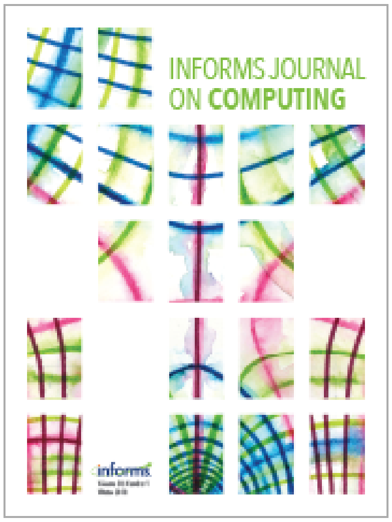

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INFORMS JOURNAL ON COMPUTING
Vol. 34, No. 1, January–February 2022, pp. 656–669
ISSN 1091-9856 (print), ISSN 1526-5528 (online)
http://pubsonline.informs.org/journal/ijoc

The Maximum *k*-Colorable Subgraph Problem and Related Problems

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Received: January 27, 2020
Revised: October 22, 2020; January 22, 2021
Accepted: March 12, 2021
Published Online in Articles in Advance: August 26, 2021

https://doi.org/10.1287/ijoc.2021.1086

Abstract. The maximum *k*-colorable subgraph (M*k*CS) problem is to find an induced *k*-colorable subgraph with maximum cardinality in a given graph. This paper is an in-depth analysis of the M*k*CS problem that considers various semidefinite programming relaxations, including their theoretical and numerical comparisons. To simplify these relaxations, we exploit the symmetry arising from permuting the colors, as well as the symmetry of the given graphs when applicable. We also show how to exploit invariance under permutations of the subsets for other partition problems and how to use the M*k*CS problem to derive bounds on the chromatic number of a graph. Our numerical results verify that the proposed relaxations provide strong bounds for the M*k*CS problem and that those outperform existing bounds for most of the test instances.

Summary of Contribution: The maximum *k*-colorable subgraph (M*k*CS) problem is to find an induced *k*-colorable subgraph with maximum cardinality in a given graph. The M*k*CS problem has a number of applications, such as channel assignment in spectrum sharing networks (e.g., Wi-Fi or cellular), very-large-scale integration design, human genetic research, and so on. The M*k*CS problem is also related to several other optimization problems, including the graph partition problem and the max-*k*-cut problem. The two mentioned problems have applications in parallel computing, network partitioning, floor planning, and so on. This paper is an in-depth analysis of the M*k*CS problem that considers various semidefinite programming relaxations, including their theoretical and numerical comparisons. Further, our analysis relates the M*k*CS results with the stable set and the chromatic number problems. We provide extended numerical results that verify that the proposed bounding approaches provide strong bounds for the M*k*CS problem and that those outperform existing bounds for most of the test instances. Moreover, our lower bounds on the chromatic number of a graph are competitive with existing bounds in the literature.

History: Accepted by Pascal Van Hentenryck, Area Editor for Modeling: Methods and Analysis.
Supplemental Material: The online supplement is available at https://doi.org/10.1287/ijoc.2021.1086.

Keywords: *k*-colorable subgraph problem • stable set • chromatic number of a graph • generalized theta number • semidefinite programming • Johnson graphs • Hamming graphs

1. Introduction

The maximum *k*-colorable subgraph (M*k*CS) problem is to find the largest induced subgraph in a given graph that can be colored in *k* colors such that no two adjacent vertices have the same color. The M*k*CS problem is also known as the *maximum k-partite induced subgraph problem* since the *k*-coloring corresponds to a *k*-partition of the subgraph. The M*k*CS problem for $k = 2$ is also known as the *maximum bipartite subgraph problem*.

In the literature, the name "maximum *k*-colorable subgraph problem" is sometimes used for the maximum *k*-cut problem (see, e.g., Papadimitriou and Yannakakis 1991, Frieze and Jerrum 1995). In the latter problem, one searches for the partition of the graph into *k* subsets such that the number of edges crossing the subsets is maximized. If one colors vertices in the resulting subsets in different colors, then the crossing edges are properly colored; that is, the end points of these edges have different colors. However, the M*k*CS and the maximum *k*-cut are different problems, and we do not consider the latter problem in this paper. We refer interested readers to Goemans and Williamson (1994), Frieze and Jerrum (1995), de Klerk et al. (2004), Sotirov (2014), Rendl (2016), and van Dam and Sotirov (2016) for more information on the maximum *k*-cut problem and related semidefinite programming (SDP) relaxations.

