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Cost allocation rules for elastic single-attribute situations

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Abstract

Many cooperative games, especially ones stemming from resource pooling in queueing or inventory systems, are based on situations in which each player is associated with a single attribute (a real number representing, say, a demand) and in which the cost to optimally serve any sum of attributes is described by an elastic function (which means that the per-demand cost is non-increasing in the total demand served). For this class of situations, we introduce and analyze several cost allocation rules: the proportional rule, the serial cost sharing rule, the benefit-proportional rule, and various Shapley-esque rules. We study their appeal with regard to fairness criteria such as coalitional rationality, benefit ordering, and relaxations thereof. After showing the impossibility of combining coalitional rationality and benefit ordering, we show for each of the cost allocation rules which fairness criteria it satisfies.

Keywords: Games/group decisions: cooperative, mathematics: convexity, inventory/production: applications, queues: applications.

1 Introduction

Consider a set N of maintenance firms, each responsible for maintaining trains and railway infrastructure in their own geographical region. Each firm $i \in N$ faces a demand rate for

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resources (e.g., spare parts) of λ_i per year. Currently, the firms have separate resources, but they could also set up a pool with shared resources to jointly serve the sum of their demand rates. Either way, the total expected yearly operating costs for a pool that optimally serves demand rate Λ in total is described by $K(\Lambda)$. Of course, the precise shape of this function K will depend on whether or not demand is stochastic, whether or not there are economies of scale in the resupply process, and what happens in case of a shortage. What's important is that K is *elastic*, i.e., $K(\Lambda)/\Lambda$ is non-increasing in Λ. Accordingly, there are *economies of scale* that can motivate the firms to collaborate.

The situation described above is an example of what we will call an *elastic single-attribute situation*. There are two key features. First, each player is associated with a single attribute: a real number, which might represent, e.g., a demand rate. Second, the costs for any group of players are described by a non-decreasing, elastic function that only depends on the players' attributes through their sum.

In recent years, several papers have appeared on resource pooling in canonical inventory and queueing models. As pointed out by Özen et al. (2011), many of them fit the framework of an elastic single-attribute situation. We mention EOQ inventory situations (Meca et al., 2004), $(S-1, S)$ inventory situations (Karsten and Basten, 2014), $M/M/s$ queueing situations (Guo et al., 2013), and $M/G/s/s$ queueing situations (Özen et al., 2011; Karsten et al., 2014). They are described in more detail in Section 2.3. These models differ in the structure of the underlying cost function, but all belong to the class of elastic single-attribute situations.

Before a resource pooling collaboration can take place in practice, there are several issues to agree upon. A key problem is that of cost allocation: If all firms join forces, then how can the total expected yearly costs $K(\sum_{i \in N} \lambda_i)$ be divided amongst the participants in a fair way? All of the above-mentioned papers provided insight into this question by formulating and analyzing a cooperative game, and we will adhere to this approach in the present paper as well. That is, we assume that any coalition $M \subseteq N$ would face costs $K(\sum_{i \in M} \lambda_i)$ and study cost allocation rules within the context of the resulting game.

In the aforementioned papers, much attention has been paid to the proportional rule, which assigns costs in proportion to the players' attributes. (Although most of these papers considered this rule on the limited domain of their specific situations, we will study the obvious extension to the domain of elastic single-attribute situations.) The proportional rule is easy to understand and easy to compute. Moreover, due to elasticity of K, it always results in an allocation in the core of the associated cooperative game (as proven by Özen et al., 2011). This is a nice property that ensures that no subset of players has an incentive to split off and act separately. However, we show that under the proportional rule, a player with a lower attribute may reap more benefits (defined as the costs a player would incur when acting alone minus the cost assignment when collaborating with everyone) than a player with a larger attribute. We view this as a downside because larger players, who contribute more, may feel like they don't receive a fair share of the benefits.

With this downside in mind, we set out to find an alternative allocation rule that always accomplishes core allocations *and* always gives larger benefits to players with larger attributes. This, however, turned out to be impossible: there is no rule on the class of elastic single-attribute situations that satisfies both requirements simultaneously.

Not to be oudone, we relax our fairness properties to weaker versions, and show that those are compatible. Subsequently, we introduce four new allocation rules on elastic single-attribute situations and evaluate their performance with respect to (the relaxations of) core inclusion and benefit ordering. The first rule is essentially the serial cost sharing rule of Moulin and Shenker (1992) but now applied to elastic single-attribute situations. The second rule is a variation on the proportional rule that allocates the benefits, rather than the costs, proportional to the players' attributes. The third and fourth rule are inspired by the seminal works of Shapley (1953, 1971) and are based on marginal allocations for adjusted situations in which the elastic function is approximated by a concave function as close as possible. The difference between these two so-called concavicated marginal rules lies in the set of marginal allocations that are being averaged.

In contrast to the proportional rule, our four newly proposed rules explicitly take into account the specific shape of the cost function. For each rule, we show which fairness properties it does and which it does not satisfy, as summarized in Table 4 on page 30. The concavicated marginal rules in particular are appealing because they always accomplish core allocations and, in contrast to the proportional rule, they are guaranteed to give larger players more benefits whenever the cost function is concave.

On the whole, this paper provides the first overview of cost allocation rules and their properties on the domain of elastic single-attribute situations. The organization is as follows. We start in Section 2 with preliminaries on cooperative game theory, elasticity, and elastic single-attribute situations. Section 3 presents our impossibility result that benefit ordering and core inclusion are not compatible in general. Subsequently, we introduce and analyze several new allocation rules: Section 4 considers the serial rule, Section 5 considers the benefit-proportional rule, and Section 6 considers the concavicated marginal rules. We conclude in Section 7.

2 Preliminaries

In this section, we first introduce several concepts from cooperative game theory that are relevant to our work. Subsequently, we define and characterize elasticity. Finally, we introduce elastic single-attribute situations and their associated games, and we provide several examples from the operations research literature.

2.1 Cooperative games and the Shapley value

A cooperative cost game with transferable utility, which we will simply refer to as a *game*, is a pair (N, c), where N is the non-empty finite *set of players* and $c : 2^N \to \mathbb{R}$ is the *characteristic cost function*, which assigns to every *coalition* $M \subseteq N$ the cost $c(M)$ that it would face if cooperation would be limited to only the players in M. By convention, $c(\emptyset) = 0$. A game is called *concave* if any player's marginal cost contribution does not increase when joining a larger coalition, i.e., $c(M \cup \{i\}) - c(M) \geq c(L \cup \{i\}) - c(L)$ for all $i \in N$ and all $M, L \subseteq N \setminus \{i\}$ with $M \subseteq L$.

A central problem in cooperative game theory is how to allocate $c(N)$, the costs of the *grand coalition* N, to the individual players. An *allocation* for a game (N, c) is a vector $x \in \mathbb{R}^N$ satisfying $\sum_{i \in N} x_i = c(N)$. The value x_i is then interpreted as the costs assigned to player i. An allocation x for a game (N, c) is called *individually rational* if every player is allocated no more costs than what he would face by staying alone, i.e., $x_i \leq c(\{i\})$ for all $i \in N$. The set of all individually rational allocations for a game (N, c) is called the *imputation set*, denoted by $\mathcal{I}(N, c)$. By extending individual rationality to all coalitions, we obtain coalitional rationality: an allocation x for a game (N, c) is called *coalitionally rational* if $\sum_{i \in M} x_i \leq c(M)$ for all $M \subseteq N$. The set of all coalitionally rational allocations for a game (N, c) is called the *core*, denoted by $\mathcal{C}(N, c)$.

One well-known cost allocation rule is the Shapley value (Shapley, 1953). To describe it, we first need to define orderings and marginal allocations. An *ordering* on player set N is a bijection $\sigma : N \to \{1, \ldots, n\}$, which should be interpreted as saying that player i is in position $\sigma(i)$. We let $\Pi(N)$ denote the set of all orderings on N. For an ordering $\sigma \in \Pi(N)$, we let $\sigma^{-1}(j)$ denote the player that is in position $j \in \{1, \ldots, |N|\}$ and we let $P_i^\sigma = \{j \in N | \sigma(j) < \sigma(i)\}$ describe the set of players that precede i. The *marginal contribution* of player i according to σ in a game (N, c) is given by $m_i^\sigma(N, c) = c(P_i^\sigma \cup \{i\}) - c(P_i^\sigma)$, i.e., the cost difference when player i joins his predecessors. The vector $m^\sigma(N, c) = (m_i^\sigma(N, c))_{i \in N}$ is called the *marginal allocation* according to σ. Note that it is efficient, i.e., $\sum_{i \in N} m_i^\sigma(N, c) = c(N)$.

The Shapley value $\Phi(N,c)$ of a game (N,c) is then defined as the average of all marginal allocations, i.e.,

$$\Phi_i(N,c) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m_i^\sigma(N,c) \qquad \text{for all } i \in N.$$

An alternative description is that

$$\Phi_i(N,c) = \sum_{M \subseteq N \setminus \{i\}} \frac{(|M|)!(|N|-|M|-1)!}{|N|!} \cdot [c(M \cup \{i\}) - c(M)] \qquad \text{for all } i \in N.$$

Following Shapley (1971), if a game (N,c) is concave, then $m^\sigma(N,c) \in \mathcal{C}(N,c)$ for any $\sigma \in \Pi(N)$ and $\Phi(N,c) \in \mathcal{C}(N,c)$. Note that as every marginal allocation is efficient, so is their average.

2.2 Elastic functions

Elasticity is a way of capturing economies of scale.[1]

Definition 2.1. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called *elastic* if $f(0) = 0$ and $f(x_1)/x_1 \geq f(x_2)/x_2$ for all $x_1, x_2 \in \mathbb{R}_{++}$ with $x_1 \leq x_2$.

Intuitively, if $f(x)$ expresses the cost of, say, serving demand level x, then elasticity of f says that the per-demand cost is non-increasing in the total demand served. The name "elasticity" is based on the economics literature, as motivated by Özen et al. (2011, p. 386).

The following example shows that an elastic function can be built up from convex segments, which is also the case for the elastic functions underlying the Erlang loss games and spare parts games that will be described in Examples 2.3 and 2.5.

Example 2.1. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} 5x & \text{if } x \in [0,1]; \\ x^2 + 4 & \text{if } x \in (1,2]; \\ 0.5x^2 + 6 & \text{if } x \in (2,3]. \\ 10.5 & \text{if } x > 3; \end{cases}$$

This function is elastic because $f(0) = 0$ and because $f(x)/x$, which is equal to 5 if $x \in [0,1)$, equal to $x + 4/x$ if $x \in (1,2]$, equal to $0.5x + 6/x$ if $x \in (2,3]$, and equal to $10.5/x$ if $x > 3$, is non-increasing for $x > 0$. ◊

[1] We write $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_{++} = (0,\infty)$.

The next example of an elastic function will form the basis for the impossibility result of Section 3.

Example 2.2. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ 5 & \text{if } x \in (0, 5]; \\ x & \text{if } x > 5. \end{cases}$$

This function is elastic because $f(0) = 0$ and because $f(x)/x$, which is equal to $5/x$ if $x \in [0, 5)$ and equal to 1 if $x \geq 5$, is non-increasing for $x > 0$. $\diamondsuit$

The following proposition characterizes an elastic function as a function f for which any straight line segment drawn through a point $(a, f(a))$ on its graph and the origin lies completely below or on the graph of f. This has a clear link to concavity. We omit the obvious proof.

Proposition 2.1. *Consider any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$.*
(i) The function f is elastic if and only if $f(x) \geq f(a)x/a$ for all $a \in \mathbb{R}_{++}$ and all $x \in (0, a]$.
(ii) If f is concave, then f is elastic.

As shown by Examples 2.1 and 2.2, the converse of Part (ii) of this theorem is not true. We next state a continuity property.

