the *null player property* if for any $(N,v) \in \mathcal{G}_N$ and null player $i \in N$ in (N,v) it holds that $\xi_i(N,v) = 0$.

According to the null player property, players who have zero contribution to every coalition should receive zero payoff.

Given a TU-game $(N,v) \in \mathcal{G}_N$, two players $i,j \in N$ are called *symmetric* in (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.

Definition 2.3.5 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies *symmetry* if for any $(N,v) \in \mathcal{G}_N$ and symmetric players $i,j \in N$ in (N,v) it holds that $\xi_i(N,v) = \xi_j(N,v)$.

According to the symmetry axiom, two players who have the same contribution to every coalition not containing both of them, should receive the same payoff. Symmetry is often referred to as the equal treatment property, see Peleg and Südhölter (2007).

Theorem 2.3.6 ***(Shapley, 1953)*** *The Shapley value is the unique solution on $\mathcal{G}_N$ that satisfies efficiency, additivity, the null player property, and symmetry.*

Young (1985) provides another characterization of the Shapley value by using strong monotonicity together with efficiency and symmetry.

Definition 2.3.7 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies *strong monotonicity* if $\xi_i(N, v) \geq \xi_i(N, w)$ holds for any $(N, v), (N, w) \in \mathcal{G}_N$ and $i \in N$ such that

$$v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$$

for all $S \subseteq N \setminus \{i\}$.

Theorem 2.3.8 ***(Young, 1985)*** *The Shapley value is the unique solution on $\mathcal{G}_N$ that satisfies efficiency, symmetry, and strong monotonicity.*

Remark 2.3.9 In fact the characterization of Young (1985) is valid under a weaker condition which is obtained by replacing the inequalities in the definition of strong monotonicity with equalities. Strong monotonicity is often referred to as marginality.

In van den Brink (2002), a fairness axiom is used together with efficiency and the null player property to get an alternative characterization of the Shapley value.

Definition 2.3.10 A solution $\xi : \mathcal{G}_N \to \mathbb{R}^n$ satisfies *fairness* if for any (N, v), $(N, w) \in \mathcal{G}_N$ and $i, j \in N$ that are symmetric in (N, w) it holds that

$$\xi_i(N, v + w) - \xi_i(N, v) = \xi_j(N, v + w) - \xi_j(N, v).$$

Theorem 2.3.11 ***(van den Brink, 2002)*** *The Shapley value is the unique solution on $\mathcal{G}_N$ that satisfies efficiency, the null player property, and fairness.*

For TU-games it is assumed that any subset of players is able to cooperate and earn the worth of this coalition. Following Myerson (1977), for TU-games with communication structure the collection of feasible coalitions is restricted by an undirected graph on the set of players. It is assumed that only connected sets of players are able to form a coalition. The Myerson value of a TU-game with communication structure is the Shapley value of the so-called Myerson restricted game.

For a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$, Myerson (1977) defines the corresponding *Myerson restricted game* v^L by

$$v^L(S) = \sum_{Q \in \widehat{C}^L(S)} v(Q), \; S \in 2^N.$$

Given a TU-game with communication structure, in the Myerson restricted game, the worth of any set of players is the sum of the worths of the components of the subgraph on this set of players.

Definition 2.3.12 On the class of TU-games with communication structure, the *Myerson value* assigns to any $(N, v, L) \in \mathcal{G}_N^{cs}$ the payoff vector $\mu(N, v, L)$ given by

$$\mu(N, v, L) = Sh(N, v^L).$$

Two axioms that fully characterize the Myerson value are component efficiency and fairness.

Definition 2.3.13 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *component efficiency* if for any $(N, v, L) \in \mathcal{G}$ it holds that

$$\sum_{i \in Q} \xi_i(N, v, L) = v(Q) \;\text{ for all } Q \in \widehat{C}^L(N).$$

A solution on a subclass of TU-games with communication structure satisfies component efficiency if it distributes to each component of the graph exactly its worth.

Definition 2.3.14 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *fairness* if for any $(N, v, L) \in \mathcal{G}$ and $\{i, j\} \in L$ it holds that

$$\xi_i(N, v, L) - \xi_i(N, v, L \setminus \{i, j\}) = \xi_j(N, v, L) - \xi_j(N, v, L \setminus \{i, j\}).$$

The fairness axiom requires that if an edge is deleted from the undirected graph, then this yields the same payoff change for the players who are involved in this edge.

Theorem 2.3.15 ***(Myerson, 1977)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cs}$ that satisfies component efficiency and fairness.*

There are several alternative characterizations of the Myerson value for TU-games with communication structure as well as for some subclasses. Borm et al. (1992) characterizes the Myerson value for TU-games with cycle-free communication structure. Together with component efficiency and additivity, Borm et al. (1992) uses the superfluous link property and the communication ability property.

Definition 2.3.16 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *additivity* if for any (N, v, L), $(N, w, L) \in \mathcal{G}$, it holds that $\xi(N, v+w, L) = \xi(N, v, L) + \xi(N, w, L)$.

Given a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$, an edge $\{i, j\} \in L$ is called *superfluous* if it holds that $r^v(A) = r^v(A \setminus \{i, j\})$ for all $A \subseteq L$, where $r^v(A)$ is defined as

$$r^v(A) = v^A(N) = \sum_{Q \in \widehat{C}^A(N)} v(Q), \;\; A \subseteq L.$$

Definition 2.3.17 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *superfluous link property* if for any $(N, v, L) \in \mathcal{G}$ and superfluous edge $\{i, j\} \in L$ it holds that

$$\xi(N, v, L) = \xi(N, v, L \setminus \{i, j\}).$$

For a communication structure (N, L), let $D(N, L) = \{i \in N \mid \{i, j\} \in L \text{ for some } j \in N\}$. A TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cf}$ is called *point anonymous* if $v^L(S) = v^L(T)$ for all $S, T \subseteq N$ with $|S \cap D(N, L)| = |T \cap D(N, L)|$. For a TU-game with communication structure, in order to be point anonymous each coalition's worth in the Myerson restricted game only depends on the number of players in the coalition that have links with other players.

Definition 2.3.18 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *communication ability property* if for any point anonymous $(N, v, L) \in \mathcal{G}$ it holds that $\xi_i(N, v, L) = \xi_j(N, v, L)$ for all $i, j \in D(N, L)$ and $\xi_i(N, v, L) = 0$ for all $i \in N \setminus D(N, L)$.

Theorem 2.3.19 ***(Borm et al., 1992)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cf}$ that satisfies component efficiency, additivity, the superfluous link property, and the communication ability property.*

On the class of TU-games with cycle-free communication structure, van den Brink et al. (2011) provides a characterization of the Myerson value by replacing the communication ability property of Borm et al. (1992) with a weaker property.

A TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ is called *point unanimous* if $v^L(S) = v^L(N) u_{D(N,L)}(S)$ for all $S \subseteq N$.

Definition 2.3.20 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *weak communication ability property* if for any point unanimous $(N, v, L) \in \mathcal{G}$ it holds that $\xi_i(N, v, L) = \xi_j(N, v, L)$ for all $i, j \in D(N, L)$ and $\xi_i(N, v, L) = 0$ for all $i \in N \setminus D(N, L)$.

Theorem 2.3.21 ***(van den Brink et al., 2011)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cf}$ that satisfies component efficiency, additivity, the superfluous link property, and the weak communication ability property.*

The average tree solution is introduced in Herings et al. (2008) on the class of TU-games with cycle-free communication structure and generalized to the class of TU-games with arbitrary communication structure in Herings et al. (2010). Given a TU-game with cycle-free communication structure, for each component of the communication structure, all spanning trees on that component and all marginal contribution vectors corresponding to these spanning trees are considered. To each player, the average tree solution assigns the average of his marginal contributions in all spanning trees defined on the component containing this player. Formally, given a TU-game with cycle-free communication structure $(N, v, L) \in \mathcal{G}_N^{cf}$, each $i \in N$ induces a unique spanning tree $(\widehat{C}_i^L, T(i))$ with the node i being the root in the following way. For any $j \in \widehat{C}_i^L \setminus \{i\}$, take the unique path in (N, L) from i to j, then change the edges on this path to arcs in such a way that the first node in any ordered pair is the node that comes first on the path from i to j. Given a TU-game with cycle free communication structure $(N, v, L) \in \mathcal{G}_N^{cf}$ and $Q \in \widehat{C}^L(N)$, since each $i \in Q$ induces a unique spanning tree $(Q, T(i))$, the number of spanning trees on Q is equal to $|Q|$. For a TU-game with cycle-free communication structure $(N, v, L) \in \mathcal{G}_N^{cf}$ and $i \in N$, the marginal contribution of player $j \in \widehat{C}_i^L$ corresponding to the spanning tree $(\widehat{C}_i^L, T(i))$ is defined as

$$m_j^{T(i)}(N, v) = v(\bar{S}_{T(i)}(j)) - \sum_{h \in N:(j,h) \in T(i)} v(\bar{S}_{T(i)}(h)).$$

Definition 2.3.22 On the class of TU-games with cycle-free communication structure, *the average tree solution* (AT) assigns to any $(N, v, L) \in \mathcal{G}_N^{cf}$ the payoff vector $AT(N, v, L)$ given by

$$AT_i(N, v, L) = \frac{1}{|\widehat{C}_i^L|} \sum_{j \in \widehat{C}_i^L} m_i^{T(j)}(N, v),\ i \in N.$$

The average tree solution is originally defined for TU-games with cycle-free communication structure. Since the spanning trees and corresponding marginal contribution vectors are defined componentwise, one may also define the solution for TU-games with connected cycle-free communication structure and if the communication structure is not connected, then for each component the average tree solution can be defined separately.

Definition 2.3.23 On the class of TU-games with connected cycle-free communication structure, *the average tree solution* (AT) assigns to any $(N, v, L) \in \mathcal{G}_N^{ccf}$ the payoff vector $AT(N, v, L)$ given by

$$AT(N, v, L) = \frac{1}{n} \sum_{j \in N} m^{T(j)}(N, v).$$

Example 2.3.24 Consider a TU-game with connected cycle-free communication structure $(N, v, L) \in \mathcal{G}_N^{ccf}$ where $N = \{1, \ldots, 7\}$ and $L = \{\{1,3\}, \{2,3\}, \{3, 4\}, \{4,5\}, \{4,6\}, \{6,7\}\}$ and let $v(S) = |S|^2$ for all $S \in 2^N$. The graphical representation of (N, L) is given in Figure 2.1.

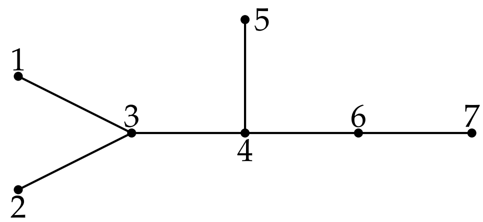


Figure 2.1: The graph (N, L) in Example 2.3.24.

There are seven spanning trees, $(N, T(1)), \ldots, (N, T(7))$, for (N, L) as depicted in Figure 2.2.

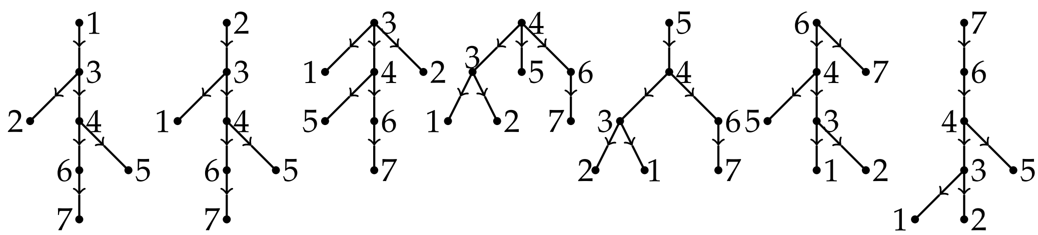


Figure 2.2: The spanning trees of (N, L) in Example 2.3.24.

To each of these spanning trees a marginal contribution vector corresponds with $m^{T(1)}(N, v) = (13, 1, 19, 11, 1, 3, 1)$, $m^{T(2)}(N, v) = (1, 13, 19, 11, 1, 3, 1)$, $m^{T(3)}(N, v) = (1, 1, 31, 11, 1, 3, 1)$, $m^{T(4)}(N, v) = (1, 1, 7, 35, 1, 3, 1)$, $m^{T(5)}(N, v) = (1, 1, 7, 23, 13, 3, 1)$, $m^{T(6)}(N, v) = (1, 1, 7, 15, 1, 23, 1)$, $m^{T(7)}(N, v) =$

$(1,1,7,15,\ 1,11,13)$. Since the average tree solution is the average of these marginal contribution vectors, we have $AT(N,v,L) = (19/7, 19/7, 125/7, 121/7, 19/7, 49/7, 19/7)$.

For the class of TU-games with cycle-free communication structure, Herings et al. (2008) characterizes the average tree solution with component efficiency and component fairness axioms. Note that in a cycle-free graph, if an edge is deleted from the graph, then there emerge two more components (replacing one component) additional to already existing ones. For a cycle free graph (N,L) and $\{i,j\} \in L$, let K^i and K^j be the components of $(N, L \setminus \{\{i,j\}\})$ containing i and j, respectively.

Definition 2.3.25 A solution $\xi : \mathcal{G}_N^{cf} \to \mathbb{R}^n$ satisfies *component fairness* if for any $(N,v,L) \in \mathcal{G}_N^{cf}$ and $\{i,j\} \in L$ it holds that

$$\frac{1}{|K^i|}\sum_{h \in K^i}\big(\xi_h(N,v,L) - \xi_h(N,v,L \setminus \{\{i,j\}\})\big) =$$

$$\frac{1}{|K^j|}\sum_{h \in K^j}\big(\xi_h(N,v,L) - \xi_h(N,v,L \setminus \{\{i,j\}\})\big).$$

Different than the fairness axiom used to characterize the Myerson value, component fairness requires that deletion of an edge causes the same average payoff change for both components resulting from this deletion.

Theorem 2.3.26 ***(Herings et al., 2008)*** *The average tree solution is the unique solution on $\mathcal{G}_N^{cf}$ that satisfies component efficiency and component fairness.*

In van den Brink (2009) another characterization for the average tree solution for TU-games with cycle-free communication structure is provided. This characterization is based on component efficiency, additivity, collusion neutrality, the communication ability property, component independence, and the equal gain/loss property.

For a TU-game $(N,v) \in \mathcal{G}_N$, when players $i,j \in N$, $i \neq j$, collude, then instead of (N,v), the TU-game $(N,v^{ij}) \in \mathcal{G}_N$ is considered where

$$v^{ij}(S) = \begin{cases} v(S \setminus \{i,j\}) & \text{if } \{i,j\} \not\subseteq S, \\ v(S) & \text{if } \{i,j\} \subseteq S. \end{cases}$$

Definition 2.3.27 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *collusion neutrality* if for any $(N,v,L) \in \mathcal{G}$ and $\{i,j\} \in L$ it holds that $\xi_i(N,v^{ij},L) + \xi_j(N,v^{ij},L) = \xi_i(N,v,L) + \xi_j(N,v,L)$.

Given a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$, a player $i \in N$ is called *superfluous* if this player is a null player in the Myerson restricted game, i.e., $v^L(S) - v^L(S \setminus \{i\}) = 0$ for all $S \subseteq N$, $S \ni i$.

Definition 2.3.28 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *superfluous player property* if for any $(N, v, L) \in \mathcal{G}$ and superfluous player $i \in N$ it holds that $\xi_i(N, v, L) = 0$.

Definition 2.3.29 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *equal gain/loss property* if for any $(N, v, L) \in \mathcal{G}$ and $\{i, j\} \in L$ it holds that $\xi_h(N, v^{ij}, L) - \xi_h(N, v, L) = \xi_g(N, v^{ij}, L) - \xi_g(N, v, L)$ for all $h, g \in N \setminus \{i, j\}$.

Definition 2.3.30 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *component independence* if for any $(N, v, L), (N, w, L') \in \mathcal{G}$ and $Q \in \widehat{C}^L(N) \cap \widehat{C}^{L'}(N)$ satisfying $(Q, L|_Q) = (Q, L'|_Q)$ and $v(S) = w(S)$ for all $S \subseteq Q$, it holds that $\xi_i(N, v, L) = \xi_i(N, w, L')$ for all $i \in Q$.

Theorem 2.3.31 ***(van den Brink, 2009)*** *The average tree solution is the unique solution on $\mathcal{G}_N^{cf}$ that satisfies component efficiency, additivity, collusion neutrality, the communication ability property, the superfluous player property, the equal gain/loss property, and component independence.*

Mishra and Talman (2010) provides a different characterization of the average tree solution for the class of TU-games with connected cycle-free communication structure. Together with efficiency and linearity, Mishra and Talman (2010) imposes strong symmetry, the dummy property, and independence in unanimity games.

Definition 2.3.32 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *efficiency* if for any $(N, v, L) \in \mathcal{G}$ it holds that $\sum_{i \in N} \xi_i(N, v, L) = v(N)$.

Efficiency means that exactly the worth of the grand coalition is distributed among the players. If the undirected graph in a TU-game with communication structure is connected then component efficiency and efficiency are equivalent to each other.

Definition 2.3.33 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *linearity* if for any (N, v, L), $(N, w, L) \in \mathcal{G}$ and $a, b \in \mathbb{R}$ it holds that $\xi(N, av + bw, L) = a\xi(N, v, L) + b\xi(N, w, L)$.

According to linearity, if there are two TU-games with the same communication structure, then applying the solution to each of them and adding up multiples of the two resulting payoff vectors gives the same outcome when the solution is applied to the TU-game which is the sum of the same multiples of the two TU-games with the same communication structure. For TU-games with communication structure linearity is stronger then additivity.

Definition 2.3.34 A solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies *strong symmetry* if for any $(N, v, L) \in \mathcal{G}_N^{ccf}$ with $v(S) = 0$ for all $S \in C^L(N)$, $S \neq N$, it holds that $\xi_i(N, v, L) = \xi_j(N, v, L)$ for all $i, j \in N$.

According to strong symmetry, in a TU-game with connected cycle-free communication structure, if all proper connected subsets of the grand coalition have zero worth, then all players should receive the same payoff. For TU-games with connected cycle-free communication structure, the weak communication ability property of van den Brink et al. (2011) is equivalent to strong symmetry.

A player $i \in N$ is called *dummy player* in a TU-game with connected communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ if $v(S) - \sum_{Q \in \widehat{C}^L(S \setminus \{i\})} v(Q) = 0$ for all $S \in C^L(N)$ and $S \ni i$.

Definition 2.3.35 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cs}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *dummy property* if for any $(N, v, L) \in \mathcal{G}$ and dummy player $i \in N$ it holds that $\xi_i(N, v, L) = 0$.

According to the dummy property, for a TU-game with communication structure, a player with zero marginal contribution in any connected set should receive zero payoff.

Definition 2.3.36 A solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies *independence in unanimity games* if for any $(N, v, L) \in \mathcal{G}_N^{ccf}$ and $Q, Q \cup \{j\} \in C^L(N)$ with $j \in N \setminus Q$, it holds that $\xi_i(N, u_Q, L) = \xi_i(N, u_{Q \cup \{j\}}, L)$ for all $i \in Q$ with $\{i, j\} \notin L$.

Theorem 2.3.37 ***(Mishra and Talman, 2010)*** *The average tree solution is the unique solution on $\mathcal{G}_N^{ccf}$ that satisfies efficiency, linearity, strong symmetry, the dummy property, and independence in unanimity games.*

2.4 A new axiomatic characterization of the average tree solution

In this section, we axiomatize the average tree solution on the class of TU-games with connected cycle-free communication structure based on axioms that are in the same spirit of the ones used to characterize the Shapley value.

On the class of TU-games, according to the Shapley value a player receives zero payoff if this player has no contribution when joining to coalitions that do not contain him. For the average tree solution, since only spanning trees and corresponding marginal contribution vectors are considered, not all of the marginal contributions of a player are taken into account. In a connected cycle-free graph (N, L), for any $i \in N$ an element $S \in \widehat{C}^L(N \setminus \{i\})$ is called a *satellite* of player i. A satellite of a player is a component of the subgraph on the set of remaining players. Each satellite of a player is connected to this player and the complement of any satellite of a player is a connected set.

A player $i \in N$ is called a *restricted null player* in a TU-game with connected cycle-free communication structure $(N, v, L) \in \mathcal{G}_N^{ccf}$ if this player never contributes when he joins to any subcollection of his satellites, i.e.,

$$v(\bigcup_{S\in Q} S \cup \{i\}) - \sum_{S\in Q} v(S) = 0$$

for all $Q \subseteq \widehat{C}^L(N \setminus \{i\})$.

Definition 2.4.1 A solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies the *restricted null player property* if for any $(N, v, L) \in \mathcal{G}_N^{ccf}$ and restricted null player $i \in N$ in (N, v, L), it holds that $\xi_i(N, v, L) = 0$.

A solution on TU-games with connected cycle-free communication structure satisfies the restricted null player property if restricted null players receive zero payoff. The restricted null player property is stronger than the dummy property of Mishra and Talman (2010). From linearity and the restricted null player property we have the following proposition.

Proposition 2.4.2 *Let a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfy linearity and the restricted null player property. Then for any $(N, v, L), (N, v', L) \in \mathcal{G}_N^{ccf}$ it holds that $\xi(N, v, L) = \xi(N, v', L)$ whenever $v(S) = v'(S)$ for all $S \in C^L(N)$.*

Proof Take any $(N, v, L), (N, v', L) \in \mathcal{G}_N^{ccf}$ such that $v(S) = v'(S)$ for all $S \in C^L(N)$. Consider the TU-game with connected cycle-free communication structure $(N, v - v', L)$. In $(N, v - v', L)$ every player is a restricted null player because $(v - v')(S) = 0$ for all $S \in C^L(N)$ and therefore receives zero payoff, that is, $\xi_i(N, v - v', L) = 0$ for all $i \in N$. By linearity, this implies $\xi(N, v, L) = \xi(N, v', L)$. ∎

Linearity and the restricted null player property of a solution on $\mathcal{G}_N^{ccf}$ together imply that the solution is completely determined by the worths of the connected sets.

The symmetry axiom for the characterization of the Shapley value states that if two players are symmetric in a TU-game, i.e., they have the same marginal contribution to any set of players which does not contain them, then the two players must receive the same payoff. On the class of TU-games with cycle-free communication structure, where connectedness of players is accounted for the set of feasible coalitions, none of the players need to be symmetric with someone else in terms of this definition, since the set of coalitions a player can join to is typically not the same as that of other players. Therefore, we consider a different kind of symmetry axiom to replace it which is strong symmetry as in Definition 2.3.33. According to strong symmetry of a solution on $\mathcal{G}_N^{ccf}$, if the worth of any connected proper subset of the grand coalition is zero, then there should be no payoff difference between the players.

The other axiom we use for the characterization of the average tree solution is restricted marginality which puts restrictions on the payoff of a single player in two different TU-games with the same connected cycle-free communication structure.

Definition 2.4.3 A solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies *restricted marginality* if for any $(N, v, L), (N, w, L) \in \mathcal{G}_N^{ccf}$ and $i \in N$, it holds that $\xi_i(N, v, L) = \xi_i(N, w, L)$ whenever

$$v(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} v(K) = w(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} w(K)$$

for $Q = N$ and $Q \in 2^N$ satisfying $N \setminus Q \in \widehat{C}^L(N \setminus \{i\})$.

According to restricted marginality, a player should receive the same payoff in two TU-games with the same connected cycle-free communication structure if this player has the same marginal contribution in both games when joining to all of his satellites and to all but one of his satellites.

Now we show that efficiency, linearity, restricted null player property, strong symmetry, and restricted marginality characterize the average tree solution on the class of TU-game with connected cycle-free communication structure. The proof uses a number of lemmata.

It is well known that every TU-game (N, v) can be written as a unique linear combination of unanimity games, i.e., $v = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S u_S$, where $\alpha_S \in \mathbb{R}$ is the so-called Harsanyi dividend of coalition $S \in 2^N \setminus \{\emptyset\}$ in the TU-game (N, v), see Harsanyi (1959). Due to linearity, given a TU-game with connected cycle-free communication structure, in order to show that there is a unique solution that satisfies linearity, efficiency, the restricted null player property, strong symmetry, and restricted marginality, it is sufficient to show that for every unanimity game with this connected cycle-free communication structure, there is a unique solution satisfying the other four axioms.

Lemma 2.4.4 *Given any connected cycle-free graph (N, L), if a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies efficiency and strong symmetry, then $\xi_i(N, u_N, L) = 1/n$ for all $i \in N$.*

Proof It holds that $u_N(N) = 1$ and $u_N(S) = 0$ for all $S \in 2^N$, $S \neq N$. By strong symmetry, this implies that $\xi_i(N, u_N, L) = \xi_j(N, u_N, L)$ for all $i, j \in N$. Since ξ is efficient, we have $\sum_{i \in N} \xi_i(N, u_N, L) = 1$. So, $\xi_i(N, u_N, L) = 1/n$ for all $i \in N$. ∎

In a graph (N, L), for each $S \in C^L(N)$, a node $i \in S$ is called an *extreme node* of S if there exists a node outside S to which i is connected. For each $S \in C^L(N)$, let $E^L(S)$ be the set of extreme nodes of S in (N, L), i.e., $E^L(S) = \{i \in S \mid \{i, j\} \in L \text{ for some } j \in N \setminus S\}$.

Lemma 2.4.5 *Given any connected cycle-free graph (N, L), if a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies efficiency, the restricted null player property, strong symmetry, and restricted marginality, then*

$$\xi_i(N, u_S, L) = \begin{cases} 0 & \text{if } i \in N \setminus S, \\ 1/n & \text{if } i \in S \setminus E^L(S), \\ 1 - \frac{|S|-1}{n} & \text{if } i \in E^L(S), \end{cases}$$

for all $S \in C^L(N)$ such that $|E^L(S)| = 1$.

Proof Let $E^L(S) = \{j\}$ for some $S \in C^L(N)$ and $j \in N$. All players $i \in N \setminus S$ are restricted null players in (N, u_S, L), and therefore from the restricted null player property it follows that $\xi_i(N, u_S, L) = 0$ for all $i \in N \setminus S$. Regarding any $i \in S \setminus \{j\}$, we have $u_S(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_S(K) = u_N(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_N(K)$ and $u_S(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_S(K) = u_N(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_N(K)$ for any $Q \in 2^N$ such that $N \setminus Q \in \widehat{C}^L(N \setminus \{i\})$. From restricted marginality and Lemma 2.4.4, we obtain that $\xi_i(N, u_S, L) = \xi_i(N, u_N, L) = 1/n$ for all $i \in S \setminus \{j\}$. Finally, by efficiency we have $\xi_j(N, u_S, L) = 1 - (|S| - 1)/n$. ∎

For a connected set with more than one extreme player and a unanimity game defined on that set, the reasoning in the proof of Lemma 2.4.5 is directly applicable to the players who are not members of that coalition and to the members of the coalition who are not extreme players of the coalition. The former players receive zero payoffs because of the restricted null player property, and because of restricted marginality the latter ones receive the same payoff they receive in the unanimity game defined on N. So, the problem is to see how those axioms assign payoffs to the extreme players of the coalition.

Lemma 2.4.6 *Given any connected cycle-free graph (N, L) and $S \in C^L(N)$, for every $j \in E^L(S)$ there exists $S' \in C^L(N)$ such that $S' \supseteq S$ and $E^L(S') = \{j\}$.*

Proof Take any $j \in E^L(S)$ and define $S' = N \setminus M$ where $M = \{i \in N \mid i \in Q$ for some $Q \in \widehat{C}^L(N \setminus \{j\})$ with $Q \cap S = \emptyset\}$. Since $M \cap S = \emptyset$, we have $S' \supseteq S$. Also, it follows that $S' \in C^L(N)$ because it is obtained by subtracting a number of satellites of j from the grand coalition. By construction and since (N, L) is a cycle-free graph it holds that $E^L(S') = \{j\}$. ∎

Note that Lemma 2.4.6 may not hold when the graph is not cycle-free. Given a connected cycle-free graph (N, L), $S \in C^L(N)$ and $i \in E^L(S)$, let S_i^L be the (unique) smallest (with respect to set inclusion) connected set in (N, L) such that $S \subseteq S_i^L$ and $E^L(S_i^L) = \{i\}$. Note that $S_i^L = S$ if $E^L(S) = \{i\}$.

Lemma 2.4.7 *Given any connected cycle-free graph (N, L), if a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies efficiency, the restricted null player property, strong symmetry, and restricted marginality, then*

$$\xi_i(N, u_S, L) = \begin{cases} 0 & \text{if } i \in N \setminus S, \\ 1/n & \text{if } i \in S \setminus E^L(S), \\ 1 - \frac{|S_i^L|-1}{n} & \text{if } i \in E^L(S), \end{cases}$$

for all $S \in C^L(N)$.

Proof Take any $S \in C^L(N)$. All players $i \in N \setminus S$ are restricted null players in (N, u_S, L), and therefore from the restricted null player property it follows that $\xi_i(N, u_S, L) = 0$ for all $i \in N \setminus S$. For any $i \in S \setminus E^L(S)$, we have $u_S(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_S(K) = u_N(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_N(K)$ and $u_S(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_S(K) = u_N(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_N(K)$ for any $Q \in 2^N$ such that $N \setminus Q \in \widehat{C}^L(N \setminus \{i\})$. Hence, by Lemma 2.4.4 and restricted marginality we have $\xi_i(N, u_S, L) = 1/n$ for all $i \in S \setminus E^L(S)$. Now take any $i \in E^L(S)$ and consider

(N, u_S, L) and $(N, u_{S_i^L}, L)$. Note that $u_S(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_S(K) = u_{S_i^L}(N) - \sum_{K \in \widehat{C}^L(N \setminus \{i\})} u_{S_i^L}(K)$ and $u_S(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_S(K) = u_{S_i^L}(Q) - \sum_{K \in \widehat{C}^L(Q \setminus \{i\})} u_{S_i^L}(K)$ for any $Q \in 2^N$ such that $N \setminus Q \in \widehat{C}^L(N \setminus \{i\})$. By restricted marginality, this implies $\xi_i(N, u_S, L) = \xi_i(N, u_{S_i^L}, L)$ for all $i \in E^L(S)$. Since $E^L(S_i^L) = \{i\}$, by Lemma 2.4.5 we have $\xi_i(N, u_S, L) = \xi_i(N, u_{S_i^L}, L) = 1 - (|S_i^L| - 1)/n$, which completes the proof. ∎

Lemma 2.4.8 *Given any connected cycle-free graph (N, L) and $S \in 2^N \setminus \{\emptyset\}$, if a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfies efficiency, the restricted null player property, strong symmetry, and restricted marginality, then $\xi(N, u_S, L)$ is uniquely determined.*

Proof If $S \in C^L(N)$, then Lemma 2.4.7 implies that $\xi(N, u_S, L)$ is uniquely determined. Suppose $S \notin C^L(N)$ and let $S^c \in C^L(N)$ be the smallest (in terms of set inclusion) connected set containing S. Since the graph (N, L) is cycle-free, S^c is uniquely determined. Consider (N, u_S, L) and (N, u_{S^c}, L). Since $u_S(Q) = u_{S^c}(Q)$ for all $Q \in C^L(N)$, from Proposition 2.4.2, it follows that $\xi(N, u_S, L) = \xi(N, u_{S^c}, L)$. Since $S^c \in C^L(N)$, Lemma 2.4.7 implies the uniqueness of the payoff vector $\xi(N, u_{S^c}, L)$, which implies the uniqueness of the payoff vector $\xi(N, u_S, L)$. ∎

Theorem 2.4.9 *The average tree solution is the unique solution on $\mathcal{G}_N^{ccf}$ that satisfies efficiency, linearity, the restricted null player property, strong symmetry, and restricted marginality.*

Proof First, we show that the average tree solution for any TU-game with connected cycle-free communication structure $(N, v, L) \in \mathcal{G}_N^L$ satisfies all axioms. Efficiency follows because the marginal vector $m^{T(i)}(N, v)$ is efficient for any $i \in N$ by construction. Since $m^{T(i)}(N, v)$ is linear in the worths of the connected sets for every $i \in N$ and the same holds for the average of those vectors, the average tree solution satisfies linearity. If player $j \in N$ is a restricted null player, then $m_j^{T(i)}(N, v) = 0$ for all $i \in N$ and therefore the average of his marginal contributions is also zero. If (N, v, L) is such that $v(S) = 0$ for all $S \subset N$, then for any $i \in N$ the marginal vector $m^{T(i)}(N, v)$ allocates $v(N)$ to the player i and zero to the rest of the players. Therefore the average of these vectors assigns $v(N)/n$ to each player and therefore strong symmetry holds. To show the average tree solution satisfies restricted marginality, consider two TU-games with the same cycle-free communication structure (N, v, L)

and (N, w, L). For some $j \in N$, let $v(N) - \sum_{K \in \widehat{C}^L(N\setminus\{j\})} v(K) = w(N) - \sum_{K \in \widehat{C}^L(N\setminus\{j\})} w(K)$ and $v(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{j\})} v(K) = w(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{j\})} w(K)$ for all $Q \in 2^N$ satisfying $N \setminus Q \in \widehat{C}^L(N \setminus \{j\}$. Then it holds that $m_j^{T(i)}(N, v) = m_j^{T(i)}(N, w)$ for all $i \in N$, and therefore player j receives the same payoff at the average tree solution in both games.

Second, we show that there exists a unique solution that satisfies all five axioms. Since a TU-game can be uniquely expressed as a linear combination of unanimity games, by linearity it is sufficient to show that for a solution $\xi : \mathcal{G}_N^{ccf} \to \mathbb{R}^n$ satisfying the other axioms $\xi(N, u_S, L)$ is a unique payoff vector for all $S \in 2^N \setminus \{\emptyset\}$. Uniqueness of $\xi(N, u_S, L)$ for all $S \in 2^N \setminus \{\emptyset\}$ is a direct result of Lemma 2.4.8, which completes the proof. ∎

Remark 2.4.10 In Theorem 2.4.9, together with strong symmetry and efficiency we use linearity and restricted marginality. For the axioms used for the characterization of the average tree solution in Theorem 2.4.9, we have no examples that show the logical independence. Unlike the Young's axiomatization (Young (1985)) of the Shapley value by efficiency, symmetry, and strong monotonicity without a priori requirement of linearity, for the axiomatization of the average tree solution on the class of TU-games with connected cycle-free communication structure, we use both linearity and restricted marginality. The reason why the induction argument of Young does not work in the latter case is that while the decomposition of a TU-game is considered via the unanimity basis determined by all possible coalitions, restricted marginality (as opposed to marginality) considers only marginal contributions of a player while joining some specific coalitions. In Young (1985) together with strong monotonicity, which is a marginality axiom, symmetry is used as one of the other axioms. In case the TU-game is a unanimity game on an arbitrary set, symmetry implies equal payoff allocation to all members of the set on which the unanimity game is defined. In our characterization together with restricted marginality we use strong symmetry which only tells how to distribute the payoff in case the TU-game is the unanimity game on the grand coalition.

The following example illustrates the reasoning used to show that the solution is a unique payoff vector.