The MkCS problem falls into the class of NP-hard problems considered by Lewis and Yannakakis (1980). Moreover, even approximating this problem is NP-hard (see Lund and Yannakakis 1993). For $k = 1$, the MkCS problem reduces to the famous maximum stable set problem, which was shown to be NP-hard by Karp (1972). Another well-known problem from the list of Karp (1972) related to the MkCS problem is the chromatic number problem. The chromatic number problem is to determine whether the vertices of a given graph can be colored in k colors. If one can solve the MkCS problem for any given number of colors, then one can also solve the maximum stable set and the chromatic number problems. However, efficient algorithms for the latter two problems do not necessarily result in efficient algorithms for the MkCS problem. For instance, the chromatic number and the stability number on perfect graphs can be computed in polynomial time, whereas the MkCS problem is NP-hard on chordal graphs, which comprises a subfamily of the set of perfect graphs (see Yannakakis and Gavril 1987). However, there are special classes of graphs for which the MkCS problem is polynomial-time solvable. Some examples are graphs where every odd cycle has two noncrossing chords for any k (Addario-Berry et al. 2010), clique-separable graphs for $k = 2$ (Addario-Berry et al. 2010), chordal graphs for fixed k (Yannakakis and Gavril 1987), interval graphs for any k (Yannakakis and Gavril 1987), and circular-arc graphs and tolerance graphs for $k = 2$ (Narasimhan 1989). It is known that the size of the maximum stable set of the Kneser graphs $K(v, d)$, where $v \geq 2d$, equals $\binom{v-1}{d-1}$ (see Erdős et al. 1961). For the Kneser graphs $K(v, 2)$, the size of the maximum k-colorable subgraph is also known (see, e.g., Füredi 1983).

The MkCS problem for $k = 2$ has been studied by, among others, Lee et al. (1992), Godsil and Royle (2001), Hüffner (2005), Fouilhoux and Mahjoub (2006), and Bresar and Valencia-Pabon (2019). However, the MkCS problem for $k > 2$ is rarely considered in the literature. Januschowski and Pfetsch (2011a) notice that such a lack of attention might be related to the connection of the MkCS problem to the earlier mentioned prominent problems. One of the few significant sources of information of the MkCS problem for $k > 2$, but also $k = 2$, are Narasimhan (1989) and Narasimhan and Manber (1990), where the authors introduce an upper bound on the optimal value of the MkCS problem called the *generalized ϑ-number*. Namely, they generalize the concept of the famous ϑ-number by Lovász (1979), which is an upper bound on the size of the maximum stable set of a graph. Alizadeh (1995) formulated the generalized ϑ-number problem using SDP. Mohar and Poljak (1993) present the bound by Narasimhan and Manber (1990) among important applications of eigenvalues of graphs in combinatorial optimization. To our knowledge, the quality of the generalized ϑ-number has never been evaluated.

Other related work considers integer programming (IP) formulations of the MkCS problem (see, e.g., Campêlo and Corrêa 2010; Januschowski and Pfetsch 2011a, b). Januschowski and Pfetsch (2011a) and Campêlo and Corrêa (2010) provide computational results for $k \geq 2$. In particular, Januschowski and Pfetsch (2011a) consider graphs with symmetry, which enables them to provide numerical results for large graphs up to 1,085 vertices. Hertz et al. (2018) analyze the performance of various existing online algorithms for the MkCS problem, and Bresar and Valencia-Pabon (2019) provide theoretical lower and upper bounds for the MkCS problem for $k \geq 2$ on the Kneser graphs.

It is worth mentioning that the MkCS problem has a number of applications, such as channel assignment in spectrum-sharing networks (e.g., Wi-Fi or cellular) (Halldórsson et al. 2004, Koster and Scheffel 2007, Subramanian et al. 2007, Hertz et al. 2016, Bentert et al. 2019), very-large-scale integration design (Marek-Sadowska 1984, Fouilhoux and Mahjoub 2012), and human genetic research (Lippert et al. 2002, Fouilhoux and Mahjoub 2012). Let us provide a brief description of one of the applications, namely, the wavelength assignment problem in optical networks (Koster and Scheffel 2007). The problem is to assign wavelengths to as many light paths as possible, considering that intersecting light paths do not use the same wavelength. Hence, light paths correspond to vertices and wavelengths to colors. There is an edge between two vertices if the corresponding light paths intersect. The number of wavelengths is restricted by the capacity of the network.

1.1. Outline and Main Results

This paper is an in-depth analysis of the MkCS problem that includes various semidefinite programming relaxations and their theoretical and numerical comparisons. This analysis also extends results for the MkCS problem to other graph partition problems and also relates the MkCS results with the stable set and the chromatic number problems.

We begin our study with the eigenvalue bound by Narasimhan and Manber (1990), which is also known as the *generalized ϑ-number*. We present the generalized ϑ-number as the optimal solution of an SDP relaxation (see also Alizadeh 1995). In order to strengthen the mentioned SDP relaxation, we add nonnegativity constraints to the matrix variable and call the solution of the resulting SDP relaxation the *generalized ϑ'-number*. This number can be seen as the generalization of the Schrijver ϑ'-number (Schrijver 1979). Both the generalized ϑ-number and the ϑ'-number require solving an SDP relaxation with one matrix variable of the order n, where n is the number of vertices in the graph.