Proposition 2.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing, elastic function. Then f is continuous on $\mathbb{R}_{++}$.*

Proof. Let $a \in \mathbb{R}_{++}$ and let $\epsilon > 0$. If $f(a) > 0$, then fix $\delta = \min\{a, \epsilon a/[2f(a)]\}$; otherwise, if $f(a) = 0$, then fix $\delta = a$. Let $x \in (a - \delta, a + \delta)$. If $x < a$, then

$$|f(a) - f(x)| = f(a) - f(x) \leq f(a) - f(a - \delta) \leq f(a) - f(a)\frac{a - \delta}{a} = f(a)\frac{\delta}{a} \leq \epsilon/2 < \epsilon,$$

where the first inequality holds because f is non-decreasing and the second inequality holds by Part (i) of Proposition 2.1. Similarly, if $x \geq a$, then

$$|f(a) - f(x)| = f(x) - f(a) \leq f(a + \delta) - f(a) \leq f(a)\frac{a + \delta}{a} - f(a) = f(a)\frac{\delta}{a} \leq \epsilon/2 < \epsilon.$$

We conclude that f is continuous at a. $\square$

2.3 Elastic single-attribute situations and associated games

In elastic single-attribute situations, every player is associated with a certain resource endowment—his *attribute*, described by a positive real number. These endowments can be pooled to attain cost savings, as described by a non-decreasing, elastic cost function.

Definition 2.2. An *elastic single-attribute situation* is a triple $(N, \tilde{K}, \lambda)$, where

- N is a non-empty, finite set of players;
- $\tilde{K}$ is a non-decreasing, elastic function mapping $\mathbb{R}_+$ to $\mathbb{R}_+$;
- λ is an element of $\mathbb{R}^N_{++}$.

The function $\tilde{K}$ expresses the costs to serve any level of attributes. The requirement that $\tilde{K}$ is non-decreasing is imposed to highlight the applications that we have in mind: in these applications, $\tilde{K}$ expresses the costs to serve any level of demand in a service system, and such a cost does not shrink as demand increases. The vector λ represents the attributes of the various players; we will systematically write $\lambda_M = \sum_{i \in M} \lambda_i$ for any $M \subseteq N$. The set of all elastic single-attribute situations with finite but variable N is denoted by $\mathcal{E}$.

An elastic single-attribute situation naturally leads to a corresponding game.

Definition 2.3. Let $\varphi = (N, \tilde{K}, \lambda)$ be an elastic single-attribute situation. The game (N, c^φ) defined by $c^\varphi(M) = \tilde{K}(\lambda_M)$ for $M \subseteq N$ is called the *associated single-attribute game.*

We are interested in rules that assign to any elastic single-attribute situation an allocation for the associated single-attribute game.

Definition 2.4. An *allocation rule on elastic single-attribute situations* (or *rule* for short) is a mapping $\mathscr{F}$ on $\mathcal{E}$ such that for every $(N, \tilde{K}, \lambda) \in \mathcal{E}$ it holds that $\mathscr{F}(N, \tilde{K}, \lambda) \in \mathbb{R}^N$ and that $\sum_{i \in N} \mathscr{F}_i(N, \tilde{K}, \lambda) = \tilde{K}(\lambda_N)$.

Allocation rules can satisfy various interesting properties. We next define one such property.

Definition 2.5. A rule $\mathscr{F}$ on $\mathcal{E}$ is said to have the *coalitional rationality property* (CR) if $\mathscr{F}(\varphi) \in \mathcal{C}(N, c^\varphi)$ for any elastic single-attribute situation $\varphi = (N, \lambda, \tilde{K})$.

Coalitional rationality says that a rule should always generate a core element for the associated single-attribute game. If players can freely form coalitions (e.g., if they could either use dedicated resources to serve their own demands, or set up a joint facility with pooled resources to serve several players together) then coalitional rationality will be an important property.

A simple rule, which is often referred to as "average pricing" in the cost sharing literature (e.g., Moulin and Shenker, 1992; Sudhölter, 1998), is to assign the costs of the grand coalition proportional to the attributes of the individual players.

Definition 2.6. The *proportional rule* $\mathscr{P}$ on $\mathcal{E}$ is defined by allocating $\mathscr{P}_i(N, \tilde{K}, \lambda) = \tilde{K}(\lambda_N) \cdot \lambda_i/\lambda_N$ to player $i \in N$ in situation $(N, \tilde{K}, \lambda) \in \mathcal{E}$.

The following theorem states that the proportional rule always yields core allocations. It is due to Özen et al. (2011).

Theorem 2.3. *The proportional rule $\mathscr{P}$ satisfies the coalitional rationality property.*

We remark that this theorem implies that all single-attribute games have core elements. Accordingly, they represent a subclass of the class of games with non-empty cores. In a recent paper, Anily and Haviv (2014) detail various other classes of games with non-empty cores; none of them coincide with the class of single-attribute games.

We next present four illustrative examples, all based on collaboration in canonical queueing or inventory models, to indicate the wide applicability of the class of elastic single-attribute situations.

Example 2.3. (Erlang loss games) Consider a set N of players (e.g., hospital departments) who require servers (e.g., beds) to serve randomly arriving customers (e.g., patients). Customer arrivals for any player $i \in N$ are governed by a Poisson process with rate $\lambda_i > 0$. If a customer finds a free server upon arrival, then he immediately goes into service, which takes one unit of time on average. If a customer finds no free server upon arrival, then he is blocked and lost to the system. The players may collaborate by pooling servers: any coalition will set up a shared service facility which, due to the blocking of customers, will behave as an $M/G/s/s$ queue (also known as Erlang loss system) and in which a number of servers is used that minimizes the long-term average costs per time unit. We consider resource costs $h > 0$ per server and penalty costs $p > 0$ per lost customer.

We can capture this situation as an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ where the cost function $\tilde{K} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\tilde{K}(0) = 0$ and

$$\tilde{K}(\ell) = \min_{s \in \{0,1,\ldots\}} \{hs + B(s, \ell)\ell p\}$$

for all $\ell > 0$, where $B(s, \ell) = [\ell^s/s!]/[\sum_{y=0}^{s} \ell^y/y!]$ represents the steady-state probability that an arriving customer is lost in an Erlang loss system with s servers and arrival rate

ℓ. As shown in Özen et al. (2011), the function $\tilde{K}$ is non-decreasing and elastic, so this is indeed an elastic single-attribute situation. Hence, by Theorem 2.3, the proportional allocation $\mathscr{P}(\varphi)$ is a coalitionally rational allocation for the associated single-attribute game (N, c^{φ}). This result was also derived by Karsten et al. (2014) using an alternative proof. ♢

Example 2.4. (EOQ games) Consider a set N of players (e.g., retailers) who have to meet deterministic demand for items (e.g., office supplies) occurring at a constant rate. The demand rate for any player $i \in N$ is $\lambda_i \geq 0$ items per unit of time. For any order that is placed, a fixed ordering cost $a > 0$ is incurred. Furthermore, there are holding cost of $h > 0$ per item on stock per unit of time. The players in a coalition may save on the fixed ordering costs by ordering jointly: they will use an optimal joint replenishment policy that meets all of their demands at minimal total average costs per time unit.

We can capture this situation as an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $\tilde{K}(\ell) = \sqrt{2ah\ell}$ for all $\ell \geq 0$. Here, $\tilde{K}$ follows from the basic Economic Order Quantity model. Clearly, $\tilde{K}$ is concave, hence elastic, and non-decreasing. So, by Theorem 2.3, the proportional allocation $\mathscr{P}(\varphi)$ is coalitionally rational. Meca et al. (2004) obtained the same conclusion via a different approach. ♢

Example 2.5. (Spare parts games) Consider a set N of players (e.g., airlines) who face demands for a low-demand, expensive item (e.g., a spare airplane part). Demands for any player $i \in N$ are governed by a Poisson process with rate $\lambda_i > 0$. Because the item is expensive and infrequently demanded, a base-stock policy with one-for-one replenishments is followed. A coalition of players may collaborate by pooling inventory: they will set up a shared stockpoint which, due to the one-for-one replenishments, will behave as an $(S-1, S)$ inventory system and in which a base stock level is used that minimizes the long-term average costs per time unit. We consider holding costs $h > 0$ per unit time per item in the on-hand stock and penalty costs $b > 0$ per unit time per backordered demand.

We can capture this situation as an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ where the cost function $\tilde{K} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\tilde{K}(0) = 0$ and

$$\tilde{K}(\ell) = \min_{s \in \{0,1,\ldots\}} \{h\mathbb{E}I(S, \ell) + b\mathbb{E}B(S, \ell)\}$$

for all $\ell > 0$, where $\mathbb{E}I(S, \ell)$ and $\mathbb{E}B(S, \ell)$ represent the expected stock on hand and the expected backorders, respectively, for an $(S-1, S)$ inventory system with base stock level S and demand rate ℓ in steady state. As shown in Karsten and Basten (2014), the function $\tilde{K}$ is non-decreasing and elastic, so this is indeed an elastic single-attribute situation. Hence, by Theorem 2.3, the proportional allocation $\mathscr{P}(\varphi)$ is a coalitionally rational allocation for the associated single-attribute game (N, c^φ). They also discussed how this property could be derived from a link with newsvendor games. $\Diamond$

Example 2.6. (Call center games) Consider a set N of players (e.g., call centers) who require staff (e.g., agents) to serve large populations of customers (e.g., calls). Customer arrivals for any player $i \in N$ are governed by a Poisson process with rate $\lambda_i > 0$. Service times are exponentially distributed and take one unit of time on average. There is infinite queueing room, and we consider linear staffing and waiting costs. Because arrival rates are large, the (close-to-optimal) square-root safety staffing principle is used to determine staffing levels. The players may collaborate by pooling servers: a coalition will set up a shared service facility that serves the union of their customer streams.

We can capture this situation as an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $\tilde{K}(\ell) = \ell + \beta\sqrt{\ell}$, where $\beta > 0$ is a parameter that depends on the staffing and waiting costs. Here, $\tilde{K}$ expresses the staffing levels according to the square-root safety staffing principle. Clearly, $\tilde{K}$ is concave, hence elastic, and non-decreasing. So, by Theorem 2.3, the proportional allocation $\mathscr{P}(\varphi)$ is coalitionally rational. The corollary that the associated single-attribute game (N, c^φ) has a non-empty core was also obtained by Guo et al. (2013), and the result is in line with the finding of Karsten et al. (2015) that when $M/M/s$ queues join forces under optimized real-valued number of servers, the corresponding game in their model admits a proportional core allocation. $\Diamond$

3 An impossibility result

In the previous section, we saw that the proportional rule $\mathscr{P}$ satisfies the appealing coalitional rationality property. In this section, we start by using an example to indicate a disadvantage of $\mathscr{P}$.

Example 3.1. Suppose that player 1 (with a montly demand rate of 9) and player 2 (with a monthly demand rate of 16) aim to set up a joint service system, whose total monthly costs increase concavely according to the square root of the total demand rate served. This may be modeled via the elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $N = \{1, 2\}$, $\lambda_1 = 9$, $\lambda_2 = 16$, and $\tilde{K}(\ell) = \sqrt{\ell}$ for all $\ell \in \mathbb{R}_+$. The single-attribute game associated with this situation, (N, c^φ), is given by $c^\varphi(\{1\}) = 3$, $c^\varphi(\{2\}) = 4$, and $c^\varphi(N) = 5$.

Clearly, $\mathscr{P}_1(\varphi) = 5 \cdot 9/25 = 9/5$ and $\mathscr{P}_2(\varphi) = 5 \cdot 16/25 = 16/5$. This means that the cost savings allocated to player 1 by the proportional rule, $c^\varphi(\{1\}) - \mathscr{P}_1(\varphi) = 6/5$, are larger than the cost savings allocated to player 2, $c^\varphi(\{2\}) - \mathscr{P}_2(\varphi) = 4/5$. So, even though collaboration under $\mathscr{P}$ does in fact produce a small saving for player 2, his cost savings are less than those of player 1. One could argue that this is unfair because the total savings were only made possible because player 2 allowed player 1 to piggyback on his large attribute. ♢

Similar objections have been observed in practice by Frisk et al. (2010) when trying to implement cost allocation rules for collaborative forest transportation. They found that when relative savings were dissimilar (as is the case when small players reap more benefits than large players) the cost allocations were difficult to accept by the participating companies.