Example 2.4.11 Consider a cycle free graph (N, L) as given in Example 2.3.24. From strong symmetry it follows that $\xi_i(N, u_N, L) = 1/7$ for all $i \in N$. Now consider the TU-game with cycle-free communication structure (N, u_S, L) where $S = \{3, 4, 5, 6\}$. It holds that $E^L(\{3, 4, 5, 6\}) = \{3, 6\}$. Consider player 4 and the two unanimity games with cycle-free communication structure (N, u_S, L) and (N, u_N, L). Note that $u_S(N) - \sum_{K \in \widehat{C}^L(N\setminus\{4\})} u_S(K) = u_N(N) - \sum_{K \in \widehat{C}^L(N\setminus\{4\})} u_N(K)$ and $u_S(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{4\})} u_S(K) = u_N(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{4\})} u_N(K)$ for all $Q \in 2^N$ satisfying $N \setminus Q \in \widehat{C}^L(N \setminus \{4\}) = \{\{1,2,3\},\{5\},\{6,7\}\}$. Since $\xi_4(N, u_N, L) = 1/7$, by restricted marginality we have $\xi_4(N, u_S, L) = 1/7$. Similarly, we have $\xi_5(N, u_S, L) = 1/7$. Now consider player 3, then $S_3^L = \{3,4,5,6,7\}$, which is shown by the dashed set in Figure 2.3. Note that $E^L(\{3,4,5,6,7\}) = \{3\}$. For $i = 4,5,6,7$, $u_{S_3^L}(N) - \sum_{K \in \widehat{C}^L(N\setminus\{i\})} u_{S_3^L}(K) = u_N(N) - \sum_{K \in \widehat{C}^L(N\setminus\{i\})} u_N(K)$ and $u_{S_3^L}(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{i\})} u_{S_3^L}(K) = u_N(Q) - \sum_{K \in \widehat{C}^L(Q\setminus\{i\})} u_N(K)$ for all $Q \in 2^N$ satisfying $N \setminus Q \in \widehat{C}^L(N \setminus \{i\})$. So, by restricted marginality we have $\xi_i(N, u_{S_3^L}, L) = \xi_i(N, u_N, L) = 1/7$ for $i = 4,5,6,7$. By the restricted null player property we have $\xi_i(N, u_{S_3^L}, L) = 0$ for $i = 1,2$. By efficiency this implies $\xi_3(N, u_S, L) = \xi_3(N, u_{S_3^L}, L) = 3/7$. Similarly, for player 6, it holds that $S_6^L = \{1,2,3,4,5,6\}$ and $\xi_6(N, u_S, L) = \xi_6(N, u_{S_6^L}, L) = 2/7$.

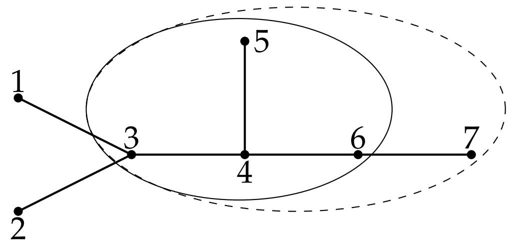


Figure 2.3: The set S_3^L for $S = \{3,4,5,6\}$ in Example 2.4.11.

Mishra and Talman (2010) provides a characterization of the average tree solution for TU-games with connected cycle-free communication structure by using linearity, the dummy property, strong symmetry, and independence in unanimity games. In Mishra and Talman (2010), the dummy property requires zero payoff allocation for a player if this player has zero marginal contribution to any set that does not contain him and together they form a connected set. For the new characterization of the average tree solution, we take the restricted null player property. The restricted null player property requires zero payoff allocation to a player if this player has zero marginal contribution to any subcollection of his satellites and it is therefore stronger than the dummy property. In Mishra and Talman (2010) independence of unanimity games, which relates two unanimity games with the same communication structure, is used for the characterization. We use restricted marginality by which it is possible to relate any two TU-games with the same connected cycle-free communication structure. The Shapley value is originally defined for TU-games where cooperation among the players is not restricted. The axioms used in this chapter to characterize the average tree solution for the class of TU-games with connected cycle-free communication structure have the same sprit with the axioms of Shapley (1953) and Young (1985). In case the graph standing for the communication structure is a connected cycle-free graph, the Myerson value satisfies efficiency, linearity and strong symmetry. On the other hand, the restricted null player property and restricted marginality are not satisfied by the Myerson value on this class of TU-games with communication structure.

CHAPTER 3

SOLUTIONS FOR TU-GAMES WITH DOMINANCE STRUCTURE

3.1 Introduction

Chapter 2 considers TU-games with communication structure which is represented by means of an undirected graph on the set of players. In this chapter, we consider TU-games with dominance structure which is represented by a directed graph on the set of players. In the literature, restricted cooperation by means of a directed graph is modeled in different ways. TU-games with permission structure refer to the situations where the players need the permission of their superiors in order to cooperate. In Gilles et al. (1992), Derks and Gilles (1995), and van den Brink and Gilles (1996) conjunctive approach is employed. For the conjunctive approach it is assumed that each player needs the permission of all of his superiors to cooperate. In Gilles and Owen (1999) and van den Brink (1997), disjunctive approach is employed where it is assumed that the permission of one direct superior is sufficient to cooperate. In both cases, by taking the permission structure into account a new TU-game is defined and the Shapley value of this game is taken as solution.

As a similar structure, Faigle and Kern (1992) considers TU-games with precedence constraints where the players are partially ordered by some precedence relation. For TU-games with precedence constraints, only the coalitions that satisfy the precedence constraints are considered to be feasible. Faigle and Kern (1992) defines a type of Shapley value for such situations and provides a characterization for this value.

In this chapter, we consider TU-games with dominance structure. Similar to TU-games with permission structure and TU-games with precedence constraints, the dominance structure in the game is modeled by a directed graph on the set of players. We consider an arbitrary digraph representing the dominance structure, hence we allow for the existence of cycles which is not covered by TU-games with precedence constraints. For the class of TU-games with dominance structure, we introduce the average covering tree solution and the dominance value.

For an arc in a digraph, there are several different basic interpretations. One interpretation is that an arc represents a way of communication and indicates which player has initiated the communication but at the same time it represents a fully developed communication link where players are able to communicate in both directions with each other. In such a case, following Myerson (1977), it is natural to assume that there is no subordination of players and to focus on component efficient values. According to an alternative interpretation, an arc represents only one-way communication situation. In this case, we still have different options for the interpretation. The first option is when the communication between players is supposed to be possible only along the directed paths in the digraph. This assumption leads to the solution concepts of web values, in particular the tree value, and the average web value for cycle-free digraph games introduced in Khmelnitskaya and Talman (2014) and the covering values for cycle-free digraph games studied in Li and Li (2011). Another option is to assume that the digraph represents the subordination of players such that after each player any of his subordinates may follow as long as this does not hurt the subordination among the players prescribed by the digraph. An example of such a situation is considering a set of tasks as the set of players where the tasks that have to be performed are not linearly ordered but the partial ordering of the tasks is represented by the arcs of a digraph.

Both for the average covering tree solution and the dominance value, we abide by the latter interpretation of an arc. The difference between these two solutions is the assumption whether independent tasks can be performed at the same time or not. For the dominance value, it is assumed that at every moment only one task can be performed. When some task is completed, the next task can be any of the tasks whose performance is not violating the subordination among the remaining tasks. For the average covering tree solution for TU-games with dominance structure, it is assumed that at every moment several tasks can be performed as long as those performed ones belong to independent groups of tasks. So, after some task is completed, from each connected group of the remaining tasks, one task is performed as long as it does not violate the subordination of the tasks in that group.

The average covering tree solution for TU-games with dominance structure is based on so-called covering trees of a digraph and the corresponding marginal contribution vectors. For a digraph, an induced covering tree preserves the domination relation in the digraph and the average covering tree solution is defined as the average of the marginal contribution vectors corresponding to all covering trees.

The dominance value for TU-games with dominance structure is based on some specific permutations. To define the dominance value, for a digraph we introduce the set of consistent permutations on the set of players. The dominance value is defined as the average of the marginal contribution vectors corresponding to all consistent permutations. We define the dominance value for TU-games with arbitrary dominance structure. On the other hand, the Shapley value introduced by Faigle and Kern (1992) is only defined for cycle-free cases. When the digraph representing the dominance structure in the game is cycle-free, the dominance value and the Shapley value of Faigle and Kern (1992) coincide.

For both solution concepts, several properties are derived and a comparison is made with other solution concepts. Also convexity type of conditions that guarantee the core stability of the solution concepts are given. TU-games with specific dominance structure, like directed cycles, directed stars, and trees, are considered and characterizations are provided.

This chapter is based on Khmelnitskaya et al. (2012) and Khmelnitskaya et al. (2014) and the structure of this chapter is as follows. Basic definitions and notation are introduced in Section 2. Section 3 introduces the average covering tree solution for TU-games with connected dominance structure and studies its properties including core stability. In Section 4, the dominance value is defined for TU-games with dominance structure and properties of this solution are studied. Special digraphs as dominance structure are considered in Section 5.

3.2 Preliminaries

In this chapter, together with a TU-game, we assume the existence of a dominance structure which restricts the cooperation among the players. The dominance structure on the finite player set $N = \{1, \ldots, n\}$ is specified by a directed graph on N.

Given a digraph (N, D) and coalition $S \in 2^N$, the *subgraph* of (N, D) on S is the digraph $(S, D|_S)$ where $D|_S = \{(i, j) \in D \mid i, j \in S\}$. A coalition $S \in 2^N$ is *connected* in a digraph (N, D) if for any two different players $i, j \in S$ there is a path in $(S, D|_S)$ between i and j. For a digraph (N, D), $S \in 2^N$ is a *component* of (N, D) if S is a connected set and for any $j \in N \setminus S$, the set $S \cup \{j\}$ is not connected. For a digraph (N, D) and $S \in 2^N$, $C^D(S)$ denotes the collection of connected subsets of S in (N, D) and $\widehat{C}^D(S)$ denotes the collection of components of the subgraph $(S, D|_S)$.

Given a digraph (N, D) and coalition $S \in 2^N$, node $i \in S$ *dominates* node $j \in S$ in $(S, D|_S)$, denoted $i \succ_{D|_S} j$, if $j \in S_{D|_S}(i)$ and $i \notin S_{D|_S}(j)$. Similarly, node $i \in S$ *immediately dominates* node $j \in S$ in $(S, D|_S)$ if i dominates j and $(i, j) \in D$. Node $i \in S$ is an *undominated* node of $(S, D|_S)$ if for every predecessor j of i in $(S, D|_S)$ there exists a directed path in $(S, D|_S)$ from i to j, i.e., $j \in S_{D|_S}(i)$ whenever $i \in S_{D|_S}(j)$. Notice that, an undominated node of $(S, D|_S)$ is either a node in S without any predecessors in $(S, D|_S)$ or a member of at least one directed cycle in $(S, D|_S)$. Since N is assumed to be finite, any digraph (N, D) and any subgraph of (N, D) has at least one undominated node. For a digraph (N, D) and a coalition $S \in 2^N$, $U_D(S)$ denotes the set of undominated nodes of the subgraph $(S, D|_S)$. A tree (N, T) on N is a *spanning tree* of a digraph (N, D) if $T \subseteq D$. The root of a tree (N, T) is denoted by $r(N, T)$

The combination of a TU-game and a digraph results in a *TU-game with dominance structure* which is denoted by a triple (N, v, D), where N is the set of players, (N, v) is a TU-game, and (N, D) is a digraph on N. $\mathcal{G}_N^{ds}$ denotes the set of TU-games with dominance structure on a fixed player set N and $\mathcal{G}_N^{cds}$ denotes the set of TU-games with connected dominance structure on a fixed player set N.

A single valued *solution* on $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$ is a function $\xi \colon \mathcal{G} \to \mathbb{R}^n$ that assigns to every TU-game with dominance structure $(N, v, D) \in \mathcal{G}$ a payoff vector $\xi(N, v, D) \in \mathbb{R}^n$.

3.3 The average covering tree solution for TU-games with dominance structure

In this section it is assumed that for a TU-game with dominance structure the connected coalitions in the digraph are the feasible coalitions. Considering the nodes in the digraph as tasks that need to be completed, we assume that at every moment several of the remaining tasks can be performed as long as they belong to independent groups of tasks. A group of tasks is independent if it forms a component in the subgraph on the remaining tasks. After completing a task, from each subgroup of independent remaining tasks a task can be performed that does not violate the subordination in that subgroup. The average covering tree solution of a TU-game with dominance structure is the average of the marginal contribution vectors corresponding to all covering trees of the underlying digraph where each covering tree describes a feasible partial ordering of the tasks to be completed. Without loss of generality, in this section we assume that the digraph is connected, i.e., N is a feasible coalition. In case the digraph representing the dominance structure is not connected, the average covering tree solution can be defined separately for each component of the digraph.

The formal definition of a covering tree of a connected digraph is as follows.

Definition 3.3.1 Given a connected digraph (N, D), a tree (N, T) is a *covering tree* of (N, D) if it holds that $(i, j) \in T$ implies $i \in U_D(\bar{S}_T(i))$ and $\bar{S}_T(j) \in \widehat{C}^D(S_T(i))$.

The root of a covering tree is one of the undominated nodes of the digraph and each other node is an undominated node of the subgraph on its successor set and this latter set is a component of the set of successors of its immediate predecessor in the tree. Since the grand coalition is assumed to be a connected set, the set of nodes in a covering tree coincides with the set of nodes of the digraph. Notice that a covering tree of a digraph may contain arcs that do not belong to the digraph, i.e., a covering tree is not necessarily a spanning tree of the digraph.

Given a connected digraph (N, D), applying the following algorithm gives the set of all covering trees (N, T) of (N, D).

Algorithm 3.3.2

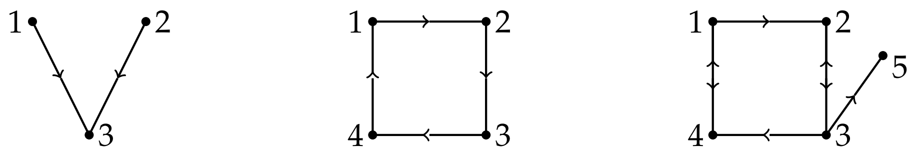
0. Input (N, D). Choose $i \in U_D(N)$. Set $T = \emptyset$, $Q_i = N \setminus \{i\}$, and $Q_j = \emptyset$ for $j \neq i$.

1. Let $\widehat{C}^D(Q_i) = \{K_1, \ldots, K_m\}$. For $k = 1, \ldots, m$, choose $j_k \in U_D(K_k)$ and set $Q_{j_k} = K_k \setminus \{j_k\}$. Set $T = T \cup \{(i, j_1), \ldots, (i, j_m)\}$ and $Q_i = \emptyset$.

2. If $Q_j = \emptyset$ for all $j \in N$, then stop. Otherwise, choose $i \in N$ such that $Q_i \neq \emptyset$, and return to Step 1.

In Step 0, the root $r(N,T)$ of the covering tree is chosen among the undominated nodes of (N,D), i.e., $r(N,T) \in U_D(N)$. We arrive to Step 1 with some node i selected in the previous step. Node i is an undominated node of some connected set in (N,D) where Q_i is the set of remaining nodes in this connected set, in particular, when coming from Step 0 node i is the already chosen root $r(N,T)$ and $Q_i = N \setminus \{r(N,T)\}$. The set of nodes in Q_i is the union of one or more components, denoted by $K_1, \ldots, K_m$. In each component K_k, $k = 1, \ldots, m$, an undominated node j_k is chosen, which becomes an immediate successor of i in the tree (N,T), and by Q_{j_k} we denote the set of remaining nodes in K_k, i.e., $Q_{j_k} = K_k \setminus \{j_k\}$. If all sets Q_j, $j \in N$, are empty, then there are no nodes left and the construction of the covering tree (N,T) is completed. Otherwise, some node i with a nonempty set Q_i is chosen and repeat the procedure. For any digraph (N,D), applying Algorithm 3.3.2 on (N,D) gives the set of all covering trees of (N,D) and any covering tree of (N,D) can be constructed by Algorithm 3.3.2. Let $\mathcal{T}^D$ denote the collection of covering trees of a connected digraph (N,D).

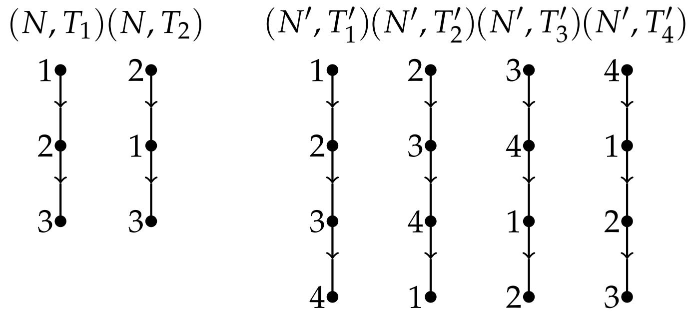
Example 3.3.3 Consider the digraphs (N,D), (N',D') and (N'',D'') where $N = \{1,2,3\}$, $D = \{(1,3),(2,3)\}$, $N' = \{1,2,3,4\}$, $D' = \{(1,2),(2,3),(3,4),(4,1)\}$, and $N'' = \{1,2,3,4,5\}$, $D'' = \{(1,2),(2,3),(3,2),(3,4),(4,1),(1,4),(3,5)\}$, as depicted in Figure 3.1.



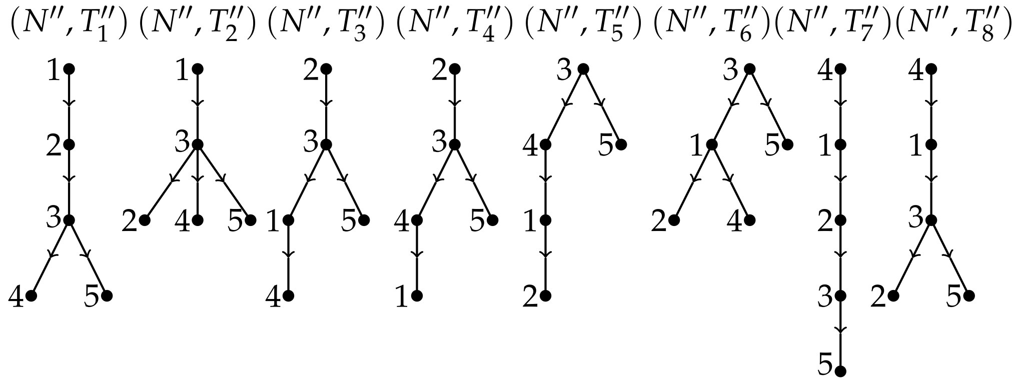
a) The digraph (N,D). *b*) The digraph (N',D'). *c*) The digraph (N'',D'').

Figure 3.1: The digraphs in Example 3.3.3

The sets of undominated nodes in digraphs (N,D), (N',D'), and (N'',D'') are $\{1,2\}$, $\{1,2,3,4\}$, and $\{1,2,3,4\}$, respectively. Following Algorithm 3.3.2 we may construct the covering trees of the digraphs as depicted in Figure 3.2.



a) The covering trees of (N,D). *b*) The covering trees of (N',D').



c) The covering trees of (N'',D'').

Figure 3.2: The covering trees in Example 3.3.3

We explain in detail the construction of the covering trees of (N,D). In (N,D) both nodes 1 and 2 are undominated and can be chosen as the root of a covering tree. If node 1 is taken as the root, the remaining nodes 2 and 3 form a connected set with only node 2 being undominated, yielding covering tree (N,T_1) where $T_1 = \{(1,2),(2,3)\}$. If node 2 is taken as the root, the remaining nodes 1 and 3 form a connected set with node 1 being undominated, yielding covering tree (N,T_2) where $T_2 = \{(2,1),(1,3)\}$.

The two covering trees of the digraph (N,D) are not spanning trees of (N,D). In fact, (N,D) has no spanning trees. However, all four covering trees of (N',D') are spanning trees. Among the covering trees of (N'',D''), (N'',T''_1), (N'',T''_4), (N'',T''_5), and (N'',T''_7) are spanning trees while the others are not.

The next theorem states a structural relation between a connected digraph and its covering trees.

Theorem 3.3.4 *Let (N,T) be a covering tree of a connected digraph (N,D). If $(i,j) \in D$ and $i \notin S_D(j)$, then $\bar{S}_D(j) \subseteq S_T(i)$.*

Proof Let $i, j \in N$ be such that $(i, j) \in D$ and $i \notin S_D(j)$ and consider any covering tree (N, T). If $i = r(N, T)$, then $i \notin S_D(j)$ and $i \neq j$ imply $\bar{S}_D(j) \subseteq N \setminus \{i\} = S_T(i)$. Suppose $i \neq r(N, T)$. Let $k \in N$ be the node where $\bar{S}_T(k) \supseteq \bar{S}_D(j) \cup \{i\}$ and $\bar{S}_T(k') \not\supseteq \bar{S}_D(j) \cup \{i\}$ for any $k' \in S_T(k)$. Suppose $k \in \bar{S}_D(j)$. Since for every $S \in 2^N$ such that $S \supseteq \bar{S}_D(j) \cup \{i\}$, node i dominates any node of $\bar{S}_D(j)$ in the subgraph $(S, D|_S)$, this contradicts with the definition of covering tree because $k \notin U_D(\bar{S}_T(k)$. Now suppose $k \in N \setminus (\bar{S}_D(j) \cup \{i\})$. Since $\bar{S}_T(k') \not\supseteq \bar{S}_D(j) \cup \{i\}$ for any $k' \in S_T(k)$, this implies $\bar{S}_D(j) \cup \{i\} \notin C^D(S_T(k))$ which contradicts with the fact that $\bar{S}_D(j) \cup \{i\}$ is a connected set in (N, D). So, $k = i$ which completes the proof. ∎

Theorem 3.3.4 says that a covering tree of a connected digraph preserves the subordination between players prescribed by the digraph. More precisely, if in a connected digraph (N, D) node i directly dominates node j, then in any covering tree of (N, D) node i is a predecessor of both node j and all successors of j in (N, D). A covering tree of a digraph possesses also the following properties.

Lemma 3.3.5 *Let (N, T) be a covering tree of a connected digraph (N, D), then it holds that*

(i) $\bar{S}_T(i) \in C^D(N)$ for all $i \in N$;

(ii) for any $i, j \in N$, if $\bar{S}_T(i) \cap \bar{S}_T(j) = \emptyset$, then $\bar{S}_T(i) \cup \bar{S}_T(j) \notin C^D(N)$.

Proof (i) Let $i \in N$. If $i = r(N, T)$, then $\bar{S}_T(i) = N$ and by assumption N is a connected set in (N, D). If $i \neq r(N, T)$, there exists $j \in N$ such that $(j, i) \in T$ which implies $\bar{S}_T(i) \in \widehat{C}^D(S_T(j))$. Hence, $\bar{S}_T(i) \in C^D(N)$.

(ii) Let $i, j \in N$ such that $\bar{S}_T(i) \cap \bar{S}_T(j) = \emptyset$. Since (N, T) is a tree, there exist $h, k, m \in N$ with $k \neq m$ satisfying $(h, k), (h, m) \in T$, $\bar{S}_T(i) \subseteq \bar{S}_T(k)$, and $\bar{S}_T(j) \subseteq \bar{S}_T(m)$. Since $\bar{S}_T(k)$ and $\bar{S}_T(m)$ are two different components of the subgraph $(S_T(h), D|_{S_T(h)})$, it holds that $\bar{S}_T(k) \cup \bar{S}_T(m) \notin C^D(N)$. Since $\bar{S}_T(i) \subseteq \bar{S}_T(k)$ and $\bar{S}_T(j) \subseteq \bar{S}_T(m)$, also $\bar{S}_T(i) \cup \bar{S}_T(j) \notin C^D(N)$. ∎

Property (i) of Lemma 3.3.5 says that in every covering tree of a connected digraph each node together with all its successors forms a connected set in the digraph. Property (ii) of Lemma 3.3.5 states that the union of different branches of a covering tree is not connected in the digraph.

Given a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$, the *marginal contribution vector* corresponding to a covering tree (N, T) of (N, D)

is the payoff vector $m^T(N, v)$ given by

$$m_i^T(N,v) = v(\bar{S}_T(i)) - \sum_{K \in \widehat{C}^D(S_T(i))} v(K), \qquad \text{for all } i \in N. \tag{3.1}$$

At the marginal contribution vector corresponding to a covering tree, as payoff a player receives the difference between the worth of the set composed by himself together with all his successors in the covering tree and the total worths of the components of the set of all his successors in the covering tree. This difference is the contribution of the player when he joins his successors in the covering tree to form a connected set.

Definition 3.3.6 On the class of TU-games with connected dominance structure $\mathcal{G}_N^{cds}$, the *average covering tree solution* (ACT) of $(N, v, D) \in \mathcal{G}_N^{cds}$ is the average of the marginal contribution vectors corresponding to all covering trees of the digraph (N, D), i.e.,

$$ACT(N,v,D) = \frac{1}{|\mathcal{T}^D|} \sum_{(N,T) \in \mathcal{T}^D} m^T(N,v).$$

Example 3.3.7 Consider a TU-game with connected dominance structure (N, v, D) where $N = \{1, 2, 3, 4, 5\}$, $v(S) = |S|^2$, $S \in 2^N$, and $D = D''$ as depicted in Figure 3.1(c). The marginal contribution vectors corresponding to the eight covering trees depicted in Figure 3.2(c) are given by

$m^{T''_1}(N,v) = (9,7,7,1,1)$, $m^{T''_2}(N,v) = (9,1,13,1,1)$, $m^{T''_3}(N,v) = (3,9,11,1,1)$, $m^{T''_4}(N,v) = (1,9,11,3,1)$, $m^{T''_5}(N,v) = (3,1,15,5,1)$, $m^{T''_6}(N,v) = (7,1,15,1,1)$, $m^{T''_7}(N,v) = (7,5,3,9,1)$, $m^{T''_8}(N,v) = (7,1,7,9,1)$. From this, we obtain that $ACT(N,v,D) = (\frac{23}{4}, \frac{17}{4}, \frac{41}{4}, \frac{15}{4}, 1)$.

When the digraph underlying a TU-game with dominance structure is a tree, there is only one covering tree, which coincides with the digraph itself. In this case, the average covering tree solution is equal to the hierarchical outcome as introduced in Demange (2004), which is later axiomatized in Khmelnitskaya (2010). When the digraph is complete, the average covering tree solution is the average of the marginal contribution vectors corresponding to $n!$ covering trees, which are all line trees, and therefore coincides with the Shapley value of the underlying TU-game.

3.3.1 Properties of the average covering tree solution

In this subsection, we study some properties that are satisfied by the average covering tree solution. Namely, we show that the average covering tree solution is efficient and satisfies linearity, the superfluous player property, hierarchical efficiency, the weak player property, and the inessential arc property. Although we do not have a characterization for the average covering tree solution, based on the properties satisfied by the solution, we compare it with some existing solution concepts. We also provide a convexity type condition under which the average covering tree solution is stable.

For TU-games with permission structure, which can be modeled by a digraph on the set of players, Gilles and Owen (1999) studies the disjunctive permission value and van den Brink (1997) provides a characterization for hierarchical permission structures. Similarly, Gilles et al. (1992) introduces the conjunctive permission value for TU-games with permission structure and van den Brink and Gilles (1996) provides a characterization for this value. Both disjunctive and conjunctive permission values define an appropriate restricted game based on the permission structure and take the Shapley value of the restricted game as solution. First we provide the definition of these values on the class of TU-games with dominance structure.

For a digraph (N, D) and $i \in N$, let $IP_D(i)$ denote the set of immediate predecessors of player i in (N, D).

Definition 3.3.8 Given a digraph (N, D), $\Psi_D^d = \{S \subseteq N \mid i \in S \text{ and } IP_D(i) \neq \emptyset \text{ imply } j \in S \text{ for some } j \in IP_D(i)\}$ is the collection of *disjunctive formable coalitions* in (N, D).

For a digraph (N, D), a coalition $S \subseteq N$ is disjunctive formable if for any member of S there exists at least one immediate predecessor (if it exists) which is also a member of S.

Definition 3.3.9 Given a digraph (N, D), for any $S \subseteq N$ the *disjunctive sovereign part* of S in (N, D) is the coalition given by

$$\sigma_D^d(S) = \cup\{S' \in \Psi_D^d \mid S' \subseteq S\}.$$

For a digraph (N, D), the disjunctive sovereign part of any coalition S is the union of all disjunctive formable coalitions in (N, D) that are subsets of S.

Definition 3.3.10 For $(N, v, D) \in \mathcal{G}_N^{ds}$, the *disjunctive restriction* of v on D is the game $\mathcal{R}_D^d(v) \in \mathcal{G}_N$ given by $\mathcal{R}_D^d(v)(S) = v(\sigma_D^d(S))$ for all $S \subseteq N$.

For a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, the *disjunctive permission value* (DPV) is the Shapley value of the corresponding disjunctive restricted game, i.e., $DPV(N, v, D) = Sh(N, \mathcal{R}_D^d(v))$.

Definition 3.3.11 Given a digraph (N, D), $\Psi_D^c = \{S \subseteq N \mid i \in S$ implies $j \in S$ for all $j \in IP_D(i)\}$ is the collection of *conjunctive formable coalitions* in (N, D).

For a digraph (N, D), a coalition $S \subseteq N$ is conjunctive formable if for each member of S, all immediate predecessors are also members of S.

Definition 3.3.12 Given a digraph (N, D), for any $S \subseteq N$ the *conjunctive sovereign part* of S in (N, D) is the coalition given by

$$\sigma_D^c(S) = \cup\{S' \in \Psi_D^c \mid S' \subseteq S\}.$$

Given a digraph (N, D), the conjunctive sovereign part of any coalition S is the union of all conjunctive formable coalitions in (N, D) that are subsets of S.

Definition 3.3.13 For $(N, v, D) \in \mathcal{G}_N^{ds}$ the *conjunctive restriction* of v on S is the game $\mathcal{R}_D^c(v) \in \mathcal{G}_N$ given by $\mathcal{R}_D^c(v)(S) = v(\sigma_D^c(S))$ for all $S \subseteq N$.

For a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, the *conjunctive permission value* (CPV) is the Shapley value of the corresponding conjunctive restricted game, i.e., $CPV(N, v, D) = Sh(N, \mathcal{R}_D^c(v))$.

Now, we go through the properties satisfied by the average covering tree solution. The first property we consider is efficiency.

Definition 3.3.14 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ is *efficient* if for any $(N, v, D) \in \mathcal{G}$, it holds that $\sum_{i \in N} \xi_i(N, v, D) = v(N)$.

On the class of TU-games with connected dominance structure, the web values, in particular the tree value and the average web value, introduced in Khmelnitskaya and Talman (2014), and the covering values studied in Li and Li (2011), are not efficient. The average covering tree solution, however, satisfies efficiency on the class of TU-games with connected dominance structure. Recall that, given a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$, the average covering tree solution is the average of the marginal contribution vectors corresponding to all covering trees of digraph (N, D). By (3.1) the marginal contribution vector corresponding to any covering tree of (N, D) distributes the worth $v(N)$ over all players in N. Whence the efficiency of the average covering tree solution follows.

For a TU-game with dominance structure, the grand coalition N is a disjunctive and conjunctive formable coalition. Since the disjunctive permission value is the Shapley value of the disjunctive restricted game and the conjunctive permission value is the Shapley value of the conjunctive restricted game, both the disjunctive and conjunctive permission values satisfy efficiency on the class of TU-games with (connected) dominance structure. On the other hand, if the dominance structure is not connected and the average covering tree solution is applied separately to each component of the digraph, then the average covering tree solution is component efficient.

The second property that is satisfied by the average covering tree solution is linearity.

Definition 3.3.15 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ is *linear* if for any $(N, v, D), (N, w, D) \in \mathcal{G}$ and for any $a, b \in \mathbb{R}$, it holds that

$$\xi(N, av + bw, D) = a\xi(N, v, D) + b\xi(N, w, D).$$

The linearity of the average covering tree solution follows straightforwardly from its definition because the solution is defined as a linear combination of the marginal contribution vectors corresponding to all covering trees. Since both the disjunctive and conjunctive permission values are defined as the Shapley value of some restricted games, linearity is also satisfied by both of these solutions.

Definition 3.3.16 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *superfluous player property* if for any $(N, v, D) \in \mathcal{G}$ it holds that $\xi_i(N, v, D) = 0$ whenever $v(S) - \sum_{Q \in \widehat{C}^D(S \setminus \{i\})} v(Q) = 0$ for all $S \in C^D(N)$ such that $i \in S$.

The superfluous player property says that if a player has no contribution in any connected coalition, then this player receives zero payoff. On the class of TU-games with (connected) dominance structure, the superfluous player property of the average covering tree solution follows immediately from its definition because for any marginal contribution vector corresponding to a covering tree, every player is receiving his marginal contribution to his set of successors with whom, according to Lemma 3.3.5, he forms a connected set. However, the disjunctive and conjunctive permission values do not satisfy the superfluous player property. For the the disjunctive and conjunctive permission values, although a player has no contribution in a connected coalition, it might be the case that some players with a nonzero contribution require his permission. To illustrate this point, consider the following example.

Example 3.3.17 Consider a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$ where $N = \{1,2,3\}$, $D = \{(1,2),(2,3)\}$, and $v = u_{\{3\}}$. For (N, D), we have $\Psi_D^d = \Psi_D^c = \{\{1\},\{1,2\},\{1,2,3\}\}$ and $\mathcal{R}_D^d(v)(S) = \mathcal{R}_D^c(v)(S) = 0$ for all $S \neq \{1,2,3\}$ and $\mathcal{R}_D^d(v)(\{1,2,3\}) = \mathcal{R}_D^c(v)(\{1,2,3\}) = 1$. Hence, $DPV(N, v, D) = CDV(N, v, D) = (1/3, 1/3, 1/3)$. Although player 1 and player 2 are superfluous players in (N, v, D), for the disjunctive and conjunctive permission values they still receive a positive payoff because player 3 needs their permission in order to be able to cooperate with other players. On the other hand, for the average covering tree solution we have $ACT(N, v, D) = (0, 0, 1)$ which shows that superfluous players receive zero payoff.

In van den Brink (1997) and van den Brink and Gilles (1996) it is stated that the inessential player property is satisfied by the disjunctive permission value. According to the inessential player property, a player receives zero payoff if this player and all of his successors have no contribution to any coalition. The inessential player property is weaker than the superfluous player property and is therefore satisfied by the average covering tree solution.

In van den Brink and Gilles (1996), one of the axioms that is used for the characterization of the conjunctive permission value for TU-games with cycle-free permission structures is the strongly inessential player property. According to the strongly inessential player property a player should receive zero payoff if he has no contribution to any set of players and the set of his immediate successors is empty. The strongly inessential player property is also weaker than the superfluous player property and is therefore satisfied by the average covering tree solution.

Definition 3.3.18 For a digraph (N, D), a coalition $S \in 2^N$ is a *closed hierarchy* if it satisfies the following conditions:

(i) $S = \bar{S}_D(i)$ for some $i \in N$;

(ii) $j \in N \setminus S$ and $h \in S \setminus U_D(S)$ imply $(j, h) \notin D$.