Next, we derive vector-lifting- and matrix-lifting-based SDP relaxations. The sizes of matrix variables in the resulting relaxations depend on n and k. We reduce the sizes of the SDP relaxations by exploiting the invariance of the MkCS problem under permutations of the colors. In particular, we exploit the fact that all constraints in the relaxations are satisfied for any color labeling, and the objective does not change if the labeling changes. This property is inherited by our SDP relaxations from the IP formulations of the MkCS problem. By exploiting color invariance, our matrix-lifting SDP relaxation reduces to a model with one SDP constraint of the order $n+1$, and our strongest SDP relaxation reduces to a model with two SDP constraints of the order $n+1$ and n, respectively, for a graph with n vertices. Thus, it turns out that matrix sizes in the vector and matrix-lifting relaxations are independent of k.

To strengthen our relaxations, we add inequalities from the Boolean quadric polytope. We also show how to further reduce our SDP relaxations for highly symmetric graphs. This reduction results in a linear program arising from the generalized ϑ'-number, or in programs with a linear objective, one second-order cone constraint, and many linear constraints.

Since the k-colorable subgraph problem is also a graph partition problem, we are able to apply our symmetry reduction approach based on the invariance under permutations of the subsets to other partition problems. In particular, we prove in an elegant way that the vector- and matrix-lifting relaxations for the k-equipartition problem are equivalent. We obtain a similar result for the max-k-cut problem.

Finally, we evaluate the quality of all here presented SDP upper bounds on instances from Januschowski and Pfetsch (2011a) and Campêlo and Corrêa (2010). We also propose two heuristic approaches to compute lower bounds for the MkCS problem. Our computational results show that our lower and upper bounds for the MkCS problem are strong and can be computed efficiently for dense graphs or highly symmetric graphs. Our lower bounds on the chromatic number of a graph are competitive with existing bounds in the literature.

This paper is organized as follows. We present several equivalent integer programming formulations of the MkCS problem in Section 2. In Section 3, we present first the generalized ϑ-number by Narasimhan and Manber (1990) and then the related strengthened bound that we call the *generalized* ϑ'*-number*. In Section 4, we first propose two vector-lifting SDP relaxations and compare them, and then we show how to apply symmetry reduction on colors in order to reduce their sizes. In order to further tighten our SDP relaxations, we consider adding inequalities from the Boolean quadric polytope in Section 4.2. In Section 4.3, we show that the Schrijver's ϑ'-number on the Cartesian product of the complete graph on k vertices and the graph under consideration equals the optimal value of our weaker vector-lifting relaxation for the original graph (see also the online supplement). A matrix-lifting SDP relaxation and its symmetry-reduced version are given in Section 5. In Section 6, we show how to further simplify and reduce our SDP relaxations by exploiting symmetry of graphs. The application of symmetry reduction on colors is extended to other graph partition problems in Section 7. Numerical results are in the online supplement. We summarize the results in Section 8.

1.1.1. Notation. The space of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$. For $A, B \in \mathbb{S}^n$, the trace inner product of A and B is denoted by $\langle A, B\rangle := \mathrm{trace}(AB)$. For two matrices $A, B \in \mathbb{R}^{n\times n}$, $A \geq B$ means $a_{ij} \geq b_{ij}$, for all i, j.

We use the notation I_n (resp., J_n) for the identity matrix (resp., the matrix of all ones) of order n, and we use e_n to denote the vector of all ones. If the dimension of the matrices is clear from the context, then we omit the subscript. We use u_i to denote the ith standard basis vector. Here, we use $[m]$ to denote the set $\{1, \dots, m\}$.

The "vec" operator stacks the columns of a matrix, whereas the "diag" operator maps an $n \times n$ matrix to the n-vector given by its diagonal. The adjoint operator of "diag" is denoted by "Diag." The Kronecker product $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $A_{i,j}B$, $i = 1, \dots, p$, $j = 1, \dots, q$.

2. Problem Formulation

Let $G = (V, E)$ be a simple undirected graph with the vertex set V and the edge set E. Let $|V| = n$, and let k be a given integer such that $1 \leq k \leq n-1$. We say that G is k-*colorable* if one can assign to each vertex in G one of the k colors such that adjacent vertices do not have the same color. A graph $G' = (V', E')$ is called an *induced subgraph* of a given graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ is the set of all edges in E connecting the vertices in V'.

The maximum k-colorable subgraph problem is to find an induced k-colorable subgraph with maximum cardinality in the given graph G. We denote by $\alpha_k(G)$ the number of vertices in the maximum k-colorable subgraph of G. When $k = 1$, the MkCS problem corresponds to the stable set problem, that is, $\alpha_1(G) = \alpha(G)$, where $\alpha(G)$ denotes the stability number of G (the maximum cardinality of a stable set in G).