The following definition formalizes the idea that an allocation rule should avoid the issue described in Example 3.1.

Definition 3.1. A rule $\mathscr{F}$ on $\mathcal{E}$ is said to have the *benefit ordering property* (BO) if, for every elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ and every $i, j \in N$ with $\lambda_i \leq \lambda_j$, we have that $c^\varphi(\{i\}) - \mathscr{F}_i(\varphi) \leq c^\varphi(\{j\}) - \mathscr{F}_j(\varphi)$.

Benefit ordering means that a player with a larger attribute should always reap at least as much benefit from the collaboration as a player with a smaller attribute. Here, "benefit" refers to cost savings.

As shown in Example 3.1, the proportional rule does not satisfy the benefit ordering property. However, by Theorem 2.3, the proportional rule does satisfy the coalitional rationality property. A natural follow-up question is whether there is a rule on $\mathcal{E}$ that satisfies both the benefit ordering property and the coalitional rationality property. The following example, however, shows that such a rule does not exist: *no* rule on $\mathcal{E}$ can satisfy both CR and BO simultaneously.

Example 3.2. Consider the elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $N = \{1, 2, 3\}$, $\lambda_1 = \lambda_2 = 2.5$, $\lambda_3 = 5$, and cost function $\tilde{K}$ described by

$$\tilde{K}(\ell) = \begin{cases} 0 & \text{if } \ell = 0; \\ 5 & \text{if } \ell \in (0, 5]; \\ \ell & \text{if } \ell > 5. \end{cases}$$

We have seen that this function is elastic in Example 2.2. For convenience, this function is graphically represented in Figure 1. The associated single-attribute game (N, c^φ) is given by

$$c^\varphi(M) = \begin{cases} 5 & \text{if } |M| = 1 \text{ or } M = \{1, 2\}; \\ 7.5 & \text{if } M = \{1, 3\} \text{ or } M = \{2, 3\}; \\ 10 & \text{if } M = N. \end{cases}$$

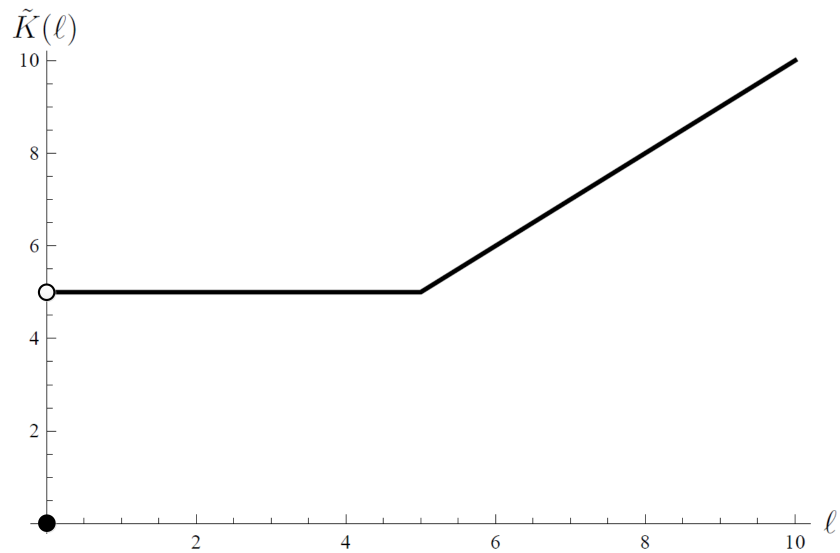

Figure 1: *A plot of the function $\tilde{K}$ in Example 3.2.*

This game's core has only one element: $\mathcal{C}(N, c^{\varphi}) = \{(2.5, 2.5, 5)\}$.[2] The benefits for player 3 under the core allocation are zero, while the benefits to players 1 and 2 are (strictly) positive. Since the attributes of players 1 and 2 are smaller than the attribute of player 3, we conclude that any rule satisfying the coalitional rationality property cannot satisfy the benefit ordering property since it must select the unique core element for this game.[3] $\diamondsuit$

Example 3.2 shows that the properties CR and BO are incompatible on $\mathcal{E}$. If we want to arrive at properties that *are* compatible, then we have to relax some of our requirements. The following two definitions propose relaxations of CR and BO, respectively.

Definition 3.2. A rule $\mathscr{F}$ is said to have the *individual rationality property* (IR) on $\mathcal{E}$ if $\mathscr{F}(\varphi) \in \mathcal{I}(N, c^{\varphi})$ for any elastic single-attribute situation φ.

Definition 3.3. A rule $\mathscr{F}$ on $\mathcal{E}$ is said to have the *benefit ordering property under concavity* (BOC) if, for every elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with concave $\tilde{K}$ and every $i, j \in N$ with $\lambda_i \leq \lambda_j$, we have that $c^{\varphi}(\{i\}) - \mathscr{F}_i(\varphi) \leq c^{\varphi}(\{j\}) - \mathscr{F}_j(\varphi)$.

[2] To see this, first note that $(2.5, 2.5, 5)$ is a core allocation, so the core is non-empty. Let x be a core allocation. This implies that $x_3 \leq c(\{3\}) = 5$, $x_1 + x_2 \leq c(\{1,2\}) = 5$, and $x_1 + x_2 + x_3 = 10$, which together yield $x_3 = x_1 + x_2 = 5$. Coalitional rationality of x in combination with $x_3 = 5$ yields $x_1 \leq c^{\varphi}(\{1,3\}) - x_3 = 2.5$ and $x_2 \leq c^{\varphi}(\{1,3\}) - x_3 = 2.5$. But since we established that $x_1 + x_2 = 5$, we must have $x_1 = x_2 = 2.5$.

[3] The same incompatibility would occur if, e.g., $\tilde{K}(\ell) = \sqrt{5\ell}$ on $[0, 5]$. The discontinuity of the cost function at 0 in the example does not drive the incompatibility; it is merely for expositional ease.

It is easy to see that CR implies IR and that BO implies BOC. Indeed, CR extends the rationality requirement of IR from single players to coalitions, and BO extends the ordering requirement of BOC from concave cost functions to arbitrary non-decreasing, elastic cost functions. These relaxations help to illustrate the extent to which CR and BO are incompatible. In Section 5, we construct a rule on $\mathcal{E}$ satisfying both IR and BO. In Section 6, we construct a rule on $\mathcal{E}$ satisfying both CR and BOC.

We remark that in many situations, such as in Examples 2.4 and 2.6, the cost function is concave. However, even in such concave situations, the proportional rule $\mathscr{P}$ is not guaranteed to dish out larger benefits to larger players, as shown in Example 3.1. This implies that $\mathscr{P}$ does not satisfy BOC. Accordingly, a rule that *does* satisfy BOC can be rightfully said to do "better" than $\mathscr{P}$ with regard to the ordering of players' benefits. In the next three sections, we introduce several new rules and analyze their properties.

4 The serial rule

Elastic single-attribute situations have the same mathematical structure as so-called cost sharing situations, which are well-studied in the literature. Indeed, a cost sharing situation is a tuple (N, q, C), with a given set $N = \{1, \ldots, n\}$ of users, user i having demand $q_i \in \mathbb{R}_+$, who share a joint production process described by a cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$. The argument of C is interpreted as the sum of demands to be served.

There are two key differences, however. First, in contrast to elastic single-attribute situations, the cost function C is not required to be elastic. Second, the typical *story* behind cost sharing situations differs from our elastic single-attribute situations. In a cost sharing situation (as studied in, e.g., Moulin and Shenker, 1992) there is only one production process that is jointly owned by all the players, and they are basically forced to collaborate with each other—no group of users can get their desired output if they split off and act independently. In an elastic single-attribute situation, collaboration via resource pooling is optional—any coalition might set up a separate service system if they so please. This different interpretation means that certain considerations which are natural for elastic single-attribute situations, such as individual rationality or stability, do not apply to cost sharing situations.

The literature on cost sharing situations, however, contains interesting cost sharing mechanisms. We translate the serial rule (Moulin and Shenker, 1992), one of the most-studied mechanisms, to a rule on $\mathcal{E}$. This rule takes into account some of the intermediate behavior of the cost function, as opposed to the proportional rule which entirely ignores the behavior of the cost function between zero and the total demand of the grand coalition.

Definition 4.1. The *serial rule* $\mathscr{S}$ on $\mathcal{E}$ is defined by allocating

$$\mathscr{S}_i(N,\tilde{K},\lambda) = \sum_{j=1}^{\sigma(i)} \frac{\tilde{K}(\Lambda_j) - \tilde{K}(\Lambda_{j-1})}{|N| + 1 - j}$$

to player $i \in N$ in the elastic single-attribute situation $(N, \tilde{K}, \lambda)$, where σ is an ordering on N such that $\lambda_{\sigma^{-1}(1)} \leq \lambda_{\sigma^{-1}(2)} \leq \ldots \leq \lambda_{\sigma^{-1}(|N|)}$, which orders the players from small to large attributes[4], where $\Lambda_0 = 0$, and where

$$\Lambda_j = (n + 1 - j)\lambda_{\sigma^{-1}(j)} + \sum_{k=1}^{\sigma(j)-1} \lambda_{\sigma^{-1}(k)}$$

for any $j \in \{1, \ldots, |N|\}$. Note that $0 < \Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_{|N|}$.

To illustrate, suppose for notational ease that $N = \{1, 2, \ldots, n\}$ and that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then, serial cost sharing says that player 1, with the lowest attribute λ_1, pays $(1/n)$th of the cost of $\Lambda_1 = n\lambda_1$, i.e., the total costs if everyone would have player 1's attribute. Player 2, with the next lowest attribute λ_2, pays player 1's cost share plus $1/(n-1)$th of the incremental cost from $\Lambda_1 = n\lambda_1$ to $\Lambda_2 = (n-1)(\lambda_2 - \lambda_1) + n\lambda_1$, i.e., to the total costs if everyone except for player 1 would have player 2's attribute. Player 3, with the next lowest attribute λ_3, pays player 2's cost share, plus $1/(n-2)$th of the incremental cost from $\Lambda_2 = (n-1)(\lambda_2 - \lambda_1) + n\lambda_1$ to $\Lambda_3 = (n-2)(\lambda_3 - \lambda_2) + (n-1)(\lambda_2 - \lambda_1) + n\lambda_1$. And so on.

The following theorem deals with two properties of the serial rule $\mathscr{S}$.