A coalition $S \in 2^N$ is a closed hierarchy in a digraph if it is a set composed by one of the players in N together with all his successors in the digraph and there is no player outside S that is an immediate predecessor of a dominated player in S. If a digraph has only one undominated player, then the grand coalition is a closed hierarchy.

Definition 3.3.19 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies *hierarchical efficiency* if for every $(N, v, D) \in \mathcal{G}$ and closed hierarchy S for (N, D) it holds that

$$\sum_{i \in S} \xi_i(N, v, D) = v(S).$$

The hierarchical efficiency of a solution implies efficiency for a TU-game with dominance structure with only one undominated player in the digraph. For a TU-game with dominance structure with the digraph being a tree, a hierarchically efficient value assigns to every coalition composed by some player together with all his successors in the tree exactly its worth.

For a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, a player is a *weak player* if $S_D(i) = \emptyset$

Definition 3.3.20 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *weak player property* if for any $(N, v, D) \in \mathcal{G}$, $\xi_i(N, v, D) = v(\{i\})$ whenever player $i \in N$ is a weak player for (N, D).

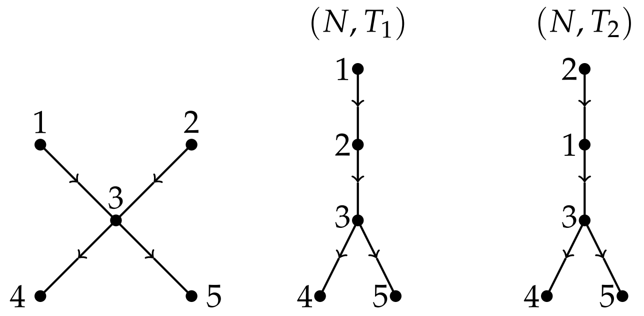
The weak player property says that a player without any successors in the digraph receives his own worth. In case the digraph representing the dominance structure is a tree, the weak player property requires that every player being a leaf of the tree receives his own worth.

Theorem 3.3.21 *The average covering tree solution on $\mathcal{G}_N^{cds}$ satisfies hierarchical efficiency.*

Proof Take any $(N, v, D) \in \mathcal{G}_N^{cds}$ and let $S \in 2^N$ be a closed hierarchy. Clearly, $S = \bar{S}_D(u)$ for any $u \in U_D(S)$. Moreover, for all $Q \in C^D(N)$ with $Q \supsetneq S$ we have $U_D(Q) \cap S = \emptyset$ and for any $i \in Q \setminus S$ there exists $Q' \in \widehat{C}^D(Q \setminus \{i\})$ such that $S \subseteq Q'$. Since the number of players is finite, for all $(N, T) \in \mathcal{T}^D$ we must have that $S = \bar{S}_T(u)$ for some $u \in U_D(S)$, which implies $\sum_{i \in S} ACT_i(N, v, D) = v(S)$. ∎

Note that any coalition consisting of exactly one weak player is a closed hierarchy. Since a weak player forms a closed hierarchy by its own, a value satisfying hierarchical efficiency also satisfies the weak player property.

Example 3.3.22 Consider a TU-game with dominance structure (N, v, D) where $N = \{1, 2, 3, 4, 5\}$, $v(S) = |S|^2$ and $D = \{(1, 3), (2, 3), (3, 4), (3, 5)\}$. For the digraph (N, D) there exist two covering trees, (N, T_1) and (N, T_2). The digraph (N, D) and the corresponding covering trees are depicted in Figure 3.3.



a) The digraph (N, D). b) The covering trees of (N, D).

Figure 3.3: Example 3.3.22

For the digraph (N, D), the coalitions $\{3, 4, 5\}$, $\{4\}$ and $\{5\}$ are the closed hierarchies. Moreover, the players 4 and 5 are the weak players for (N, D). For the two marginal contribution vectors corresponding to the covering trees, we have $m^{T_1}(N, v) = (9, 7, 7, 1, 1)$ and $m^{T_2}(N, v) = (7, 9, 7, 1, 1)$. So, $ACT(N, v, D) = (8, 8, 7, 1, 1)$. For $ACT(N, v, D)$, it holds that $\sum_{i \in \{3,4,5\}} ACT_i(N, v, D) = v(\{3, 4, 5\}) = 9$, $ACT_4(N, v, D) = v(\{4\}) = 1$, and $ACT_5(N, v, D) = v(\{5\}) = 1$. This illustrates that the average covering tree solution for (N, v, D) satisfies hierarchical efficiency and the weak player property.

We remark that the average covering tree solution also satisfies hierarchical efficiency if the dominance structure is not connected. Hierarchical efficiency is not satisfied by the disjunctive and conjunctive permission values. To see this, consider Example 3.3.17, where the conjunctive and disjunctive permission values coincide. The coalition containing only player 3 is a closed hierarchy and player 3 is a weak player. In this example, both the disjunctive and conjunctive permission values allocate $1/3$ to this player as payoff which is less than the worth of the coalition.

The next property that is satisfied by the average covering tree solution is the inessential arc property.

Definition 3.3.23 Given a digraph (N, D), an arc $(i, j) \in D$ is *inessential* if $i \notin S_D(j)$ and there exists $i' \in N$ such that $(i, i') \in D$, $i \notin S_D(i')$, and $j \in S_D(i')$.

An arc (i, j) in a digraph (N, D) is inessential if it is possible to reach node j from i also by using a directed path in (N, D) different than the arc (i, j). Moreover, the first node coming after i in this alternative path must be dominated by i. For a digraph, the absence of an inessential arc does not change the set of predecessors of any player and does not change the connectedness of the digraph.

Definition 3.3.24 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *inessential arc property* if for any $(N, v, D) \in \mathcal{G}$ and inessential arc $(i, j) \in D$ it holds that $\xi(N, v, D) = \xi(N, v, D \setminus \{(i, j)\})$.

Theorem 3.3.25 *The average covering tree solution on $\mathcal{G}_N^{cds}$ satisfies the inessential arc property.*

Proof Take any $(N, v, D) \in \mathcal{G}_N^{cds}$ and let $(i, j) \in D$ be an inessential arc, so there exists $i' \in N$ such that $(i, i') \in D$, $i \notin S_D(i')$, and $j \in S_D(i')$. Let $D' = D \setminus \{(i, j)\}$. We claim that $\mathcal{T}^D = \mathcal{T}^{D'}$.

Take any $(N, T) \in \mathcal{T}^D$. Since $(i, i') \in D$ and $i \notin S_D(i')$, Theorem 3.3.4 implies $\bar{S}_D(i') \subseteq S_T(i)$. Hence, for all $S \supseteq S_T(i)$, $U_D(S) = U_{D'}(S)$ and $\widehat{C}^D(S) = \widehat{C}^{D'}(S)$. Moreover, $D|_{S_T(i)} = D'|_{S_T(i)}$, which implies that $(N, T) \in \mathcal{T}^{D'}$.

Conversely, take any $(N, T') \in \mathcal{T}^{D'}$. The only difference between D and D' is the absence of the arc (i, j). So, we have $(i, i') \in D'$ and $i \notin S_{D'}(i')$. Again from Theorem 3.3.4 it follows that $\bar{S}_{D'}(i') \subseteq S_{T'}(i)$. Hence, for all $S \supseteq S_{T'}(i)$, $U_{D'}(S) = U_D(S)$ and $\widehat{C}^{D'}(S) = \widehat{C}^D(S)$. Moreover, $D'|_{S_{T'}(i)} = D|_{S_{T'}(i)}$, which implies that $(N, T') \in \mathcal{T}^D$. ∎

Notice that, the average covering tree solution satisfies the inessential arc property even if the dominance structure is not connected. On the class of TU-games with dominance structure, the disjunctive permission value does not satisfy the independence of inessential arcs. Because given a digraph (N, D) and an inessential arc $(i, j) \in D$, if this inessential arc is deleted from the digraph, the collection of disjunctive formable coalitions do not need to stay the same. On the other hand, for the conjunctive permission value if an inessential arc is deleted from the digraph, the collection of conjunctive formable coalitions stays the same. This is because the digraph contains another directed path from i to j. Since the collection of formable coalitions stays the same, the conjunctive restricted game is also the same, which implies that the conjunctive permission value allocates the same payoffs to every player.

Now we study stability of the average covering tree solution. In a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$, if only connected sets of players are able to cooperate, then the core is defined as the set of efficient payoff vectors that are not dominated by any connected set, i.e.,

$$CORE_c(N, v, D) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S),\ \forall S \in C^D(N)\}.$$

Definition 3.3.26 For a digraph (N, D), a connected set $S \in C^D(N)$ is a *hierarchical network* in (N, D) if for any $i \in S$ and $j \in N$ such that $(i, j) \in D$ and $i \notin S_D(j)$, it holds that $\bar{S}_D(j) \subseteq S$.

A connected set in a digraph is a hierarchical network if whenever a node of the connected set dominates an immediate successor, then this immediate successor together with all his successors in the digraph also belong to this connected set. Let $H^c(D)$ be the set of hierarchical networks in a digraph (N, D).

From Theorem 3.3.4 we immediately obtain the following corollary.

Corollary 3.3.27 *For a digraph (N, D), if $(N, T) \in \mathcal{T}^D$, then $\bar{S}_T(i) \in H^c(D)$ holds for all $i \in N$.*

A TU-game (N, v) is called *convex* if $v(S \cup Q) + v(S \cap Q) \geq v(S) + v(Q)$, for all $S, Q \in 2^N$. For a TU-game $(N, v) \in \mathcal{G}_N$, it is well know that the Shapley value is in the core of the game if the game satisfies convexity. In Chapter 2 of this monograph the Myerson value, which is the Shapley value of the corresponding Myerson restricted game, is discussed. For a game with communication structure, $(N, v, L) \in \mathcal{G}_N^{cs}$, the Myerson value is in the core of the game if convexity is satisfied by the Myerson restricted game, i.e.,

$$\sum_{K \in \widehat{C}^L(S)} v(K) + \sum_{K \in \widehat{C}^L(Q)} v(K) \leq \sum_{K \in \widehat{C}^L(S \cup Q)} v(K) + \sum_{K \in \widehat{C}^L(S \cap Q)} v(K)$$

holds for any $S, Q \in 2^N$.

As the following result shows, for a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$ a weaker convexity condition is sufficient for the stability of the average covering tree solution.

Theorem 3.3.28 *For a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$, $ACT(N, v, D) \in CORE_c(N, v, D)$ if*

$$v(S) + v(Q) \leq v(S \cup Q) + \sum_{K \in \widehat{C}^D(S \cap Q)} v(K)$$

holds for any $S, Q \in C^D(N)$ satisfying:
(i) $S \cup Q \in C^D(N)$;
(ii) $S \in H^c(D)$ or $Q \in H^c(D)$;
(iii) $K \in H^c(D)$ for all $K \in \widehat{C}^D(S \cap Q)$.

Proof Consider a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$ that satisfies the condition in the theorem. We show that for every covering tree $(N, T) \in \mathcal{T}^D$, the corresponding marginal contribution vector $m^T(N, v)$ is an element of the core and therefore also its average must be. Take any

$(N,T) \in \mathcal{T}^D$. The efficiency of $ACT(N,v,D)$ is shown above. Take any $S \in C^D(N)$ and consider the subgraph $(S, T|_S)$. It has components $S_1, \ldots, S_{k'}$. Note that $(S_1, T|_{S_1}), \ldots, (S_{k'}, T|_{S_{k'}})$ are all subtrees of (N,T). For $k = 1, \ldots, k'$, let r_k denote the root of subtree $(S_k, T|_{S_k})$. Without loss of generality, let $r_1, \ldots, r_{k'}$ be such that $k^1 < k^2$ implies $\bar{S}_T(r_{k^1}) \subset \bar{S}_T(r_{k^2})$ or $\bar{S}_T(r_{k^1}) \cap \bar{S}_T(r_{k^2}) = \emptyset$. For $k = 1, \ldots, k'$, let G_{r_k} be the set of immediate successors of the nodes of S_k in (N,T) that are not in S, i.e., $G_{r_k} = \{j \in N \setminus S \mid (i,j) \in T \text{ for some } i \in S_k\}$. Let $R = \{r_1, \ldots, r_{k'}\}$ and $I = \cup_{r \in R} G_r$. We define a tree T^* with root $r_{k'}$ on the set of nodes $R \cup I$, where the set of immediate successors of a node $r \in R$ is given by G_r and the set of immediate successors of a node $i \in I$ is given by the set

$$G_i = \{r \in R |\ \bar{S}_T(r) \subset \bar{S}_T(i), \nexists r' \in R \setminus \{r\} \text{ with } \bar{S}_T(r) \subset \bar{S}_T(r') \subset \bar{S}_T(i)\}.$$

Let $I = \{i_1, \ldots, i_l\}$. Without loss of generality, let $i_1, \ldots, i_{l'}$ be such that $l^1 < l^2$ implies $k^1 \leq k^2$ where k^h, $h = 1,2$, is such that $(r_{k^h}, i_{l^h}) \in T^*$. For $l = 1, \ldots, l'$ consider the sets $\bar{S}_T(i_l)$ and $B_{i_{l-1}} = S \cup (\bar{S}_T(i_1) \cup \cdots \cup \bar{S}_T(i_{l-1}))$. By Corollary 3.3.27, $\bar{S}_T(i_l)$ is a hierarchical network in (N,D) for any $l = 1, \ldots, l'$. To apply the induction argument on l to show that the set B_{i_l} is a connected set, suppose that $B_{i_{l-1}}$ is a connected set. Notice that for $l = 1$ the set $B_{i_{l-1}} = S$ is a connected set. Let $i \in N$ be the unique immediate predecessor of i_l in (N,T), then from the construction of T^* it follows that $i \in S$ and from (ii) of Definition 3.3.1 it follows that $\bar{S}_T(i_l) \in \widehat{C}^D(S_T(i))$. Due to (i) of Lemma 3.3.5 $\bar{S}_T(i)$ is a connected set, which implies that $(i,j) \in D$ for some $j \in \bar{S}_T(i_l)$. Because $j \in \bar{S}_T(i_l)$ and $i \in B_{i_{l-1}}$, $(i,j) \in D$ implies that their union, which is equal to B_{i_l}, is indeed a connected set. Moreover, by construction of T^*, the components of their possibly empty intersection are the hierarchical networks $\bar{S}_T(r)$, $r \in G_{i_l}$. From the condition in the theorem it follows that

$$v(S \cup (\bar{S}_T(i_1) \cup \cdots \cup \bar{S}_T(i_{l-1}))) + v(\bar{S}_T(i_l)) \leq$$
$$v(S \cup (\bar{S}_T(i_1) \cup \cdots \cup \bar{S}_T(i_l))) + \sum_{r \in G_{i_l}} v(\bar{S}_T(r)).$$

By repeated application of this inequality for $l = 1, \ldots, l'$ and since

$$S \cup (\bigcup_{l=1}^{l'} \bar{S}_T(i_l)) = \bar{S}_T(r_{k'})$$

it follows that

$$v(S) + \sum_{l=1}^{l'} v(\bar{S}_T(i_l)) \leq v(\bar{S}_T(r_{k'})) + \sum_{l=1}^{l'} \sum_{r \in G_{i_l}} v(\bar{S}_T(r)).$$

Because $\{i_1, \ldots, i_{l'}\} = \bigcup_{k=1}^{k'} G_{r_k}$, the latter inequality can be rewritten as

$$v(S) + \sum_{k=1}^{k'} \sum_{i \in G_{r_k}} v(\bar{S}_T(i)) \leq v(\bar{S}_T(r_{k'})) + \sum_{k=1}^{k'} \sum_{i \in G_{r_k}} \sum_{r \in G_i} v(\bar{S}_T(r)).$$

Since T^* is a tree, every hierarchical network $\bar{S}_T(r_k)$, $k = 1, \ldots, k'$, appears exactly once in the right hand side and we obtain

$$v(S) + \sum_{k=1}^{k'} \sum_{i \in G_{r_k}} v(\bar{S}_T(i)) \leq \sum_{k=1}^{k'} v(\bar{S}_T(r_k)).$$

Since for $k = 1, \ldots, k'$, $S_k = \bar{S}_T(r_k) \setminus (\bigcup_{i \in G_{r_k}} \bar{S}_T(i))$, we have

$$\sum_{i \in S} m_i{}^T(N, v) = \sum_{k=1}^{k'} [v(\bar{S}_T(r_k)) - \sum_{i \in G_{r_k}} v(\bar{S}_T(i))].$$

From the last two equations it follows that $\sum_{i \in S} m_i{}^T(N, v) \geq v(S)$, which completes the proof. ∎

3.4 The dominance value for TU-games with dominance structure

In this section, given a TU-game with dominance structure we assume that not the connected coalitions but the coalitions that do not violate the subordination of players in the dominance structure are able to form. Considering the nodes in the digraph as tasks that need to be completed, for the dominance value, we assume that at every moment exactly one of the remaining tasks can be performed as long as it does not violate the subordination of the remaining tasks in the digraph. We introduce a single-valued solution concept, the so-called dominance value for the class of TU-games with dominance structure.

For TU-games the Shapley value is the average of the marginal contribution vectors corresponding to all permutations on the player set. At such a vector, every player receives as payoff the marginal contribution when he joins his predecessors in the permutation. For TU-games with dominance structure, we define the dominance value as the average of the marginal contribution vectors corresponding to all permutations on the player set that are consistent with the dominance structure prescribed by the digraph. To each of these consistent permutations corresponds a sequence of feasible coalitions such that every player together with his predecessors in the permutation, if they exist, forms a feasible coalition. Faigle and Kern (1992) considers TU-games with precedence constraints which is represented by a partial ordering on the set of players. The dominance structure considered in this section is more general and can be any binary relation on the set of players and is represented by an arbitrary digraph.

In this section, under the dominance relation induced by a digraph, the set of feasible coalitions is assumed to be determined by the set of so-called hierarchical coalitions.

Definition 3.4.1 For a digraph (N, D), a coalition $S \in 2^N$ is a *hierarchical coalition* in (N, D) if for any $i \in S$ and $j \in N$, such that $(i, j) \in D$ and $i \notin S_D(j)$, it holds that $\bar{S}_D(j) \subset S$.

Recall that a hierarchical network is a connected set that satisfies the condition in Definition 3.4.1. On the other hand, a hierarchical coalition does not need to be a connected set. So, each hierarchical network is a hierarchical coalition but a hierarchical coalition does not need to be a hierarchical network.

If a player in a hierarchical coalition dominates an immediate successor in the digraph, then the coalition also contains this latter player and all his successors in the digraph. The set of hierarchical coalitions contains all coalitions that preserve the domination relations between the players and is assumed to be the set of feasible coalitions that are able to cooperate. For a cycle-free digraph (N, D), a coalition $S \in 2^N$ is hierarchical if and only if for any $i \in S$, it holds that the successors set of i in (N, D) belongs to S, i.e., $\bar{S}_D(i) \subseteq S$. So, for a cycle-free digraph the set of hierarchical coalitions coincides with the set of feasible coalitions considered in Faigle and Kern (1992) when the precedence constraints are induced by the same digraph. Notice that both the empty coalition and the grand coalition of all players are always hierarchical. If the digraph is empty, then every coalition is hierarchical. The set of all hierarchical coalitions of a digraph (N, D) is denoted by $H(D)$. Observe that $S, T \in H(D)$ implies that $S \cup T \in H(D)$ and $S \cap T \in H(D)$.

Definition 3.4.2 For a digraph (N, D), a permutation π on N is *consistent* with (N, D) if $i \in U_D(\bar{P}_\pi(i))$ holds for all $i \in N$.

Given a digraph, a permutation is consistent if each player is an undominated player of the subgraph on the set of players consisting of this player and his predecessors in the permutation. The set of permutations on N consistent with the digraph (N, D) is denoted by Π^D. Since N is finite and therefore any subgraph of (N, D) has at least one undominated node, $\Pi^D \neq \emptyset$ for any digraph (N, D). A consistent permutation keeps the subordination of players prescribed by the digraph in the sense that for any permutation $\pi \in \Pi^D$ consistent with (N, D) it holds that $\pi(j) < \pi(i)$ whenever there exists $h \in N$ such that $(i, h) \in D$, $i \notin S_D(h)$ and $j \in \bar{S}_D(h)$. As the next proposition shows, in any consistent permutation, every player together with his predecessors and also his predecessors themselves form hierarchical coalitions in the digraph.

Proposition 3.4.3 *For a digraph (N, D), if $\pi \in \Pi^D$, then both $\bar{P}_\pi(i) \in H(D)$ and $P_\pi(i) \in H(D)$ for all $i \in N$.*

Proof Take any $\pi \in \Pi^D$ and suppose $\overline{P}_\pi(i) \notin H(D)$ for some $i \in N$. Then there exists $i' \in \overline{P}_\pi(i)$ and $j' \in N$ such that $(i', j') \in D$, $i' \notin S_D(j')$ but $k \notin \overline{P}_\pi(i)$ for some $k \in \overline{S}_D(j')$. Let $k' \in S_D(j')$ be the node such that $k' \in N \setminus \overline{P}_\pi(i)$ and $\overline{P}_\pi(k') \supseteq \overline{P}_\pi(k)$ for all $k \in S_D(j')$. So, $\overline{P}_\pi(k') \supseteq S_D(j')$ and $i, i', j' \in \overline{P}_\pi(k')$. Since $(i', j') \in D$, $S_D(j') \subseteq \overline{P}_\pi(k')$, and $k' \in S_D(j')$, it holds that $k' \in S_{D|_{\overline{P}_\pi(k')}}(i')$. On the other hand, $k' \in S_D(j')$ and $i' \notin S_D(j')$ imply $i' \notin S_{D|_{\overline{P}_\pi(k')}}(k')$. This implies $k' \notin U_D(\overline{P}_\pi(k'))$, which contradicts $\pi \in \Pi^D$. Since, for any $i \in N$, $P_\pi(i) = \overline{P}_\pi(j)$ where $j \in P_\pi(i)$ is such that $\pi(j) = \max_{k \in P_\pi(i)} \pi(k)$, it also holds that $P_\pi(i) \in H(D)$ for all $i \in N$. ∎

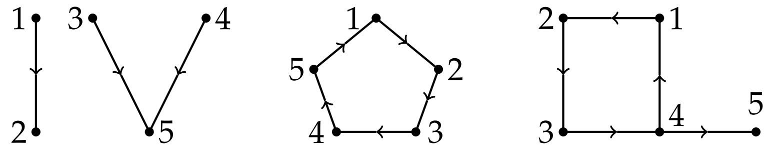
Given a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, for each consistent permutation $\pi \in \Pi^D$, there corresponds a marginal contribution vector $m^\pi(N, v)$ given by

$$m_i^\pi(N, v) = v(\bar{P}_\pi(i)) - v(P_\pi(i)),\ i \in N.$$

Definition 3.4.4 On the class of TU-games with dominance structure $\mathcal{G}_N^{ds}$, the *dominance value* (DOM) of $(N, v, D) \in \mathcal{G}_N^{ds}$ is the average of the marginal contribution vectors that correspond to all permutations consistent with (N, D), i.e.,

$$DOM(N, v, D) = \frac{1}{|\Pi^D|} \sum_{\pi \in \Pi^D} m^\pi(N, v).$$

Example 3.4.5 Consider the 5-player TU-games with dominance structure (N, v, D), (N, v, D') and (N, v, D'') with characteristic function $v(S) = |S|^2$ for all $S \subseteq N$ and $D = \{(1,2), (3,5), (4,5)\}$, $D' = \{(1,2), (2,3), (3,4), (4,5), (5,1)\}$ and $D'' = \{(1,2), (2,3), (3,4), (4,1), (4,5)\}$, as depicted in Figure 3.5.



a) The digraph (N, D). b) The digraph (N, D'). c) The digraph (N, D'').

Figure 3.5 : The digraphs in Example 3.4.5

There are 20 permutations consistent with the digraph (N, D): $\pi^1 = (5,4,3,2,1)$, $\pi^2 = (5,3,4,2,1)$, $\pi^3 = (5,4,2,3,1)$, $\pi^4 = (5,2,4,3,1)$, $\pi^5 = (2,5,4,3,1)$, $\pi^6 = (5,3,2,4,1)$, $\pi^7 = (5,2,3,4,1)$, $\pi^8 = (2,5,3,4,1)$, $\pi^9 = (5,2,4,1,3)$, $\pi^{10} = (2,5,4,1,3)$, $\pi^{11} = (5,4,2,1,3)$, $\pi^{12} = (5,2,1,4,3)$, $\pi^{13} = (2,5,1,4,3)$, $\pi^{14} = (2,1,5,4,3)$, $\pi^{15} = (5,2,3,1,4)$, $\pi^{16} = (2,5,3,1,4)$, $\pi^{17} = (5,3,2,1,4)$, $\pi^{18} = (5,2,1,3,4)$, $\pi^{19} = (2,5,1,3,4)$, $\pi^{20} = (2,1,5,3,4)$, and $DOM(N, v, D) = (7,3,13/2,13/2,2)$. For the digraph (N, D') there are 5 permutations consistent with (N, D'): $\pi^1 = (5,4,3,2,1)$, $\pi^2 = (4,3,2,1,5)$, $\pi^3 = (3,2,1,5,4)$, $\pi^4 = (2,1,5,4,3)$, $\pi^5 = (1,5,4,3,2)$, and $DOM(N, v, D') = (5,5,5,5,5)$. For the digraph (N, D'') there are 10 permutations consistent with (N, D''): $\pi^1 = (5,4,3,2,1)$, $\pi^2 = (1,5,4,3,2)$, $\pi^3 = (5,1,4,3,2)$, $\pi^4 = (2,1,5,4,3)$, $\pi^5 = (5,2,1,4,3)$, $\pi^6 = (2,5,1,4,3)$, $\pi^7 = (3,2,1,5,4)$, $\pi^8 = (5,3,2,1,4)$, $\pi^9 = (3,5,2,1,4)$, $\pi^5 = (3,2,5,1,4)$, and $DOM(N, v, D') = (52/10, 46/10, 52/10, 70/10, 30/10)$.

When there is no dominance relation between the players in the digraph (N, D), i.e., $(j, i) \in D$ whenever $(i, j) \in D$, the dominance value of the TU-game with dominance structure (N, v, D) and the Shapley value of the TU-game (N, v) coincide. Since a cycle-free digraph on the player set corresponds to a partial ordering of the players, on the subclass of TU-games with cycle-free dominance structure the dominance value coincides with the Shapley value defined in Faigle and Kern (1992) for cooperative games under precedence constraints.

3.4.1 Properties of the dominance value

In this section we discuss several properties of the dominance value for TU-games with dominance structure. Since on the class of TU-games with cycle-free dominance structure, the Shapley value introduced in Faigle and Kern (1992) and the dominance value coincides, for this class the properties introduced below are also satisfied by the Shapley value of Faigle and Kern (1992). The first two of these properties are efficiency and linearity. Recall that, for

TU-games with dominance structure, efficiency requires the total payoff allocated to the players to be equal to the worth of the grand coalition. Efficiency of the dominance value immediately follows from the fact that each marginal contribution vector corresponding to a consistent permutation is efficient. Moreover, since the marginal contribution vectors corresponding to consistent permutations are linear in the worths of coalitions and the solution is a linear combination of these vectors, the dominance value also satisfies linearity. Notice that efficiency of the dominance value also holds if the digraph is not connected, whereas the average covering tree solution in that case is not efficient but component efficient.

Given a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, a player $i \in N$ is called a *restricted null player* in (N, v, D) if $v(S) - v(S \setminus \{i\}) = 0$ for all $S \in H(D)$ such that $i \in S$ and $S \setminus \{i\} \in H(D)$. Such a player should receive zero payoff.

Definition 3.4.6 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *restricted null player property* if for any $(N, v, D) \in \mathcal{G}$ it holds that $\xi_i(N, v, D) = 0$ whenever $i \in N$ is a restricted null player in (N, v, D).

According to the restricted null player property a player receives zero payoff if this player has no marginal contribution to the hierarchical coalitions that are still hierarchical after joining them. The superfluous player property, however, requires a player to receive zero payoff if this player has no marginal contribution when joining a collection of connected sets such that after joining they all together form a connected set in the digraph. Recall that a hierarchical coalition does not need to be connected and a connected set does not need to be a hierarchical coalition.

Proposition 3.4.7 *The dominance value on $\mathcal{G}_N^{ds}$ satisfies the restricted null player property.*

Proof Take any TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$ and restricted null player i in (N, v, D). Consider any $\pi \in \Pi^D$. By Proposition 3.4.3, both $\bar{P}_\pi(i)$ and $P_\pi(i)$ are hierarchical coalitions. Since i is a restricted null player in (N, v, D), $m_i^\pi(N, v) = v(\bar{P}_\pi(i)) - v(P_\pi(i)) = 0$. Hence, $DOM_i(N, v, D) = 0$. ∎

The restricted null player property is not satisfied by both the disjunctive and conjunctive permission values. In order to see this, consider Example 3.3.17 where player 2 is a restricted null player and both the disjunctive and conjunctive permission values distribute 1/3 as payoff to player 2. The average covering tree solution does not satisfy the restricted null player property and the dominance value does not satisfy the restricted superfluous property. This is because the two properties are based on the marginal contributions while joining to different collection of coalitions.

Given a digraph (N, D), $i, j \in N$ are *symmetric in* (N, D) if i and j have the same set of immediate successors and immediate predecessors, i.e., $\{k \mid (i,k) \in D\} = \{k \mid (j,k) \in D\}$ and $\{k \mid (k,i) \in D\} = \{k \mid (k,j) \in D\}$. Given a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, players $i, j \in N$ are *symmetric in worth* if for all hierarchical coalitions $S \subseteq N \setminus \{i, j\}$ for which coalitions $S \cup \{i\}$, $S \cup \{j\}$ and $S \cup \{i, j\}$ are also hierarchical, $v(S \cup \{i\}) = v(S \cup \{j\})$. Given a TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$, two different players in N are *symmetric in* (N, v, D) if they are symmetric in (N, D) and symmetric in worth. Such players should receive the same payoff.

Definition 3.4.8 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{ds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$ satisfies the *restricted equal treatment property* if for any $(N, v, D) \in \mathcal{G}_N^{ds}$ and $i, j \in N$ symmetric in (N, v, D), it holds that $\xi_i(N, v, D) = \xi_j(N, v, D)$.

Remark 3.4.9 For players $i, j \in N$ being symmetric in a digraph (N, D) it holds that $\pi \in \Pi^D$ if and only if $\pi' \in \Pi^D$, where $\pi'(i) = \pi(j)$, $\pi'(j) = \pi(i)$ and $\pi'(k) = \pi(k)$ for all $k \in N \setminus \{i, j\}$.

Proposition 3.4.10 *The dominance value on $\mathcal{G}_N^{ds}$ satisfies the restricted equal treatment property.*

Proof Take any TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$ and two players $i, j \in N$ symmetric in (N, v, D). We show that $m_i^\pi(N, v) = m_j^{\pi'}(N, v)$ and $m_j^\pi(N, v) = m_i^{\pi'}(N, v)$ for any $\pi, \pi' \in \Pi^D$ such that $\pi'(i) = \pi(j)$, $\pi'(j) = \pi(i)$, and $\pi'(k) = \pi(k)$ for all $k \in N \setminus \{i, j\}$. Then Remark 3.4.9 implies $DOM_i(N, v, D) = DOM_j(N, v, D)$, because the dominance value for a TU-game with dominance structure is the average of the marginal vectors corresponding to all consistent permutations. Without loss of generality assume that $\pi(i) > \pi(j)$. To show $m_i^\pi(N, v) = m_j^{\pi'}(N, v)$, note that $\pi'(i) = \pi(j)$ and $\pi'(k) = \pi(k)$ for all $k \in N \setminus \{i, j\}$ implies $\bar{P}_\pi(i) = \bar{P}_{\pi'}(j)$ and $P_\pi(i) \setminus \{j\} = P_{\pi'}(j) \setminus \{i\}$. Let $S = P_\pi(i) \setminus \{j\} = P_{\pi'}(j) \setminus \{i\}$. By Proposition 3.4.3, $S \cup \{i\}$, $S \cup \{j\}$ and $S \cup \{i, j\}$ are hierarchical coalitions. Since i and j are symmetric in worth, $v(S \cup \{i\}) = v(S \cup \{j\})$, which means $v(P_\pi(i)) = v(P_{\pi'}(j))$. Together with $\bar{P}_\pi(i) = \bar{P}_{\pi'}(j)$, we obtain $m_i^\pi(N, v) = v(\bar{P}_\pi(i)) - v(P_\pi(i)) =$

$v(\bar{P}_{\pi'}(j)) - v(P_{\pi'}(j)) = m_j^{\pi'}(N,v)$. In order to show $m_j^{\pi}(N,v) = m_i^{\pi'}(N,v)$, note that $P_{\pi}(j) = P_{\pi'}(i)$. Let $S = P_{\pi}(j) = P_{\pi'}(i)$. By Proposition 3.4.3, $S \cup \{i\}$, $S \cup \{j\}$ and S are hierarchical coalitions. Since i and j are symmetric in worth, $v(S \cup \{i\}) = v(S \cup \{j\})$, which means $v(\bar{P}_{\pi}(j)) = v(\bar{P}_{\pi'}(i))$. So, $m_j^{\pi}(N,v) = v(\bar{P}_{\pi}(j)) - v(P_{\pi}(j)) = v(\bar{P}_{\pi'}(i)) - v(P_{\pi'}(i)) = m_i^{\pi'}(N,v)$. ∎

For the average covering tree solution on the class of TU-games with dominance structure, the inessential arc property is discussed in Section 3. As the next proposition shows, on the class of TU-game with dominance structure this property is also satisfied by the dominance value.

Proposition 3.4.11 *The dominance value on $\mathcal{G}_N^{ds}$ satisfies the inessential arc property.*

Proof Take any TU-game with dominance structure $(N,v,D) \in \mathcal{G}_N^{ds}$ and let $(i,j) \in D$ be inessential. Then there exists $i' \in N$ such that $(i,i') \in D$, $i \notin S_D(i')$, and $j \in S_D(i')$. Let $D' = D \setminus \{(i,j)\}$. We show that $\Pi^D = \Pi^{D'}$, which implies $DOM(N,v,D) = DOM(N,v,D')$.

Take any $\pi \in \Pi^D$. Since $(i,i') \in D$ and $i \notin S_D(i')$, Proposition 3.4.3 implies $\bar{P}_{\pi}(i') \subseteq P_{\pi}(i)$. Hence, for all $S \supseteq P_{\pi}(i)$, $U_D(S) = U_{D'}(S)$. Moreover, $D|_{P_{\pi}(i)} = D'|_{P_{\pi}(i)}$. This implies that $\pi \in \Pi^{D'}$.