For any $k \geq 1$, the MkCS problem can be formulated as an integer problem. Let $X \in \{0,1\}^{n\times k}$ be the matrix with one in the entry (i, r) if vertex $i \in [n]$ is colored with color $r \in [k]$ and zero otherwise. An IP formulation for the MkCS problem is given by (1a)–(1c):

$$\alpha_k(G) = \max_{X \in \{0,1\}^{n\times k}} \sum_{i\in[n], r\in[k]} X_{ir} \tag{1a}$$

$$\text{(s.t.)} \quad X_{ir}X_{jr} = 0, \ \text{ for all } \{ij\} \in E, r \in [k] \tag{1b}$$

$$\sum_{r\in[k]} X_{ir} \le 1, \ \text{ for all } i \in [n]. \tag{1c}$$

Here, constraints ((1b)) ensure that two adjacent vertices are not colored with the same color, and constraints (1c) ensure that each vertex is colored with at most one color.

Alternative IP formulation for the MkCS problem can be obtained by adding a binary slack variable to each inequality constraint in (1c), that is, by replacing those constraints with the following ones:

$$\sum_{r\in[k+1]} X_{ir} = 1, \ \text{ for all } i \in [n]. \tag{2}$$

It is known that a model with equality constraints may provide stronger relaxations than an alternative one (see, e.g., Burer 2009, Rendl and Sotirov 2018). In Section 4, we use the IP model with equality constraints (2) to derive our strongest SDP relaxation for the MkCS problem. Another IP model for the MkCS problem is obtained by replacing ((1b)) constraints with the following ones:

$$X_{ir} + X_{jr} \le 1, \quad \{ij\} \in E, \ r \in [k]. \tag{3}$$

The IP formulation (1a), (1c), and (3) is exploited in Januschowski and Pfetsch (2011a, b).

3. The Generalized ϑ- and ϑ'-Numbers

In this section, we present the generalized ϑ-number by Narasimhan and Manber (1990), which is an upper bound for the MkCS problem. This eigenvalue upper bound was reformulated as an SDP relaxation in Alizadeh (1995). Here, we strengthen the mentioned SDP relaxation by adding nonnegativity constraints and introduce the generalized ϑ'-number.

3.1. Eigenvalue and SDP Formulations of the Generalized ϑ-number

Narasimhan and Manber (1990) introduce $\vartheta_k(G)$, the generalized ϑ-number, as an upper bound for $\alpha_k(G)$:

$$\alpha_k(G) \le \vartheta_k(G) = \min_{A\in\mathbb{S}^n} \left\{ \sum_{i=1}^k \lambda_i(A) : A_{ij} = 1 \text{ for } \{ij\} \notin E \text{ or } i = j \right\}, \tag{4}$$

where $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$ are eigenvalues of A. The bound $\vartheta_k(G)$ improves the obvious bound $\alpha_k(G) \le k\vartheta(G)$. The minimum in (4) is attained (see theorem 12 in Narasimhan 1989). To show that $\vartheta_k(G)$ is an upper bound on $\alpha_k(G)$, we follow the reasoning of Mohar and Poljak (1993), who use Fan's theorem.

Theorem 1 (Fan 1949). *Let A be a symmetric matrix with eigenvalues $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$. Then*

$$\sum_{i=1}^k \lambda_i(A) = \max_{X\in\mathbb{R}^{n\times k}} \{\langle A, XX^\top\rangle : X^\top X = I_k\}. \tag{5}$$

Let $X^* \in \{0,1\}^{n\times k}$ be an optimal solution to the IP problem (1). That is, for every $r \in [k]$, the rth column of X^* is the incidence vector of the stable set colored in color r. Let $\hat{X}$ be the matrix whose columns are the columns of X^* normalized to one. Thus, $\hat{X}^\top\hat{X} = I_k$ by construction. Now, using Theorem 1, for any matrix A feasible for problem (4), we have

$$\alpha_k(G) = \langle A, \hat{X}\hat{X}^\top\rangle \le \max_{X\in\mathbb{R}^{n\times k}} \{\langle A, XX^\top\rangle : X^\top X = I_k\} = \sum_{i=1}^k \lambda_i(A).$$

Hence, $\alpha_k(G) \le \vartheta_k(G)$. For $k = 1$, $\vartheta_k(G)$ is the eigenvalue formulation of the ϑ-number by Lovász (1979). In the original paper by Narasimhan and Manber (1990), the generalized ϑ-number is introduced for the generalized clique number of G, which implies that the generalized ϑ-number there is defined for the complement of G. The generalized ϑ-number in Narasimhan and Manber (1990) is introduced by showing that the sum of the k largest eigenvalues of the adjacency matrix (with ones along the diagonal) of a graph is at least as large as the size of the largest induced subgraph that can be covered with k cliques. This result is a generalization of the well-known result that the largest eigenvalue of the adjacency matrix (with ones along the diagonal) of a graph is at least as large as the size of the largest clique in the graph. Here, we are interested in the maximum number of vertices in a k-partite subgraph of G and therefore define $\vartheta_k(G)$ for G.