Theorem 4.1. *The serial rule $\mathscr{S}$ on $\mathcal{E}$ satisfies the individual rationality property and the benefit ordering property under concavity.*

Proof. Let $\varphi = (N, \tilde{K}, \lambda) \in \mathcal{E}$. Without loss of generality, assume for notational conve-

[4]If several players have the same attribute, then all such orderings lead to the same allocation.

nience that $N = \{1, 2, \ldots, n\}$ and that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $i \in N$. Then,

$$\begin{aligned}
\mathscr{S}_i(\varphi) &= \sum_{j=1}^{i} \frac{\tilde{K}(\Lambda_j) - \tilde{K}(\Lambda_{j-1})}{n+1-j} \\
&= \frac{\tilde{K}(\Lambda_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{\tilde{K}(\Lambda_j)}{(n-j+1)(n-j)} \\
&= \frac{\Lambda_i}{n-i+1} \cdot \frac{\tilde{K}(\Lambda_i)}{\Lambda_i} - \sum_{j=1}^{i-1} \frac{\Lambda_j}{(n-j+1)(n-j)} \cdot \frac{\tilde{K}(\Lambda_j)}{\Lambda_j} \\
&\leq \frac{\Lambda_i}{n-i+1} \cdot \frac{\tilde{K}(\max\{\Lambda_{i-1}, \lambda_i\})}{\max\{\Lambda_{i-1}, \lambda_i\}} - \sum_{j=1}^{i-1} \frac{\Lambda_j}{(n-j+1)(n-j)} \cdot \frac{\tilde{K}(\max\{\Lambda_{i-1}, \lambda_i\})}{\max\{\Lambda_{i-1}, \lambda_i\}} \\
&= \frac{\tilde{K}(\max\{\Lambda_{i-1}, \lambda_i\})}{\max\{\Lambda_{i-1}, \lambda_i\}} \cdot \left(\frac{\Lambda_1}{n} + \sum_{j=2}^{i} \frac{\Lambda_j - \Lambda_{j-1}}{n+1-j} \right) \\
&= \frac{\tilde{K}(\max\{\Lambda_{i-1}, \lambda_i\})}{\max\{\Lambda_{i-1}, \lambda_i\}} \cdot \left(\lambda_1 + \sum_{j=2}^{i} (\lambda_j - \lambda_{j-1}) \right) \\
&= \frac{\tilde{K}(\max\{\Lambda_{i-1}, \lambda_i\})}{\max\{\Lambda_{i-1}, \lambda_i\}} \cdot \lambda_i \\
&\leq \frac{\tilde{K}(\lambda_i)}{\lambda_i} \cdot \lambda_i = c^{\varphi}(\{i\}),
\end{aligned}$$

where both inequalities hold by elasticity of $\tilde{K}$. In the first inequality, we use that $\Lambda_i \geq \max\{\Lambda_{i-1}, \lambda_i\}$, while $\Lambda_j \leq \max\{\Lambda_{i-1}, \lambda_i\}$ for any $j \in \{1, \ldots, i-1\}$. In the second inequality, we use that $\lambda_i \leq \max\{\Lambda_{i-1}, \lambda_i\}$. We conclude that $\mathscr{S}(\varphi) \in \mathcal{I}(N, c^{\varphi})$.

To study benefit ordering under concavity, assume that $\tilde{K}$ is concave. Suppose that $i \in \{1, 2, \ldots, n-1\}$. By assumption, $\lambda_i \leq \lambda_{i+1}$. It suffices to prove that $c^{\varphi}(\{i\}) - \mathscr{S}_i(\varphi) \leq c^{\varphi}(\{i+1\}) - \mathscr{S}_{i+1}(\varphi)$. To this end, we first derive that

$$\begin{aligned}
\tilde{K}(\lambda_{i+1}) - \tilde{K}(\lambda_i) &= \frac{\tilde{K}(\lambda_i + \lambda_{i+1} - \lambda_i) - \tilde{K}(\lambda_i)}{\lambda_{i+1} - \lambda_i} \cdot (\lambda_{i+1} - \lambda_i) \\
&\geq \frac{\tilde{K}\big(\Lambda_i + (n-i)(\lambda_{i+1} - \lambda_i)\big) - \tilde{K}(\Lambda_i)}{(n-i)(\lambda_{i+1} - \lambda_i)} \cdot (\lambda_{i+1} - \lambda_i) \\
&= \frac{\tilde{K}\big(\Lambda_i + (n-i)(\lambda_{i+1} - \lambda_i)\big) - \tilde{K}(\Lambda_i)}{n-i} \\
&= \frac{\tilde{K}(\Lambda_{i+1}) - \tilde{K}(\Lambda_i)}{n-i},
\end{aligned}$$

where the inequality holds because $\tilde{K}$ is concave and thus the difference quotient obtained when adding an amount (in particular, $\lambda_{i+1} - \lambda_i$) to λ_i is at least as large as the difference quotient obtained when adding an amount (in particular, $(n-i)(\lambda_{i+1}-\lambda_i)$) to Λ_i, since $\lambda_i \leq \Lambda_i$.

Subtracting $\tilde{K}(\lambda_{i+1})$ from both sides and multiplying by -1 yields

$$\tilde{K}(\lambda_i) \leq \tilde{K}(\lambda_{i+1}) - \frac{\tilde{K}(\Lambda_{i+1}) - \tilde{K}(\Lambda_i)}{n-i}.$$

Using this inequality, we obtain

$$\begin{aligned} c^{\varphi}(\{i\}) - \mathscr{S}_i(\varphi) &= \tilde{K}(\lambda_i) - \sum_{j=1}^{i} \frac{\tilde{K}(\Lambda_j) - \tilde{K}(\Lambda_{j-1})}{n+1-j} \\ &\leq \tilde{K}(\lambda_{i+1}) - \sum_{j=1}^{i+1} \frac{\tilde{K}(\Lambda_j) - \tilde{K}(\Lambda_{j-1})}{n+1-j} = c^{\varphi}(\{i+1\}) - \mathscr{S}_{i+1}(\varphi). \end{aligned}$$

We conclude that $\mathscr{S}$ satisfies the benefit ordering property under concavity. □

The following example shows that $\mathscr{S}$ neither satisfies the coalitional rationality property nor the benefit ordering property.

Example 4.1. Consider the single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $N = \{1, 2, 3\}$, $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 5$, and the elastic, but not concave, cost function $\tilde{K}$ as in Example 3.2. The associated single-attribute game (N, c^{φ}) is given by

$$c^{\varphi}(M) = \begin{cases} 5 & \text{if } |M| = 1; \\ \sum_{i \in M} \lambda_i & \text{otherwise.} \end{cases}$$

The serial rule allocates $\mathscr{S}_1(\varphi) = \tilde{K}(3)/3 = 5/3$, $\mathscr{S}_2(\varphi) = \tilde{K}(3)/3 + (\tilde{K}(9) - \tilde{K}(3))/2 = 11/3$, and $\mathscr{S}_3(\varphi) = 14/3$. Since $\mathscr{S}_1(\varphi) + \mathscr{S}_2(\varphi) = 16/3 > 5 = c^{\varphi}(\{1, 2\})$, we conclude that $\mathscr{S}(\varphi) \notin \mathcal{C}(N, c^{\varphi})$. Hence, $\mathscr{S}$ does not satisfy the coalitional rationality property.

Furthermore, since $c^{\varphi}(\{1\}) - \mathscr{S}_1(\varphi) = 10/3$ and $c^{\varphi}(\{2\}) - \mathscr{S}_2(\varphi) = 4/3$, player 1 obtains a larger cost saving than player 2, which implies that $\mathscr{S}$ does not satisfy the benefit ordering property either. ◊

We conclude that $\mathscr{S}$ satisfies IR and BOC, but lacks CR and BO.

5 The benefit-proportional rule

Our next alternative allocation rule is a variation on the proportional rule. Rather than allocating the total *costs* proportional to the attribute of each player, we allocate the *benefits* proportionally instead.

Definition 5.1. The *benefit-proportional rule* $\mathscr{B}$ on $\mathcal{E}$ is defined by allocating $\mathscr{B}_i(N, \tilde{K}, \lambda) = \tilde{K}(\lambda_i) - \left[\sum_{k\in N} \tilde{K}(\lambda_k) - \tilde{K}(\lambda_N)\right] \cdot \lambda_i/\lambda_N$ to player $i \in N$ in $(N, \tilde{K}, \lambda) \in \mathcal{E}$.

The following example illustrates this rule and shows that it does not satisfy the coalitional rationality property.

Example 5.1. Reconsider the single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ of Example 3.2. The allocation of the benefit-proportional rule $\mathscr{B}(\varphi)$, which is given by $(3.75, 3.75, 2.5)$, differs from the unique core allocation, $(2.5, 2.5, 5)$; hence $\mathscr{B}(\varphi)$ is not in the core of the associated single-attribute game. This implies that it does not satisfy the coalitional rationality property. ◊

The following theorem deals with two more properties.

Theorem 5.1. *The benefit-proportional rule $\mathscr{B}$ satisfies the benefit ordering property and the individual rationality property.*

Proof. Let $\varphi = (N, \tilde{K}, \lambda) \in \mathcal{E}$. Let $i, j \in N$ with $\lambda_i \leq \lambda_j$. Then,

$$\begin{aligned}
c^{\varphi}(\{i\}) - \mathscr{B}_i(\varphi) &= \tilde{K}(\lambda_i) - \Big(\tilde{K}(\lambda_i) - \left[\textstyle\sum_{k\in N} \tilde{K}(\lambda_k) - \tilde{K}(\lambda_N)\right] \cdot \lambda_i/\lambda_N\Big) \\
&= \left[\textstyle\sum_{k\in N} \tilde{K}(\lambda_k) - \tilde{K}(\lambda_N)\right] \cdot \lambda_i/\lambda_N \\
&\leq \left[\textstyle\sum_{k\in N} \tilde{K}(\lambda_k) - \tilde{K}(\lambda_N)\right] \cdot \lambda_j/\lambda_N \\
&= \tilde{K}(\lambda_j) - \Big(\tilde{K}(\lambda_j) - \left[\textstyle\sum_{k\in N} \tilde{K}(\lambda_k) - \tilde{K}(\lambda_N)\right] \cdot \lambda_j/\lambda_N\Big) \\
&= c^{\varphi}(\{j\}) - \mathscr{B}_j(\varphi),
\end{aligned}$$

where the inequality holds because $\lambda_i \leq \lambda_j$. Hence, $\mathscr{B}$ satisfies BO. Furthermore, $\mathscr{B}$ satisfies IR because $\sum_{j\in N} \tilde{K}(\lambda_j) \geq \tilde{K}(\lambda_N)$ by Theorem 2.3; hence, by definition, $\mathscr{B}_i(\varphi) \leq c^{\varphi}(\{i\})$ for each $i \in N$. □

Hence, $\mathscr{B}$ shows that the properties IR and BO are compatible.

6 Concavicated marginal rules

This section introduces and analyzes two new rules on elastic single-attribute situations: the concavicated increasing marginal rule and the concavicated average marginal rule. Section 6.1 focuses on marginal allocations and the Shapley value in single-attribute situations with concave cost functions. Section 6.2 describes how to construct an order-specific concave function under an elastic function. Section 6.3 defines the two concavicated rules and analyzes their properties.

6.1 Concave single-attribute situations

This subsection considers single-attribute situations with concave cost functions. For the corresponding single-attribute games, we will study marginal allocations and the Shapley value. We remark that these are rules for games, not on elastic single-attribute situations. We start with a simple preliminary result.

Lemma 6.1. *Let $\varphi = (N, \tilde{K}, \lambda)$ be a single-attribute situation with concave $\tilde{K}$. Then,*

(i) *(N, c^{φ}) is concave.*

(ii) *For all orderings σ on N, $m^{\sigma}(N, c^{\varphi}) \in \mathcal{C}(N, c^{\varphi})$.*

(iii) *$\Phi(N, c^{\varphi}) \in \mathcal{C}(N, c^{\varphi})$.*

Proof. (i). Let $i \in N$ and let $M, L \subseteq N \setminus \{i\}$ with $M \subseteq L$. Then,

$$\begin{aligned} c^{\varphi}(M \cup \{i\}) - c^{\varphi}(M) &= \tilde{K}(\lambda_M + \lambda_i) - \tilde{K}(\lambda_M) \\ &\geq \tilde{K}(\lambda_M + \lambda_{L \setminus M} + \lambda_i) - \tilde{K}(\lambda_M + \lambda_{L \setminus M}) \\ &= c^{\varphi}(L \cup \{i\}) - c^{\varphi}(L), \end{aligned}$$

where the inequality holds because $\tilde{K}$ is concave. This means that (N, c^{φ}) is concave.

(ii). Follows from Part (i) since, by Shapley (1971), any marginal vector is in the core of a concave game.