Conversely, take any $\pi' \in \Pi^{D'}$. Since $D' = D \setminus \{(i,j)\}$, it holds that $(i,i') \in D'$ and $i \notin S_{D'}(i')$. From Proposition 3.4.3 it follows that $\bar{P}_{\pi'}(i') \subseteq P_{\pi'}(i)$. Hence, for all $S \supseteq P_{\pi'}(i)$, $U_{D'}(S) = U_D(S)$. Moreover, $D'|_{P_{\pi'}(i)} = D|_{P_{\pi'}(i)}$. This implies $\pi' \in \Pi^D$. ∎

Now we study the stability of the dominance value and provide a convexity type condition under which the dominance value is stable. In a TU-game with dominance structure $(N,v,D) \in \mathcal{G}_N^{ds}$, if only hierarchical coalitions are able to cooperate, then the core is defined as the set of efficient payoff vectors that are not dominated by any hierarchical coalition, i.e.,

$$CORE_h(N,v,D) = \{x \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S)\ \forall S \in H(D)\}.$$

Theorem 3.4.12 *For a TU-game with dominance structure $(N,v,D) \in \mathcal{G}_N^{ds}$, if*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

holds for any $S, T \in H(D)$, then $DOM(N,v,D) \in CORE_h(N,v,D)$.

Proof Take any TU-game with dominance structure $(N, v, D) \in \mathcal{G}_N^{ds}$ that satisfies the condition in the theorem. Due to efficiency and since the value is the average of all consistent marginal contribution vectors, it suffices to show that $\sum_{i \in S} m_i^\pi(N, v) \geq v(S)$ for any $S \in H(D)$ and $\pi \in \Pi^D$. Let $S_1, \ldots, S_k$ be the unique maximal partition of S such that $S_h = \{i \in S \mid b_h \leq \pi(i) \leq a_h\}$, $h = 1, \ldots, k$, where a_h and b_h, $h = 1, \ldots, k$, satisfy $a_{h-1} < b_h \leq a_h$, with $a_0 = 0$. We define $\bar{P}_\pi(0) = \emptyset$. For any given $h \in \{1, \ldots, k\}$ consider the sets $S \cup \bar{P}_\pi(a_{h-1})$ and $P_\pi(b_h)$. By Proposition 3.4.3 and since S is hierarchical, both sets are hierarchical coalitions. Moreover, their intersection is equal to $\bar{P}_\pi(a_{h-1})$ and their union is equal to $S \cup \bar{P}_\pi(a_h)$ which are both hierarchical coalitions. From the condition in the theorem, it then follows that

$$v(S \cup \bar{P}_\pi(a_h)) + v(\bar{P}_\pi(a_{h-1})) \geq v(S \cup \bar{P}_\pi(a_{h-1})) + v(P_\pi(b_h)).$$

By repeated application of this inequality for $h = 1, \ldots k$, we obtain

$$v(S \cup \bar{P}_\pi(a_k)) + \sum_{h=1}^{k} v(\bar{P}_\pi(a_{h-1})) \geq v(S \cup \bar{P}_\pi(a_0)) + \sum_{h=1}^{k} v(P_\pi(b_h)).$$

Because $\bar{P}_\pi(a_0) = \emptyset$ and $S \cup \bar{P}_\pi(a_k) = \bar{P}_\pi(a_k)$, it follows that

$$\sum_{h=1}^{k} v(\bar{P}_\pi(a_h)) \geq v(S) + \sum_{h=1}^{k} v(P_\pi(b_h)).$$

Since for $h = 1, \ldots, k$ it holds that

$$\sum_{i \in S_h} m_i^\pi(N, v) = v(\bar{P}_\pi(a_h) - v(P_\pi(b_h))$$

and $\sum_{i \in S} m_i^\pi(N, v) = \sum_{h=1}^{k} \sum_{i \in S_h} m_i^\pi(N, v)$, we obtain $\sum_{i \in S} m_i^\pi(N, v) \geq v(S)$. ∎

The condition for the stability in the theorem is weaker than convexity of the underlying TU-game, because the convexity relation is only required for hierarchical coalitions.

3.5 Special cases for dominance structure

In this section, we consider some special digraphs representing the dominance structure in a TU-game with dominance structure. Although generally the average covering tree solution and the dominance value are different from each other, as the following proposition states, for TU-games with some specific connected dominance structures these two solution concepts coincide.

Proposition 3.5.1 For a TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$, if each covering tree $(N, T) \in \mathcal{T}^D$ is a line tree, then $ACT(N, v, D) = DOM(N, v, D)$.

Proof Take any TU-game with connected dominance structure $(N, v, D) \in \mathcal{G}_N^{cds}$ such that every $(N, T) \in \mathcal{T}^D$ is a line tree. For each $(N, T) \in \mathcal{T}^D$, let $\pi_T : N \to N$ be the permutation such that $P_{\pi_T}(i) = S_T(i)$ for all $i \in N$. We claim that $\Pi^D = \{\pi_T \mid (N, T) \in \mathcal{T}^D\}$. Take any $(N, T) \in \mathcal{T}^D$ and suppose for a contradiction $\pi_T \notin \Pi^D$. So, there exists $i \in N$ such that $i \notin U_D(\overline{P}_{\pi_T}(i))$. Since $\overline{P}_{\pi_T}(i) = \overline{S}_T(i)$, this contradicts with the fact that $i \in U_D(\overline{S}_T(i))$. Now take any $\pi \in \Pi^D$. If $P_\pi(i) \in C^D(N)$ for all $i \in N$, then it holds that $(N, T) \in \mathcal{T}^D$ such that $S_T(i) = P_\pi(i)$ for all $i \in N$. Now suppose $P_\pi(i) \notin C^D(N)$ for some $i \in N$. Let $i' \in N$ be such that $\widehat{C}^D(P_\pi(i')) = \{K_1, \ldots, K_m\}$ for some $m > 1$ and $P_\pi(j) \in C^D(N)$ for all $j \in N \setminus \overline{P}_\pi(i')$. This implies the existence of a covering tree $(N, T) \in \mathcal{T}^D$ such that $(i', i_k) \in T$ and $i_k \in U_D(K_k)$ for $k = 1, \ldots, m$. This contradicts that every tree $\mathcal{T}^D$ is a line tree. ∎

3.5.1 Directed cycle as dominance structure

In this subsection, we consider TU-games with dominance structure which is represented by a directed cycle on the set of players. Faigle and Kern (1992) considers TU-games with precedence constraints which are modeled by some partially ordered set of players and provide a characterization of their Shapley value for such situations. In case the precedence constraints are modeled by a cycle-free digraph on the set of players, the dominance value coincides with the Shapley value defined in Faigle and Kern (1992) for cooperative games under precedence constraints. In this subsection, we provide a characterization of the dominance value for TU-games with directed cycle dominance structure which are not covered by the model in Faigle and Kern (1992).

Definition 3.5.2 A connected digraph (N, D) is a *directed cycle* if there exists a permutation $\pi : N \to N$ such that

$$D = \{(\pi(1), \pi(2)), (\pi(2), \pi(3)), \ldots, (\pi(n-1), \pi(n)), (\pi(n), \pi(1))\}.$$

A directed cycle is a connected digraph where each player has exactly one immediate successor and exactly one immediate predecessor. A TU-game with directed cycle dominance structure is a combination of a TU-game and a dominance structure represented by a directed cycle on the set of players. Let $\mathcal{G}_N^{cycle}$ be the set of all TU-games with directed cycle dominance structure, with the player set N.

Proposition 3.5.3 *For any TU-game with directed cycle dominance structure* $(N, v, D) \in \mathcal{G}_N^{cycle}$*, it holds that* $ACT(N, v, D) = DOM(N, v, D)$.

Since for a TU-game with directed cycle dominance structure each $(N, T) \in \mathcal{T}^D$ is a line tree, the equivalence of the average covering tree solution and the dominance value on the class of TU-games with directed cycle dominance structure directly follows from Proposition 3.5.1.

In Chapter 2, we introduce strong symmetry for TU-games with connected cycle-free communication structures which implies equal payoff distribution whenever any proper subset of the grand coalition has zero worth. For TU-games with connected dominance structure, now we introduce strong symmetry among undominated players which requires equal payoff allocation to the undominated players of the connected digraph when any proper subset of the grand coalition has zero worth.

Definition 3.5.4 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$, satisfies *strong symmetry among undominated players* if for any $(N, v, D) \in \mathcal{G}$ with $v(S) = 0$ for all $S \in 2^N$, $S \neq N$, it holds that $\xi_i(N, v, D) = \xi_j(N, v, D)$ for all $i, j \in U_D(N)$.

Definition 3.5.5 A solution $\xi : \mathcal{G}_N^{cds} \to \mathbb{R}^n$ satisfies *restricted marginality* if for any (N, v, D), $(N, w, D) \in \mathcal{G}_N^{cds}$ and $i \in N$, it holds that $\xi_i(N, v, D) = \xi_i(N, w, D)$ whenever $v(S) - v(S \setminus \{i\}) = w(S) - w(S \setminus \{i\})$ for all $S \in H^c(D)$ such that $i \in U_D(S)$ and $S \setminus \{i\} \in H^c(D)$.

According to restricted marginality, a player should receive the same payoff in two TU-games with the same connected dominance structure if this player has the same marginal contribution in both games when joining to a hierarchical network as an undominated node such that after joining it is still a hierarchical network.

Theorem 3.5.6 *The average covering tree solution is the unique solution on* $\mathcal{G}_N^{cycle}$ *that satisfies efficiency, linearity, the restricted null player property, strong symmetry among undominated players, and restricted marginality.*

Proof First, we show that the average covering tree solution for any TU-game with directed cycle dominance structure $(N, v, D) \in \mathcal{G}_N^{cycle}$ satisfies all axioms. Efficiency and linearity are shown in Section 3. If player $i \in N$ is a restricted null player in (N, v, D), then $m_i^T(N, v) = 0$ for any $(N, T) \in \mathcal{T}^D$ and therefore the average of his marginal contributions is also zero. To see the average covering tree solution satisfies strong symmetry among undominated players, take any $(N, v, D) \in \mathcal{G}_N^{cycle}$ such that $v(S) = 0$ for all $S \in 2^N$, $S \neq N$. Since $U_N(D) = N$, we need to show $\xi_i(N, v, D) = \xi_j(N, v, D)$ for all $i, j \in N$. Consider any $(N, T) \in \mathcal{T}^D$ and let $r(N, T) = r$ for some $r \in N$. Since $v(S) = 0$ for all $S \subset N$, we have $m_r^T(N, v) = v(N)$ and $m_i^T(N, v) = 0$ for all $i \in N \setminus \{r\}$. Since each $i \in N$ is the root of exactly one covering tree and the number of covering trees is equal to n, this implies $\xi_i(N, v, D) = v(N)/n$ for all $i \in N$. Finally, consider two TU-games with the same directed cycle dominance structure (N, v, D) and (N, w, D), and for some $i \in N$ let $v(S) - v(S \setminus \{i\}) = w(S) - w(S \setminus \{i\})$ holds for all $S \in H^c(D)$ such that $i \in U_D(S)$ and $S \setminus \{i\} \in H^c(D)$. Since in each corresponding covering tree, player i is joining to a hierarchical network as an undominated player and after joining the coalition is still hierarchical network, then $m_i^T(N, v) = m_i^T(N, w)$ for all $(N, T) \in \mathcal{T}^N$, and therefore player i receives the same payoff at the average covering tree solution in both (N, v, D) and (N, w, D).

Second, we show that there exists a unique solution that satisfies all five axioms. For this, because of linearity it is sufficient to show that for a solution $\xi : \mathcal{G}_N^{cycle} \to \mathbb{R}^n$ satisfying all five axioms $\xi(N, u_S, D)$ is a unique payoff vector for all $S \in 2^N \setminus \{\emptyset\}$. Take any $S \in 2^N \setminus \emptyset$ and consider (N, u_S, D).

If $S = N$, then $U_D(S) = N$ and strong symmetry among undominated players and efficiency imply $\xi_i(N, u_N, D) = \frac{1}{n}$ for all $i \in N$.

If $S \in C^D(N)$, $S \neq N$, then $U_D(S) = \{r\}$ for some $r \in S$. For each $i \in S \setminus U_D(S)$, it holds that $u_S(Q) - u_S(Q \setminus \{i\}) = u_N(Q) - u_N(Q \setminus \{i\})$ for all $Q \in H^c(D)$ such that $i \in U_D(Q)$ and $Q \setminus \{i\} \in H^c(D)$. By restricted marginality, this implies $\xi_i(N, u_S, D) = \xi_i(N, u_N, D) = \frac{1}{n}$ for all $i \in S \setminus U_D(S)$. Since each $i \in N \setminus S$ is a restricted null player in (N, u_S, D), by the restricted null player property we have $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus S$. Efficiency implies $\xi_r(N, u_S, D) = 1 - \frac{|S|-1}{n}$.

Finally, take any $S \notin C^D(N)$. Let $\widehat{C}^D(S) = \{S_1, \ldots, S_k\}$, then $U_D(S_h) = \{r_h\}$ for some $r_h \in S_h$, $h = 1, \ldots, k$. Note that each $i \in N \setminus S$ is a restricted null player in (N, u_S, D). By the restricted null player property, we have $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus S$. For each $i \in S \setminus \{r_1, \ldots, r_k\}$, it holds that $u_S(Q) - u_S(Q \setminus \{i\}) = u_N(Q) - u_N(Q \setminus \{i\})$ for all $Q \in H^c(D)$ such that $i \in U_D(Q)$ and $Q \setminus \{i\} \in H^c(D)$. By restricted marginality, this implies $\xi_i(N, u_S, D) = \xi_i(N, u_N, D) = \frac{1}{n}$ for all $i \in S \setminus \{r_1, \ldots, r_k\}$. For any $h \in \{1, \ldots, k\}$, let $S^h \in C^D(N)$ be the unique smallest connected set containing S such that $U_D(S^h) = r_h$. For each $i \in S^h \setminus \{r_h\}$, it holds that $u_{S^h}(Q) -$

$u_{S^h}(Q \setminus \{i\}) = u_N(Q) - u_N(Q \setminus \{i\})$ for all $Q \in H^c(D)$ such that $i \in U_D(Q)$ and $Q \setminus \{i\} \in H^c(D)$. By restricted marginality, this implies $\xi_i(N, u_{S^h}, D) = \frac{1}{n}$ for all $i \in S^h \setminus \{r_h\}$. Since each $i \in N \setminus S^h$ is a restricted null player in (N, u_{S^h}, D), by the restricted null player property we have $\xi_i(N, u_{S^h}, D) = 0$ for all $i \in N \setminus S^h$. By efficiency, this implies $\xi_{r_h}(N, u_{S^h}, D) = 1 - \frac{|S^h|-1}{n}$. Since $u_{S^h}(Q) - u_{S^h}(Q \setminus \{r_h\}) = u_S(Q) - u_S(Q \setminus \{r_h\})$ for all $Q \in H^c(D)$ such that $r_h \in U_D(Q)$ and $Q \setminus \{i\} \in H^c(D)$, we have $\xi_{r_h}(N, u_{S^h}, D) = \xi_{r_h}(N, u_S, D) = 1 - \frac{|S^h|-1}{n}$, which completes the proof. ∎

Unlike Young's axiomatization (Young (1985)) of the Shapley value for TU-games by efficiency, symmetry and strong monotonicity without a priori requirement of additivity, for the axiomatization of the average covering tree solution for TU-games with directed cycle dominance structure we use both linearity and restricted marginality. The reason why the induction argument of Young does not work in the latter case is that while the decomposition of a TU-game is considered via the unanimity basis determined by all possible coalitions, restricted marginality as opposed to marginality considers only some specific coalitions.

Given a TU-game with directed cycle dominance structure, since all covering trees are line trees, by Proposition 3.5.3 we have the following corollary.

Corollary 3.5.7 *The dominance value is the unique solution on $\mathcal{G}_N^{cycle}$ that satisfies efficiency, linearity, the restricted null player property, strong symmetry among undominated players, and restricted marginality.*

On the class of TU-games with directed cycle dominance structure the Shapley value introduced in Faigle and Kern (1992) is not defined. Considering the permission values, given a TU-game with directed cycle dominance structure $(N, v, D) \in \mathcal{G}_N^{cycle}$, it holds that $\Psi_D^d = \Psi_D^c = \{N\}$. This implies $\mathcal{R}_D^d(v)(S) = \mathcal{R}_D^c(v)(S) = 0$ for all $S \neq N$ and $\mathcal{R}_D^d(v)(N) = \mathcal{R}_D^c(v)(N) = v(N)$. Hence, for any TU-game with directed cycle dominance structure $(N, v, D) \in \mathcal{G}_N^{cycle}$, it holds that $DPV_i(N, v, D) = CPV_i(N, v, D) = v(N)/n$ for all $i \in N$. On the other hand, both the average covering tree solution and the dominance value take into account the players' marginal contributions when joining to hierarchical networks and allocate the payoff accordingly.

On the class of TU-games with directed cycle dominance structure, the disjunctive and conjunctive permission values do not satisfy restricted marginality. In order to see this, consider two TU-games with directed cycle dominance structure $(N, v, D), (N, w, D) \in \mathcal{G}_N^{cycle}$ where $D = \{(1,2), (2,3), (3,1)\}$.

Let $v = u_N$ and $w(N) = 2$, $w(\{2,3\}) = 1$, and $w(S) = 0$ for all $S \in 2^N$, $S \neq N$, $S \neq \{2,3\}$. Note that $v(S) - v(S \setminus \{1\}) = w(S) - w(S \setminus \{1\})$ holds for all $S \in H^c(D)$ such that $1 \in U_D(S)$ and $S \setminus \{1\} \in H^c(D)$. But we have $DPV_1(N,v,D) = CPV_1(N,v,D) = 1/3$ and $DPV_1(N,w,D) = CPV_1(N,w,D) = 2/3$.

3.5.2 Directed star as dominance structure

In this subsection we consider TU-games with dominance structure which is represented by a directed star and we provide a characterization of the average covering tree solution for such situations.

Definition 3.5.8 A connected digraph (N,D) is a *directed star* if there exists a unique node $h \in N$, called the *hub*, for which either $(i,h) \in D$ or $(h,i) \in D$ holds for all $i \in N \setminus \{h\}$ and D contains no other arcs.

Note that a directed star is a connected and strongly cycle-free digraph. A TU-game with directed star dominance structure is a combination of a TU-game and a domination structure represented by a directed star on the set of players. Let $\mathcal{G}_N^{star}$ be the set of TU-games with directed star dominance structure.

For a directed star (N,D), let $h(N,D)$ be the hub of the digraph, i.e., either $(i,h(N,D)) \in D$ or $(h(N,D),i) \in D$ holds for all $i \in N$, $i \neq h(N,D)$. Moreover, given a directed star (N,D), let $A(N,D) = \{i \in N \mid (i,h(N,D)) \in D\}$ and $B(N,D) = \{i \in N \mid (h(N,D),i) \in D\}$ be the set of predecessors and successors of the hub, respectively. Note that for a directed star (N,D), $U_D(N) = A(N,D)$ if $A(N,D) \neq \emptyset$. If $A(N,D) = \emptyset$, then $U_D(N) = \{h(N,D)\}$ and (N,D) is a tree with root $h(N,D)$.

Given a TU-game with connected dominance structure $(N,v,D) \in \mathcal{G}_N^{cds}$, a player $i \in N$ is called a *hierarchical null player* in (N,v,D) if $v(S) - \sum_{Q \in \widehat{C}^D(S \setminus \{i\})} v(Q) = 0$ for all $S \in H^c(D)$ such that $i \in S$ and $S \setminus \{i\} \in H(D)$. For a TU-game with connected dominance structure, a player is a hierarchical null player if his marginal contribution to the components of any hierarchical coalition is zero such that after joining to this coalition it becomes a hierarchical network. Such a player should receive zero payoff.

Definition 3.5.9 On a subclass $\mathcal{G} \subseteq \mathcal{G}_N^{cds}$, a solution $\xi : \mathcal{G} \to \mathbb{R}^n$, satisfies the *hierarchical null player property* if for any $(N,v,D) \in \mathcal{G}$ and hierarchical null player $i \in N$ in (N,v,D), it holds that $\xi_i(N,v,D) = 0$.

Given a TU-game with connected dominance structure $(N,v,D) \in \mathcal{G}_N^{cds}$, two players $i,j \in N$ are called *symmetric* in (N,v,D) if $S_D(i) = S_D(j)$ and

$v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \in H^c(D)$ such that $i, j \notin S$, $S \cup \{i\} \in H^c(D)$ and $S \cup \{j\} \in H^c(D)$. For a TU-game with connected dominance structure two players are symmetric if they have the same set of successors and their marginal contributions are the same when joining to a hierarchical network such that after joining the it is still a hierarchical network.

Definition 3.5.10 A solution $\xi : \mathcal{G}_N^{star} \to \mathbb{R}^n$ satisfies *star-symmetry* if for any $(N, v, D) \in \mathcal{G}_N^{star}$ it holds that $\xi_i(N, v, D) = \xi_j(N, v, D)$ whenever $i \in N$ and $j \in N$ are symmetric in (N, v, D).

Theorem 3.5.11 *The average covering tree solution is the unique solution on $\mathcal{G}_N^{star}$ that satisfies efficiency, linearity, the hierarchical null player property, and star-symmetry.*

Proof First, we show that the average covering tree solution for any TU-game with directed star dominance structure $(N, v, D) \in \mathcal{G}_N^{star}$ satisfies all axioms. Efficiency and linearity are shown in Section 3. If player $i \in N$ is a hierarchical null player in (N, v, D), then $m_i^T(N, v) = 0$ for any $(N, T) \in \mathcal{T}^D$ and therefore the average of his marginal contributions is also zero. To see the average covering tree solution satisfies star-symmetry, take any $(N, v, D) \in \mathcal{G}_N^{star}$ and $i, j \in N$, $i \neq j$, that are symmetric in (N, v, D). Since $i, j \in N$ are symmetric players in (N, v, D), either $i, j \in A(N, D)$ or $i, j \in B(N, D)$. If $i, j \in A(N, D)$, then for any $(N, T) \in \mathcal{T}^D$ there exists a unique $(N, T') \in \mathcal{T}^D$ such that $S_T(i) = S_{T'}(j)$, $S_T(j) = S_{T'}(i)$, and $S_T(k) = S_{T'}(k)$ for all $k \in N \setminus \{i, j\}$. Since $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for all $S \in H^c(D)$ such that $S \cup \{i\} \in H^c(D)$, $S \cup \{j\} \in H^c(D)$ and the average covering tree solution is the average of the marginal contribution vectors corresponding to all covering trees, this implies $ACT_i(N, v, D) = ACT_j(N, v, D)$. If $i, j \in B(N, D)$, then for any $(N, T) \in \mathcal{T}^D$, it holds that $S_T(i) = S_T(j) = \emptyset$. Since $v(\{i\}) = v(\{j\})$, this implies $m_i^T(N, v) = m_j^T(N, v)$ for any $(N, T) \in \mathcal{T}^D$, hence $ACT_i(N, v, D) = ACT_j(N, v, D)$.

Second, we show that there exists a unique solution that satisfies all axioms. For this, because of linearity it is sufficient to show that for a solution $\xi : \mathcal{G}_N^{star} \to \mathbb{R}^n$ satisfying all axioms $\xi(N, u_S, D)$ is a unique payoff vector for any $S \in 2^N \setminus \{\emptyset\}$.

Take $S = N$. If $A(N, D) \neq \emptyset$, each $i \in N \setminus A(N, D)$ is a hierarchical null player in (N, u_N, D) and by the hierarchical null player property this implies $\xi_i(N, u_N, D) = 0$ for all $i \in N \setminus A(N, D)$. Moreover, any $i \in A(N, D)$ and $j \in A(N, D)$ are symmetric in (N, u_N, D). By efficiency and star-symmetry, this implies $\xi_i(N, u_N, D) = 1/|A(N, D)|$ for all $i \in A(N, D)$. If $A(N, D) = \emptyset$, each $i \in N \setminus \{h(N,D)\}$ is a hierarchical null player in (N, u_N, D) and by the hierarchical null player property this implies $\xi_i(N, u_N, D) = 0$ for all $i \in N \setminus \{h(N,D)\}$ and by efficiency $\xi_{h(N,D)}(N, v, D) = 1$.

Next, take any $S \in C^D(N)$, $S \neq N$. Since (N, D) is a directed star, $S \in C^D(N)$ implies $|S| = 1$ or $h(N,D) \in S$. If $S = \{j\}$ for some $j \in N$, by the hierarchical null player property $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus \{j\}$ and by efficiency $\xi_j(N, u_N, D) = 1$. Now consider the case where $S \in C^D(N)$ and $|S| > 1$, then $h(N,D) \in S$ which implies each $i \in B(N,D)$ is a hierarchical null player in (N, u_S, D) and by the hierarchical null player property $\xi_i(N, u_S, D) = 0$ for all $i \in B(N,D)$. If $S \cap A(N,D) = \emptyset$, then each $i \in A(N,D)$ is also a hierarchical null player in (N, u_S, D) and by the hierarchical null player property we have $\xi_i(N, u_S, D) = 0$ for all $i \in A(N,D)$ and by efficiency $\xi_{h(N,D)}(N, u_S, D) = 1$. Now consider the case where $S \cap A(N,D) \neq \emptyset$. Let $S \cap A(N,D) = A'$. Then $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus A'$ because any $i \in N \setminus A'$ is a hierarchical null player in (N, u_S, D). Since any $i, j \in A'$ are symmetric in (N, u_S, D), by efficiency and star-symmetry, this implies $\xi_i(N, u_S, D) = 1/|A'|$ for all $i \in A'$.

Now consider the case where $S \notin C^D(N)$ which implies $h(N,D) \notin S$. Let $S \cap A(N,D) = A'$ and $S \cap B(N,D) = B'$. If $A' \neq \emptyset$, then each $i \in N \setminus A'$ is a hierarchical null player in (N, u_S, D) and any $i, j \in A'$ are symmetric in (N, u_S, D). Hence, by the hierarchical null player property we have $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus A'$ and efficiency together with star-symmetry for the players in A' implies $\xi_i(N, u_S, D) = 1/|A'|$ for all $i \in A'$. If $A' = \emptyset$, then $|B'| \geq 2$ and each $i \in N \setminus \{h(N,D)\}$ is a hierarchical null player in (N, u_S, D). By the hierarchical null player property this implies $\xi_i(N, u_S, D) = 0$ for all $i \in N \setminus \{h(N,D)\}$ and by efficiency we have $\xi_{h(N,D)}(N, u_S, D) = 1$. ∎

On the class of TU-games with directed star dominance structure, both the disjunctive and conjunctive permission values and the dominance value do not satisfy the hierarchical null player property and star-symmetry.

3.5.3 Tree as dominance structure

In this subsection we consider TU-games with dominance structure that is represented by a tree on the set of players. Demange (2004) introduces the marginal contribution vector corresponding to a tree as the hierarchical outcome. Khmelnitskaya (2010) provides a characterization of the tree value, which is the hierarchical outcome, for TU-games with dominance structure represented by a tree on the set of players. For this characterization, Khmelnitskaya (2010) uses component efficiency and successor equivalence. According to successor equivalence, if an arc from one player to another one is deleted from the tree, then the payoff remains the same for the latter player and all of his successors. The tree value and the average covering tree solution coincide on the class of TU-games with dominance structure represented by a tree. In this subsection, we provide a characterization of the average covering tree solution which can be seen as an alternative characterization of the hierarchical outcome introduced in Demange (2004) and the tree value introduced in Khmelnitskaya (2010).

A TU-game with tree dominance structure is a combination of a TU-game and a dominance structure represented by a tree on the set of players. Let $\mathcal{G}_N^{tree}$ be the set of all TU-games with tree dominance structure.

Lemma 3.5.12 *Given two TU-games with tree dominance structure* $(N, v, D), (N, w, D) \in \mathcal{G}_N^{tree}$*, if a solution* $\xi : \mathcal{G}_N^{tree} \to \mathbb{R}^n$ *satisfies linearity and the hierarchical null player property, then* $\xi(N, v, D) = \xi(N, w, D)$ *whenever* $v(S) = w(S)$ *holds for all* $S \in H^c(D)$*.*

Proof Take any $(N, v, D), (N, w, D) \in \mathcal{G}_N^{tree}$ such that $v(S) = w(S)$ for all $S \in H^c(D)$. Consider the TU-game with tree dominance structure $(N, v - w, D)$. In $(N, v - w, D)$ every player is a hierarchical null player because $(v - w)(S) = 0$ for all $S \in H^c(D)$. Hence, by the hierarchical null player property $\xi_i(N, v - w, D) = 0$ for all $i \in N$. By linearity, this implies $\xi(N, v, D) = \xi(N, w, D)$, which completes the proof. ∎

Theorem 3.5.13 *The average covering tree solution is the unique solution on* $\mathcal{G}_N^{tree}$ *that satisfies efficiency, linearity, and the hierarchical null player property.*

Proof First, we show that the average covering tree solution for any TU-game with tree dominance structure $(N, v, D) \in \mathcal{G}_N^{tree}$ satisfies all axioms. Since the dominance structure is represented by the tree (N, D), the only covering tree is (N, D). Efficiency and linearity are shown in Section 3. If player $i \in N$ is a hierarchical null player in (N, v, D), then $ACT_i(N, v, D) = m_i^D(N, v) = 0$. Hence, the average covering tree solution for (N, v, D) satisfies the hierarchical null player property.

Second, we show that there exists a unique solution that satisfies all three axioms. For this, because of linearity it is sufficient to show that for a solution

$\xi : \mathcal{G}_N^{tree} \to \mathbb{R}^n$ satisfying all three axioms $\xi(N, u_S, D)$ is a unique payoff vector for all $S \in 2^N \setminus \{\emptyset\}$.

If $S = N$, then each $i \in N \setminus \{r(N, D)\}$ is a hierarchical null player in (N, u_N, D) and together with efficiency the hierarchical null player property implies $\xi_i(N, v, D) = 0$ for all $i \in N \setminus \{r(N, D)\}$ and $\xi_{r(N,D)}(N, v, D) = 1$.

If $S \in H^c(D)$, then each $i \in N \setminus \{r(S, D|_S)\}$ is a hierarchical null player in (N, u_S, D) and together with efficiency the hierarchical null player property implies $\xi_i(N, v, D) = 0$ for all $i \in N \setminus \{r(S, D|_S)\}$ and $\xi_{r(S,D|_S)}(N, v, D) = 1$.

If $S \notin H^c(D)$, let $S' \in H^c(D)$ be the unique smallest hierarchical network (in terms of set inclusion) that contains S. Since $u_S(Q) = u_{S'}(Q)$ holds for all $Q \in H^c(D)$, by Lemma 3.5.12 we have $\xi(N, u_S, D) = \xi(N, u_{S'}, D)$. Since $S' \in H^c(D)$ and each $i \in N \setminus \{r(S', D|_{S'})\}$ is a hierarchical null player in $(N, u_{S'}, D)$, this implies $\xi_i(N, u_S, D) = \xi_i(N, u_{S'}, D) = 0$ for all $i \in N \setminus \{r(S', D|_{S'})\}$ and by efficiency $\xi_{r(S',D|_{S'})}(N, u_S, D) = \xi_{r(S',D|_{S'})}(N, u_{S'}, D) = 1$. ∎

CHAPTER 4

TU-GAMES WITH COALITIONAL STRUCTURE

4.1 Introduction

In the standard cooperative game theory literature it is assumed that all coalitions of players are able to form. However, in many practical situations the collection of feasible coalitions is restricted by some social, economical, hierarchical or technical structure. Chapter 2 of this monograph considers undirected communication graphs which restrict cooperation among players. For TU-games with communication structure, only the members of a connected set of players are assumed to be able to cooperate. Although TU-games with communication structure are more general than TU-games and can be applied to many situations, they are still not enough to explain some real life phenomenon. To illustrate this insufficiency, consider a situation defined on country A, country B, country C, and country D. The restriction on the cooperation of these countries is as follows. Country A and country D do not have any diplomatic relation and in order to be able to participate in a feasible coalition they need country B which is the only country that has a diplomatic relation with both countries. In this example, all coalitions are feasible except the coalition containing country A and country D and the coalition containing country A, country C and country D. This situation can not be represented by a graph, because country A is able to cooperate with country C and country C is able to cooperate with country D, but countries A, C, and D are not able to form a feasible coalition. As this example shows, in some situations modeling restricted cooperation by means of a graph on the set of players may not be an adequate representation of the restricted cooperation. In the literature, a large collection of papers considers specific classes of set systems as a way to represent limited cooperation among the players. In all of these research, some restrictions are assumed on the set systems. Algaba et al. (2001) considers union stable cooperation structures, Bilbao and Edelman (2000a) considers convex geometries, Bilbao et al. (2001) considers matroids, Algaba et al. (2003) considers antimatroids, Bilbao and Ordónez (2009a) considers augmenting systems, and Ui et al. (2011a) considers complete coalition structures. The set system given in the example above that is defined on countries, does not fit in these models. In the example, the coalition containing country A and country C, and the coalition containing country D and country C are feasible. However, the union of these two feasible coalitions is not feasible. The set system representing this situation in the example is neither a union closed set system nor an antimatroid, because both union closed set systems and antimatroids require the union of any two feasible coalition to be also feasible. Similarly, for an augmenting set system if two feasible coalitions have nonempty intersection, then the union of these coalitions is also feasible. Since the coalition containing the countries A, C, and D is not feasible, the set system in the example is not an augmenting set system. Since any subset of the feasible coalition containing country A, country B and country D is not feasible in the example, it is also not a matroid. Moreover, for the example above if it is also assumed that singleton coalitions are not feasible, then the resulting situation will not be a convex geometry.

Aumann and Dréze (1974) considers TU-games with coalition structure, which is represented with a partition of the grand coalition. Aumann and Dréze (1974) assumes that it is not possible to transfer payoff among two different members of the partition of the grand coalition and a solution should allocate the total worth of each member of the partition to the players in that coalition. With this approach, cooperation is not restricted within each member of the partition, but on the other hand, for the players that belong to different elements of the partition it is impossible to cooperate. Aumann and Dréze (1974) studies well-known solution concepts, including the Shapley value, and establishes relations of each solution when applied to a TU-game with coalition structure (represented by a partition on the set of players) and when applied to appropriately defined games on each of the members of the partition.

Owen (1977) defines the Owen value for TU-games with priori unions which is also represented by a partition of the grand coalition. The model in Owen (1977) differs from the approach of Aumann and Dréze (1974) by dropping the assumption which states that the total payoff available for the players in each member of the partition should be equal to the worth of that coalition. Owen (1977) assumes the grand coalition will form and requires that the worth of the grand coalition should be distributed among all players. The Owen value considers all permutations on the set of players in which all players of the same partition member appears successively and assigns to each player his expected marginal contribution with respect to those permutations.

In this chapter, we assume that the grand coalition of all players is always able to form a coalition. Together with the grand coalition, we consider an arbitrary collection of subsets of the grand coalition as the collection of feasible coalitions that are able to cooperate and obtain some worth. Considering the example above, instead of the connected sets in a graph, one could assume feasibility of all coalitions except the coalitions that contains country A and country D but not country B. For such situations, as solution concept we propose the average coalitional tree solution, being the average of the marginal contribution vectors corresponding to all maximal nested sets of the set system representing the coalitional structure. A nested set is a collection of feasible coalitions such that for any two different coalitions in the collection, either one of them is a subset of the other or they are disjoint, and, moreover, the union of two or more disjoint coalitions in the collection is not a feasible coalition. To each maximal nested set a coalitional tree corresponds. In a coalitional tree each node is a coalition, which may not be feasible by its own, but together with its set of successors in the coalitional tree it is feasible and it is a member of the maximal nested set. For each of these coalitional trees, a marginal contribution vector is defined, at which the players at a node receive together as payoff the marginal contribution when they join to their successors in the coalitional tree and this payoff is equally distributed among them. We discuss several properties of the solution and consider some special cases of coalitional structures.