It is known that problem (5) can be formulated as an SDP relaxation (see Alizadeh 1995). This implies that $\vartheta_k(G)$ can be obtained as the optimal solution of the following SDP relaxation:

$$\vartheta_k(G) = \min_{Y\in\mathbb{S}^n, \mu, \{x_{ij}\}_{\{ij\}\in E}} \left\{ \langle I, Y\rangle + \mu k : \sum_{\{ij\}\in E} \mathcal{E}^{ij} x_{ij} - J + \mu I + Y \succcurlyeq 0, \ Y \succcurlyeq 0 \right\}, \tag{6}$$

where $\mathcal{E}^{ij} = u_i u_j^\top + u_j u_i^\top$ for $i, j \in [n]$ s.t. $i \ne j$. The dual of (6) is

$$\vartheta_k(G) = \max_{Z\in\mathbb{S}^n} \{\langle J, Z\rangle : Z_{ij} = 0 \ \forall \{ij\} \in E, \ \langle I, Z\rangle = k, \ Z \succcurlyeq 0, \ I - Z \succcurlyeq 0\}. \tag{7}$$

Note that $Z = \frac{k}{n} I$ is a strictly feasible point for the SDP relaxation (7); in particular, strong duality holds for the primal-dual pair (6)–(7).

Alternatively, the SDP relaxation (7) can be obtained directly from the IP model (1). As before, let $X \in \{0,1\}^{n\times k}$ be an optimal solution to problem (1), and let $\hat{X}$ be the matrix whose columns are the columns of X normalized to one. Then the matrix $Z := \hat{X}\hat{X}^\top$ is feasible for the relaxation (7) with the objective value $\alpha_k(G)$. In particular, this follows since for any $r = 1,\dots,k$ we have $(\hat{X}\hat{X}^\top)_{ij} = \frac{1}{c_r}$ if vertices i and j are colored with color r, where c_r is the total number of vertices colored in color r; $(\hat{X}\hat{X}^\top)_{ij} = 0$ otherwise. The first constraint in (7) is satisfied by construction of $\hat{X}$ and clearly $\hat{X}\hat{X} \succcurlyeq 0$. The second and the last constraints in (7) are satisfied since the columns of $\hat{X}$ are normalized to one, $\hat{X}\hat{X}^\top$ has k eigenvalues equal to one, and $n-k$ eigenvalues equal to zero.

For $k = 1$ constraint $I - Z \succcurlyeq 0$ in (7) becomes redundant since Z is positive semidefinite and its eigenvalues sum up to one. In this case, the SDP relaxation (7) reduces to the following formulation of the ϑ-number by Lovász (1979):

$$\vartheta(G) = \max_{Z\in\mathbb{S}^n} \{\langle J,Z\rangle :\ Z_{ij} = 0\ \{ij\}\in E,\ \langle I,Z\rangle = 1,\ Z \succcurlyeq 0\}. \tag{8}$$

3.2. Strengthening the Generalized ϑ-number

Note that all entries of an optimal solution to the integer programming problem (1) are nonnegative. Therefore, we can add nonnegativity constraints to the matrix variable in (7) to strengthen the relaxation. This leads to the following SDP relaxation:

$$\begin{aligned}\vartheta'_k(G) = \max_{Z\in\mathbb{S}^n}\{&\langle J,Z\rangle : Z_{ij} = 0\\ &\forall\,\{ij\}\in E,\ \langle I,Z\rangle = k,\ Z \succcurlyeq 0, I - Z \succcurlyeq 0,\ Z \geq 0\}.\end{aligned} \tag{9}$$

We refer to the solution of this SDP relaxation as the *generalized ϑ'-number*. Note that, for $k = 1$, $\vartheta'_k(G)$ equals the $\vartheta'(G)$ upper bound on $\alpha(G)$ by Schrijver (1979).

4. Vector-Lifting SDP Relaxations

In this section, we derive two new SDP relaxations for the MkCS problem. To derive relaxations, we use the vector-lifting approach (see, e.g., Wolkowicz and Zhao 1999, Ding et al. 2011, Sotirov 2014). In Section 4.1, we reduce the SDP models by exploiting the fact that the MkCS problem is invariant under any permutation of the colors. To strengthen the relaxations, we add inequalities from the Boolean quadric polytope in Section 4.2. In Section 4.3, we relate the stable set problem on the Cartesian product of the complete graph on k vertices and G, with the MkCS problem. In particular, we show that our weaker vector-lifting relaxation is equivalent to the SDP model for the Schrijver ϑ'-number on the Cartesian product of the two mentioned graphs.