(iii). Follows from Part (ii) since the Shapley value is the average of the marginal vectors and the core is a convex set. □

We next describe a restriction on orderings.

Definition 6.1. Given an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda) \in \mathcal{E}$, the set $\Theta(\varphi)$ is defined as the set of all orderings on N for which players are ordered in non-decreasing order of their attributes, i.e., $\Theta(\varphi) = \{\sigma \in \Pi(N) \mid \lambda_{\sigma^{-1}(1)} \leq \lambda_{\sigma^{-1}(2)} \leq \ldots \leq \lambda_{\sigma^{-1}(|N|)}\}$.

The following theorem considers such orderings and states that, under any corresponding marginal allocation, the players' benefits will be ordered in the same way as their attributes.

Theorem 6.2. *Let $\varphi = (N, \tilde{K}, \lambda)$ be a single-attribute situation with concave $\tilde{K}$. Let $\sigma \in \Theta(\varphi)$. Then, $c^{\varphi}(\{i\}) - m_i^{\sigma}(N, c^{\varphi}) \leq c^{\varphi}(\{j\}) - m_j^{\sigma}(N, c^{\varphi})$ for each $i, j \in N$ with $\sigma(i) \leq \sigma(j)$.*

σ^{-1}	$m_1^\sigma(N, c^\varphi)$	$m_2^\sigma(N, c^\varphi)$	$m_3^\sigma(N, c^\varphi)$
(1,2,3)	1	$\sqrt{2}-1$	$2-\sqrt{2}$
(1,3,2)	1	$2-\sqrt{3}$	$\sqrt{3}-1$
(2,1,3)	$\sqrt{2}-1$	1	$2-\sqrt{2}$
(2,3,1)	$2-\sqrt{3}$	1	$\sqrt{3}-1$
(3,2,1)	$2-\sqrt{3}$	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$
(3,1,2)	$\sqrt{3}-\sqrt{2}$	$2-\sqrt{3}$	$\sqrt{2}$
Sum	$5-\sqrt{3}$	$5-\sqrt{3}$	$2+2\sqrt{3}$
Average	$\frac{5}{6}-\frac{1}{6}\sqrt{3}$	$\frac{5}{6}-\frac{1}{6}\sqrt{3}$	$\frac{1}{3}+\frac{1}{3}\sqrt{3}$

Table 1: *All marginal allocations and their average for the game in Example 6.1.*

Proof. Let $i, j \in N$ with $\sigma(i) \leq \sigma(j)$, which implies that $\lambda_i \leq \lambda_j$. Then,

$$\begin{aligned}
c^\varphi(\{i\}) - m_i^\sigma(N, c^\varphi) &= c^\varphi(\{i\}) - c(P_i^\sigma \cup \{i\}) + c(P_i^\sigma) \\
&= \tilde{K}(\lambda_i) - \tilde{K}(\lambda_{P_i^\sigma} + \lambda_i) + \tilde{K}(\lambda_{P_i^\sigma}) \\
&\leq \tilde{K}(\lambda_i + (\lambda_j - \lambda_i)) - \tilde{K}(\lambda_{P_i^\sigma} + \lambda_i + (\lambda_j - \lambda_i)) + \tilde{K}(\lambda_{P_i^\sigma}) \\
&\leq \tilde{K}(\lambda_i + (\lambda_j - \lambda_i)) - \tilde{K}(\lambda_{P_j^\sigma} + \lambda_i + (\lambda_j - \lambda_i)) + \tilde{K}(\lambda_{P_j^\sigma}) \\
&= c^\varphi(\{j\}) - m_j^\sigma(N, c^\varphi).
\end{aligned}$$

The first inequality holds because $\tilde{K}$ is concave and $\lambda_j \geq \lambda_i$. The second inequality holds because $\tilde{K}$ is concave and, since $\sigma(i) \leq \sigma(j)$ in the ordering, $P_i^\sigma \subseteq P_j^\sigma$. □

The following example provides an illustration and shows that a marginal allocation need not assign the same costs to players with identical attributes.

Example 6.1. Consider the elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $N = \{1, 2, 3\}$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$, and $\tilde{K}(\ell) = \sqrt{\ell}$ for all $\ell \in \mathbb{R}_+$. Note that $\tilde{K}$ is concave. The associated single-attribute game (N, c^φ) is described by

$$c^\varphi(M) = \begin{cases} 1 & \text{if } M = \{1\} \text{ or } M = \{2\}; \\ \sqrt{2} & \text{if } M = \{1,2\} \text{ or } M = \{3\}; \\ \sqrt{3} & \text{if } M = \{1,3\} \text{ or } M = \{2,3\}; \\ 2 & \text{if } M = N. \end{cases}$$

This game is concave. All marginal allocations and their average (i.e., the Shapley value) for this game are described in Table 1. Note that they are in the core of (N, c^φ).

The set $\Theta(\varphi)$ contains two orderings: one whose inverse is $(1,2,3)$ and one whose inverse is $(2,1,3)$. Let us consider the first-mentioned one and denote it by σ. So, in first position is player 1, in second position is player 2, and in third position is player 3. The marginal allocation $m^{\sigma}(N,c^{\varphi})$ according to this ordering is given by $(1,\sqrt{2}-1,2-\sqrt{2})$. Note that even though players 1 and 2 have identical attributes, they get a different cost assignment. Due to the ordering, player 1's benefit of $c^{\varphi}(\{1\})-m_1^{\sigma}(N,c^{\varphi})=0$ is smaller than player 2's benefit of $c^{\varphi}(\{2\})-m_2^{\sigma}(N,c^{\varphi})=2-\sqrt{2}$, which in turn is smaller than player 3's benefit of $c^{\varphi}(\{3\})-m_3^{\sigma}(N,c^{\varphi})=2\sqrt{2}-2$. ◊

This example raises the issue that a marginal allocation corresponding to an ordering in $\Theta(\varphi)$ may treat identical players differently. This can be avoided by averaging over $\Theta(\varphi)$, as shown in the following theorem.

Theorem 6.3. *Let $\varphi=(N,\tilde{K},\lambda)\in\mathcal{E}$ with concave $\tilde{K}$. Then,*

$$c^{\varphi}(\{i\})-\frac{1}{|\Theta(\varphi)|}\sum_{\sigma\in\Theta(\varphi)}m_i^{\sigma}(N,c^{\varphi})\leq c^{\varphi}(\{j\})-\frac{1}{|\Theta(\varphi)|}\sum_{\sigma\in\Theta(\varphi)}m_j^{\sigma}(N,c^{\varphi})$$

for each $i,j\in N$ with $\lambda_i\leq\lambda_j$.

Proof. Let $i,j\in N$ with $\lambda_i\leq\lambda_j$. We distinguish between two cases.

Case 1: $\lambda_i<\lambda_j$. Then, $\sigma(i)<\sigma(j)$ for each $\sigma\in\Theta(\varphi)$. The desired inequality then holds by Theorem 6.2.

Case 2: $\lambda_i=\lambda_j$. Then, obviously, $c^{\varphi}(\{i\})=c^{\varphi}(\{j\})$. Moreover, players i and j are symmetric, and thus for every ordering $\check{\sigma}\in\Theta(\varphi)$ with $\check{\sigma}(i)=a$ and $\check{\sigma}(j)=b$ there exists another ordering $\hat{\sigma}\in\Theta(\varphi)$ with $\hat{\sigma}(i)=b$, $\hat{\sigma}(j)=a$, and $\hat{\sigma}(k)=\sigma(k)$ for all $k\in N\setminus\{i,j\}$. For these orderings, $m_i^{\check{\sigma}}(N,c^{\varphi})=m_j^{\hat{\sigma}}(N,c^{\varphi})$, so $\sum_{\sigma\in\Theta(\varphi)}m_i^{\sigma}(N,c^{\varphi})=\sum_{\sigma\in\Theta(\varphi)}m_j^{\sigma}(N,c^{\varphi})$. Hence, the desired inequality holds with equality. □

Remarkably, the average of all marginal allocations (i.e., the Shapley value) also has the players' benefits ordered in the same way as their attributes.

Theorem 6.4. *Let $\varphi=(N,\tilde{K},\lambda)\in\mathcal{E}$ with concave $\tilde{K}$. Then,*

$$c^{\varphi}(\{i\})-\Phi_i(N,c^{\varphi})\leq c^{\varphi}(\{j\})-\Phi_j(N,c^{\varphi})$$

for each $i,j\in N$ with $\lambda_i\leq\lambda_j$.

Proof. Define $\alpha(M)=(|M|)!\cdot(|N|-|M|-1)!$ for all $M\subset N$. This number $\alpha(M)$ may be interpreted as follows: given a fixed player $k\in N\setminus M$, $\alpha(M)$ is the number of different orderings of N where positions 1 through $|M|$ are are taken by players in M, position $|M|+1$ is taken by player k, and any remaining positions are taken by players in $N \setminus (M \cup \{k\})$. So, $\sum_{M \subseteq N\setminus\{k\}} \alpha(M) = |N|!$ for all $k \in N$. Note that $\alpha(M)$ depends on M only through M's cardinality.

Let $i, j \in N$ with $\lambda_i \leq \lambda_j$. Using the definition of the Shapley value, we obtain

$$
\begin{aligned}
& c^{\varphi}(\{i\}) - \Phi_i(N, c^{\varphi}) \\
&= c^{\varphi}(\{i\}) - \sum_{M \subseteq N\setminus\{i\}} \frac{\alpha(M)}{|N|!} \Big[c^{\varphi}(M \cup \{i\}) - c^{\varphi}(M) \Big] \\
&= \frac{1}{|N|!} \cdot \sum_{M \subseteq N\setminus\{i\}} \alpha(M) \Big[c^{\varphi}(\{i\}) - c^{\varphi}(M \cup \{i\}) + c^{\varphi}(M) \Big] \\
&= \frac{1}{|N|!} \cdot \Big[\sum_{M \subseteq N\setminus\{i\}: j \notin M} \alpha(M) \Big(\tilde{K}(\lambda_i) - \tilde{K}(\lambda_M + \lambda_i) + \tilde{K}(\lambda_M) \Big) \\
&\qquad + \sum_{M \subseteq N\setminus\{i\}: j \in M} \alpha(M) \Big(\tilde{K}(\lambda_i) - \tilde{K}(\lambda_M + \lambda_i) + \tilde{K}(\lambda_M) \Big) \Big] \\
&\leq \frac{1}{|N|!} \cdot \Big[\sum_{M \subseteq N\setminus\{i\}: j \notin M} \alpha(M) \Big(\tilde{K}(\lambda_i + (\lambda_j - \lambda_i)) - \tilde{K}(\lambda_M + \lambda_i + (\lambda_j - \lambda_i)) + \tilde{K}(\lambda_M) \Big) \\
&\qquad + \sum_{M \subseteq N\setminus\{i\}: j \in M} \alpha(M) \Big(\tilde{K}(\lambda_i) - \tilde{K}(\lambda_M + \lambda_i) + \tilde{K}(\lambda_M) \Big) \Big] \\
&\leq \frac{1}{|N|!} \cdot \Big[\sum_{M \subseteq N\setminus\{i\}: j \notin M} \alpha(M) \Big(\tilde{K}(\lambda_j) - \tilde{K}(\lambda_M + \lambda_j) + \tilde{K}(\lambda_M) \Big) \\
&\qquad + \sum_{M \subseteq N\setminus\{i\}: j \in M} \alpha(M) \Big(\tilde{K}(\lambda_j) - \tilde{K}(\lambda_M + \lambda_i) + \tilde{K}(\lambda_M - \lambda_j + \lambda_i) \Big) \Big] \\
&= c^{\varphi}(\{j\}) - \frac{1}{|N|!} \cdot \Big[\sum_{M \subseteq N\setminus\{i\}: j \notin M} \alpha(M) \Big(c^{\varphi}(M \cup \{j\}) - c^{\varphi}(M) \Big) \\
&\qquad + \sum_{M \subseteq N\setminus\{i\}: j \in M} \alpha(M) \Big(c^{\varphi}(M \cup \{i\}) - c^{\varphi}((M \setminus \{j\}) \cup \{i\}) \Big) \Big] \\
&= c^{\varphi}(\{j\}) - \frac{1}{|N|!} \cdot \Big[\sum_{M \subseteq N\setminus\{j\}: i \notin M} \alpha(M) \Big(c^{\varphi}(M \cup \{j\}) - c^{\varphi}(M) \Big) \\
&\qquad + \sum_{M \subseteq N\setminus\{j\}: i \in M} \alpha(M) \Big(c^{\varphi}(M \cup \{j\}) - c^{\varphi}(M) \Big) \Big] \\
&= c^{\varphi}(\{j\}) - \frac{1}{|N|!} \cdot \sum_{M \subseteq N\setminus\{j\}} \alpha(M) \Big(c^{\varphi}(M \cup \{j\}) - c^{\varphi}(M) \Big) \\
&= c^{\varphi}(\{j\}) - \Phi_j(N, c^{\varphi}),
\end{aligned}
$$

where both inequalities hold by concavity of $\tilde{K}$. In the first inequality, we use that $\lambda_i \leq \lambda_j$. In the second inequality, we also use $\tilde{K}(\lambda_M - \lambda_j + \lambda_j) - \tilde{K}(\lambda_j) \leq \tilde{K}(\lambda_M - \lambda_j + \lambda_i) - \tilde{K}(\lambda_i)$, which implies that $\tilde{K}(\lambda_i) + \tilde{K}(\lambda_M) \leq \tilde{K}(\lambda_j) + \tilde{K}(\lambda_M - \lambda_j + \lambda_i)$. □

The following example provides an illustration of Theorems 6.3 and 6.4.