For the average coalitional tree solution, we assume the grand coalition is feasible. This assumption is quite restrictive and even not satisfied by the communication structures that are considered in Chapter 2, i.e., the undirected graph representing a communication structure does not need to be connected. In case the grand coalition is not a feasible coalition, the average coalitional tree solution can still be applied if the set of all players can be partitioned into feasible coalitions and each feasible coalition in the set system is a subset of one of these partition members. Then the average coalitional tree solution can be applied separately to each member of the partition. This also covers the case if we consider the collection of connected coalitions in an undirected graph which is not connected.

Now we provide a numerical example where the average coalitional tree solution is applied. Consider a communication structure represented by a graph (N, L) where $N = \{1,2,3,4,5\}$ is the set of players and $L = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$ is the set of edges. Since (N, L) is a connected communication structure as discussed in Chapter 2, $C^L(N)$ is the collection of feasible coalitions. Additional to being connected set in the communication structure, in order to be a feasible winning coalition also the size of the coalition must be greater than or equal to 3. The collection of feasible winning coalitions is given by the set system $\mathcal{F} = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{1,2,3,4\},\{2,3,4,5\},\{1,2,3,4,5\}\}$. Let the worth of any feasible coalition be equal to 1, i.e., $v(S) = 1$ for all $S \in \mathcal{F}$. There exist four maximal nested sets to be considered for the average coalitional tree solution, $\mathcal{X}^1 = \{\{1,2,3\},\{1,2,3,4\},N\}$, $\mathcal{X}^2 = \{\{2,3,4\},\{1,2,3,4\},N\}$, $\mathcal{X}^3 = \{\{2,3,4\},\{2,3,4,5\},N\}$, $\mathcal{X}^4 = \{\{3,4,5\},\{2,3,4,5\},N\}$. For the corresponding marginal contribution vectors, we have $(1/3,1/3,1/3,0,0)$ for $\mathcal{X}^1$, $(0,1/3,1/3,1/3,0)$ for $\mathcal{X}^2$, $(0,1/3,1/3,1/3,0)$ for $\mathcal{X}^3$, and $(0,0,1/3,1/3,1/3)$ for $\mathcal{X}^4$. The average of these marginal contribution vectors is the average coalitional tree solution, which is $(1/12, 3/12, 4/12, 3/12, 1/12)$. For this example, the collection of feasible coalitions is induced by the communication structure and a predetermined quota. Now consider a situation where the feasibility of a coalition depends only on the size. Given the set of players N as the set of n voters and some q, $\frac{1}{2} \leq q \leq 1$, let $\mathcal{F}^q = \{S \in 2^N \mid |S| \geq qn\}$ be the set of winning coalitions under the quota majority rule with quota q. Let $(N, v, \mathcal{F})$ be a TU-game (N, v) with coalitional structure $\mathcal{F} \subseteq \mathcal{F}^q$ and worth $v(S) = 1$ for all $S \in \mathcal{F}$. The average coalitional tree solution (ACOT) allocates to each player $i \in N$ a payoff which is given by

$$ACOT_i(N, v, \mathcal{F}) = \frac{1}{|\mathcal{M}^{\mathcal{F}}|} \sum_{S \in \mathcal{M}^{\mathcal{F}}(i)} \frac{1}{|S|}$$

where $\mathcal{M}^{\mathcal{F}}$ is the collection of minimal feasible winning coalitions and $\mathcal{M}^{\mathcal{F}}(i)$ is the collection of minimal feasible winning coalitions that contain player i.

Aguilera et al. (2010) also considers TU-games with arbitrary coalitional structures and defines a Shapley value for such situations. As solution, they consider the average of the marginal contribution vectors corresponding to all maximal chains. A chain is a collection of feasible coalitions such that for any two different members one of them is a subset of the other. Hence, chains do not consider the cases where one or more players are able to join simultaneously to two or more feasible disjoint coalitions whose union is not feasible, to form a larger feasible coalition. A chain only considers the marginal contribution of a set of players when they simultaneously join to a single feasible coalition to form a larger feasible coalition. For a maximal nested set, no proper subset of any set of players that simultaneously join is a feasible coalition and therefore is able to obtain some contribution by its own. This property makes a maximal nested set a natural concept to define marginal contributions for the players. A similar approach with maximal chains is employed in Lange and Grabisch (2009) for more restricted structures. They consider regular set systems where the feasible coalitions form a poset whose maximal chains all have the same length. They propose an axiomatization of the Shapley value for this class of games.

For games with building sets as coalitional struture, the gravity center (GC) solution is introduced by Koshevoy and Talman (2014). Koshevoy and Talman (2014) proposes to use the GC solution for games with arbitrary coalitional structure by taking its building cover, which is the smallest building set that contains the coalitional structure. By using the Möbius inversion, they extend the game to this building cover and take the GC solution of the extended game as the GC solution of the game. For games with building sets as coalitional structure, the average coalitional tree solution coincides with the GC solution

This chapter is based on Selçuk and Talman (2013) and the structure of the chapter is as follows. Basic definitions and notation are given in Section 2. Section 3 introduces the new solution concept for TU-games with coalitional structure. Some properties of the solution concept are given in Section 4. Section 5 considers some special coalitional structures.

4.2 Preliminaries

A TU-game with coalitional structure is a triple $(N, v, \mathcal{F})$, where $N = \{1, ..., n\}$ is a finite set of players, $\mathcal{F} \subseteq 2^N$ is a *set system* on N representing the coalitional structure with $N \in \mathcal{F}$, and $v : \mathcal{F} \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$.[1] A set $S \in \mathcal{F}$ is a *feasible coalition* and the real number $v(S)$ represents the *worth* of S, which can be freely distributed among its members.

[1]The empty set is always assumed to be a member of the set system representing the coalitional structure. However, we do not mention it every time we refer to a coalitional structure.

We denote the set of TU-games with coalitional structure with a fixed player set N by $\mathcal{G}_N^{cos}$.

A *solution* on $\mathcal{G}_N^{cos}$ is a function $\xi\colon \mathcal{G}_N^{cos} \to \mathbb{R}^n$ that assigns to any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ a payoff vector $\xi(N, v, \mathcal{F}) \in \mathbb{R}^n$. In the sequel, we use notation $x(S) = \sum_{i \in S} x_i$ for any payoff vector $x \in \mathbb{R}^n$ and $S \subseteq N$.

Given a partition $\mathcal{P}$ of the set of players N, $(\mathcal{P}, T)$ stands for a *coalitional directed graph*, or *coalitional digraph*, on $\mathcal{P}$, where $T \subseteq \{(P_1, P_2) \mid P_1, P_2 \in \mathcal{P}, P_1 \neq P_2\}$ is a collection of directed links on the set of members of the partition $\mathcal{P}$ of N. A coalitional digraph may be seen as a generalization of a *digraph*, where the nodes, being elements of N, are replaced with subsets of N that together form a partition of N and the directed links are defined on these subsets. Given a partition $\mathcal{P}$ of N and a coalitional digraph $(\mathcal{P}, T)$ on $\mathcal{P}$, a sequence of different members of $\mathcal{P}$, $(P_1, \ldots, P_k)$ with $k \geq 2$, is a *directed coalitional path* in $(\mathcal{P}, T)$ from P_1 to P_k if $(P_h, P_{h+1}) \in T$ for $h = 1, \ldots, k-1$. If there exists a directed coalitional path in $(\mathcal{P}, T)$ from $P \in \mathcal{P}$ to $P' \in \mathcal{P}$, then P' is a *successor* of P and P is a *predecessor* of P'. If $(P, P') \in T$ then P' is an *immediate successor* of P and P is an *immediate predecessor* of P'. For any $P \in \mathcal{P}$, $S_T(P)$ denotes the union of all successors of P in the coalitional digraph $(\mathcal{P}, T)$ and $\overline{S}_T(P)$ denotes the union of all successors of P together with the members of P, i.e., $\overline{S}_T(P) = S_T(P) \cup P$. Furthermore, $\mathcal{I}_T(P)$ denotes the set of immediate successors of P in $(\mathcal{P}, T)$, i.e., $\mathcal{I}_T(P) = \{P' \in \mathcal{P} | (P, P') \in T\}$.

A coalitional digraph $(\mathcal{P}, T)$ on a given partition $\mathcal{P}$ of N is a *coalitional tree* if there exists a unique member of the partition $\mathcal{P}$, called the *coalitional root* of $(\mathcal{P}, T)$ and denoted by $r(\mathcal{P}, T)$, having no predecessors in $(\mathcal{P}, T)$ and there is a unique directed coalitional path in $(\mathcal{P}, T)$ from $r(\mathcal{P}, T)$ to every other member of the partition.

4.3 Average coalitional tree solution

For a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, the idea is that the grand coalition N will form and the problem is how to distribute its worth $v(N)$ among the agents. Among the solution concepts we discussed earlier, Bilbao and Edelman (2000b) which considers the Shapley value for convex geometries, Bilbao et al. (2001) which considers the Shapley value for matroids, and Ui et al. (2011b) which considers the Myerson value for complete coalition structures also require this assumption. On the other hand, Bilbao and Ordónez (2009b) which considers the Shapley value for augmenting systems does not assume the feasibility of the grand coalition. Although an antimatroid does not need to contain the grand coalition, Algaba et al. (2003) considers antimatroids satisfying a normality assumption which requires the grand coalition as a feasible coalition. As solution concept, we propose to take the average of the marginal contribution vectors induced by all maximal nested sets of the set system representing the coalitional structure. In case the grand coalition is not feasible but there is a partition of it into feasible coalitions such that each feasible coalition is a subset of one of the partition members, the solution concept can be applied separately to each partition member. This includes the case where the set system is the collection of connected coalitions in an undirected graph which is not connected. Also when in an augmenting system each player belongs to at least one feasible coalition, such a partition exists and the solution can be applied in the same way. Nested sets of a set system are introduced by Postnikov (2009).

Definition 4.3.1 Given a coalitional structure $\mathcal{F}$ on N, a subset $\mathcal{X}$ of $\mathcal{F}$ is a *nested set* of $\mathcal{F}$ if it satisfies the following conditions:
(i) For any two different $X_1, X_2 \in \mathcal{X}$ it holds that either $X_1 \subset X_2$ or $X_2 \subset X_1$ or $X_1 \cap X_2 = \emptyset$;
(ii) For any collection of h, $h \geq 2$, disjoint nonempty subsets $X_1, \ldots, X_h$ in $\mathcal{X}$ it holds that $X_1 \cup \cdots \cup X_h \notin \mathcal{F}$;
(iii) $\emptyset \notin \mathcal{X}$ and $N \in \mathcal{X}$.

A nested set of a coalitional structure is a collection of non-empty feasible coalitions, including the set of all players, such that for any two different members either one of them is a subset of the other one or they are disjoint, and, moreover, the union of two or more disjoint members is not a feasible coalition. Notice that every chain of a coalitional structure $\mathcal{F}$ is a nested set of $\mathcal{F}$, where $\mathcal{Y}$ is called a *chain* of $\mathcal{F}$ if $N \in \mathcal{Y}$ and for any two different $Y_1, Y_2 \in \mathcal{Y}$ it holds that either $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$.

A nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ is *maximal* if there does not exist any other nested set $\mathcal{X}'$ of $\mathcal{F}$ that contains $\mathcal{X}$. Each maximal nested set defines a unique way to build the grand coalition by letting one or simultaneously several players join to one or more disjoint feasible coalitions to form bigger feasible coalitions, starting from disjoint minimal (by set inclusion) coalitions in the set system. For a maximal nested set of a coalitional structure, no proper subset of any set of simultaneously joining players is a feasible coalition, otherwise we can add this subcoalition and obtain a larger nested set. This means that no proper subset of a set of joining players can form a feasible coalition by its own. A maximal chain may not have this natural property, because players can only simultaneously join one feasible coalition and therefore may contain players that together form a feasible coalition. In general, any maximal chain is a nested set, but may not be a maximal nested set, and a maximal nested set may not be a maximal chain. If a maximal chain is not a maximal nested set, then there exists at least one maximal nested set that contains the chain as a proper subset.

Example 4.3.2 Consider the coalitional structure $\mathcal{F} = \{\{1\}, \{1,2\}, \{2,3\}, \{1,2,3,4\}\}$ on $N = \{1,2,3,4\}$. This set system has two maximal nested sets, $\mathcal{X}^1 = \{\{1\}, \{1,2\}, \{1,2,3,4\}\}$ and $\mathcal{X}^2 = \{\{1\}, \{2,3\}, \{1,2,3,4\}\}$. In $\mathcal{X}^1$, player 2 joins feasible singleton player 1 to form feasible coalition $\{1,2\}$ and players 3 and 4 join simultaneously the latter coalition to form the grand coalition. In $\mathcal{X}^2$, player 4 joins to both feasible singleton player 1 and minimal feasible coalition $\{2,3\}$ to form immediately the grand coalition. The two feasible coalitions $\{1,2\}$ and $\{2,3\}$ cannot be members of a same maximal nested set because one is not a subset of the other and their intersection is nonempty. On the other hand, the two disjoint feasible coalitions $\{1\}$ and $\{2,3\}$ can be members of the same maximal nested set, because their union, $\{1,2,3\}$, is not a feasible coalition. The maximal nested set $\mathcal{X}^1$ is also a maximal chain, but $\mathcal{X}^2$ is not. On the other hand, $\mathcal{Y} = \{\{2,3\}, \{1,2,3,4\}\}$ is a maximal chain that is not a maximal nested set, because it is a proper subset of $\mathcal{X}^2$. In $\mathcal{Y}$, although singleton player 1 is a feasible coalition, players 1 and 4 simultaneously join to the feasible minimal coalition $\{2,3\}$ to form the grand coalition.

Any coalitional structure on N contains at least one maximal nested set. To see this, note that N itself is a nested set, which does not need to be maximal. If it is not a maximal nested set, then we can include any other feasible coalition. If this new collection of two feasible coalitions is again not maximal, we continue with including feasible coalitions which do not violate the definition of a nested set, and so on. Because the number of feasible coalitions is finite, at some point we end up with a maximal nested set. This argument also shows that every feasible coalition is member of at least one maximal nested set.

For a coalitional structure $\mathcal{F}$ on N, $\overline{\mathcal{X}}^{\mathcal{F}}$ denotes the collection of maximal nested sets of $\mathcal{F}$. Notice that in case the coalitional structure $\mathcal{F}$ contains all subsets of N, i.e., $\mathcal{F} = 2^N$, then the number of maximal nested sets is maximal and equal to $n!$ and all maximal nested sets are maximal chains.

Given a coalitional structure $\mathcal{F}$ on N, for a nested set $\mathcal{X}$ of $\mathcal{F}$ and $i \in N$, the set $M_{\mathcal{X}}(i)$ denotes the unique minimal element of $\mathcal{X}$ that contains player

i. Notice that this set is well defined.

Lemma 4.3.3 *For any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ of a coalitional structure $\mathcal{F}$ on N, it holds that for every $X \in \mathcal{X}$ there exists $i \in N$ such that $M_{\mathcal{X}}(i) = X$.*

Proof Suppose there exists a feasible coalition $X \in \mathcal{X}$ for which there is no $i \in N$ with $M_{\mathcal{X}}(i) = X$. Since $\mathcal{X}$ is a maximal nested set and $M_{\mathcal{X}}(i) \neq X$ for all $i \in X$, it holds that $M_{\mathcal{X}}(i) \subset X$, $M_{\mathcal{X}}(i) \in \mathcal{X}$ and $i \in M_{\mathcal{X}}(i)$ for all $i \in X$. Since $M_{\mathcal{X}}(i) \in \mathcal{X}$ for all $i \in X$, there exist $i_1, \ldots, i_k \in X$ such that $M_{\mathcal{X}}(i_1), \ldots, M_{\mathcal{X}}(i_k)$ are disjoint and $\cup_{h=1}^{k} M_{\mathcal{X}}(i_h) = X$. Since X is a feasible coalition, this violates condition (ii) of Definition 4.3.1. ∎

Example 4.3.4 Consider the coalitional structure $\mathcal{F} = \{\{1,2\},\{3\},\{2,3,4\},\{1,2,3,4,5\}\}$ on $\{1,2,3,4,5\}$. It has two maximal nested sets, $\mathcal{X}^1 = \{\{1,2\},\{3\},\{1,2,3,4,5\}\}$ and $\mathcal{X}^2 = \{\{3\},\{2,3,4\},\{1,2,3,4,5\}\}$. For these maximal nested sets we have $M_{\mathcal{X}^1}(1) = M_{\mathcal{X}^1}(2) = \{1,2\}$, $M_{\mathcal{X}^1}(3) = \{3\}$, $M_{\mathcal{X}^1}(4) = M_{\mathcal{X}^1}(5) = \{1,2,3,4,5\}$, and $M_{\mathcal{X}^2}(1) = M_{\mathcal{X}^2}(5) = \{1,2,3,4,5\}$, $M_{\mathcal{X}^2}(2) = M_{\mathcal{X}^2}(4) = \{2,3,4\}$, $M_{\mathcal{X}^2}(3) = \{3\}$.

As the example shows, for distinct players the minimal elements of a maximal nested set that contain these players can be the same.

Definition 4.3.5 For a collection $\mathcal{S}$ of subsets of N, two players $i, j \in N$ are *equivalent* with respect to $\mathcal{S}$ if $\{S \in \mathcal{S} \mid i \in S\} = \{S \in \mathcal{S} \mid j \in S\}$.

For a collection $\mathcal{S}$ of subsets of N and $i \in N$, $P_{\mathcal{S}}(i)$ denotes the set of equivalent players of i with respect to $\mathcal{S}$.

Remark 4.3.6 Given a coalitional structure $\mathcal{F}$ on N, for a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, two players $i, j \in N$ are equivalent with respect to $\mathcal{X}$ if and only if $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$.

Remark 4.3.7 For a coalitional structure $\mathcal{F}$ on N, if two players $i, j \in N$ are equivalent with respect to $\mathcal{F}$, then i and j are equivalent with respect to every maximal nested set in $\overline{\mathcal{X}}^{\mathcal{F}}$.

Remark 4.3.6 follows from condition (i) of Definition 4.3.1. Remark 4.3.7 is an immediate result of the fact that each maximal nested set is a subset of the set system representing the coalitional structure.

A maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ of a coalitional structure $\mathcal{F}$ on N induces a partition $\mathcal{P}^{\mathcal{X}}$ of N into sets of equivalent players with respect to $\mathcal{X}$, with

$P_{\mathcal{X}}(i)$ the partition member containing the equivalent players of player $i \in N$. Since any $P \in \mathcal{P}^{\mathcal{X}}$ is a set of equivalent players with respect to $\mathcal{X}$, it holds that $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$ for all $i, j \in P$. For $P \in \mathcal{P}^{\mathcal{X}}$, we denote $M_{\mathcal{X}}(P)$ to be the set $M_{\mathcal{X}}(i)$ for any $i \in P$.

Given a maximal nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ on N and the corresponding partition $\mathcal{P}^{\mathcal{X}}$, the coalitional digraph $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is given by $(P_1, P_2) \in T^{\mathcal{X}}$ if $M_{\mathcal{X}}(P_1) \supset M_{\mathcal{X}}(P_2)$ and there exists no $X \in \mathcal{X}$ with $M_{\mathcal{X}}(P_1) \supset X \supset M_{\mathcal{X}}(P_2)$. The next theorem shows that this coalitional digraph is a coalitional tree.

Theorem 4.3.8 *Given a a coalitional structure $\mathcal{F}$ on N, for any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, the coalitional digraph $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree satisfying the following properties:*
(i) Its coalitional root $r(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is equal to $\{i \in N \mid M_{\mathcal{X}}(i) = N\}$;
(ii) For any $P \in \mathcal{P}^{\mathcal{X}}$ it holds that $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$;
(iii) For any $P \in \mathcal{P}^{\mathcal{X}}$ it holds that $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid (P, P') \in T^{\mathcal{X}}\}$ is the unique maximal partition of $S_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P) \setminus P$ into elements of $\mathcal{X}$.

Proof To show that $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree we need to prove the uniqueness of a coalitional root and the uniqueness of a directed coalitional path in the tree from the coalitional root to any other member of $\mathcal{P}^{\mathcal{X}}$. By Lemma 4.3.3 the set $R = \{i \in N \mid M_{\mathcal{X}}(i) = N\}$ is nonempty and consists of equivalent players with respect to $\mathcal{X}$. Hence, $R \in \mathcal{P}^{\mathcal{X}}$. Since there exists no $P \in P^{\mathcal{X}}$ with $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(R) = N$, it holds that R has no predecessor in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$. Since $\mathcal{X}$ is a maximal nested set there exists a unique directed path in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ from R to any other member of $\mathcal{P}^{\mathcal{X}}$. This implies that $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ is a coalitional tree with the coalitional root being the set R, which also proves property (i).

Property (ii) is shown by induction. Take any $P \in \mathcal{P}^{\mathcal{X}}$ without successor in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$, then $\overline{S}_{T^{\mathcal{X}}}(P) = P$ and $P \subseteq M_{\mathcal{X}}(P)$. Suppose there exists $i \in M_{\mathcal{X}}(P) \setminus P$, then i is not equivalent to the players in P with respect to $\mathcal{X}$ and therefore $M_{\mathcal{X}}(i) \subset M_{\mathcal{X}}(P)$. This implies the existence of a directed coalitional path in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ from P to $P_{\mathcal{X}}(i)$, which contradicts that P has no successors. Next, we show $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$ if $\overline{S}_{T^{\mathcal{X}}}(P') = M_{\mathcal{X}}(P')$ for all P' satisfying $(P, P') \in T^{\mathcal{X}}$. Let $P_1, \ldots, P_k$ be the collection of immediate successors of P in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$, then $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(P_i) = \overline{S}_{T^{\mathcal{X}}}(P_i)$ for all $i = 1, \ldots, k$. Since $\overline{S}_{T^{\mathcal{X}}}(P) = (\cup_{i=1}^{i=k} \overline{S}_{T^{\mathcal{X}}}(P_i)) \cup P$ and $P \subseteq M_{\mathcal{X}}(P)$, this implies $M_{\mathcal{X}}(P) \supseteq \overline{S}_{T^{\mathcal{X}}}(P)$. Suppose $j \in M_{\mathcal{X}}(P) \setminus \overline{S}_{T^{\mathcal{X}}}(P)$. Since j is not equivalent to the players in P with respect to $\mathcal{X}$ and $j \in M_{\mathcal{X}}(P)$, we have $M_{\mathcal{X}}(j) \subset M_{\mathcal{X}}(P)$. This implies the existence of a directed coalitional path in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ from P to $P_{\mathcal{X}}(j)$, which contradicts that $j \notin \overline{S}_{T^{\mathcal{X}}}(P)$.

To show property (iii), let $P_1, \ldots, P_k$ be the collection of immediate successors of $P \in \mathcal{P}^{\mathcal{X}}$ in $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$. Property (ii) implies that $\overline{S}_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P)$ and $\overline{S}_{T^{\mathcal{X}}}(P_j) = M_{\mathcal{X}}(P_j)$ for all $j = 1\ldots,k$. Hence, $M_{\mathcal{X}}(P) \setminus P = \overline{S}_{T^{\mathcal{X}}}(P) \setminus P = S_{T^{\mathcal{X}}}(P) = \cup_{j=1}^{k} \overline{S}_{T^{\mathcal{X}}}(P_j) = \cup_{j=1}^{k} M_{\mathcal{X}}(P_j)$. Since $\mathcal{X}$ is a maximal nested set, $M_{\mathcal{X}}(P_j) \cap M_{\mathcal{X}}(P_h) = \emptyset$ for all $j \neq h$, and $M_{\mathcal{X}}(P_j) \in \mathcal{X}$ for $j = 1, \ldots, k$. This implies that $\{\overline{S}_{T^{\mathcal{X}}}(P_1), \ldots, \overline{S}_{T^{\mathcal{X}}}(P_k)\}$ is the unique maximal partition of $S_{T^{\mathcal{X}}}(P) = M_{\mathcal{X}}(P) \setminus P$ into members of $\mathcal{X}$. ∎

A coalitional tree on a set of players N that is induced by a maximal nested set may be seen as a generalization of a tree on N in which the nodes of the tree are sets of equivalent players with respect to the maximal nested set instead of individual players.

Since each maximal nested set $\mathcal{X}$ of a coalitional structure $\mathcal{F}$ on N contains the grand coalition N and by applying Lemma 4.3.3, there exist players for which the grand coalition is the minimal set in $\mathcal{X}$ containing them. According to property (i) the root of the coalitional tree $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ induced by $\mathcal{X}$ precisely consists of these players. So, if a maximal nested set is considered as a way to build the grand coalition, the root of the induced coalitional tree is the final set of equivalent players that simultaneously join after all other sets of equivalent players have joined each other. According to property (ii), the players at a node of $(\mathcal{P}^{\mathcal{X}}, T^{\mathcal{X}})$ together with the players in all succeeding nodes is the minimal set in $\mathcal{X}$ that contains any player at that node. A direct implication of this property is that the union of the successors of any partition member $P \in \mathcal{P}^{\mathcal{X}}$ together with the elements of P is a member of the set system $\mathcal{F}$, and hence is a feasible coalition. Moreover, no proper subset of any $P \in \mathcal{P}^{\mathcal{X}}$ is a feasible coalition. Property (iii) says that for each member X of a maximal nested set, it holds that if we delete from X all players for which X is the minimal set containing them, then there is a unique maximal partition of the set of remaining players in X into members of the maximal nested set.

Example 4.3.9 Consider the coalitional structures $\mathcal{F} = \{\{1\}, \{2,3\}, \{1,2,3\}\}$ and $\mathcal{F}' = \{\{1\}, \{2\}, \{2,3\}, \{1,2,3\}\}$ on $N = \{1,2,3\}$. Both $\mathcal{F}$ and $\mathcal{F}'$ contain two maximal nested sets, $\mathcal{X}^1 = \{\{1\}, \{1,2,3\}\}$ and $\mathcal{X}^2 = \{\{2,3\}, \{1,2,3\}\}$ for $\mathcal{F}$, and $\mathcal{Y}^1 = \{\{1\}, \{2\}, \{1,2,3\}\}$ and $\mathcal{Y}^2 = \{\{2\}, \{2,3\}, \{1,2,3\}\}$ for $\mathcal{F}'$. For the partitions induced by these maximal nested sets we have $\mathcal{P}^{\mathcal{X}^1} = \{\{1\}, \{2,3\}\}$, $\mathcal{P}^{\mathcal{X}^2} = \{\{2,3\}, \{1\}\}$ and $\mathcal{P}^{\mathcal{Y}^1} = \{\{1\}, \{2\}, \{3\}\}$, $\mathcal{P}^{\mathcal{Y}^2} = \{\{2\}, \{3\}, \{1\}\}$. The coalitional trees on these partitions are equal to $(\mathcal{P}^{\mathcal{X}^1}, T^{\mathcal{X}^1})$, $(\mathcal{P}^{\mathcal{X}^2}, T^{\mathcal{X}^2})$, $(\mathcal{P}^{\mathcal{Y}^1}, T^{\mathcal{Y}^1})$ and $(\mathcal{P}^{\mathcal{X}^2}, T^{\mathcal{Y}^2})$ where $T^{\mathcal{X}^1} = \{(\{2,3\}, \{1\})\}$, $T^{\mathcal{X}^2} = \{(\{1\}, \{2,3\})\}$ and $T^{\mathcal{Y}^1} = \{(\{3\}, \{1\}), (\{3\}, \{2\})\}$, $T^{\mathcal{Y}^2} = \{(\{1\}, \{3\}), (\{3\}, \{2\})\}$.

Note that $M_{\mathcal{X}^1}(\{1\}) = \{1\}$ and $M_{\mathcal{X}^1}(\{2,3\}) = \{1,2,3\}$. Since $M_{\mathcal{X}^1}(\{1\}) \subset M_{\mathcal{X}^1}(\{2,3\})$ we have $(\{2,3\},\{1\}) \in T^{\mathcal{X}^1}$. Similarly for $\mathcal{X}^2$, we have $M_{\mathcal{X}^2}(\{1\}) = \{1,2,3\}$ and $M_{\mathcal{X}^2}(\{2,3\}) = \{2,3\}$ which means $M_{\mathcal{X}^2}(\{2,3\}) \subset M_{\mathcal{X}^2}(\{1\})$. Hence, $(\{1\},\{2,3\}) \in T^{\mathcal{X}^2}$. The graphical representation of these coalitional trees is depicted in Figure 4.1. The root of $(\mathcal{P}^{\mathcal{X}^1}, T^{\mathcal{X}^1})$ is $\{2,3\}$ with succeeding set $S_{T^{\mathcal{X}^1}}(\{2,3\}) = \{1\}$, whereas $\{1\}$ is the root of $(\mathcal{P}^{\mathcal{X}^2}, T^{\mathcal{X}^2})$ with succeeding set $S_{T^{\mathcal{X}^2}}(\{1\}) = \{2,3\}$. In $(\mathcal{P}^{\mathcal{Y}^1}, T^{\mathcal{Y}^1})$, the set $\{3\}$ is the root and the sets $\overline{S}_{T^{\mathcal{Y}^1}}(\{1\}) = \{1\}$ and $\overline{S}_{T^{\mathcal{Y}^1}}(\{2\}) = \{2\}$ partition the set of players in the succeeding sets of the root, $S_{T^{\mathcal{Y}^1}}(\{3\}) = \{1,2\}$, into members of $\mathcal{Y}^1$. Coalitional tree $(\mathcal{P}^{\mathcal{Y}^2}, T^{\mathcal{Y}^2})$ has $\{1\}$ as root and feasible coalition $S_{T^{\mathcal{Y}^2}}(\{1\}) = \{2,3\}$ is the only succeeding set of the root. Since both $\{2\}$ and $\{2,3\}$ are feasible coalitions in $\mathcal{F}'$ but $\{3\}$ is not, only $\{3\}$ can be an immediate successor of $\{1\}$ and therefore $S_{T^{\mathcal{Y}^2}}(\{3\}) = \{2\}$.

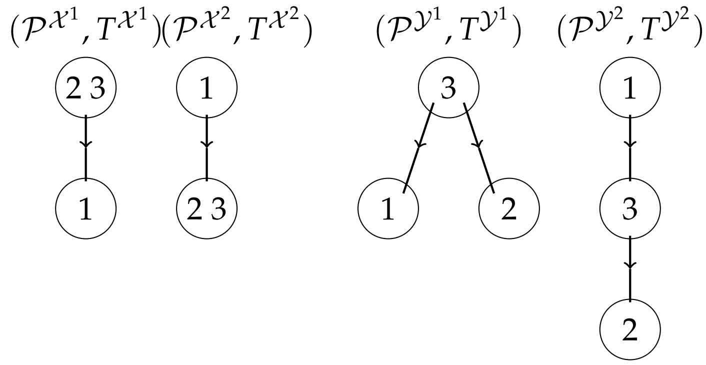


Figure 4.1: The coalitional trees of $\mathcal{F}$ and $\mathcal{F}'$ in Example 4.3.9.

The following example shows that the sets of equivalent players may differ in different coalitional trees of the same set system.

Example 4.3.10 Consider the coalitional structure $\mathcal{F} = \{\{3\},\{8\},\{1,2\},\{1,8\}, \{2,3,4,5\},\{1,2,3,4,5\},N\}$ on $N = \{1,2,3,4,5,6,7,8\}$. $\mathcal{F}$ contains two maximal nested sets, $\mathcal{X}^1 = \{\{1,2\},\{3\},\{8\},\{1,2,3,4,5\},\{1,2,3,4,5,6,7,8\}\}$ and $\mathcal{X}^2 = \{\{3\},\{8\},\{1,8\},\{2,3,4,5\},\{1,2,3,4,5,6,7,8\}\}$. For the maximal nested set $\mathcal{X}^1$, we have $M_{\mathcal{X}^1}(1) = M_{\mathcal{X}^1}(2) = \{1,2\}$, $M_{\mathcal{X}^1}(3) = \{3\}$, $M_{\mathcal{X}^1}(4) = M_{\mathcal{X}^1}(5) = \{1,2,3,4,5\}$, $M_{\mathcal{X}^1}(6) = M_{\mathcal{X}^1}(7) = \{1,2,3,4,5,6,7,8\}$, $M_{\mathcal{X}^1}(8) = \{8\}$, and for the maximal nested set $\mathcal{X}^2$, we have $M_{\mathcal{X}^2}(1) = \{1,8\}$, $M_{\mathcal{X}^2}(2) = M_{\mathcal{X}^2}(4) = M_{\mathcal{X}^2}(5) = \{2,3,4,5\}$, $M_{\mathcal{X}^2}(3) = \{3\}$, $M_{\mathcal{X}^2}(6) = M_{\mathcal{X}^2}(7) = \{1,2,3,4,5,6,7,8\}$, $M_{\mathcal{X}^2}(8) = \{8\}$. With respect to $\mathcal{X}^1$, player 1 is equivalent to player 2, player 4 is equivalent to player 5, and player 6 is equivalent to player 7. However, with respect to $\mathcal{X}^2$, player 6 is equivalent to player

7, and players 2, 4, 5 are equivalent to each other. For the induced partitions of equivalent players we have $\mathcal{P}^{\mathcal{X}^1} = \{\{1,2\},\{3\},\{8\},\{4,5\},\{6,7\}\}$ and $\mathcal{P}^{\mathcal{X}^2} = \{\{1\},\{3\},\{2,4,5\},\{6,7\},\{8\}\}$. The graphical representation of the corresponding coalitional trees $(\mathcal{P}^{\mathcal{X}^1}, T^{\mathcal{X}^1})$ and $(\mathcal{P}^{\mathcal{X}^2}, T^{\mathcal{X}^2})$ is depicted in Figure 4.2. Coalition $S_{T^{\mathcal{X}^1}}(\{6,7\}) = \{1,2,3,4,5,8\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^1}}(\{4, 5\}) = \{1,2,3,4,5\}$ and $\overline{S}_{T^{\mathcal{X}^1}}(\{8\}) = \{8\}$, and coalition $S_{T^{\mathcal{X}^1}}(\{4,5\}) = \{1,2,3\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^1}}(\{1,2\}) = \{1,2\}$ and $\overline{S}_{T^{\mathcal{X}^1}}(\{3\}) = \{3\}$. Similarly, coalition $S_{T^{\mathcal{X}^2}}(\{6,7\}) = \{1,2,3,4,5,8\}$ is partitioned into feasible coalitions $\overline{S}_{T^{\mathcal{X}^2}}(\{2,4,5\}) = \{2,3,4,5\}$ and $\overline{S}_{T^{\mathcal{X}^2}}(\{1\}) = \{1,8\}$, $S_{T^{\mathcal{X}^2}}(\{2,4,5\}) = \overline{S}_{T^{\mathcal{X}^2}}(\{3\}) = \{3\}$, and $S_{T^{\mathcal{X}^2}}(\{1\}) = \overline{S}_{T^{\mathcal{X}^2}}(\{8\}) = \{8\}$.