To derive our first SDP relaxation in this section, we consider the IP model for the MkCS problem with all equality constraints. Thus, assume that $X \in \{0,1\}^{n\times(k+1)}$ is feasible for the IP model (1a), ((1b)), and (2); that is, X is the matrix with one in the entry (i, r) if vertex i is colored with color r and zero otherwise, where "color" $k+1$ represents the uncolored vertices. Let $x = \operatorname{vec}(X)$ and $Y = xx^\top$. As $x \in \{0,1\}^{n(k+1)}$, it follows that $Y = \operatorname{diag}(Y)\operatorname{diag}(Y)^\top$, which can be relaxed to $Y - \operatorname{diag}(Y)\operatorname{diag}(Y)^\top \succcurlyeq 0$, which is equivalent to the following convex constraint:

$$\begin{bmatrix} 1 & \operatorname{diag}(Y)^\top \\ \operatorname{diag}(Y) & Y \end{bmatrix} \succcurlyeq 0. \tag{10}$$

Matrix Y consists of $(k+1)^2$ blocks of the size $n\times n$. We denote by Y^{rl} the $n\times n$ block of Y located in position $(r,l)\in[k+1]\times[k+1]$. From here on, we use the subscripts i, j to indicate vertices and use superscripts r, l to indicate colors in matrix variables. From (1b), it follows that $Y^{rr}_{ij} = 0$, $\forall\{ij\}\in E, r\in[k]$, and, from (2), $\sum_{r\in[k+1]} Y^{rr}_{ii} = 1$, for all $i\in[n]$. Also, from (2) and the fact that X is binary, we have $Y^{rl}_{ii} = 0$, for all $i\in[n]$, $r,l\in[k+1]$, $r\neq l$.

We collect all of the aforementioned constraints, add nonnegativity constraints, and arrive to the following SDP relaxation for the MkCS problem:

$$\begin{aligned}\max_{Y\in\mathbb{S}^{n(k+1)}}\ & \sum_{r\in[k]}\sum_{i\in[n]} Y^{rr}_{ii}\\ \text{s.t.}\ & Y^{rr}_{ij} = 0,\ \text{ for all } \{ij\}\in E,\ r\in[k]\\ & \sum_{r\in[k+1]} Y^{rr}_{ii} = 1,\ \text{ for all } i\in[n]\\ & Y^{rl}_{ii} = 0,\ \text{ for all } i\in[n],\ r,l\in[k+1],\ r\neq l\\ & Y\geq 0, \begin{bmatrix} 1 & \operatorname{diag}(Y)^\top \\ \operatorname{diag}(Y) & Y \end{bmatrix} \succcurlyeq 0.\end{aligned} \tag{11}$$

The SDP relaxation (11) is not strictly feasible. Namely, one can verify that the columns of $[-e_n, e^\top_{k+1}\otimes I_n]^\top$ are contained in the null space of the barycenter point, that is, a point in the relative interior of the minimal face containing the feasible set of (11) (see, e.g., Wolkowicz and Zhao 1999). Similarly, we derive a vector-lifting SDP relaxation from the IP problem (1). Namely, by exploiting $X\in\{0,1\}^{n\times k}$ that is feasible for (1), we obtain the following SDP relaxation:

$$\max_{Y\in\mathbb{S}^{nk}}\ \sum_{r\in[k]}\sum_{i\in[n]} Y^{rr}_{ii} \tag{12a}$$

$$\text{s.t. } Y^{rr}_{ij} = 0,\ \text{ for all } \{ij\}\in E,\ r\in[k] \tag{12b}$$

$$Y^{rl}_{ii} = 0,\ \text{ for all } i\in[n],\ r,l\in[k],\ r\neq l \tag{12c}$$

$$Y\geq 0, \begin{bmatrix} 1 & \operatorname{diag}(Y)^\top \\ \operatorname{diag}(Y) & Y \end{bmatrix} \succcurlyeq 0. \tag{12d}$$

Note that it is not necessary to include in (12) the constraints $\sum_{r\in[k]} Y^{rr}_{ii} \leq 1$ $(i\in[n])$ that arise naturally from (1c), as they are redundant. Namely, we have the following result.

Lemma 1. For each $i \in [n]$, the constraint $\sum_{r\in[k]} Y_{ii}^{rr} \le 1$ is redundant for the SDP relaxation (12).

Proof. See the online supplement.

The SDP relaxation (12) is strictly feasible. To show this, one can use an argument that is similar to the one that shows that the symmetry-reduced SDP relaxation (15) is strictly feasible (see Lemma 2 and Theorem 3). It is not difficult to verify that the SDP relaxation (12) is dominated by the SDP relaxation (11). We will show that those two relaxations are equivalent after adding the following inequality constraints to the relaxation (12):

$$1 - \sum_{r\in[k]} Y_{ii}^{rr} - \sum_{r\in[k]} Y_{jj}^{rr} + \sum_{r\in[k]}\sum_{l\in[k]} Y_{ij}^{rl} \ge 0, \quad \text{for all } i > j \tag{13}$$

$$Y_{ii}^{ll} - \sum_{r\in[k]} Y_{ij}^{rl} \ge 0, \quad \text{for all } i \ne j,\ l, \tag{14}$$

where $i, j \in [n]$ and $l, r \in [k]$. These inequalities are based on the reformulation-linearization technique by Sherali and Adams (1994). In particular, inequalities (13) are linearizations of the products of pairs of constraints (1c). Inequalities (14) represent multiplication of elementwise nonnegativity constraint on X with each individual constraint in (1c). Similar inequalities are used in Rendl and Sotirov (2018) and Rendl et al. (2019) to improve SDP relaxations for the min-cut problem and the bandwidth problem, respectively.