Example 6.2. Reconsider the elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ of Example 6.1. The allocation $\sum_{\sigma \in \Theta(\varphi)} m^\sigma(N, c^\varphi)/|\Theta(\varphi)|$ is given by $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 2 - \sqrt{2})$. Under this allocation, the benefit to player 1, $1 - \frac{1}{2}\sqrt{2} \approx 0.29$, is the same as the benefit to player 2, which in turn is smaller than the benefit to player 3, $2\sqrt{2} - 2 \approx 0.83$.

The Shapley value of the game (N, c^φ) is given by $(\frac{5}{6} - \frac{1}{6}\sqrt{3}, \frac{5}{6} - \frac{1}{6}\sqrt{3}, \frac{1}{3} + \frac{1}{3}\sqrt{3})$; see Table 1. Under the Shapley value, the benefit to player 1, $c^\varphi(\{1\}) - \Phi_1(N, c^\varphi) = \frac{1}{6}(1 + \sqrt{3}) \approx 0.46$, is the same as the benefit to player 2, which in turn is smaller than the benefit to player 3, $c^\varphi(\{1\}) - \Phi_1(N, c^\varphi) = \sqrt{2} - \frac{1}{3}(1 + \sqrt{3}) \approx 0.50$. ◊

We now know that when the cost function is concave, players with larger attributes get larger benefits under the Shapley value. Since the Shapley value is the average of all marginal vectors, a natural question is whether or not this result extends to all marginal vectors. The following example shows that this is not the case.

Example 6.3. Reconsider the elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ of Example 6.1. Consider the ordering σ on N described by $\sigma^{-1} = (3, 2, 1)$. So, player 3 is in first position, player 2 is in second position, and player 1 is in last position. The marginal allocation $m^\sigma(N, c^\varphi)$ according to this ordering is given by $(2 - \sqrt{3}, \sqrt{3} - \sqrt{2}, \sqrt{2})$. We see that $c^\varphi(\{1\}) - m_1^\sigma(N, c^\varphi) = -1 + \sqrt{3} > 0 = c^\varphi(\{3\}) - m_3^\sigma(N, c^\varphi)$ even though $\lambda_1 < \lambda_3$. ◊

Marginal allocations for single-attribute games are not guaranteed to be coalitionally rational if the cost function is merely elastic (as opposed to concave). Indeed, as shown in Karsten and Basten (2014, Example 5.1), the Shapley value can lie outside the core in such a case. In the remainder, we aim to remedy this.

6.2 Concave functions under elastic functions

The preceding subsection focused on concave cost functions. We now return to elastic cost functions. Given the positive results of Theorems 6.3 and 6.4 for concave cost functions, we will propose a number of ways—one per ordering of the players—to approximate an elastic function with a concave function. As we will show, if this is done in a proper way, then marginal allocations for the single-attribute games induced by these concave functions can yield allocations for the original game that retain the nice properties of their concavicated counterparts.

Any marginal allocation depends on the cost function only through the value of that function at $|N|$ distinct arguments, and we will construct a concave function that approximates the original elastic function as closely as possible at these $|N|$ arguments. These arguments may differ across marginal allocations. However, no marginal allocation depends on the cost function beyond the maximum argument λ_N, which allows us to restrict ourselves to constructing a function with domain $[0, \lambda_N]$. Hence, we will construct a concave function on $[0, \lambda_N]$ for every possible ordering of the players. This function will be made up of straight, consecutive line segments.

Definition 6.2. For any $\lambda, \mu, q, r \in \mathbb{R}_+$ with $\lambda < \mu$ and $q \leq r$, we define the function $L_{(\lambda,q)}^{(\mu,r)} : \mathbb{R} \to \mathbb{R}$ by

$$L_{(\lambda,q)}^{(\mu,r)}(x) = \frac{r-q}{\mu-\lambda}(x-\mu) + r \qquad \text{for all } x \in \mathbb{R} \ .$$

This function represents the straight line through the points (λ, q) and (μ, r).

We next describe how to draw an order-specific concave function under an elastic function. Figure 2 may prove helpful as an illustration.

Procedure 6.1. Let $(N, \tilde{K}, \lambda)$ be an elastic single-attribute situation and let σ be an ordering on N. For notational ease, write $n = |N|$. Define $\Lambda_0^\sigma = 0$ and $\Lambda_i^\sigma = \lambda_{\sigma^{-1}(1)} + \ldots + \lambda_{\sigma^{-1}(i)}$ for all $i \in \{1, 2, \ldots, n\}$. Note that $\Lambda_n = \lambda_N$. We now aim to construct a continuous function that is made up of line segments between points $(\Lambda_0^\sigma, Q_0^\sigma)$, $(\Lambda_1^\sigma, Q_1^\sigma)$, …, $(\Lambda_n^\sigma, Q_n^\sigma)$ such that the resulting function is non-negative, concave, non-decreasing, and not above $\tilde{K}$ on $[0, \Lambda_n^\sigma]$. We now present the procedure for determining the numbers $Q_0^\sigma, Q_1^\sigma, \ldots, Q_n^\sigma$.

We start from the right and set $Q_n^\sigma = \tilde{K}(\Lambda_n^\sigma)$. We then fix Q_{n-1}^σ by drawing the highest possible line through $(\Lambda_n^\sigma, Q_n^\sigma)$ and $(\Lambda_{n-1}^\sigma, Q_{n-1}^\sigma)$ that is not above $\tilde{K}$ on the interval $[\Lambda_{n-1}^\sigma, \Lambda_n^\sigma]$. That is, we take

$$Q_{n-1}^\sigma = \max\left\{ q \in \left[L_{(0,0)}^{(\Lambda_n^\sigma, Q_n^\sigma)}(\Lambda_{n-1}^\sigma), \tilde{K}(\Lambda_{n-1}^\sigma) \right] \,\middle|\, L_{(\Lambda_{n-1}^\sigma, q)}^{(\Lambda_n^\sigma, Q_n^\sigma)}(\ell) \leq \tilde{K}(\ell) \ \forall\, \ell \in [\Lambda_{n-1}^\sigma, \Lambda_n^\sigma] \right\} .$$

The number Q_{n-1}^σ is well-defined because $\tilde{K}$ is elastic.[5]

[5]The set over which we take the maximum is non-empty because if we would take $q = L_{(0,0)}^{(\Lambda_n^\sigma, Q_n^\sigma)}(\Lambda_{n-1})$, then the resulting line is not above $\tilde{K}$ on $[\Lambda_{n-1}^\sigma, \Lambda_n^\sigma]$ by Proposition 2.2. The maximum actually exists because if $n > 1$ then $\tilde{K}$ is continuous on $[\Lambda_{n-1}^\sigma, \Lambda_n^\sigma]$ by Proposition 2.2; otherwise, if $n = 1$ then $Q_0 = 0$ because the interval from which we are to pick q only includes 0.

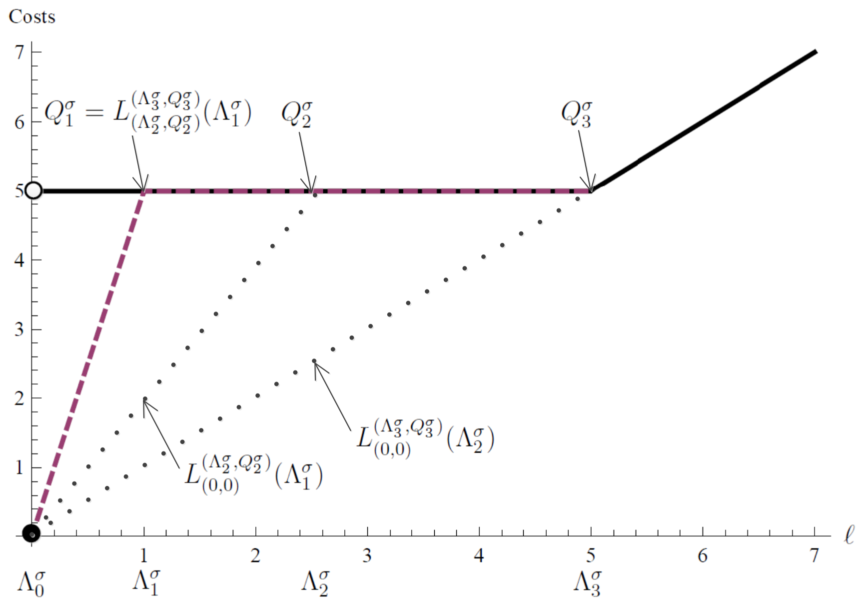

Figure 2: *The function $\tilde{K}$ from Example 6.4 and a σ-concavication (dashed).*

For $j \in \{0, \ldots, n-2\}$, recursively, we then fix Q_j^σ by drawing the highest possible line through $(\Lambda_{j+1}^\sigma, Q_{j+1}^\sigma)$ and $(\Lambda_j^\sigma, Q_j^\sigma)$ that is not above $\tilde{K}$ on the interval $[\Lambda_j^\sigma, \Lambda_{j+1}^\sigma]$ and, moreover, that is at least as steep as the previous line. That is, we take

$$Q_j^\sigma = \max\left\{q \in \left[L_{(0,0)}^{(\Lambda_{j+1}^\sigma,Q_{j+1}^\sigma)}(\Lambda_j^\sigma), L_{(\Lambda_{j+1}^\sigma,Q_{j+1}^\sigma)}^{(\Lambda_{j+2}^\sigma,Q_{j+2}^\sigma)}(\Lambda_j^\sigma)\right] \middle| L_{(\Lambda_j^\sigma,q)}^{(\Lambda_{j+1}^\sigma,Q_{j+1}^\sigma)}(\ell) \le \tilde{K}(\ell)\ \forall\ \ell \in [\Lambda_j^\sigma, \Lambda_{j+1}^\sigma]\right\}.$$

The number Q_j^σ is well-defined because, again, $\tilde{K}$ is elastic.[6]

Given the numbers $Q_0^\sigma, Q_1^\sigma, \ldots, Q_n^\sigma$ as defined above, the σ-concavication $\tilde{K}_\sigma^{\text{conc}} : [0, \lambda_N] \to \mathbb{R}_+$ is given by

$$\tilde{K}_\sigma^{\text{conc}}(\ell) = L_{(\Lambda_{j-1}^\sigma,Q_{j-1}^\sigma)}^{(\Lambda_j^\sigma,Q_j^\sigma)}(\ell) \text{ for } j \in \{1, \ldots, n\} \text{ with } \ell \in [\Lambda_{j-1}^\sigma, \Lambda_j^\sigma].$$

for all $\ell \in (0, \lambda_N]$ and $\tilde{K}_\sigma^{\text{conc}}(0) = 0$.