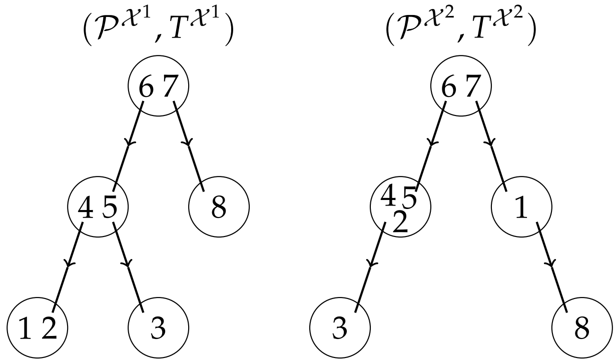


Figure 4.2: The coalitional trees of $\mathcal{F}$ in Example 4.3.10.

Definition 4.3.11 Given a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, for each maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ the marginal contribution vector $m^{\mathcal{X}}(N, v, \mathcal{F})$ is the payoff vector given by

$$m_i^{\mathcal{X}}(N,v,\mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(v(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} v(M_{\mathcal{X}}(P))\Big),\ i \in N.$$

For a marginal contribution vector corresponding to the coalitional tree induced by a maximal nested set, each set of equivalent players receives as total payoff the marginal contribution when these players simultaneously join to the players in the sets of their successors in the tree. The total payoff available for a set of equivalent players is distributed equally among the players of the set. The intuition behind this marginal contribution vector is as follows. Given a maximal nested set and the corresponding coalitional tree, each set of equivalent players contains no feasible subcoalitions and is the smallest set that can join to its successors to form a bigger feasible coalition. So, each set of equivalent players should receive its marginal contribution when joining to its successors. On the individual level, since all members of a set of equivalent players join simultaneously, this marginal contribution is equally divided among them. We assume that every maximal nested set of the underlying coalitional structure of the game is as likely to occur. This gives the following definition of a solution.

Definition 4.3.12 For a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, the *average coalitional tree solution* assigns the payoff vector $ACOT(N, v, \mathcal{F})$ which is given by

$$ACOT(N, v, \mathcal{F}) = \frac{1}{|\overline{\mathcal{X}}^{\mathcal{F}}|} \sum_{\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}} m^{\mathcal{X}}(N, v, \mathcal{F}).$$

The average coalitional tree solution of a TU-game with coalitional structure is the average of the marginal contribution vectors that correspond to the coalitional trees induced by all maximal nested sets of the coalitional structure. Like the Shapley value, the solution considers players' marginal contributions. However, for some players it may not be possible to join a feasible coalition individually in order to form a larger feasible coalition. To illustrate such a case, consider the coalitional structure $\mathcal{F}$ given in Example 4.3.9. For this coalitional structure, there exists no feasible coalition to which player 2 can join, make it a larger feasible coalition and receive his own marginal contribution. However, together with player 3, player 2 is able to join to the feasible coalition consisting of singleton player 1. This contribution is realized in the coalitional tree $(\mathcal{P}^{\mathcal{X}^1}, T^{\mathcal{X}^1})$ of the example. The joint marginal contribution of players 2 and 3 while joining to singleton player 1 is divided equally among the two players.

Example 4.3.13 Consider a TU-game with coalitional structure $(N, v, \mathcal{F})$ where the coalitional structure $\mathcal{F}$ is the one given in Example 4.3.10. For the characteristic function $v : 2^N \to \mathbb{R}$, let $v(S) = |S|^2$ for all $S \in 2^N$. The two maximal nested sets, $\mathcal{X}^1$ and $\mathcal{X}^2$, induce the two marginal contribution vectors $m^{\mathcal{X}^1} = m^{\mathcal{X}^1}(N, v, \mathcal{F})$ and $m^{\mathcal{X}^2} = m^{\mathcal{X}^2}(N, v, \mathcal{F})$. For $m^{\mathcal{X}^1}$ it holds that $m_1^{\mathcal{X}^1} = m_2^{\mathcal{X}^1} = (v(\{1,2\}) - v(\emptyset))/2 = v(\{1,2\})/2 = 2$, $m_3^{\mathcal{X}^1} = v(\{3\}) - v(\emptyset) = 1$, $m_4^{\mathcal{X}^1} = m_5^{\mathcal{X}^1} = (v(\{1,2,3,4,5\}) - v(\{1,2\}) - v(\{3\}))/2 = 10$, $m_6^{\mathcal{X}^1} = m_7^{\mathcal{X}^1} = (v(N) - v(\{1,2,3,4,5\}) - v(\{8\}))/2 = 19$, $m_8^{\mathcal{X}^1} = v(\{8\}) - v(\emptyset) = 1$, and so $m^{\mathcal{X}^1} = (2,2,1,10,10,19,19,1)$. Similarly, for $m^{\mathcal{X}^2} = m^{\mathcal{X}^2}(N, v, \mathcal{F})$ it holds that $m_1^{\mathcal{X}^2} = v(\{1,8\}) - v(\{8\}) = 3$, $m_2^{\mathcal{X}^2} = m_4^{\mathcal{X}^2} = m_5^{\mathcal{X}^2} = (v(\{2,3,4,5\}) - v(\{3\}))/3 = 5$, $m_3^{\mathcal{X}^2} = v(\{3\}) - v(\emptyset) = 1$, $m_6^{\mathcal{X}^2} = m_7^{\mathcal{X}^2} = (v(N) - v(\{2,3,4,5\}) - v(\{1,8\}))/2 = 22$, $m_8^{\mathcal{X}^2} = v(\{8\}) - v(\emptyset) = 1$, and so $m^{\mathcal{X}^2} = (3,5,1,5,5,22,22,1)$. The average coalitional tree solution of the game is the average of these two marginal contribution vectors, so $ACOT(N, v, \mathcal{F}) = \frac{1}{2}(m^{\mathcal{X}^1} + m^{\mathcal{X}^2}) = (5/2, 7/2, 1, 15/2, 15/2, 41/2, 41/2, 1)$.

We remark that the average coalitional tree solution is defined for cooperative games with any coalitional structure that contains the grand coalition. For a TU-game with complete coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ where $\mathcal{F} = 2^N$, any set of players forms a feasible coalition. This means there is no restriction on cooperation and the TU-game with coalitional structure is in fact a TU-game.

Lemma 4.3.14 *Given a complete coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S, Q \in \mathcal{X}$ it holds that $S \subseteq Q$ or $Q \subseteq S$.*

Proof Suppose there exists a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and distinct S, $Q \in \mathcal{X}$ with $S \cap Q = \emptyset$. Since $\mathcal{F}$ is complete, it holds that $S \cup Q \in \mathcal{F}$. But this contradicts with condition (ii) of Definition 4.3.1. Since $\mathcal{X}$ is a maximal nested set and we can rule out the case $S \cap Q = \emptyset$ for distinct members S and Q, we end up with $S \subseteq Q$ or $Q \subseteq S$ for all $S, Q \in \mathcal{X}$. ∎

Lemma 4.3.15 *Given a complete coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S \in \mathcal{X}$ with $|S| \geq 2$, it holds that there exists $i \in S$ such that $S \setminus \{i\} \in \mathcal{X}$.*

Proof Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and $S \in \mathcal{X}$ with $|S| \geq 2$. Let $Q \in \mathcal{X}$ be a maximal proper subset of S in $\mathcal{X}$, i.e., either $Q = \emptyset$ or $Q \in \mathcal{X}$, $Q \subset S$ and there exists no $Q' \in \mathcal{X}$ with $S \supset Q' \supset Q$. Suppose $|S \setminus Q| \geq 2$. So there exists $i, j \in S \setminus Q$, $i \neq j$. Then $Q \subset Q \cup \{i\} \subset S$ and $Q \cup \{i\} \notin \mathcal{X}$. Since $\mathcal{F}$ is complete, $Q \cup \{i\} \in \mathcal{F}$, which contradicts that $\mathcal{X}$ is a maximal nested set of $\mathcal{F}$. ∎

Remark 4.3.16 For a complete coalitional structure $\mathcal{F}$ on N, it holds that $|\overline{\mathcal{X}}^{\mathcal{F}}| = n!$.

Remark 4.3.16 is an immediate result of Lemma 4.3.14 and Lemma 4.3.15. To form a maximal nested set, first we can include the grand coalition. By Lemma 4.3.15 we should also include a subset of the grand coalition with cardinality one less. The number of possibilities is n for this step. At the next step we will have $n - 1$ possibilities and so on. Hence, in total there are $n!$ possibilities each of which corresponds to a maximal nested set.

For TU-games, the Shapley value is the best known single-valued solution concept, see Shapley (1953). To find the Shapley value, all permutations over the players are considered and a marginal contribution vector is associated to each permutation, at which each player receives his marginal contribution when joining to his set of successors in the permutation. The Shapley value is the average of all these marginal contribution vectors. Clearly, these permutations correspond one-to-one to the set of maximal nested sets. Let $Sh(N,v)$ stand for the Shapley value of a TU-game (N,v).

Proposition 4.3.17 *For a TU-game with coalitional structure $(N,v,\mathcal{F}) \in \mathcal{G}_N^{cos}$ with complete coalitional structure $\mathcal{F}$ on N, i.e., $\mathcal{F} = 2^N$, it holds that $ACOT(N,v,\mathcal{F}) = Sh(N,v)$.*

4.4 Properties of the average coalitional tree solution

In this section we provide some properties that are satisfied by the average coalitional tree solution. The first property we consider is efficiency.

For a TU-game with coalitional structure $(N,v,\mathcal{F}) \in \mathcal{G}_N^{cos}$, a payoff vector $x \in \mathbb{R}^n$ is *efficient* if x distributes the worth $v(N)$ of the grand coalition, i.e. $x(N) = v(N)$. A value ξ on $\mathcal{G}_N^{cos}$ is *efficient* if for any TU-game with coalitional structure $(N,v,\mathcal{F}) \in \mathcal{G}_N^{cos}$ the payoff vector $\xi(N,v,\mathcal{F})$ is efficient.

Since each marginal contribution vector corresponding to each coalitional tree distributes exactly $v(N)$ over all players and the average coalitional tree solution is the average of those vectors, the efficiency of the average coalitional tree solution immediately follows.

The second property that is satisfied by the average coalitional solution is linearity. A value ξ on $\mathcal{G}_N^{cos}$ satisfies *linearity* if for any $(N,v,\mathcal{F})$ and $(N,w,\mathcal{F}) \in \mathcal{G}_N^{cos}$ and for any $a,b \in \mathbb{R}$, it holds that

$$\xi(N, av + bw, \mathcal{F}) = a\xi(N,v,\mathcal{F}) + b\xi(N,w,\mathcal{F}),$$

where the characteristic function $av + bw$ is defined as $(av + bw)(S) = av(S) + bw(S)$ for all $S \in \mathcal{F}$.

Given two TU-games with coalitional structure $(N,v,\mathcal{F})$ and $(N,w,\mathcal{F})$ in $\mathcal{G}_N^{cos}$ and real numbers $a,b \in \mathbb{R}$, since the coalitional structure is represented by $\mathcal{F}$ for $(N,v,\mathcal{F})$, $(N,w,\mathcal{F})$ and $(N,av+bw,\mathcal{F})$, the collection of maximal nested sets is the same for each of these three TU-games with coalitional structure. Since the average coalitional tree solution of a TU-game with coalitional structure is a linear combination of the marginal contribution vectors induced by all maximal nested sets and each marginal contribution vector is a linear combination of the worths of the coalitions, the average coalitional tree solution satisfies linearity.

Now we consider the equal treatment of equivalent players property. For a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, two players $i, j \in N$ are *equivalent* if they are equivalent with respect to $\mathcal{F}$. A solution $\xi\colon \mathcal{G}_N^{cos} \to \mathbb{R}^n$ satisfies the *equal treatment of equivalent players property* if for any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ it holds that $\xi_i(N, v, \mathcal{F}) = \xi_j(N, v, \mathcal{F})$ whenever $i \in N$ and $j \in N$ are equivalent players for $(N, v, \mathcal{F})$. The intuition behind this property is that since equivalent players for a TU-game with coalitional structure are members of the same feasible coalitions, these players should receive the same payoff.

Proposition 4.4.1 *The average coalitional tree solution on $\mathcal{G}_N^{cos}$ satisfies the equal treatment of equivalent players property.*

Proof Take any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, and two players $i, j \in N$ being equivalent for $(N, v, \mathcal{F})$. Let $(P^{\mathcal{X}}, T^{\mathcal{X}})$ be the coalitional tree induced by $\mathcal{X}$. According to Remark 4.3.7, players i and j are equivalent also with respect to $\mathcal{X}$. This implies the existence of $P \in \mathcal{P}^{\mathcal{X}}$ such that $i, j \in P$. Hence,

$$m_i^{\mathcal{X}}(N, v, \mathcal{F}) = m_j^{\mathcal{X}}(N, v, \mathcal{F}) = \frac{1}{|P|}\Big(v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P'))\Big).$$

Since

$$ACOT(N, v, \mathcal{F}) = \frac{1}{|\overline{\mathcal{X}}^{\mathcal{F}}|} \sum_{\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}} m^{\mathcal{X}}(N, v, \mathcal{F}),$$

it implies that $ACOT_i(N, v, \mathcal{F}) = ACOT_j(N, v, \mathcal{F})$. ∎

For TU-games, since any subset of players is feasible, a player is able to join any subset of players he doesn't belong to. For TU-games, a null player is defined to be a player whose marginal contribution is zero when joining to any set of players. For our setting, since not all coalitions are feasible, a player may not be able to join all coalitions. Moreover, a player may need other players to be able to join to a feasible coalition and also we may have cases where players are not joining to a single feasible coalition but to several disjoint feasible coalitions.

Definition 4.4.2 Given a coalitional structure $\mathcal{F}$ on N and $S \in \mathcal{F}$, $\{S_1, \ldots, S_k\}$ is a *maximal subpartition* of S if it satisfies the following conditions:

(i) $S_h \subset S$ for all $h \in \{1, \ldots, k\}$ and $S \setminus (\bigcup_{h=1}^{k} S_h) \neq \emptyset$;

(ii) $S_h \in \mathcal{F}$ and $S_h \cap S_m = \emptyset$ holds for all distinct $h, m \in \{1, \ldots, k\}$;

(iii) $S' \cup (\bigcup_{m \in M} S_m) \notin \mathcal{F}$ for all $S' \subset (S \setminus \bigcup_{h=1}^{k} S_h)$ and $M \subseteq \{1, \ldots, k\}$.

For a feasible coalition $S \in \mathcal{F}$, a collection of disjoint feasible coalitions $\{S_1, \ldots, S_k\}$ is a maximal subpartition of S if their union is a proper subset of S and it is not possible to find any other such collection that is obtained by combining some of the members of the collection with some other players in S. A maximal subpartition of a feasible coalition S does not need to be unique and is the empty set if there exists no $S' \in \mathcal{F}$ such that $S' \subset S$. For $S \in \mathcal{F}$, $\overline{\mathcal{D}}_{\mathcal{F}}(S)$ denotes the collection of maximal subpartitions of S. Note that if the empty set is a maximal subpartition of a feasible coalition S, then there exists no other maximal subpartition of S.

Example 4.4.3 Consider the coalitional structure $\mathcal{F} = \{\{1\}, \{2\}, \{2,3\}, \{4,5\}, \{1,2,3,4,5\}\}$ on $N = \{1,2,3,4,5\}$. For the grand coalition N there exist three maximal subpartitions, $\{\{1\}, \{2,3\}\}$, $\{\{1\}, \{2\}, \{4,5\}\}$, and $\{\{2,3\}, \{4,5\}\}$. Furthermore, $\overline{\mathcal{D}}_{\mathcal{F}}(\{2,3\}) = \{\{\{2\}\}\}$ and $\overline{\mathcal{D}}_{\mathcal{F}}(\{4,5\}) = \{\emptyset\}$.

In a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, player $i \in N$ is called a *null player* if $v(S) - \sum_{Q \in \mathcal{D}} v(Q) = 0$ for all $S \in \mathcal{F}$ and $\mathcal{D} \in \overline{\mathcal{D}}_{\mathcal{F}}(S)$ satisfying $i \in S \setminus \bigcup_{Q \in \mathcal{D}} Q$.

A player is a null player if his contribution in worth is zero when he and possibly other players join to any maximal subpartition of a feasible coalition he belongs to. Notice that if a player $i \in N$ is a null player and $\{i\} \in \mathcal{F}$, then $v(\{i\}) = 0$.

Definition 4.4.4 A value ξ on $\mathcal{G}_N^{cos}$ satisfies the *null player property* if for any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, it holds that $\xi_i(N, v, \mathcal{F}) = 0$ whenever $i \in N$ is a null player in $(N, v, \mathcal{F})$.

Proposition 4.4.5 *The average coalitional tree solution on $\mathcal{G}_N^{cos}$ satisfies the null player property.*

Proof Take any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ and $i \in N$ such that player i is a null player in $(N, v, \mathcal{F})$. Consider any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and let $P_{\mathcal{X}}(i) = P$. By property (iii) of Theorem 4.3.8, $\{\overline{S}_{T^{\mathcal{X}}}(P') \mid P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)\}$ is a maximal subpartition of $S_{T^{\mathcal{X}}}(P)$ not containing i. Since i is a null player in $(N, v, \mathcal{F})$ and $\overline{S}_{T^{\mathcal{X}}}(Q) = M_{\mathcal{X}}(Q)$ for any $Q \in \mathcal{P}^{\mathcal{X}}$, we have

$$v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P')) = v(\overline{S}_{T^{\mathcal{X}}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(\overline{S}_{T^{\mathcal{X}}}(P')) = 0.$$

This implies

$$m_i^{\mathcal{X}}(N, v, \mathcal{F}) = \frac{1}{|P|}\Big(v(M_{\mathcal{X}}(P)) - \sum_{P' \in \mathcal{I}_{T^{\mathcal{X}}}(P)} v(M_{\mathcal{X}}(P'))\Big) = 0.$$

Since the average coalitional tree solution is the average of the marginal contribution vectors corresponding to the coalitional trees induced by all maximal nested sets of $\mathcal{F}$, it holds that $ACOT_i(N, v, \mathcal{F}) = 0$. ∎

Now we state that the average coalitional tree solution is independent of closed coalitions. If a feasible coalition of a coalitional structure is disjoint to any other feasible coalition that does not contain it or is not a subset of it and is not able to join to other feasible coalitions to form a larger feasible coalition, then this coalition is called a closed coalition.

Definition 4.4.6 Given a coalitional structure $\mathcal{F}$ on N, a feasible coalition $Q \in \mathcal{F}$ is a *closed coalition* if it satisfies the following conditions:

(i) For all $S \in \mathcal{F}$, $S \neq Q$, either $Q \subset S$ or $Q \supset S$ or $Q \cap S = \emptyset$;

(ii) For any $S_1, \ldots, S_k \in \mathcal{F}$ satisfying $S_h \cap Q = \emptyset$ for $h = 1, \ldots, k$ it holds that $(\bigcup_{j=1}^{k} S_j) \cup Q \notin \mathcal{F}$.

Subsets of a closed coalition are not able to join to coalitions that also contain players outside the coalition. In some sense, a closed coalition performs on its own. Hence, the members of a closed coalition should receive together just the worth of that coalition, the payoffs of players inside a closed coalition should only depend on the worths of the feasible subcoalitions of the closed coalition, and the payoffs of players outside the closed coalition should be independent of these worths.

Definition 4.4.7 A value ξ on $\mathcal{G}_N^{cos}$ satisfies *independence of closed coalitions* if for any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ and closed coalition $Q \in \mathcal{F}$ the following conditions hold:

(i) $\sum_{i \in Q} \xi_i(N, v, \mathcal{F}) = v(Q)$;
(ii) For any TU-game with coalitional structure $(N, w, \mathcal{F}) \in \mathcal{G}_N^{cos}$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \subseteq Q$, it holds that $\xi_i(N, v, \mathcal{F}) = \xi_i(N, w, \mathcal{F})$ for all $i \in Q$;
(iii) For any TU-game with coalitional structure $(N, w, \mathcal{F}) \in \mathcal{G}_N^{cos}$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \supseteq Q$ or $S \cap Q = \emptyset$, it holds that $\xi_i(N, v, \mathcal{F}) = \xi_i(N, w, \mathcal{F})$ for all $i \in N \setminus Q$.

Proposition 4.4.8 *The average coalitional tree solution on $\mathcal{G}_N^{cos}$ satisfies independence of closed coalitions.*

Proof Take any TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ and closed coalition $Q \in \mathcal{F}$. We first show that $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Suppose there exists a maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ such that $Q \notin \mathcal{X}$. Since Q is a closed coalition and $\mathcal{X} \subseteq \mathcal{F}$, conditions (i) and (ii) of Definition 4.4.6 imply that $\mathcal{X} \cup \{Q\}$ is a nested set, which contradicts that $\mathcal{X}$ is a maximal nested set.

To prove condition (i) of Definition 4.4.7, take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. By Lemma 4.3.3 and Theorem 4.3.8, there exists $P \in \mathcal{P}^{\mathcal{X}}$ such that $M_{\mathcal{X}}(P) = \overline{S}_{T^{\mathcal{X}}}(P) = Q$. This implies $\sum_{i \in Q} m_i^{\mathcal{X}}(N, v, \mathcal{F}) = v(M_{\mathcal{X}}(P)) = v(Q)$. Since the average coalitional tree solution is the average of the marginal contribution vectors corresponding to the coalitional trees induced by all maximal nested sets, we have $\sum_{i \in Q} ACOT_i(N, v, \mathcal{F}) = v(Q)$.

To show condition (ii) of Definition 4.4.7, take any $i \in Q$ and TU-game with coalitional structure $(N, w, \mathcal{F})$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \subseteq Q$. Both $(N, v, \mathcal{F})$ and $(N, w, \mathcal{F})$ have the same coalitional structure $\mathcal{F}$ and $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$, then

$$m_i^{\mathcal{X}}(N, v, \mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(v(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} v(M_{\mathcal{X}}(P))\Big)$$

and

$$m_i^{\mathcal{X}}(N, w, \mathcal{F}) = \frac{1}{|P_{\mathcal{X}}(i)|}\Big(w(M_{\mathcal{X}}(i)) - \sum_{P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))} w(M_{\mathcal{X}}(P))\Big).$$

$Q \in \mathcal{X}$ and $i \in Q$ imply both $M_{\mathcal{X}}(i) \subseteq Q$ and $M_{\mathcal{X}}(P) \subseteq Q$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Since $v(S) = w(S)$ for all $S \subseteq Q$, we obtain $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Hence, $m_i^{\mathcal{X}}(N, v, \mathcal{F}) = m_i^{\mathcal{X}}(N, w, \mathcal{F})$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and therefore $ACOT_i(N, v, \mathcal{F}) = ACOT_i(N, w, \mathcal{F})$.

To show condition (iii) of Definition 4.4.7, take any $i \in N \setminus Q$ and TU-game with coalitional structure $(N, w, \mathcal{F}) \in \mathcal{G}_N^{cos}$ such that $w(S) = v(S)$ for all $S \in \mathcal{F}$ satisfying $S \supseteq Q$ or $S \cap Q = \emptyset$. Again both TU-games with coalitional structure $(N, v, \mathcal{F})$ and $(N, w, \mathcal{F})$ have the same coalitional structure $\mathcal{F}$ and $Q \in \mathcal{X}$ for all $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Since $Q \in \mathcal{X}$ is a closed coalition and $i \in N \setminus Q$, either $M_{\mathcal{X}}(i) \cap Q = \emptyset$ or $M_{\mathcal{X}}(i) \supset Q$. If $M_{\mathcal{X}}(i) \cap Q = \emptyset$ then $M_{\mathcal{X}}(P) \cap Q = \emptyset$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$, which implies $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and therefore $m_i^{\mathcal{X}}(N, v, \mathcal{F}) = m_i^{\mathcal{X}}(N, w, \mathcal{F})$. If $M_{\mathcal{X}}(i) \supset Q$, then $M_{\mathcal{X}}(P') \supseteq Q$ for precisely one $P' \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and $M_{\mathcal{X}}(P) \cap Q = \emptyset$ for all other $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$. Since $v(S) = w(S)$ for all $S \supseteq Q$ or $S \cap Q = \emptyset$, this implies $v(M_{\mathcal{X}}(i)) = w(M_{\mathcal{X}}(i))$ and $v(M_{\mathcal{X}}(P)) = w(M_{\mathcal{X}}(P))$ for all $P \in \mathcal{I}_{T^{\mathcal{X}}}(P_{\mathcal{X}}(i))$ and therefore $m_i^{\mathcal{X}}(N, v, \mathcal{F}) = m_i^{\mathcal{X}}(N, w, \mathcal{F})$. The two cases together imply that $ACOT_i(N, v, \mathcal{F}) = ACOT_i(N, w, \mathcal{F})$. ∎

Since we assume $N \in \mathcal{F}$, the grand coalition N is a closed coalition. Hence, efficiency of the average coalitional tree solution also follows from independence of closed coalitions, condition (i) of Definition 4.4.7. For a TU-game with coalitional structure, a closed coalition can be seen as a set of players whose performance is not affected by the other players of the game (condition (ii)) and also does not affect the performance of the other players (condition (iii)). Condition (ii) also implies that the average coalitional tree solution of the subgame obtained by the players of a closed coalition is the same as the average coalitional tree solution for these players under the original game.

For the Shapley value introduced in Aguilera et al. (2010), the property of independence of closed coalitions is not satisfied, because a closed coalition may not be a member of every maximal chain of the coalitional structure.

Example 4.4.9 Consider the coalitional structure $\mathcal{F} = \{\{1,2\}, \{2,3\}, \{1,2,3\}, \{4,6\}, \{5,7\}, \{6,7\}, \{1,2,3,4,5\}, \{1,2,3,5,7,8\}, N\}$ on $N = \{1,2,3,4,5,6,7,8\}$. $\mathcal{F}$ contains six maximal nested sets, $\mathcal{X}^1 = \{\{1,2\}, \{1,2,3\}, \{5,7\}, \{1,2,3,5,7,8\}, N\}$, $\mathcal{X}^2 = \{\{2,3\}, \{1,2,3\}, \{5,7\}, \{1,2,3,5,7,8\}, N\}$, $\mathcal{X}^3 = \{\{1,2\}, \{1,2,3\}, \{6,7\}, \{1,2,3,4,5\}, N\}$, $\mathcal{X}^4 = \{\{2,3\}, \{1,2,3\}, \{6,7\}, \{1,2,3,4,5\}, N\}$, $\mathcal{X}^5 = \{\{1,2\}, \{1,2,3\}, \{4,6\}, \{5,7\}, N\}$, and $\mathcal{X}^6 = \{\{2,3\}, \{1,2,3\}, \{4,6\}, \{5,7\}, N\}$. The corresponding coalitional trees are depicted in Figure 4.3.

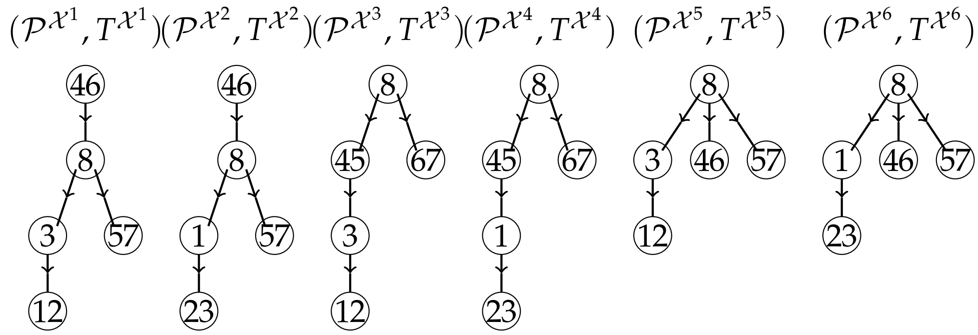


Figure 4.3: The coalitional trees of $\mathcal{F}$ in Example 4.4.9.

Coalition $\{1,2,3\}$ is a closed coalition. Therefore it is a member of all maximal nested sets and a branch in each coalitional tree. However, $\{\{5,7\},\{1,2,3,5,7,8\},N\}$ is a maximal chain which is not a maximal nested set and does not contain $\{1,2,3\}$. For each of the six maximal nested sets the total payoff for the members of the coalition $\{1,2,3\}$ at the induced marginal contribution vector is equal to its worth $v(\{1,2,3\})$. If for two TU-games with coalitional structure $(N,v,\mathcal{F})$ and $(N,w,\mathcal{F})$ it holds that $v(S) = w(S)$ for all feasible $S \subseteq \{1,2,3\}$, then at the average coalitional tree solution the payoffs for the players in $\{1,2,3\}$ are the same. Similarly, if $v(S) = w(S)$ holds for all feasible S satisfying $S \cap \{1,2,3\} = \emptyset$ or $S \supseteq \{1,2,3\}$, then at the average coalitional tree solution the payoffs for the players in $\{4,5,6,7,8\}$ are the same.

4.5 Special cases for coalitional structure

As discussed before, for complete coalitional structures, which is a special case for coalitional structure, the average coalitional tree solution and the Shapley value coincide. In this section we consider some other special coalitional structures.

4.5.1 Building set as coalitional structure

This subsection considers the case where building sets represent the coalitional structure in a TU-game. A set system on N is a building set if all singleton coalitions, the union of any intersecting feasible coalitions, and the grand coalition are feasible. For TU-games with a building set as the coalitional structure, Koshevoy and Talman (2014) uses maximal nested sets to define the gravity center solution (GC) which coincides with the average coalitional tree solution on this class of games. Koshevoy and Talman (2014) also proposes a way to use the GC solution for games with arbitrary coalitional structure by taking its building cover. Given a set system $\mathcal{F}$ on N, the building cover of $\mathcal{F}$, $\mathcal{B}(\mathcal{F})$, is defined as the smallest building set on N that contains $\mathcal{F}$. For a TU-game with coalitional structure $(N, v, \mathcal{F})$ on N, by using the Möbius inversion, they define the so called M-extension $v^{\mathcal{F}}$ of the characteristic function v and propose as GC solution of $(N, v, \mathcal{F})$ the GC solution of the game $(N, v^{\mathcal{F}}, \mathcal{B}(\mathcal{F}))$. The definition of Möbius inversion and M-extension is given below.

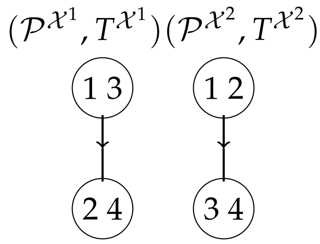
For a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, the *Möbius inversion* of v is given by

$$\mu(T) = \sum_{T' \subseteq T} (-1)^{|T|-|T'|} v(T), \;\; T \in 2^N.$$

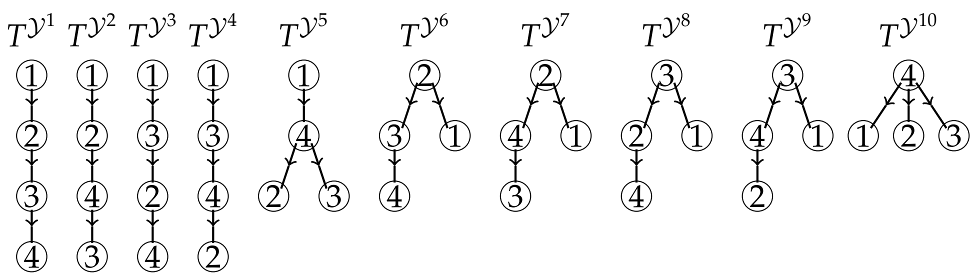
Definition 4.5.1 Given a TU-game with coalitional structure $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$, the *M-extension* $v^{\mathcal{F}} : 2^N \to \mathbb{R}$ of v is given by the following conditions:
(i) $v^{\mathcal{F}}(S) = v(S)$ for all $S \in \mathcal{F}$;
(ii) For the Möbius inversion of $v^{\mathcal{F}}$, $\mu^{\mathcal{F}}$, it holds that $\mu^{\mathcal{F}}(S) = 0$ for all $S \notin \mathcal{F}$.

Although for a TU-game with coalitional structure $(N, v, \mathcal{F})$, where $\mathcal{F}$ is a building set, the average coalitional tree solution and the GC solution coincide, for games with more general coalitional structure, they differ from each other.

Example 4.5.2 Consider the TU-game with coalitional structure $(N, v, \mathcal{F})$ with $\mathcal{F} = \{\{2,4\}, \{3,4\}, \{1,2,3,4\}\}$ and $v(S) = |S|^2$ for all $S \in \mathcal{F}$. $\mathcal{F}$ has two maximal nested sets, $\mathcal{X}^1 = \{\{2,4\}, \{1,2,3,4\}\}$ and $\mathcal{X}^2 = \{\{3,4\}, \{1,2,3,4\}\}$, with corresponding coalitional trees depicted in Figure 4.5a. Since $\mathcal{F}$ contains no singleton coalitions and also not the union $\{2,3,4\}$ of the feasible coalitions $\{2,4\}$ and $\{3,4\}$, $\mathcal{F}$ is not a building set. The building cover of $\mathcal{F}$ is the collection $\mathcal{B}(\mathcal{F}) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{2,4\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$, having ten maximal nested sets, $\mathcal{Y}^1 = \{\{4\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^2 = \{\{3\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^3 = \{\{4\}, \{2,4\}, \{2,3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^4 = \{\{2\}, \{2,4\}, \{2,3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^5 = \{\{2\}, \{3\}, \{2,3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^6 = \{\{1\}, \{4\}, \{3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^7 = \{\{1\}, \{3\}, \{3,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^8 = \{\{1\}, \{4\}, \{2,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^9 = \{\{1\}, \{2\}, \{2,4\}, \{1,2,3,4\}\}$, $\mathcal{Y}^{10} = \{\{1\}, \{2\}, \{3\}, \{1,2,3,4\}\}$, with corresponding coalitional trees depicted in Figure 4.5b.



a. The coalitional trees of $\mathcal{F}$.



b. The coalitional trees of $\mathcal{B}(\mathcal{F})$.
Figure 4.5: Example 4.5.2.

For the M-extension of v, $v^{\mathcal{F}}$, we obtain $v^{\mathcal{F}}(S) = v(S)$ if $S \in \mathcal{F}$, $v^{\mathcal{F}}(S) = 0$ if $|S| = 1$, and $v^{\mathcal{F}}(\{2,3,4\}) = v(\{2,4\}) + v(\{3,4\}) = 8$. It holds that $ACOT(N, v, \mathcal{F}) = (6,4,4,2)$ and $GC(N, v, \mathcal{F}) = GC(v^{N,\mathcal{F}}, \mathcal{B}(\mathcal{F})) = (4,4,4,4)$. Notice that some of the marginal contribution vectors that correspond to the coalitional trees of the building cover $\mathcal{B}(\mathcal{F})$ are the same.