Theorem 2. *The SDP relaxation* (12) *with additional constraints* (13) *and* (14) *is equivalent to the SDP relaxation* (11).

Proof. See the online supplement.

To strengthen relaxations (11) and (12), one may tend to add the clique constraints, that is, $\sum_{i\in C} \operatorname{diag}(Y_{ii}^{rr}) \le 1$, where $C \subseteq [n]$ denotes a set of indices corresponding to vertices in a clique, and $r \in [k]$. However, those constraints are redundant (see Remark 2 for an explanation).

4.1. Symmetry Reduction on Colors

In this section, we exploit the fact that the MkCS problem is invariant under permutations of the colors, in order to reduce the sizes of the vector-lifting SDP relaxations from the previous section. We also show that the symmetry-reduced SDP relaxations are strictly feasible.

We begin with a lemma related to strict feasibility.

Lemma 2. *Let $n \ge 1$, $k \ge 1$, and let $e \in \mathbb{R}^n$ be the vector of all ones. Then*

$$M := \begin{bmatrix} k & \frac{1}{(n+1)} e^\top \\ \frac{1}{(n+1)} e & \frac{1}{(n+1)} I \end{bmatrix} \succ 0.$$

Proof. Let $\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] \in \mathbb{R}^{n+1} \setminus \{0\}$. Then we have $\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right]^\top M \left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] = \frac{1}{n+1}(((n+1)k - n)x_0^2 + \|x_0 e + x\|^2) > 0.$ □

To reduce the size of the SDP relaxation (12) with additional constraints (13) and (14), we need the following result.

Lemma 3. (Gvozdenović and Laurent 2008, lemma 2.8). *Let $Y \in \mathbb{R}^{kn\times kn}$ be a block matrix that consists of k^2 blocks of the size $n \times n$. Let Y have a matrix $A \in \mathbb{S}^n$ as its diagonal blocks, and a matrix $B \in \mathbb{S}^n$ as its nondiagonal blocks; that is,*

$$Y = \underbrace{\begin{bmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & A \end{bmatrix}}_{k \text{ blocks}} = I \otimes A + (J - I) \otimes B.$$

Then $Y \succcurlyeq 0$ if and only if $A - B \succcurlyeq 0$ and $A + (k-1)B \succcurlyeq 0$.

Now, we are ready to prove our main result in this section.

Theorem 3. *The SDP relaxation* (11) *and the SDP relaxation* (12) *with additional constraints* (13) *and* (14) *are equivalent to the following SDP relaxation*:

$$\theta_k^1(G) = \max_{Z,\, X\in\mathbb{S}^n} \quad \langle I, Z\rangle \tag{15a}$$

$$\text{s.t.} \quad \begin{aligned} &Z_{ij} = 0, \ \text{for } \{ij\} \in E \\ &X_{ii} = 0, \ \text{for } i \in [n] \end{aligned} \tag{15b}$$

$$Z \ge 0,\ X \ge 0 \tag{15c}$$

$$Z - X \succcurlyeq 0 \tag{15d}$$

$$\begin{bmatrix} 1 & \operatorname{diag}(Z)^\top \\ \operatorname{diag}(Z) & Z + (k-1)X \end{bmatrix} \succcurlyeq 0 \tag{15e}$$

$$1 - Z_{ii} - Z_{jj} + Z_{ij} + (k-1)X_{ij} \ge 0, \ \text{for } i, j \in [n], i > j \tag{15f}$$

$$Z_{ii} - Z_{ij} - (k-1)X_{ij} \ge 0, \ \text{for } i, j \in [n], i \ne j, \tag{15g}$$

and this SDP relaxation is strictly feasible.

Proof. First, the relaxations (11) and (12) with additional constraints (13) and (14) are equivalent by Theorem 2. We show here that the SDP relaxation (12) with (13) and (14) is equivalent to (15).

Let Y be a feasible solution to the SDP relaxation (12) with additional constraints (13) and (14). If we permute the color labels, then we permute the "columns" and "rows" of blocks in Y. For instance, permuting color r and color l results in permuting blocks Y^{pl} and Y^{pr}, for all $p \in [k]$, and then permuting blocks Y^{lp} and Y^{rp}, for all $p \in [k]$. In particular, Y^{rr} and Y^{ll} are permuted.

Let $\bar{Y}$ be the average over all $k!$ permutations of the color labels. By construction, problem (12) with constraints (13) and (14) is convex and invariant under color permutations. Therefore, $\bar{Y}$ is feasible for (12) and satisfies the additional constraints (13) and (14). Notice that $\bar{Y}$ has the form

$$\bar{Y} = \frac{1}{k}\underbrace{\begin{bmatrix} Z & X & \dots & X \\ X & Z & \dots & X \\ \vdots & \vdots & \ddots & \vdots \\ X & X & \dots & Z \end{bmatrix}}_{k \text{ blocks}} = \frac{1}{k} I_k \otimes Z + \frac{1}{k}(J_k - I_k) \otimes X, \tag{16}$$

where $Z = \sum_{r=1}^{k} Y^{rr}$ and $X = \frac{1}{k-1}\sum_{r,l=1, r\neq l}^{k} Y^{rl}$.