[6] The set over which we take the maximum is non-empty, for two reasons. First, the interval from which we are to pick q is non-empty because, by construction, $L_{(0,0)}^{(\Lambda_{j+2}^\sigma,Q_{j+2}^\sigma)}(\Lambda_{j+1}) \le L_{(\Lambda_{j+1},Q_{j+1})}^{(\Lambda_{j+2},Q_{j+2})}(\Lambda_{j+1})$; this implies that $L_{(0,0)}^{(\Lambda_{j+1},Q_{j+1})}$ is steeper than $L_{(\Lambda_{j+1},Q_{j+1})}^{(\Lambda_{j+2},Q_{j+2})}$. Second, by construction, $Q_{j+1}^\sigma \le \tilde{K}(\Lambda_{j+1}^\sigma)$; hence, if we would take $q = L_{(0,0)}^{(\Lambda_{j+1}^\sigma,Q_{j+1}^\sigma)}(\Lambda_j)$, then on $[0, \Lambda_{j+1}]$ the resulting line is not above $L_{(0,0)}^{(\Lambda_{j+1}^\sigma,\tilde{K}(\Lambda_{j+1}^\sigma))}$, which in turn is not above $\tilde{K}$ by Proposition 2.2. The maximum actually exists because if $j \ge 1$ then $\tilde{K}$ is continuous on $[\Lambda_{n-1}^\sigma, \Lambda_n^\sigma]$ by Proposition 2.2; otherwise, if $j = 0$ then $\tilde{K}(0) = 0$ implies $Q_j = 0$.

The following lemma collects several properties of a σ-concavication that follow directly from its definition.

Lemma 6.5. *Let $(N, \tilde{K}, \lambda) \in \mathcal{E}$, and let σ be an ordering on N.*

(i) $\tilde{K}^{conc}_{\sigma}$ *is concave.*
(ii) $\tilde{K}^{conc}_{\sigma}(\ell) \leq \tilde{K}(\ell)$ *for all* $\ell \in [0, \lambda_N)$ *and* $\tilde{K}^{conc}_{\sigma}(\lambda_N) = \tilde{K}(\lambda_N)$.
(iii) *If* $\tilde{K}$ *is concave, then* $\tilde{K}^{conc}_{\sigma}(\Lambda^{\sigma}_i) = \tilde{K}(\Lambda^{\sigma}_i)$ *for all* $i \in \{0, 1, \ldots, n\}$.
(iv) $\tilde{K}^{conc}_{\sigma}(0) = 0$.

The following example illustrates the construction of a σ-concavication.

Example 6.4. Consider the single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ with $N = \{1, 2, 3\}$, $\lambda_1 = 1$, $\lambda_2 = 1.5$, $\lambda_3 = 2.5$, and elastic cost function $\tilde{K}$ equal to the function considered in Example 3.2, which is for convenience represented again in Figure 2. For the ordering σ with $\sigma^{-1} = (1, 2, 3)$, the σ-concavication $\tilde{K}^{\text{conc}}_{\sigma} : [0, 5] \to \mathbb{R}$ corresponding to attribute vector λ is given by

$$\tilde{K}^{\text{conc}}_{\sigma}(\ell) = \begin{cases} 5\ell & \text{if } \ell \in [0, 1]; \\ 5 & \text{if } \ell \in (1, 5]. \end{cases}$$

See Figure 2. For other orderings, the construction is similar. ◊

The following example illustrates the construction of a σ-concavication for a more complicated situation.

Example 6.5. Consider the elastic single-attribute situation $(N, \tilde{K}, \lambda)$ with $N = \{1, 2, 3\}$, $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6$, and cost function $\tilde{K} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\tilde{K}(\ell) = \begin{cases} 4\ell & \text{if } \ell \in [0, 1]; \\ 4 & \text{if } \ell \in (1, 3]; \\ 2 + \ell \cdot 2/3 & \text{if } \ell \in (3, 6]; \\ 6 + ((\ell - 6)/4)^2 & \text{if } \ell \in (6, 10]; \\ 7 & \text{if } \ell > 10; \end{cases}$$

This function and each of its σ-concavications are graphically represented in Figure 3. The function $\tilde{K}$ is elastic because

- for ℓ on $(0, 1]$, $\tilde{K}(\ell)/\ell = 4$;
- for ℓ on $(1, 3]$, $\tilde{K}(\ell)/\ell = 4/\ell$ is decreasing in ℓ and ranges from 4 to 4/3;
- for ℓ on $(3, 6]$, $\tilde{K}(\ell)/\ell = 2/3 + 2/\ell$ is decreasing in ℓ and ranges from 4/3 to 1;

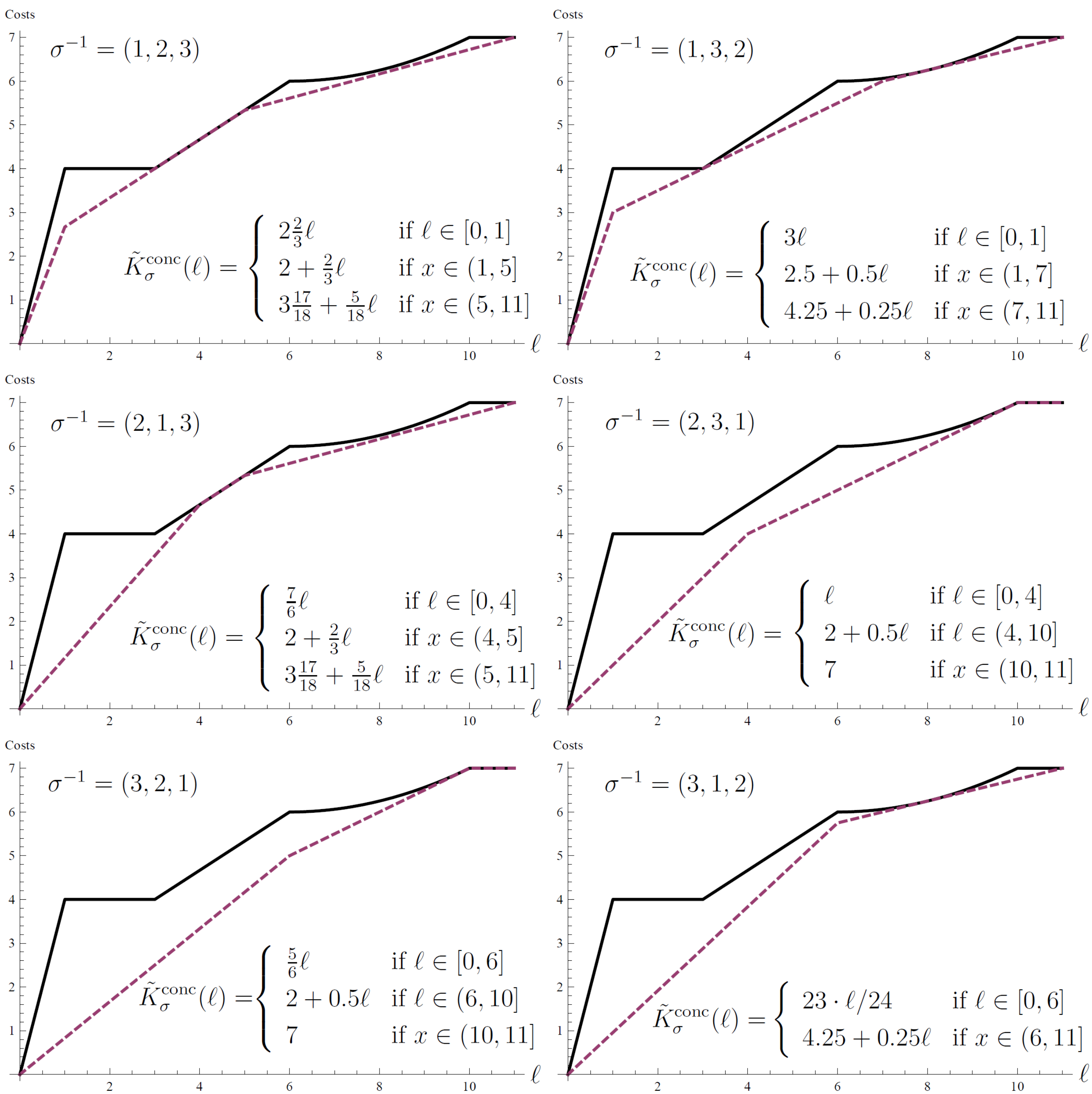

Figure 3: *The function $\tilde{K}$ from Example 6.5 and all its σ-concavications (dashed).*

- for ℓ on $(6, 10]$, $\tilde{K}(\ell)/\ell = 6/\ell + (\ell - 12 + 36/\ell)/16$ is decreasing[7] in ℓ and ranges from 1 to 7/10;

- for $\ell > 10$, $\tilde{K}(\ell)/\ell = 7/\ell$ is decreasing in ℓ with a maximal value of 7/10.

We illustrate the construction of a σ-concavication for two orderings. First, consider the ordering σ with $\sigma^{-1} = (2, 3, 1)$. So, player 2 is in first position, player 3 is in second position, and player 1 is in third position. Hence, $\Lambda_1^\sigma = 4$, $\Lambda_2^\sigma = 10$, and $\Lambda_3^\sigma = 11$. Obviously, $Q_3^\sigma = 7$, $Q_2^\sigma = 7$, and $Q_0^\sigma = 0$. Consider Q_1^σ, i.e., the largest $q \leq \tilde{K}(\Lambda_1^\sigma) = 4\frac{2}{3}$ such that $L_{(4,q)}^{(10,7)}(\ell) \leq \tilde{K}(\ell)$ for all $\ell \in [4, 10]$. Since the derivative of $6 + ((\ell - 6)/4)^2$ evaluated at $\ell = 10$ is equal to 0.5, any $q > \tilde{K}(\Lambda_2^\sigma) - 0.5 \cdot (\Lambda_2^\sigma - \Lambda_1^\sigma) = 4$ would result in going above the graph of $\tilde{K}$. Yet, $q = 4$ would not take us above the graph of $\tilde{K}$, so $Q_1^\sigma = 4$.

Next, consider the ordering σ with $\sigma^{-1} = (3, 1, 2)$. So, player 3 is in first position, player 1 is in second position, and player 2 is in third position. Hence, $\Lambda_1^\sigma = 6$, $\Lambda_2^\sigma = 7$, $\Lambda_3^\sigma = 11$. Obviously, $Q_3^\sigma = 7$ and $Q_0^\sigma = 0$. Consider Q_2^σ, i.e., the largest $q \leq \tilde{K}(\Lambda_2^\sigma) = 6\frac{1}{16}$ such that $L_{(7,q)}^{(11,7)}(\ell) \leq \tilde{K}(\ell)$ for all $\ell \in [7, 11]$. For $q \in [5, 6\frac{1}{16}]$, standard optimization techniques reveal that $6 + ((\ell - 6)/4)^2 - L_{(7,q)}^{(11,7)}(\ell)$ as a function of ℓ on $[7, 11]$ has a unique minimizer $\ell^*(q) = 20 - 2q$. Consequently, $6 + ((\ell^*(q) - 6)/4)^2 = L_{(7,q)}^{(11,7)}(\ell^*(q))$ if and only if $q = 6$. Hence, $Q_2^\sigma = 6$. Finally, the line $L_{(7,6)}^{(11,7)}$ is not above $\tilde{K}$ on $[6, 7]$, so $Q_2^\sigma = L_{(7,6)}^{11,7)}(6) = 5.75$.