Chapter 2 of this monograph studies TU-games with connected communication structure for which the collection of feasible coalitions is the set of connected coalitions of an (undirected) graph on the set of players. For a connected communication structure which is represented by a graph, the collection of feasible coalitions forms a building set, because together with the grand coalition, all singleton coalitions are connected and given any two intersecting connected sets the union is also connected. So, a TU-game with connected communication structure can be considered as a TU-game with coalitional structure which is represented by a building set.

A coalitional structure $\mathcal{F}$ on N is called *graphical*, if $\mathcal{F}$ is the collection of all connected sets of players of a graph (N, L), i.e., $\mathcal{F} = C^L(N)$. In order to have the grand coalition N as a feasible set, throughout this subsection, we assume that N forms a connected set in the graph that induces the coalitional structure. In case the set of feasible coalitions is restricted to the collection of connected sets of a connected undirected graph, the TU-game with coalitional structure is a TU-game with communication structure, see Chapter 2.

Lemma 4.5.3 *Given a graphical coalitional structure $\mathcal{F}$ on N, for any maximal nested set $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ it holds that $P_{\mathcal{X}}(i) = \{i\}$ for all $i \in N$.*

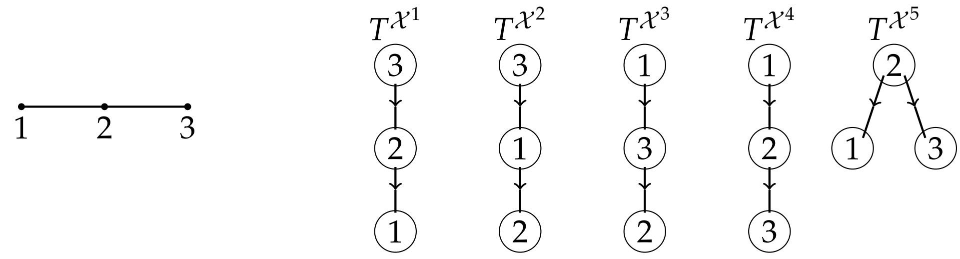
Proof Let the graphical coalitional structure $\mathcal{F}$ on N be induced by a connected graph (N, L) and take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Suppose there exists $i \in N$ for which $j \in P_{\mathcal{X}}(i)$, for some $j \neq i$. Then $M_{\mathcal{X}}(i) = M_{\mathcal{X}}(j)$. Let $K \in \widehat{C}^L(M_{\mathcal{X}}(i) \setminus \{i\})$ be the component of $M_{\mathcal{X}}(i) \setminus \{i\}$ in (N, L) containing j. Then $K \in \mathcal{F}$ and $K \notin \mathcal{X}$. Since (N, L) is a connected graph, $\mathcal{X} \cup \{K\}$ is a nested set, contradicting that $\mathcal{X}$ is a maximal nested set. ∎

An immediate result of Lemma 4.5.3 is that, for a TU-game with graphical coalitional structure, all coalitional trees corresponding to the maximal nested sets are trees.

Lemma 4.5.4 *Given a graphical coalitional structure $\mathcal{F}$ on N induced by a graph (N, L), for any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ it holds that $\overline{S}_{T^{\mathcal{X}}}(P') \in \widehat{C}^L(S_{T^{\mathcal{X}}}(P))$ if $(P, P') \in T^{\mathcal{X}}$.*

Proof Take any $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$ and let $(P, P') \in T^{\mathcal{X}}$. As a direct result of Theorem 4.3.8 and since $\mathcal{F} = C^L(N)$, we have $\overline{S}_{T^{\mathcal{X}}}(P) \in C^L(N)$ and $\overline{S}_{T^{\mathcal{X}}}(P') \in C^L(N)$. Let $S_1, \ldots, S_k$ be the components of $S_{T^{\mathcal{X}}}(P)$ in (N, L), i.e., $S_h \in \widehat{C}^L(S_{T^{\mathcal{X}}}(P))$ holds for all $h = 1, \ldots, k$. First, suppose $\overline{S}_{T^{\mathcal{X}}}(P') \subset S_h$ for some $h \in \{1, \ldots, k\}$. Since $(P, P') \in T^{\mathcal{X}}$, we have $M_{\mathcal{X}}(P) \supset M_{\mathcal{X}}(P')$ and there exists no $X \in \mathcal{X}$ such that $M_{\mathcal{X}}(P) \supset X \supset M_{\mathcal{X}}(P')$. Hence, $S_h \notin \mathcal{X}$, which contradicts that $\mathcal{X}$ is a maximal nested set. Next, suppose $\overline{S}_{T^{\mathcal{X}}}(P') \cap S_h \neq \emptyset$ and $\overline{S}_{T^{\mathcal{X}}}(P') \cap S_m \neq \emptyset$ for some $h, m \in \{1, \ldots, k\}$, $h \neq m$. Since $\overline{S}_{T^{\mathcal{X}}}(P') \in C^L(N)$, this contradicts that S_h and S_m both are components of $S_{T^{\mathcal{X}}}(P)$ in (N, L). Hence, $\overline{S}_{T^{\mathcal{X}}}(P')$ is equal to S_h for some $h \in \{1, \ldots, k\}$, which completes the proof. ∎

Example 4.5.5 Consider the graph (N, L) on $\{1, 2, 3\}$ with $L = \{\{1, 2\}, \{2, 3\}\}$ as depicted in Figure 4.6a. $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ is the corresponding graphical coalitional structure on N. $\mathcal{F}$ has five maximal nested sets, $\mathcal{X}^1 = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, $\mathcal{X}^2 = \{\{2\}, \{1, 2\}, \{1, 2, 3\}\}$, $\mathcal{X}^3 = \{\{2\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathcal{X}^4 = \{\{3\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathcal{X}^5 = \{\{3\}, \{1\}, \{1, 2, 3\}\}$, with corresponding coalitional trees as depicted in Figure 4.6b.



a. The graph (N, L). b. The coalitional trees of $\mathcal{F}$.

Figure 4.6: Example 4.5.5.

Since the collection of connected sets of a graph is a building set, it holds that on the class of TU-games with graphical coalitional structure the GC solution and the average coalitional tree solution coincide.

In Chapter 3, the average covering tree solution is introduced for TU-games with dominance structure. The average covering tree solution is applicable to TU-games with communication structure by considering the directed analogue of the communication structure. The directed analogue of an undirected graph is obtained by replacing each edge in the communication structure by two arcs with opposite directions. On the class of TU-games with graphical coalitional structure, the average coalitional tree solution and the average covering tree solution coincide when the average covering tree solution is applied to the TU-game with dominance structure whose dominance structure is the directed analogue of the graph inducing the coalitional structure.

4.5.2 Partitional coalitional structures

In this subsection we consider coalitional structures where for each player the only alternative of not participating in the grand coalition is to participate in a unique smaller coalition. A coalitional structure $\mathcal{F}$ is called *partitional* if it contains exactly the grand coalition N and a proper partition of N.

Lemma 4.5.6 *If $\mathcal{F}$ is a partitional coalitional structure on N, then* $|\overline{\mathcal{X}}^{\mathcal{F}}| = |\mathcal{F}| - 1$.

Proof Let $\mathcal{F} = \{S_1, \ldots, S_k, N\}$ where $S_1, \ldots, S_k$ forms a partition of N for some $k \geq 2$. Then $\mathcal{X}$ is a maximal nested set of $\mathcal{F}$ if and only if there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{X} = \mathcal{F} \setminus \{S_i\}$. So $|\overline{\mathcal{X}}^{\mathcal{F}}| = k$, which completes the proof. ∎

A direct result of Lemma 4.5.6 is that, given a partitional coalitional structure, each collection of feasible coalitions that excludes only one of the partition members is a maximal nested set.

Lemma 4.5.7 *Given a partitional coalitional structure* $\mathcal{F} = \{S_1, \ldots, S_k, N\}$, *if* $i \in S_h$ *for some* $h \in \{1, \ldots, k\}$, *then* $P_{\mathcal{X}}(i) = S_h$ *for all* $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$.

Proof Take any $h \in \{1, \ldots, k\}$ and $i \in S_h$ and let $\mathcal{X} \in \overline{\mathcal{X}}^{\mathcal{F}}$. Suppose $S_h \in \mathcal{X}$. Since $S_1, \ldots, S_k$ forms a partition of N and $S_h \in \mathcal{X}$, we have $M_{\mathcal{X}}(j) = S_h$ for all $j \in S_h$. So, $P_{\mathcal{X}}(i) = S_h$. Next, suppose $S_h \notin \mathcal{X}$. Since $S_1, \ldots, S_k$ forms a partition of N and $S_h \notin \mathcal{X}$, we have $M_{\mathcal{X}}(j) = N$ for all $j \in S_h$. Hence, again $P_{\mathcal{X}}(i) = S_h$, which completes the proof. ∎

Since coalitional trees are defined on the sets of equivalent players, by Lemma 4.5.7, for a partitional coalitional structure $\mathcal{F} = \{S_1, \ldots, S_k, N\}$, all of the induced coalitional trees are defined on the partition $\{S_1, \ldots, S_k\}$.

Theorem 4.5.8 *Given a TU-game with coalitional structure* $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ *with partitional coalitional structure* $\mathcal{F} = \{S_1, \ldots, S_k, N\}$, *for all* $m \in \{1, \ldots, k\}$ *it holds that*

$$\sum_{i \in S_m} ACOT_i(N, v, \mathcal{F}) = v(S_m) + \frac{1}{k}\Big(v(N) - \sum_{h=1}^{k} v(S_h)\Big).$$

Proof Take any $m \in \{1, \ldots, k\}$. By Lemma 4.5.6 we have $|\overline{\mathcal{X}}^{\mathcal{F}}| = k$. Let $\overline{\mathcal{X}}^{\mathcal{F}} = \{\mathcal{X}^1, \ldots, \mathcal{X}^k\}$. Then $k-1$ of these maximal nested sets contain S_m and only one of them does not contain S_m. Without loss of generality let $S_m \in \mathcal{X}^h$ hold for $h = 1, \ldots, k-1$. Then $\sum_{i \in S_m} m_i^{\mathcal{X}^h}(N, v, \mathcal{F}) = v(S_m)$ for $h \in \{1, \ldots, k-1\}$ and for $\mathcal{X}^k$ we have $\sum_{i \in S_m} m_i^{\mathcal{X}^k}(N, v, \mathcal{F}) = v(N) - \sum_{h=1}^{k-1} v(S_h)$. Since the average coalitional tree solution is the average of these marginal contribution vectors, we obtain $\sum_{i \in S_m} ACOT_i(N, v, \mathcal{F}) = v(S_m) + (v(N) - \sum_{h=1}^{k} v(S_h))/k$. ∎

As Theorem 4.5.8 shows, given a TU-game with a partitional coalitional structure, each member of the partition receives its worth plus an equal share of the total contribution of all members of the partition while forming the grand coalition. On the individual level, each player receives an equal share of the total payoff available to the partition member he belongs to. So, we have the following corollary.

Corollary 4.5.9 *Given a TU-game with coalitional structure* $(N, v, \mathcal{F}) \in \mathcal{G}_N^{cos}$ *with partitional coalitional structure* $\mathcal{F} = \{S_1, \ldots, S_k, N\}$, *for any* $i \in S_m$, $m \in \{1, \ldots, k\}$, *it holds that*

$$ACOT_i(N, v, \mathcal{F}) = \frac{1}{|S_m|}\Big(v(S_m) + \frac{1}{k}\Big(v(N) - \sum_{h=1}^{k} v(S_h)\Big)\Big).$$

This result is confirmed by the property of equal treatment of equivalent players, because for a TU game with partitional coalitional structure the players in any partition member are equivalent to each other.

Example 4.5.10 Consider a partitional coalitional structure $\mathcal{F} = \{\{1,2,3\}, \{4,5\}, \{6\}, \{7\}, \{1,2,3,4,5,6,7\}\}$. $\mathcal{F}$ has four maximal nested sets, $\mathcal{X}^1 = \{\{4,5\}, \{6\}, \{7\}, \{1,2,3,4,5,6,7\}\}$, $\mathcal{X}^2 = \{\{1,2,3\}, \{6\}, \{7\}, \{1,2,3,4,5,6,7\}\}$, $\mathcal{X}^3 = \{\{1,2,3\}, \{4,5\}, \{7\}, \{1,2,3,4,5,6,7\}\}$ and $\mathcal{X}^4 = \{\{1,2,3\}, \{4,5\}, \{6\}, \{1,2,3,4,5,6,7\}\}$, with corresponding coalitional trees as depicted in Figure 4.7.

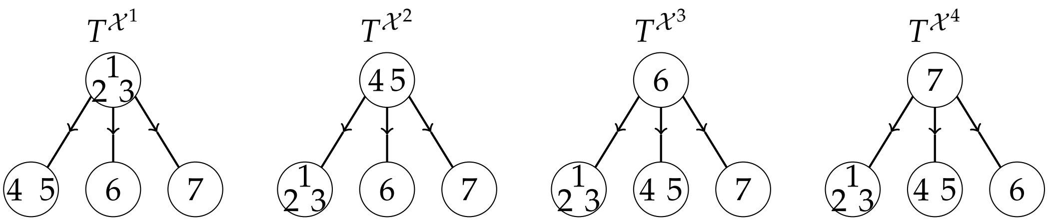


Figure 4.7: The coalitional trees of $\mathcal{F}$ in Example 4.5.10.

CHAPTER 5

CHARACTERIZATION OF THE COPELAND SOLUTION FOR TOURNAMENTS

5.1 Introduction

Chapter 3 of this monograph considers TU-games with dominance structure where the dominance structure is represented by a directed graph. In the literature a special type of directed graphs, tournaments, that contain an arc between any pair of nodes, attract special attention. In real life, such a directed graph may be the result of a sports competition where every contestant meets every other contestant. For a tournament representing the result of a sports competition, the nodes are the contestants and each arc in the directed graph represents the result of the competition between those contestants. Alternatively, given a set of candidates and a set of an odd number of individuals with preferences on these candidates, pairwise majority comparison of the candidates yields a tournament on the set of candidates. With this approach, in a tournament an arc from a candidate to another candidate means the majority of the individuals prefers the first candidate to the latter one. To solve the problem of choosing from a tournament, Copeland (1951) proposes to pick the candidates that beat the maximum number of candidates. The proposal of Copeland received attention from a variety of fields, including graph theory as in van den Brink and Gilles (2003); economics as in Paul (1997); computer science as in Singh and Kurose (1991); and social choice theory as in Moulin (1986). As a result, it has been the subject matter of thorough investigations and we know, at present, many of its properties. However, the literature is not very rich in characterizations of the Copeland solution. In fact, the original proposal of Copeland (1951) is not supported by a characterization. Later, Moon (1968) shows the equivalence between the Copeland ranking and the one generated by the maximum likelihood solution of Zermelo (1929) which assigns a strength to each candidate and derives the social ranking accordingly. The first axiomatic characterization of the Copeland solution is by Rubinstein (1980) who characterizes the Copeland welfare function as a method to rank the candidates in a tournament. Henriet (1985) extends this characterization to environments which allow for ties between candidates. Similarly, van den Brink and Gilles (2003) provides a characterization of the Copeland welfare function for an arbitrary digraph which is a more general class than tournaments.

In this chapter, we provide a new characterization of the Copeland rule based on the number of steps in which candidates beat each other. In a tournament, the Condorcet winner is the candidate which beats every other candidate in one step i.e., there exists an arc from the Condorcet winner to every other candidate. So, given n candidates and a tournament on these candidates, a Condorcet winner beats all remaining candidates in a total of $n-1$ number of steps. When choosing from a tournament, there is universal agreement on the Condorcet principle which requires to pick the Condorcet winner, whenever it exists. As a Condorcet winner may fail to exist, the Condorcet principle can be extended to what we call the minisum principle: Choose the candidate(s) that beat all remaining candidates in the smallest total number of steps. The minisum principle can be seen also as choosing the most central candidate in a tournament. We show that the minisum principle characterizes the Copeland solution. This chapter is based on Sanver and Selçuk (2010) and its structure is as follows. Section 2 introduces the basic notions. Section 3 states the results.

5.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of candidates with $n \geq 3$. A *tournament* on N is a directed graph (N, T) which is obtained by assigning a direction to each edge of a complete undirected graph on N. We write Θ_N for the set of all tournaments on N. So, for a tournament $(N, T) \in \Theta_N$ and distinct $i, j \in N$, either $(i, j) \in T$ or $(j, i) \in T$ holds. Given a tournament $(N, T) \in \Theta_N$, if $(i, j) \in T$, then we say that candidate $i \in N$ *directly beats* candidate $j \in N$ in (N, T).

A *tournament solution* is a mapping $f : \Theta_N \to 2^N \setminus \{\emptyset\}$. For $(N,T) \in \Theta_N$, let $\delta_T(i) = |\{j \in N \mid (i,j) \in T\}|$ be the *Copeland score* of $i \in N$, i.e., the number of candidates that $i \in N$ directly beats in (N,T). The *Copeland solution* (CS) is the tournament solution defined as $CS(N,T) = \{i \in N \mid \delta_T(i) \geq \delta_T(j) \text{ for all } j \in N\}$ for any tournament $(N,T) \in \Theta_N$. Given a tournament $(N,T) \in \Theta_N$ and $i,j \in N$, $i \neq j$, a *directed path* from i to j in (N,T) is a sequence $(i_1,\ldots,i_k)$ of different candidates with $i_1 = i$ and $i_k = j$ such that $(i_h, i_{h+1}) \in T$ for all $h \in \{1,\ldots,k-1\}$. We refer to k as the *length* of the directed path. Note that in a tournament, there may exist more than one directed path from a candidate to another one. For $i,j \in N$, let $\lambda_T(i,j)$ be the length of the shortest directed path from i to j in (N,T), i.e., the length of any directed path from i to j in (N,T) is at least $\lambda_T(i,j)$. Note that in a tournament there may not exist a directed path from any player to any other player. We set $\lambda_T(i,j) = n$, when (N,T) contains no directed path from $i \in N$ to $j \in N$ and $\lambda_T(i,i) = 0$ for all $i \in N$. If $\lambda_T(i,j) = 1$, this means $i \in N$ directly beats $j \in N$ in (N,T). In a tournament, the *Condorcet winner* is the candidate (if it exists) which directly beats all other candidates.

5.3 Characterization of the Copeland solution

Given a set of candidates N, an ordering is a complete, transitive, and reflexive binary relation $\mathcal{R}$ on N. For an ordering $\mathcal{R}$ on N, let $i \succeq_{\mathcal{R}} j$ if and only if $(i,j) \in \mathcal{R}$. Let Π stand for the set of all orderings on N. A ranking method $f : \Theta_N \to \Pi$ is a function that assigns an ordering $f(N,T)$ to any tournament $(N,T) \in \Theta_N$, which, with slight abuse of notation, we denote by $\succeq_{f(N,T)}$. For a tournament, the Copeland scores of the candidates can be used to rank the candidates. For a tournament $(N,T) \in \Theta_N$, the *Copeland score ranking method* (CR) is the ordering defined as $i \succeq_{CR(N,T)} j$ if $\delta(i) \geq \delta(j)$, for all $i,j \in N$.

In Zermelo (1929), a method that is based on the strengths of the candidates is introduced to rank the candidates in a tournament. For a tournament $(N,T) \in \Theta_N$ and $i \in N$, let $\sigma(i)$ be the strength of player i where $\sigma(i) > 0$ for all $i \in N$ and $\sum_{i \in N} \sigma(i) = 1$. According to Zermelo (1929), if the strengths of all players are known, it is possible to assign a probability to each arc of a tournament where for any arc $(i,j) \in T$, the probability to have that arc, $p(i,j)$, depends on the relative strengths of the candidates involved in this arc, i.e., $p(i,j) = \sigma(i)/(\sigma(i) + \sigma(j))$. With this approach, the probability of a tournament $(N,T) \in \Theta_N$ is $p(N,T) = \prod_{(i,j) \in T} p(i,j)$. According to Zermelo (1929), the strengths of players in a tournament $(N,T) \in \Theta$ should be chosen in a way to maximize $p(N, T)$. Later, Moon (1968) shows that ranking the candidates according to Zermelo's strengths is the same as ranking them according to their Copeland scores. So, the candidates with maximum strength are also the candidates with the maximum Copeland score which means they are members of the Copeland solution.

For tournaments, Rubinstein (1980) provides another characterization of the Copeland score ranking method.

For a tournament $(N, T) \in \Theta_N$ and permutation π on N, let $(N, \pi(T))$ be the tournament such that $(i, j) \in T$ implies $(\pi(i), \pi(j)) \in \pi(T)$.

Definition 5.3.1 A ranking method $f : \Theta_N \to \Pi$ satisfies *anonymity* if for any $(N, T) \in \Theta_N$ and permutation π on N, $i \succeq_{f(N,T)} j$ implies $\pi(i) \succeq_{f(N,\pi(T))} \pi(j)$.

According to Definition 5.3.1, in order to be anonymous a ranking method should not depend on the labels of the candidates.

Definition 5.3.2 A ranking method $f : \Theta_N \to \Pi$ satisfies *positive responsiveness* if for any $(N, T), (N, T') \in \Theta_N$, $i \succeq_{f(N,T)} j$ implies $i \succeq_{f(N,T')} j$ whenever $T' = (T \setminus \{(k, i)\}) \cup \{(i, k)\}$ for some $(k, i) \in T$.

According to positive responsiveness, if a candidate is ranked above another candidate for a tournament, then the same should hold for any another tournament where the above ranked candidate is favored.

Definition 5.3.3 A ranking method $f : \Theta_N \to \Pi$ is *independent of irrelevant candidates* if for any $(N, T), (N, T') \in \Theta_N$ and distinct $i, j, k, l \in N$, $i \succeq_{f(N,T)} j$ implies $i \succeq_{f(N,T')} j$ whenever $T' = (T \setminus \{(k, l)\}) \cup \{(l, k)\}$ and $(k, l) \in T$.

Independence of irrelevant candidates says that the comparison of any two candidates should be independent of the relation of any other two candidates in the tournament.

Theorem 5.3.4 ***(Rubinstein, 1980)*** *The Copeland ranking method is the only ranking method that satisfies anonymity, positive responsiveness, and independence of irrelevant candidates.*

Henriet (1985) extends the Copeland scoring method to a more general case of complete binary relations. In Henriet (1985), a complete binary relation refers to a directed graph in which there exist exactly two arcs, possibly with the same direction, between any pair of candidates. Together with modified versions of anonymity and positive responsiveness of Rubinstein (1980), for complete binary relation case, Henriet (1985) uses independence of cycles for the characterization. For the complete binary relation case, independence of cycles requires no change in the ranking if a cycle is reversed in the complete binary relation.

As a generalization of Rubinstein (1980), van den Brink and Gilles (2003) provides the characterization of the Copeland score ranking method for arbitrary digraphs. Together with a modification of the anonymity and positive responsiveness axioms which are used in Rubinstein (1980), van den Brink and Gilles (2003) uses independence of non-dominated arcs. According to independence of non-dominated arcs, the ordering between two candidates does not change as long as the set of immediate successors stays the same for both of these candidates.

Now we provide a characterization of the Copeland solution that is based on the lengths of the shortest directed paths from a candidate to other candidates. Given a tournament $(N, T) \in \Theta_N$ and $i \in N$, let $SUM_T(i) = \sum_{j \in N} \lambda_T(i, j)$ be the sum of the lengths of the shortest paths from candidate i to all remaining candidates.

Definition 5.3.5 Given a tournament $(N, T) \in \Theta_N$, $i \in N$ is a *minisum candidate* in (N, T) if $SUM_T(i) \leq SUM_T(j)$ for all $j \in N$.

For a tournament $(N, T) \in \theta_N$, $\mu(N, T)$ denotes the set of minisum candidates in (N, T). We provide a new characterization of the Copeland solution in terms of the set of minisum candidates.

Following Miller (1980), given a tournament $(N, T) \in \Theta_N$ and distinct $i, j \in N$, we say that *i covers j* in (N, T) if $(i, j) \in T$ and, moreover, $(j, j') \in T$ implies $(i, j') \in T$. We denote $UC(N, T) = \{i \in N \mid \nexists j \in N \text{ which covers } i \text{ in } (N, T)\}$ for the *uncovered set* of (N, T).

Remark 5.3.6 Given a tournament $(N, T) \in \Theta_N$, for any $i, j, k \in N$, if i covers j in (N, T) and j covers k in (N, T), then i covers k in (N, T).

Remark 5.3.6 says that the covering relation in a tournament is transitive. Together with the fact that N is finite, this implies $UC(N, T) \neq \emptyset$.

When a tournament $(N, T) \in \Theta_N$ does not admit a Condorcet winner, Shepsle and Weingast (1984) shows that for the candidates in the uncovered set of a tournament, the length of the shortest directed paths from this candidate to other candidates is at most 2.

Proposition 5.3.7 ***(Shepsle and Weingast, 1984)*** *For any tournament* $(N, T) \in \Theta_N$*, if* $i \in UC(N, T)$*, then* $\lambda_T(i, j) \in \{1, 2\}$ *holds for all* $j \in N$*,* $j \neq i$*.*

Proof Suppose there exists $i \in UC(N, T)$ and $j \in N$ such that $\lambda_T(i, j) > 2$. Since $\lambda_T(i, j) > 2$, it holds that $(j, i) \in T$ and $(j, i') \in T$ for all $i' \in N$ such that $(i, i') \in T$. This implies that j covers i, which contradicts with $i \in UC(N, T)$. ∎

According to Proposition 5.3.7, the uncovered set of a tournament consists of all candidates that beats every other candidate minimally in at most two steps. This is an extension of the Condorcet principle through the requirement of minimizing the maximum number of steps. We call this the maximin principle. In Shepsle and Weingast (1984) this is called the two step principle. So, for a tournament choosing the uncovered set is equivalent to choosing according to the maximin principle. The literature admits various solutions that refine the uncovered set. Proposition 5.3.8 quotes a result of Miller (1980) showing that the Copeland solution is a refinement of the uncovered set.

Proposition 5.3.8 ***(Miller, 1980)*** *For any tournament* $(N, T) \in \Theta_N$*,* $CS(N, T) \subseteq UC(N, T)$*.*

Proof Suppose there exists $i \in CS(N, T)$ where $i \notin UC(N, T)$. Then there exists $j \in N \setminus \{i\}$ which covers i. This means $(j, i) \in T$ and $(j, i') \in T$ holds for all $i' \in N$ such that $(i, i') \in T$. This implies $\delta_T(j) \geq \delta_T(i) + 1$, which contradicts with $i \in CS(N, T)$. ∎

The following proposition shows that also the set of minisum candidates is a refinement of the uncovered set.

Proposition 5.3.9 *For any tournament* $(N, T) \in \Theta_N$*,* $\mu(N, T) \subseteq UC(N, T)$*.*

Proof Take any $i \in \mu(N, T)$. So, $SUM_T(i) \leq SUM_T(j)$ holds for all $j \in N$. Suppose $i \notin UC(N, T)$. Then there exists $k \in N \setminus \{i\}$ that covers i in (N, T). This means $(k, i) \in T$ and $(i, m) \in T$ implies $(k, m) \in T$ for any $m \in N \setminus \{k, i\}$. By the definition of shortest path, we have $\lambda_T(k, j) \leq \lambda_T(i, j)$ for all $j \in N \setminus \{i, k\}$. Thus, $\sum_{j \in N \setminus \{i,k\}} \lambda_T(k, j) \leq \sum_{j \in N \setminus \{i,k\}} \lambda_T(i, j)$. Moreover, $(k, i) \in T$ implies $\lambda_T(k, i) = 1 < \lambda_T(i, k)$. Hence, $\sum_{j \in N} \lambda_T(k, j) < \sum_{j \in N} \lambda_T(i, j)$, which means $SUM_T(k) < SUM_T(i)$, contradicting with $SUM_T(i) \leq SUM_T(j)$ for all $j \in N$. Thus, there exists no $k \in N$ that covers $i \in N$, hence $i \in UC(N, T)$. ∎

The following theorem shows that for any tournament the set of minisum candidates coincides with the set of Copeland solution.

Theorem 5.3.10 *For any tournament* $(N, T) \in \Theta_N$, $CS(N, T) = \mu(N, T)$

Proof If (N, T) has a Condorcet winner, then the result immediately holds. Suppose (N, T) has no Condorcet winner. To show $CS(N, T) \subseteq \mu(N, T)$, take any $i \in CS(N, T)$. By Proposition 5.3.8, it follows that $i \in UC(N, T)$. Hence, by Proposition 5.3.7, for any $j \in N \setminus \{i\}$ we have $\lambda_T(i, j) = 1$ if $(i, j) \in T$ and $\lambda_T(i, j) = 2$ if $(j, i) \in T$. So, $SUM_T(i) = |\{j \in N \mid (i, j) \in T\}| + 2|\{j \in N \mid (j, i) \in T\}| = \delta_T(i) + 2(n-1-\delta_T(i)) = 2n-2-\delta_T(i)$. Moreover, $SUM_T(j) \geq 2n-2-\delta_T(j)$ for all $j \in N$. Since $i \in CS(N, T)$, we have $\delta_T(i) \geq \delta_T(j)$ for all $j \in N$. Hence, $SUM_T(i) \leq SUM_T(j)$ for all $j \in N$. This implies $i \in \mu(N, T)$. To show $\mu(N, T) \subseteq CS(N, T)$, take any $i \in \mu(N, T)$. From Proposition 5.3.9, it follows that $i \in UC(N, T)$. By Proposition 5.3.7, this implies $SUM_T(i) = 2n - 2 - \delta_T(i)$. Suppose $i \notin CS(N, T)$. Since $CS(N, T) \subseteq UC(N, T)$, this implies that there exists $j \in UC(N, T)$ such that $\delta_T(j) > \delta_t(i)$. Since $SUM_T(k) = 2n-2-\delta_T(k)$ holds for all $k \in UC(N, T)$, this implies $SUM_T(i) > SUM_T(j)$, which contradicts with the fact that $i \in \mu(N, T)$. ∎

Example 5.3.11 Consider a tournament $(N, T) \in \Theta_N$ where $N = \{1, 2, 3, 4, 5, 6\}$ and $T = \{(1,2), (1,3), (1,5), (1,6), (2,3), (2,4), (2,5), (3,4), (3,5), (3,6), (4,1), (5,4), (5,6), (6,2), (6,4)\}$ as depicted in Figure 5.1. Note that (N, T) has no Condorcet winner and candidate 1 directly beats four candidates in (N, T). Hence $\delta_T(1) = 4$. Similarly, we have $\delta_T(2) = 3$, $\delta_T(3) = 3$, $\delta_T(4) = 1$, $\delta_T(5) = 2$, and $\delta_T(6) = 2$. So, we have $CS(N, T) = \{1\}$. Moreover, $UC(N, T) = N$ and $CS(N, T) \subseteq UC(N, T)$. Note that $\lambda_T(1, j) = 1$ for $j = 2, 3, 5, 6$ and $\lambda_T(1, 4) = 2$. Hence, $SUM_T(1) = 6$. Similarly we have $SUM_T(2) = 7$, $SUM_T(3) = 7$, $SUM_T(4) = 9$, $SUM_T(5) = 10$, and $SUM_T(6) = 9$. Since minisum candidates are the ones minimising the total number of steps to reach every other alternative we have $\mu(N, T) = \{1\}$, which is also the Copeland solution.

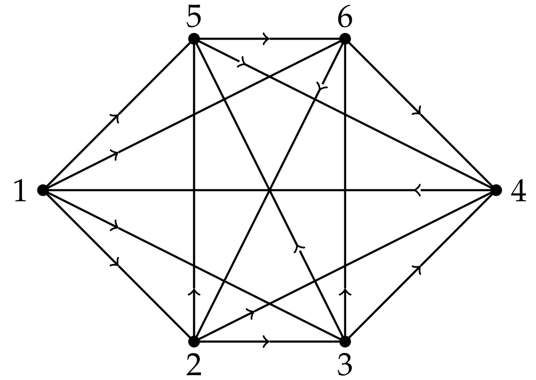


Figure 5.1: The tournament (N, T) in Example 5.3.11.

As the Copeland rule (which is equivalent to the minisum principle) refines the uncovered set (which is equivalent to the maximin principle), tournament solutions exemplify a case where the minisum principle refines the maximin principle — a fact which is not common in the literature. For a detailed discussion of these two principles, see Brams et al. (2007).

CHAPTER 6

SOPHISTICATED PREFERENCE AGGREGATION

6.1 Introduction

With the seminal work of Arrow (1951), Kenneth Arrow is considered to be the founder of modern social choice theory. Given a set of alternatives and individuals with preferences on these alternatives, social choice theory deals with the problem of creating a collective preference on these alternatives. In Arrow (1951), a social welfare function is defined as a functional relation specifying a unique social ordering for any given profile of individual orderings. A preference is a complete and transitive ordering of alternatives and a preference aggregation rule is a function which maps profiles of individual preferences into a social preference. Arrow (1951) provides an impossibility result which states the non-existence of an aggregation rule satisfying a set of desirable properties. The desirable properties in Arrow (1951) are the unrestricted domain property, Pareto optimality, independence of irrelevant alternatives, and non-dictatorship. The unrestricted domain property states that the domain of the aggregation rule is the set of all possible profiles of individual preferences. Pareto optimality requires an alternative to be socially preferred to another one if this preference is shared by all individuals. Independence of irrelevant alternatives states that social comparison of any two alternatives should only depend on the individual preferences on these alternatives. Finally, non-dictatorship requires non-existence of an individual such that whenever this individual prefers one alternative to another one the same holds for the social preference. The Arrow impossibility theorem states the non-existence of a social welfare function that satisfy all these axioms. Following Arrow's impossibility result, in the literature several other related impossibility results emerged. Among those Gibbard (1973) and Satterthwaite (1975) are the most well known which show that every voting procedure, except the dictatoriality, can be manipulated by the voters if they act strategically.

It is also possible to have a more general perspective of the preference aggregation problem by incorporating elements of ambiguity into the preferences. As there are various ways of conceiving ambiguity, there are also various ways of generalizing the aggregation model of Arrow (1951) through ambiguous preferences. In the literature, there are two major strands dealing with the ambiguity of the preferences. One group of research considers a preference as a fuzzy binary relation and the other has a probabilistic conception of preferences, see Fishburn (1998) and Barrett and Salles (2011) for a survey of the related literature. The analysis in this chapter belongs to the latter strand. In this chapter, we introduce the concept of sophisticated preference which is a weighted pairwise comparison of alternatives that allows some kind of a mixed feeling in comparing any given pair of alternatives. To illustrate this point, suppose individuals are asked to reveal their preferences on two cities, Amsterdam and Paris, based on various criteria like public transportation, air quality, business opportunities or education quality. Depending on the number of criteria for which Paris is better than Amsterdam, a sophisticated preference allows an answer of the following type: I like Paris more than Amsterdam in some respect but I like Amsterdam more than Paris in other respects. The answer is also required to quantify the rate at which Paris is better than Amsterdam and vice versa.

In this chapter, we consider situations where individuals have standard preferences and the social preference is allowed to be a sophisticated preference. We define the sophisticated social welfare function as a mapping from the profiles of individual preferences to sophisticated social preferences. We give a full characterization of Pareto optimal and pairwise independent sophisticated social welfare functions in terms of oligarchies induced by some power distribution on the set individuals. For a sophisticated social welfare function, Pareto optimality states that if all individuals prefer an alternative to another one, then according to the sophisticated social preference the first alternative is preferred to the latter one with a rate equal to 1. A sophisticated social welfare function satisfies independence of irrelevant alternatives if the sophisticated social preference between any pair of alternatives depends only on the individual preferences on this pair of alternatives.