The construction of the σ-concavication for the other orderings is similar; see Figure 3. ◊

6.3 Two concavicated rules

Based on σ-concavicated situations, we will introduce two new allocation rules on elastic single-attribute situations.

Definition 6.3. For an elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ and an ordering σ on N, we call $\varphi(\sigma) = (N, \tilde{K}_\sigma^{\text{conc}}, \lambda)$ the corresponding σ-*concavicated situation.*

We remark that although the domain of $\tilde{K}_\sigma^{\text{conc}}$ is $[0, \lambda_N]$ and not $\mathbb{R}_+$, this does not pose a problem for the two allocation rules that we define next, as they do not depend on the cost function beyond λ_N.

Definition 6.4. The *concavicated increasing marginal rule* $\mathscr{M}$ on $\mathcal{E}$ assigns to each elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ the allocation $\mathscr{M}(\varphi) = \sum_{\sigma \in \Theta(\varphi)} m^\sigma(N, c^{\varphi(\sigma)})/|\Theta(\varphi)|$.

[7] Indeed, its derivative $(1 - 132/\ell^2)/16$ is negative for ℓ on $(6, 10]$.

M	λ_M	$c^{\varphi}(M)$	$\sum_{i\in M}\mathscr{M}_i(\varphi)$	$\sum_{i\in M}\mathscr{A}_i(\varphi)$
$\{1\}$	1	1	$2\frac{2}{3}$	$1\frac{7}{72}$
$\{2\}$	4	$4\frac{2}{3}$	$2\frac{2}{3}$	$2\frac{5}{9}$
$\{3\}$	6	6	$1\frac{2}{3}$	$3\frac{25}{72}$
$\{1,2\}$	5	$5\frac{1}{3}$	$5\frac{1}{3}$	$3\frac{47}{72}$
$\{1,3\}$	7	$6\frac{1}{16}$	$4\frac{1}{3}$	$4\frac{32}{72}$
$\{2,3\}$	10	7	$4\frac{1}{3}$	$5\frac{65}{72}$
N	11	7	7	7

Table 2: *The game and concavicated marginal allocations in Example 6.6.*

Definition 6.5. The *concavicated average marginal rule* $\mathscr{A}$ on $\mathcal{E}$ assigns to each elastic single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ the allocation $\mathscr{A}(\varphi) = \sum_{\sigma\in\Pi(N)} m^{\sigma}(N, c^{\varphi(\sigma)})/|\Pi(N)|$.

The following lemma states that if the cost function is concave, then marginal allocations are unchanged by concavicated situations.

Lemma 6.6. *Let $\varphi = (N, \tilde{K}, \lambda) \in \mathcal{E}$ with concave $\tilde{K}$.*
(i) For all orderings σ on N and all $i \in N$, it holds that $m_i^{\sigma}(N, c^{\varphi}) = m_i^{\sigma}\left(N, c^{\varphi(\sigma)}\right)$.
(ii) $\mathscr{M}(\varphi) = \sum_{\sigma\in\Theta(\varphi)} m^{\sigma}(N, c^{\varphi})/|\Theta(\varphi)|$.
(iii) $\mathscr{A}(\varphi) = \Phi(N, c^{\varphi})$.

Proof. For Part (i), let $\sigma \in \Pi(N)$ and $i \in N$. By Part (iii) of Lemma 6.5, $\tilde{K}^{conc}(\Lambda^{\sigma}_{\sigma(i)}) - \tilde{K}^{conc}(\Lambda^{\sigma}_{\sigma(i)-1}) = \tilde{K}^{conc}(\lambda_{P_i^{\sigma}}+\lambda_i) - \tilde{K}^{conc}(\lambda_{P_i^{\sigma}})$, which implies that $m_i^{\sigma}(N, c^{\varphi}) = m_i^{\sigma}\left(N, c^{\varphi(\sigma)}\right)$.

Part (ii) and (iii) follow immediately from Part (i). □

The concavicated rules only evaluate a σ-concavication at the arguments Λ_0^{σ}, Λ_1^{σ}, ..., Λ_n^{σ}. The intermediate behavior of the σ-concavication, however, is still important because it affects the corresponding game. Despite not taking into account this intermediate behavior, our concavicated rules still lead to core allocations. Before proving this, we provide an illustration.

Example 6.6. Reconsider the single-attribute situation $\varphi = (N, \tilde{K}, \lambda)$ from Example 6.5. The associated single-attribute game (N, c^{φ}) is described in Table 2. For any ordering σ on N, the cost function for the corresponding σ-concavicated situation $\varphi(\sigma) = (N, \tilde{K}^{\text{conc}}_{\sigma}, \lambda)$ is given in Figure 3, and the corresponding marginal allocation is described in Table 3.

The concavicated increasing marginal allocation $\mathscr{M}(\varphi)$ is given by $(2\frac{2}{3}, 2\frac{2}{3}, 1\frac{2}{3})$ because $\Theta(\varphi)$ only consists of the ordering σ with $\sigma^{-1} = (1, 2, 3)$. The concavicated average

σ^{-1}	$m_1^\sigma\left(N, c^{\varphi(\sigma)}\right)$	$m_2^\sigma\left(N, c^{\varphi(\sigma)}\right)$	$m_3^\sigma\left(N, c^{\varphi(\sigma)}\right)$
(1,2,3)	$2\frac{2}{3}$	$2\frac{2}{3}$	$1\frac{2}{3}$
(1,3,2)	3	1	3
(2,1,3)	$\frac{2}{3}$	$4\frac{2}{3}$	$1\frac{2}{3}$
(2,3,1)	0	4	3
(3,2,1)	0	2	5
(3,1,2)	$\frac{1}{4}$	1	$5\frac{3}{4}$
Sum	$6\frac{7}{12}$	$15\frac{1}{3}$	$20\frac{1}{12}$

Table 3: *The marginal allocations corresponding to all σ-concavicated situations in Example 6.6.*

marginal allocation $\mathscr{A}(\varphi)$ is obtained by averaging all marginal vectors in Table 3, which results in $(1\frac{7}{72}, 2\frac{5}{9}, 3\frac{25}{72})$. Note that $\mathscr{A}$ is *not* the Shapley value of a "straightforwardly" derived game because each marginal allocation is based on a different σ-concavicated situation and thus on a different game.

It is easy to infer from Table 2 that both $\mathscr{M}(\varphi)$ and $\mathscr{A}(\varphi)$ are core allocations for (N, c^φ). ◊

Theorem 6.7. *Both $\mathscr{M}$ and $\mathscr{A}$ satisfy the coalitional rationality property on $\mathcal{E}$.*

Proof. Let $\varphi = (N, \tilde{K}, \lambda)$ be an elastic single-attribute situation, and let σ be any ordering on N. Then, by Part (i) of Lemma 6.1, the single-attribute game associated with the σ-concavicated situation $\varphi(\sigma)$ is concave. By Part (ii) of Lemma 6.1, it follows that $m^\sigma\left(N, c^{\varphi(\sigma)}\right) \in \mathcal{C}\left(N, c^{\varphi(\sigma)}\right)$. By Part (ii) of Lemma 6.5, it holds that $\mathcal{C}\left(N, c^{\varphi(\sigma)}\right) \subseteq \mathcal{C}(N, c^\varphi)$. Hence, $m^\sigma\left(N, c^{\varphi(\sigma)}\right) \in \mathcal{C}(N, c^\varphi)$ as well. Since $\mathscr{M}(\varphi)$ and $\mathscr{A}(\varphi)$ are defined as averages of marginal allocations for games associated with concavicated situations, their coalitional rationality is immediate. □

Both $\mathscr{M}$ and $\mathscr{A}$ lack the benefit ordering property because, by Theorem 6.7, they prescribe the unique core allocation for the situation of Example 3.2, which described our impossibility result. However, both rules satisfy the relaxation BOC.

Theorem 6.8. *Both $\mathscr{M}$ and $\mathscr{A}$ satisfy the benefit ordering property under concavity.*

Proof. Let $\varphi = (N, \tilde{K}, \lambda)$ be any elastic single-attribute situation with concave $\tilde{K}$, and let σ be any ordering on N. By Part (i) of Lemma 6.5, $\tilde{K}^{conc}$ is concave. Hence, Lemma 6.6 applies. Benefit ordering under concavity of $\mathscr{M}$ then follows from Theorem 6.3 and from

Rule	CR	IR	BO	BOC
$\mathscr{P}$	$\surd$	$\surd$	X	X
$\mathscr{S}$	X	$\surd$	X	$\surd$
$\mathscr{B}$	X	$\surd$	$\surd$	$\surd$
$\mathscr{M}$	$\surd$	$\surd$	X	$\surd$
$\mathscr{A}$	$\surd$	$\surd$	X	$\surd$

Table 4: *Overview of the various rules and their properties. Legend for rules: $\mathscr{P}$ is the proportional rule, $\mathscr{S}$ is the serial rule, $\mathscr{B}$ is the benefit-proportional rule, $\mathscr{M}$ is the concavicated increasing marginal rule, and $\mathscr{A}$ is the concavicated average marginal rule. Legend for properties: CR is the coalitional rational property, IR is the individual rationality property, BO is the benefit ordering property, and BOC is the benefit ordering property under concavity.*

Part (*ii*) of Lemma 6.6, while benefit ordering under concavity of $\mathscr{A}$ follows from Theorem 6.4 and from Part (*iii*) of Lemma 6.6. $\square$

We conclude that both $\mathscr{M}$ and $\mathscr{A}$ satisfy CR, IR, and BOC, but lack BO.

7 Conclusion

Table 4 presents an overview of our main results. We have shown that coalitional rationality and benefit ordering are incompatible. At the same time, we have found two rules that satisfy the combination of coalitional rationality and benefit ordering under concavity: the concavicated increasing marginal rule $\mathscr{M}$ and the concavicated average marginal rule $\mathscr{A}$. Accordingly, if we desire to improve on the proportional rule with regard to the ordering of players' benefits, while keeping coalitional rationality intact, then these rules would be appealing solutions. Remarkably, $\mathscr{A}$ coincides with the Shapley value—one of the most celebrated solutions for cooperative games—when the cost function is concave. Yet, $\mathscr{A}$ remedies the possible non-stability of the Shapley value when the cost function is merely elastic.

We conclude by providing four directions for future research. A first direction would be on alternative concavications. Indeed, the collection of functions described in Procedure 6.1 are not the *only* concave functions that fit under an elastic function. Although we believe that Procedure 6.1 is compelling because of the property described in Part (*iii*) of Lemma 6.5 and because, from a computational perspective, it merely requires the determination of $|N|$ straight line segments, future research may look for alternative concavications that retain these nice properties while additionally being continuous in the attribute vector.

A second direction for future research is on other fairness properties and other allocation rules. There are, of course, many other allocation cost rules possible for elastic single-attribute situations beyond the ones that we considered. The decreasing serial rule proposed in de Frutos (1998) or the equal profit method proposed in Frisk et al. (2010) might be interesting. At the same time, we have not exhausted the list of reasonable fairness criteria. For example, population monotonicity (cf. Sprumont, 1990) might be interesting because of the link between elasticity and population monotonicity established by Özen et al. (2011).

A third possible research direction is on a study of the properties exhibited by rules on a restricted domain. Indeed, all the properties we have considered deal with the domain of elastic single-attribute situations. If we would restrict the domain of a rule to a specific class of situations (e.g., whose cost function represents the optimal costs in an Erlang loss model as in Example 2.3) then it is possible that a rule would exhibit specific behavior on that restricted domain.

Finally, it would be valuable to have an axiomatic characterization of some or all of the allocation rules that we studied in this paper. It would be particularly interesting to see if the properties that we considered can be used in such a characterization.

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