The preference aggregation approach of this chapter is closely related to the collective probabilistic judgement model of Barberá and Valenciano (1983) where probabilistic judgement is defined as a function which assigns a number between zero and one to any ordered pair of alternatives. In fact, their collective probabilistic judgement functions are more general than the sophisticated social welfare functions. On the other hand we provide a strong result which does not follow from Barberá and Valenciano (1983). The characterization in this chapter generalizes two major results of the literature. The first one is the impossibility theorem of Arrow (1951). Because when the ranges of Pareto optimal and pairwise independent SSWFs are restricted to non-sophisticated preferences, that are linear orderings, the oligarchies must contain precisely one individual which is a dictator. Moreover, in case the social outcome is restricted to non-sophisticated preferences that are complete and quasitransitive, Pareto optimal and pairwise independent SSWFs are oligarchical in the sense that the oligarchy has full decision power while all proper subsets of the oligarchy have equal decision power. This result is known as the oligarchy theorem of Gibbard (2014).

This chapter is based on Sanver and Selçuk (2009). Section 2 contains the preliminaries and provides the characterization of sophisticated social welfare functions that are independent of irrelevant alternatives and satisfy Pareto optimality.

6.2 Sophisticated social welfare function

We consider a finite set of individuals $N = \{1, \ldots, n\}$ with $n \geq 2$, confronting a finite set of alternatives A with $|A| \geq 3$. A *sophisticated preference* over A is a mapping $\sigma : A \times A \to [0,1]$ such that for all distinct $x, y \in A$ we have $\sigma(x,y) + \sigma(y,x) = 1$ while $\sigma(x,x) = 0$ holds for all $x \in A$. Interpreting $\sigma(x,y)$ as the weight by which x is preferred to y, the former condition imposes a kind of completeness over σ while the latter is an irreflexivity requirement. With this approach, a (nonsophisticted) preference over A is a mapping $\pi : A \times A \to \{0,1\}$ where $\pi(x,y) = 1$ means $x \in A$ is preferred to $y \in A$. A sophisticated preference σ is *transitive* if $\sigma(x,y) = 1 \implies \sigma(x,z) \geq \sigma(y,z)$ for all $x, y, z \in A$. We write Σ for the set of transitive sophisticated preferences and $\Pi \subset \Sigma$ for the set of transitive preferences over A.

A sophisticated social welfare function (SSWF) over A is a mapping γ :

$\Pi^N \to \Sigma$, for which we assume that individual preferences belong to Π and we write $\pi_i \in \Pi$ for the preference of individual $i \in N$ over A. A preference profile over A is an n-tuple $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi^N$ of individual preferences. So, $\gamma(\underline{\pi}) \in \Sigma$ is a sophisticated preference over A which, by a slight abuse of notation, we denote $\gamma_{\underline{\pi}}$. Thus, $\gamma_{\underline{\pi}}(x,y) \in [0,1]$ stands for the weight that the sophisticated social welfare function γ assigns to $(x,y) \in A \times A$ at the preference profile $\underline{\pi} \in \Pi^N$.

Definition 6.2.1 An SSWF $\gamma : \Pi^N \to \Sigma$ is *Pareto optimal* (PO) if for any distinct $x, y \in A$ and $\underline{\pi} \in \Pi^N$ where $\pi_i(x,y) = 1$ for all $i \in N$, it holds that $\gamma_{\underline{\pi}}(x,y) = 1$.

According to PO, if all individuals prefer an alternative to another one, then the sophisticated social welfare function should do the same by assigning a weight equal to 1.

Definition 6.2.2 An SSWF $\gamma : \Pi^N \to \Sigma$ is *independent of irrelevant alternatives* (IIA) if for any distinct $x, y \in A$ and $\underline{\pi}, \underline{\pi}' \in \Pi^N$ with $\pi_i(x,y) = \pi'_i(x,y)$ for all $i \in N$, it holds that $\gamma_{\underline{\pi}}(x,y) = \gamma_{\underline{\pi}'}(x,y)$.

According to IIA, a sophisticated social preference on a pair of alternatives should only depend on the individual preferences on this pair.

Any SSWF satisfying IIA can be expressed in terms of pairwise SSWFs. To see this, take any distinct $x, y \in A$ and let Π^{xy} be the set of preferences over x, y and similarly let Σ^{xy} be the set of sophisticated preferences over x, y. Given any $\underline{\pi} \in \Pi^N$, let $\underline{\pi}^{xy} \in (\Pi^{xy})^N$ be the preference profile restricted to any distinct $x, y \in A$. A pairwise SSWF defined over distinct $x, y \in A$ is a mapping $f^{xy} : (\Pi^{xy})^N \to \Sigma^{xy}$. For any $\underline{r} \in (\Pi^{xy})^N$, $f^{xy}(\underline{r}) \in \Sigma^{xy}$ is a sophisticated preference over x, y which, by a slight abuse of notation, we denote $f^{xy}_{\underline{r}}$. Thus, every SSWF $\gamma : \Pi^N \to \Sigma$ satisfying IIA can equivalently be expressed in terms of a family of pairwise SSWFs $\{f^{xy}\}$ indexed over all distinct $x, y \in A$ such that for any $\underline{\pi} \in \Pi^N$ and distinct $x, y \in A$ it holds that $f^{xy}_{\underline{\pi}^{xy}}(x,y) = \gamma_{\underline{\pi}}(x,y)$.

We first show that given an SSWF $\gamma : \Pi^N \to \Sigma$ that can be represented by a family of pairwise SSWFs, the pairwise functions are the same for any pair of alternatives.

Given $x, y, z, t \in A$ satisfying $x \neq y$ and $z \neq t$, for any $f^{xy} : (\Pi^{xy})^N \to \Sigma^{xy}$ and $f^{zt} : (\Pi^{zt})^N \to \Sigma^{zt}$, let $f^{xy} \approx f^{zt}$ if it holds that $f^{xy}_{\underline{r}}(x,y) = f^{zt}_{\underline{s}}(z,t)$ for every $\underline{r} \in (\Pi^{xy})^N$ and $\underline{s} \in (\Pi^{zt})^N$ such that $r_i(x,y) = s_i(z,t)$ for all $i \in N$.

Proposition 6.2.3 *Take any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$ which is expressed by a family of pairwise SSWFs $\{f^{xy}\}$ over all distinct $x, y \in A$. For every $a, b, c, d \in A$ satisfying $a \neq b$ and $c \neq d$, it holds that $f^{ab} \approx f^{cd}$.*

Proof To establish $f^{ab} \approx f^{ac}$ when $|\{a, b, c, d\}| = 3$, take any distinct $a, b, c \in A$ and any $\underline{r} \in (\Pi^{ab})^N$ and $\underline{s} \in (\Pi^{ac})^N$ such that $r_i(a, b) = s_i(a, c)$ for all $i \in N$. Let $S \subseteq N$ such that $r_i(a, b) = s_i(a, c) = 1$ for all $i \in S$ and $r_i(a, b) = s_i(a, c) = 0$ for all $i \in N \setminus S$. Suppose for a contradiction $f^{ab}_{\underline{r}}(a, b) > f^{ac}_{\underline{s}}(a, c)$. Consider some $\underline{\pi} \in \Pi^N$ such that $\pi_i(a, b) = \pi_i(b, c) = \pi_i(a, c) = 1$ for all $i \in S$ and $\pi_i(b, c) = \pi_i(c, a) = \pi_i(b, a) = 1$ for all $i \in N \setminus S$. Since $\pi_i(b, c) = 1$ for all $i \in N$, by PO we have $\gamma_{\underline{\pi}}(b, c) = 1$. Since $\gamma_{\underline{\pi}}$ is transitive, we obtain $\gamma_{\underline{\pi}}(b, a) \geq \gamma_{\underline{\pi}}(c, a)$. This means $\gamma_{\underline{\pi}}(a, b) \leq \gamma_{\underline{\pi}}(a, c)$. Since $\underline{\pi}^{ab} = \underline{r}$ and $\underline{\pi}^{ac} = \underline{s}$ we have $f^{ab}_{\underline{r}}(a, b) \leq f^{ac}_{\underline{s}}(a, c)$ which is the desired contradiction.

To establish $f^{ab} \approx f^{cb}$ when $|\{a, b, c, d\}| = 3$, take any distinct $a, b, c \in A$ and any $\underline{r} \in (\Pi^{ab})^N$ and $\underline{s} \in (\Pi^{cb})^N$ such that $r_i(a, b) = s_i(c, b)$ for all $i \in N$. Let $S \subseteq N$ such that $r_i(a, b) = s_i(c, b) = 1$ for all $i \in S$ and $r_i(a, b) = s_i(c, b) = 0$ for all $i \in N \setminus S$. Suppose for a contradiction $f^{cb}_{\underline{r}}(c, b) > f^{ab}_{\underline{s}}(a, b)$. Consider some $\underline{\pi} \in \Pi^N$ such that $\pi_i(a, c) = \pi_i(c, b) = \pi_i(a, b) = 1$ for all $i \in S$ and $\pi_i(b, a) = \pi_i(a, c) = \pi_i(b, c) = 1$ for all $i \in N \setminus S$. Since $\pi_i(a, c) = 1$ for all $i \in N$, by PO we have $\gamma_{\underline{\pi}}(a, c) = 1$. Since $\gamma_{\underline{\pi}}$ is transitive, we obtain $\gamma_{\underline{\pi}}(a, b) \geq \gamma_{\underline{\pi}}(c, b)$. Since $\underline{\pi}^{ab} = \underline{r}$ and $\underline{\pi}^{cb} = \underline{s}$ we have $f^{cb}_{\underline{r}}(c, b) \leq f^{ab}_{\underline{s}}(a, b)$ which is the desired contradiction.

We have shown $f^{ab} \approx f^{ac}$ and $f^{ac} \approx f^{bc}$ and $f^{bc} \approx f^{ba}$. Hence $f^{ab} \approx f^{ba}$ when $|\{a, b, c, d\}| = 2$.

When a, b, c, d are all distinct alternatives in A, we have already shown that $f^{ab} \approx f^{ad}$ and $f^{ad} \approx f^{cd}$, which implies $f^{ab} \approx f^{cd}$. ∎

According to Proposition 6.2.3, if an SSWF satisfies PO and IIA then any pair of alternatives is compared with the same principles. Note that $f^{xy} \approx f^{yx}$ implies $f^{xy}_{\underline{r}}(x, y) = f^{xy}_{\underline{s}}(y, x)$ for all $\underline{r}, \underline{s} \in (\Pi^{xy})^N$ with $r_i(x, y) = s_i(y, x)$ for all $i \in N$. Hence the pairwise SSWFs are also neutral. So, by Proposition 6.2.3, any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$ can be expressed with a single neutral pairwise SSWF $f : (\Pi^{xy})^N \to \Sigma^{xy}$ for any given $x, y \in A$. Hence, for a PO and IIA SSWF, there is no need to refer to a specific pair when we talk about a pairwise SSWF. We now show that the pairwise SSWF satisfies monotonicity.

Definition 6.2.4 A pairwise SSWF $f : (\Pi^{xy})^N \to \Sigma^{xy}$ is *monotonic* if for any $\underline{r}, \underline{r}' \in (\Pi^{xy})^N$ with $r_i(x, y) \geq r'_i(x, y)$ for all $i \in N$, it holds that $f_{\underline{r}}(x, y) \geq f_{\underline{r}'}(x, y)$.

Proposition 6.2.5 *Let $\gamma : \Pi^N \to \Sigma$ be a PO and IIA SSWF. If $f : (\Pi^{xy})^N \to \Sigma^{xy}$ is the pairwise SSWF that expresses γ, then f is monotonic.*

Proof Suppose f fails monotonicity. So there exists $x, y \in A$ and $\underline{r}, \underline{r}' \in (\Pi^{xy})^N$ with $r_i(x,y) \geq r'_i(x,y)$ for all $i \in N$ while $f_{\underline{r}}(x,y) < f_{\underline{r}'}(x,y)$. Let $K = \{i \in N \mid r_i(x,y) = 1\}$ and $L = \{i \in N \mid r'_i(x,y) = 1\}$. Note that $L \subseteq K$. Take any distinct $a, b, c \in A$ and any $\underline{\pi} \in \Pi^N$ such that $\pi_i(a,b) = \pi_i(b,c) = \pi_i(a,c) = 1$ for all $i \in L$, $\pi_i(a,c) = \pi_i(c,b) = \pi_i(a,b) = 1$ for all $i \in K \setminus L$, and $\pi_i(c,a) = \pi_i(a,b) = \pi_i(c,b) = 1$ for all $i \in N \setminus K$. By PO, we have $\gamma_{\underline{\pi}}(a,b) = 1$. Since f is the pairwise SSWF that expresses γ, by the choice of $\underline{\pi}$, we have $\gamma_{\underline{\pi}}(a,c) = f_{\underline{r}}(x,y)$ and $\gamma_{\underline{\pi}}(b,c) = f_{\underline{r}'}(x,y)$. Thus, $\gamma_{\underline{\pi}}(a,c) < \gamma_{\underline{\pi}}(b,c)$, violating the transitivity of $\gamma_{\underline{\pi}}$. ∎

According to Proposition 6.2.5, an SSWF satisfying PO and IIA increases weight assigned to a pair $(x,y) \in A \times A$ as the number of individuals who prefers x to y increases.

Definition 6.2.6 For an SSWF $\gamma : \Pi^N \to \Sigma$, a coalition $S \subseteq N$ is *decisive* for $x \in A$ over $y \in A \setminus \{x\}$ if for some $\underline{\pi} \in \Pi^N$ such that $\pi_i(x,y) = 1$ for all $i \in S$ and $\pi_i\ (x,y) = 0$ for all $i \in N \setminus S$, it holds that $\gamma_{\underline{\pi}}(x,y) > 0$.

Note that, if an SSWF $\gamma : \Pi^N \to \Sigma$ satisfies IIA and a coalition is decisive for $x \in A$ over $y \in A \setminus \{x\}$ for some $\underline{\pi} \in \Pi^N$, then this coalition is decisive for x over y for any preference profile $\underline{\pi}' \in \Pi^N$.

Lemma 6.2.7 *For any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, if $S \subseteq N$ is decisive for some $a \in A$ over some $b \in A \setminus \{a\}$, then for any distinct $x, y \in A$, it holds that S is decisive for x over y.*

Proof Let $S \subseteq N$ be decisive for some $a \in A$ over some $b \in A \setminus \{a\}$.
Claim 1: S is decisive for a over x for any $x \in A \setminus \{a,b\}$. Take any $x \in A \setminus \{a,b\}$. Consider a preference profile $\underline{\pi} \in \Pi^N$ where $\pi_i(a,b) = \pi_i(b,x) = \pi_i(a,x) = 1$ for all $i \in S$ and $\pi_i(b,x) = \pi_i(x,a) = \pi_i(b,a) = 1$ for all $i \in N \setminus S$. Note that $\pi_i(a,b) = 1$ for all $i \in S$ and $\pi_i(a,b) = 0$ for all $i \in N \setminus S$. Since S is decisive for a over b, we have $\gamma_{\underline{\pi}}(a,b) > 0$. By PO, $\pi_i(b,x) = 1$ for all $i \in N$ implies $\gamma_{\underline{\pi}}(b,x) = 1$. By the transitivity of $\gamma_{\underline{\pi}}$, this implies $\gamma_{\underline{\pi}}(b,a) \geq \gamma_{\underline{\pi}}(x,a)$, which means $\gamma_{\underline{\pi}}(a,x) \geq \gamma_{\underline{\pi}}(a,b) > 0$. Since $\pi_i(a,x) = 1$ for all $i \in S$ and $\pi_i(a,x) = 0$ for all $i \in N \setminus S$, this means S is decisive for a over x.
Claim 2: S is decisive for x over b for any $x \in A \setminus \{a,b\}$. Take any $x \in A \setminus \{a,b\}$ and consider a preference profile $\underline{\pi} \in \Pi^N$ where $\pi_i(x,a) = \pi_i(a,b) =$

$\pi_i(x,b) = 1$ for all $i \in S$ and $\pi_i(b,x) = \pi_i(x,a) = \pi_i(b,a) = 1$ for all $i \in N \setminus S$. Since S is decisive for a over b, then $\gamma_{\underline{\pi}}(a,b) > 0$. By PO, $\pi_i(x,a) = 1$ for all $i \in N$ implies $\gamma_{\underline{\pi}}(x,a) = 1$. Transitivity of $\gamma_{\underline{\pi}}$ implies $\gamma_{\underline{\pi}}(x,b) \geq \gamma_{\underline{\pi}}(a,b) > 0$. So, S is decisive for x over b as well.

Now take any distinct $x, y \in A$ and consider the following three exhaustive cases.
CASE 1: $x \in A \setminus \{b\}$. By claim 2 S is decisive for x over b and by Claim 1 S is decisive for x over y.
CASE 2: $x = b$ and $y \in A \setminus \{a\}$. By Claim 1 S is decisive for a over y and by Claim 2 S is decisive for b over y for $y \neq b$.
CASE 3: $x = b$ and $y = a$. Take some $z \in A \setminus \{a,b\}$. By Claim 1 S is decisive for a over z, by Claim 2 S is decisive for b over z, and by Claim 1 S is decisive for b over a. $\blacksquare$

According to Lemma 6.2.7, given an SSWF for which a coalition is decisive for a pair of alternatives, by PO and IIA this coalition is decisive for any pair of alternatives. Hence, there is no need to refer to a pair of alternatives when we say that a coalition is decisive.

Lemma 6.2.8 *For any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, if disjoint $S, L \subseteq N$ are both not decisive, then $S \cup L$ is not decisive either.*

Proof Take any disjoint $S, L \subseteq N$ which are both not decisive. Consider distinct $x, y, z \in A$ and a profile $\underline{\pi} \in \Pi^N$ where $\pi_i(x,z) = \pi_i(z,y) = \pi_i(x,y) = 1$ for all $i \in S$, $\pi_i(z,y) = \pi_i(y,x) = \pi_i(z,x) = 1$ for all $i \in L$, and $\pi_i(y,x) = \pi_i(x,z) = \pi_i(y,z) = 1$ for all $i \in N \setminus (S \cup L)$. Since S and L are not decisive, we have $\gamma_{\underline{\pi}}(x,y) = 0$ and $\gamma_{\underline{\pi}}(z,x) = 0$. By transitivity of $\gamma_{\underline{\pi}}$, this implies $\gamma_{\underline{\pi}}(z,y) = 0$, showing that $S \cup L$ is not decisive. $\blacksquare$

According to Lemma 6.2.8, the union of any two coalitions is not decisive if these coalitions are not decisive by their own.

Lemma 6.2.9 *For any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, if $S \subseteq N$ is decisive and $L \subset S$, then L or $S \setminus L$ is decisive.*

Proof Suppose neither L nor $S \setminus L$ is decisive. By Lemma 6.2.8, $L \cup (S \setminus L) = S$ is not decisive either, which contradicts that S is decisive. $\blacksquare$

According to Lemma 6.2.9, if a decisive coalition is partitioned into two sets, at least one of these sets is a decisive coalition. So, every decisive coalition contains a decisive subset. Since N is finite, from Lemma 6.2.9 it can be concluded that every decisive coalition contains an individual who is decisive as a singleton coalition.

Lemma 6.2.10 *For any PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, if $S \subseteq N$ is decisive, then any $L \supseteq S$ is also decisive.*

Proof Consider any distinct $x, y, z \in A$ and take a profile $\underline{\pi} \in \Pi^N$ where $\pi_i(z,x) = \pi_i(x,y) = \pi_i(z,y) = 1$ for all $i \in S$, $\pi_i(z,y) = \pi_i(y,x) = \pi_i(z,x) = 1$ for all $i \in L \setminus S$, and $\pi_i(y,z) = \pi_i(z,x) = \pi_i(y,x) = 1$ for all $i \in N \setminus L$. Since S is decisive, we have $\gamma_{\underline{\pi}}(x,y) > 0$. By PO, $\pi_i(z,x) = 1$ for all $i \in N$ implies $\gamma_{\underline{\pi}}(z,x) = 1$. Transitivity of $\gamma_{\underline{\pi}}$ implies $\gamma_{\underline{\pi}}(z,y) \geq \gamma_{\underline{\pi}}(x,y)$. Since $\gamma_{\underline{\pi}}(x,y) > 0$, this implies $\gamma_{\underline{\pi}}(z,y) > 0$, showing that L is decisive. ∎

We now show that PO and IIA SSWFs fall into a class that is called oligarchical SSWFs.

Definition 6.2.11 An SSWF $\gamma : \Pi^N \to \Sigma$ is *oligarchical* if there exists $O \in 2^N \setminus \{\emptyset\}$, which is the oligarchy, such that for every distinct $x, y \in A$ and $\underline{\pi} \in \Pi^N$ it holds that $\gamma_{\underline{\pi}}(x,y) > 0$ if and only if $\pi_i(x,y) = 1$ for some $i \in O$.

An SSWF is oligarchical if there exists a nonempty set of individuals, which is the oligarchy, such that a social preference assigns a positive weight to a pair $(x,y) \in A \times A$ if and only if there exists at least one member of the oligarchy who prefers x to y.

Given an SSWF $\gamma : \Pi^N \to \Sigma$, let Δ^γ be the set of decisive coalitions.

Lemma 6.2.12 *For a PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, there exists $O \in 2^N \setminus \{\emptyset\}$ such that for any $S \in 2^N$ it holds that $S \in \Delta^\gamma$ if and only if $S \cap O \neq \emptyset$.*

Proof By PO, we have $N \in \Delta^\gamma$. Applying Lemma 6.2.9 successively and by the finiteness of N, the set $\{i \in N \mid \{i\} \in \Delta^\gamma\}$ is non-empty. We now claim that $O = \{i \in N \mid \{i\} \in \Delta^\gamma\}$ and $S \in \Delta^\gamma$ if and only if $S \cap O \neq \emptyset$. Take any $S \subset N$. If $S \cap O \neq \emptyset$, then $i \in S$ for some $\{i\} \in \Delta^\gamma$. So, by Lemma 6.2.10, $S \in \Delta^\gamma$ as well. If $S \in \Delta^\gamma$, then again by applying Lemma 6.2.9 successively and by the finiteness of S, there exists $i \in S$ such that $\{i\} \in \Delta^\gamma$, hence $i \in O$, establishing that $S \cap O \neq \emptyset$. ∎

According to Lemma 6.2.12, every decisive coalition contains at least one individual who is decisive by himself. As the following theorem shows, if an SSWF satisfies PO and IIA, then the set of individuals who are as a singleton coalition decisive is the oligarchy which makes the SSWF oligarchical.

Theorem 6.2.13 *If an SSWF $\gamma : \Pi^N \to \Sigma$ satisfies PO and IIA, then γ is oligarchical with oligarchy the set $O = \{i \in N \mid \{i\} \in \Delta^\gamma\}$.*

Proof From the proof of Lemma 6.2.12, it follows that the set O is nonempty. Consider any profile $\underline{\pi} \in \Pi^N$ and distinct $x, y \in A$ where $\gamma_{\underline{\pi}}(x,y) > 0$. Since γ satisfies PO, the set $K = \{i \in N \mid \pi_i(x,y) = 1\}$ is nonempty. Suppose $\pi_i(x,y) = 0$ for all $i \in O$. Then K is decisive and $K \cap O = \emptyset$, which contradicts with Lemma 6.2.12. Now consider any profile $\underline{\pi'} \in \Pi^N$ and distinct $x, y \in A$ where $\pi'_j(x,y) = 1$ for some $j \in O$. Let $S = \{i \in N \mid \pi'_i(x,y) = 1\}$. Since $j \in S$, Lemma 6.2.12 implies S is decisive and hence $\gamma_{\underline{\pi'}}(x,y) > 0$. ∎

According to Theorem 6.2.13, for a PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$, in order to assign a positive weight to a pair $(x,y) \in A \times A$, there must exist an individual, who is as a singleton coalition decisive, and prefers x to y. In case none of such individuals prefers x to y, then PO and IIA SSWF should assign zero weight to the pair $(x,y) \in A \times A$. Hence, the preferences of the oligarchy members determine whether or not a positive weight will be assigned to a pair of alternatives. Remark that the converse statement of Theorem 6.2.13 does not hold. Although an oligarchical SSWF is PO, it does not need to satisfy IIA. To transform Theorem 6.2.13 into a full characterization, we need to know more about IIA and oligarchical SSWFs. So we proceed by showing that under IIA and oligarchical SSWFs, the social outcome depends only on the preferences of the oligarchy members which is stated by the next proposition.

Proposition 6.2.14 *Take any oligarchical and IIA SSWF $\gamma : \Pi^N \to \Sigma$ which is expressed by the pairwise SSWF $f : \Pi^{xy} \to \Sigma^{xy}$. Let $O \subseteq N$ be the oligarchy induced by f. Given any $\underline{r}, \underline{s} \in (\Pi^{xy})^N$, if $r_i(x,y) = s_i(x,y)$ holds for all $i \in O$, then $f_{\underline{r}}(x,y) = f_{\underline{s}}(x,y)$.*

Proof Consider any $\underline{r}, \underline{s} \in \Pi^{xy}$ where $r_i(x,y) = s_i(x,y)$ for all $i \in O$. Without loss of generality let $O^1_{\underline{r}} = \{i \in N \mid r_i(x,y) = 1\}$, $O^0_{\underline{r}} = \{i \in N \mid r_i(x,y) = 0\}$, $K^1_{\underline{r}} = \{i \in N \setminus O \mid r_i(x,y) = 1\}$, and $K^0_{\underline{r}} = \{i \in N \setminus O \mid r_i(x,y) = 0\}$. Take any distinct $a, b, c \in A$ and consider a profile $\underline{\pi} \in \Pi^N$ where $\pi_i(a,c) = \pi_i(c,b) = \pi_i(a,b) = 1$ for all $i \in O^1_{\underline{r}}$, $\pi_i(a,b) = \pi_i(b,c) = \pi_i(a,c) = 1$ for all $i \in K^1_{\underline{r}}$, and $\pi_i(c,b) = \pi_i(b,a) = \pi_i(c,a) = 1$ for all $i \in O^0_{\underline{r}} \cup K^0_{\underline{r}}$. Note that $\pi_i(a,b) = \pi_i(a,c) = r_i(x,y)$ for all $i \in N$, which implies $\gamma_{\underline{\pi}}(a,b) = \gamma_{\underline{\pi}}(a,c)$. Now consider a profile $\underline{\pi'} \in \Pi^N$ where $\pi'_i(a,b) = \pi_i(a,b)$, $\pi'_i(b,c) = \pi_i(b,c)$, $\pi'_i(a,c) = \pi_i(a,c)$ for all $i \in O$. So for $\underline{\pi'}$ every oligarchy member has the same preferences as in $\underline{\pi}$. Moreover, let $\pi'_i(a,c) = \pi_i(a,c)$ for all $i \in N \setminus O$.

Since $\pi'_i(a,c) = \pi_i(a,c)$ holds for all $i \in N$, we have $\gamma_{\underline{\pi'}}(a,c) = \gamma_{\underline{\pi}}(a,c)$. Since $\pi'_i(c,b) = 1$ for all $i \in O$, we have $\gamma_{\underline{\pi'}}(c,b) = 1$. By transitivity this implies $\gamma_{\underline{\pi'}}(c,a) = \gamma_{\underline{\pi}}(c,a) \geq \gamma_{\underline{\pi'}}(b,a)$. Now consider another preference profile $\underline{\pi''} \in \Pi^N$ where $\pi''_i(a,b) = \pi'_i(a,b)$, $\pi''_i(b,c) = \pi'_i(b,c)$, $\pi''_i(a,c) = \pi'_i(a,c)$ for all $i \in N \setminus O$. Moreover, let $\pi''_i(b,c) = \pi'_i(c,b)$, $\pi''_i(a,b) = \pi'_i(a,b)$, and $\pi''_i(a,c) = \pi'_i(a,c)$ for all $i \in O$. Note that $\pi''_i(a,c) = \pi'_i(a,c)$ holds for all $i \in N$. This implies $\gamma_{\pi''}(a,c) = \gamma_{\pi'}(a,c)$. Since $\pi''_i(b,c) = 1$ for all $i \in O$, we have $\gamma_{\pi''}(b,c) = 1$, by transitivity this implies $\gamma_{\pi''}(b,a) \geq \gamma_{\pi''}(c,a) = \gamma_{\pi'}(c,a)$. Noting $\gamma_{\underline{\pi''}}(b,a) = \gamma_{\underline{\pi'}}(b,a)$, we establish $\gamma_{\underline{\pi'}}(b,a) = \gamma_{\underline{\pi}}(c,a) = \gamma_{\underline{\pi}}(b,a)$, which completes the proof. ∎

A *power distribution* on the set of individuals is a mapping $\omega : 2^N \to [0,1]$ such that $\omega(K) + \omega(N \setminus K) = 1$ for all $K \in 2^N$. We consider *monotonic power distributions* which satisfy $\omega(K) \leq \omega(L)$ for all $K, L \in 2^N$ with $K \subseteq L$ while $\omega(N) = 1$. A monotonic power distribution $\omega : 2^N \to [0,1]$ is called *oligarchical* if and only if $\omega(L) = 0$ implies $\omega(K \cup L) = \omega(K)$ for all $K, L \in 2^N$. Remark that when ω is oligarchical, the set $\{i \in N \mid \omega(\{i\}) > 0\}$ is nonempty. Moreover, $\omega(K) = 0$ for all $K \in 2^N$ with $K \cap \{i \in N \mid \omega(\{i\}) > 0\} = \emptyset$.

Lemma 6.2.15 *Any oligarchical power distribution $\omega : 2^N \to [0,1]$ induces a PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$ which is defined as $\gamma_{\underline{\pi}}(x,y) = \omega(\{i \in N \mid \pi_i(x,y) = 1\})$ for all $\underline{\pi} \in \Pi^N$ and $x, y \in A$. Moreover γ is oligarchical where $O = \{i \in N \mid \omega(\{i\}) > 0\}$ is the oligarchy.*

Proof Take any $\underline{\pi} \in \Pi^N$ and any $x, y \in A$. If x and y are not distinct, then $\gamma_{\underline{\pi}}(x,x) = 0$ holds by the irreflexivity of individual preferences and the fact that $\omega(\emptyset) = 0$. If x, y are distinct, then the definition of a power distribution implies $\gamma_{\underline{\pi}}(x,y) \in [0,1]$ and $\gamma_{\underline{\pi}}(x,y) + \gamma_{\underline{\pi}}(y,x) = 1$. So $\gamma_{\underline{\pi}}$ is a sophisticated preference. To see the transitivity of $\gamma_{\underline{\pi}}$, take any distinct $x, y, z \in A$ with $\gamma_{\underline{\pi}}(x,y) = 1$. Let $K_1 = \{i \in N \mid \pi_i(x,y) = \pi_i(y,z) = \pi_i(x,z) = 1\}$, $K_2 = \{i \in N \mid \pi_i(x,z) = \pi_i(z,y) = \pi_i(x,y) = 1\}$, $K_3 = \{i \in N \mid \pi_i(z,x) = \pi(x,y) = \pi(z,y) = 1\}$, $L_1 = \{i \in N \mid \pi_i(y,x) = \pi_i(x,z) = \pi_i(y,z) = 1\}$, $L_2 = \{i \in N \mid \pi_i(y,z) = \pi_i(z,x) = \pi_i(y,x) = 1\}$, and $L_3 = \{i \in N \mid \pi_i(z,y) = \pi_i(y,x) = \pi_i(z,x) = 1\}$. Note that $\{K_1, K_2, K_3, L_1, L_2, L_3\}$ is a partition of N. Moreover, the way ω induces γ implies $\gamma_{\underline{\pi}}(x,y) = \omega(K_1 \cup K_2 \cup K_3) = 1$, $\gamma_{\underline{\pi}}(y,z) = \omega(K_1 \cup L_1 \cup L_2)$ and $\gamma_{\underline{\pi}}(x,z) = \omega(K_1 \cup K_2 \cup L_1)$. As $\omega(K_1 \cup K_2 \cup K_3) = 1$, $\omega(L_1 \cup L_2 \cup L_3) = 0$ and by the monotonicity of ω, we have $\omega(L) = 0$ for all $L \subseteq (L_1 \cup L_2 \cup L_3)$. As ω is oligarchical, $\gamma_{\underline{\pi}}(y,z) = \omega(K_1 \cup L_1 \cup L_2) = \omega(K_1)$ and $\gamma_{\underline{\pi}}(x,z) = \omega(K_1 \cup K_2 \cup L_1) = \omega(K_1 \cup K2)$ and the monotonicity of ω

implies $\gamma_{\underline{\pi}}(x,z) \geq \gamma_{\underline{\pi}}(y,z)$, showing the transitivity of $\gamma_{\underline{\pi}}$. Thus, γ is an SSWF. It is trivial that γ is oligarchical and satisfies both PO and IIA. ∎

So every oligarchical power distribution $\omega : 2^N \to [0,1]$ generates a PO and IIA SSWF $\gamma : \Pi^N \to \Sigma$ where at each $\underline{\pi} \in \Pi^N$, the weight at which $x \in A$ is socially preferred to $y \in A$ equals to the power of the coalition of the individuals who prefer x to y at the preference profile $\underline{\pi}$. We refer to γ as the ω-oligarchical SSWF with $O = \{i \in N \mid \omega(\{i\}) > 0\}$ being the corresponding oligarchy.

Theorem 6.2.16 *An SSWF $\gamma : \Pi^N \to \Sigma$ is PO and IIA if and only if γ is ω-oligarchical for some oligarchical power distribution ω.*

Proof The if part follows from Lemma 6.2.15. To see the only if part, recall that by Proposition 6.2.3, γ can be expressed in terms of a single neutral pairwise SSWF $f : \Pi^{xy} \to \Sigma^{xy}$. On the other hand, f can be expressed in terms of a value function $v : 2^N \to [0,1]$ which is defined for each $K \in 2^N$ as $v(K) = f_{\underline{r}}(x,y)$ where $x, y \in A$ is an arbitrarily chosen distinct pair while $\underline{r} \in \Pi^{xy}$ is such that $r_i(x,y) = 1$ for all $i \in K$ and $r_i(x,y) = 0$ for all $i \in N \setminus K$. The fact that $f_{\underline{r}}(x,y) + f_{\underline{r}}(y,x) = 1$ for any distinct $x, y \in A$ and any $\underline{r} \in \Pi^{xy}$ results in v being a power distribution. Moreover, v is monotonic by Proposition 6.2.5 and oligarchical by Proposition 6.2.14. As v and f uniquely determine each other, v is the oligarchical power distribution that induces γ. ∎

According to Theorem 6.2.16, a sophisticated social welfare function is PO and IIA if and only if it is generated by an oligarchical power distribution on the set of individuals. This is a fairly large class ranging from dictatoriality, where power is concentrated on a single individual, to anonymous power distribution, where every subset of individuals with the same cardinality has the same power.

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