The SDP relaxation (12) with constraints (13) and (14) can be restricted without loss of generality to matrices of the form (16), which results in (15). Indeed, the objective and all linear constraints of the SDP relaxation (15) are obtained by rewriting the objective and the corresponding linear constraints of (12), (13), and (14) for matrices of the form (16). Now, consider the SDP constraint $\begin{bmatrix} 1 & \operatorname{diag}(Y)^\top \\ \operatorname{diag} Y & Y \end{bmatrix} \succcurlyeq 0$, where Y is of the form (16), which, by the Schur complement, is equivalent to $Y - \operatorname{diag}(Y)\operatorname{diag}(Y)^\top \succcurlyeq 0$. We have $Y - \operatorname{diag}(Y)\operatorname{diag}(Y)^\top = \frac{1}{k} I \otimes (Z - \frac{1}{k}\operatorname{diag}(Z)\operatorname{diag}(Z)^\top) + \frac{1}{k}(J - I) \otimes (X - \frac{1}{k}\operatorname{diag}(Z)\operatorname{diag}(Z)^\top)$. Hence, by Lemma 3, the positive semidefinite constraint holds if and only if $Z - X \succcurlyeq 0$ and $Z + (k-1)X - \operatorname{diag}(Z)\operatorname{diag}(Z)^\top \succcurlyeq 0$.

Now we show the strict feasibility of (15). Let $A_{\bar{G}}$ be the adjacency matrix of the complement of G, and let $M \succ 0$ be given as in Lemma 2. Since also $\frac{1}{k(n+1)} I \succ 0$, there exists $0 < \varepsilon < \frac{1}{kn(n+1)}$ such that

$$\frac{1}{k}M + \varepsilon \begin{bmatrix} 0 & 0^\top \\ 0 & A_{\bar{G}} + (k-1)(J - I) \end{bmatrix} \succ 0$$
$$\text{and} \quad \frac{1}{k(n+1)} I + \varepsilon A_{\bar{G}} - \varepsilon (J - I) \succ 0.$$

Define $\bar{Z} := \frac{1}{k(n+1)} I + \varepsilon A_{\bar{G}}$, $\bar{X} := \varepsilon(J - I)$. By the choice of ε, one can verify that $(\bar{Z}, \bar{X})$ strictly satisfies constraints (15c), (15f), and (15g). For example, we will verify that constraints (15f) are strictly satisfied for $(\bar{Z}, \bar{X})$: $1 - \bar{Z}_{ii} - \bar{Z}_{jj} + \bar{Z}_{ij} + (k-1)\bar{X}_{ij} \geq 1 - \frac{2}{k(n+1)} = \frac{kn+k-2}{k(n+1)} \geq \frac{1}{k(n+1)}$, for all $i, j \in [n], i > j$. □

Remark 1. We denote by $\theta_k^2(G)$ the optimal value of the SDP relaxation obtained from (15), where constraints (15f) and (15g) are removed. Thus, $\theta_k^2(G)$ equals the optimal value of the SDP relaxation (12).

For the sake of completeness, we also present the symmetry-reduced version of the relaxation (11). See Corollary 1 in the online supplement.

Note that the SDP relaxation from Corollary 1 in the online supplement has a matrix variable of order $3n$, whereas the SDP relaxation from Theorem 3 has a matrix variable of order $2n$. However, they are equivalent.

4.2. Boolean Quadric Polytope Inequalities

To further strengthen our strongest SDP relaxation, one can add inequalities from the Boolean quadric polytope (BQP) (see, e.g., Padberg 1989). Namely, let X be a feasible solution to the binary problem (1) and consider $Y = \operatorname{vec}(X)\operatorname{vec}(X)^\top$. Then, for all $i, j, p \in [nk]$, the following BQP inequalities are valid for Y:

$$0 \leq Y_{i,j} \leq Y_{i,i} \tag{17}$$
$$Y_{i,i} + Y_{j,j} \leq 1 + Y_{i,j} \tag{18}$$
$$Y_{i,p} + Y_{j,p} \leq Y_{p,p} + Y_{i,j} \tag{19}$$
$$Y_{i,i} + Y_{j,j} + Y_{p,p} \leq Y_{i,j} + Y_{i,p} + Y_{j,p} + 1. \tag{20}$$

Therefore, one can add those constraints to the SDP relaxation (12) with additional constraints (13) and (14) in order to further strengthen it. In the view of the symmetry reduction on colors, we may consider only those feasible solutions Y to the relaxation (12) that can be written as in (16). Then, (17)–(20) reduce to

$$0 \leq Z_{i,j} \leq Z_{i,i}, \quad 0 \leq X_{i,j} \leq Z_{i,i}, \tag{21}$$
$$Z_{i,i} + Z_{j,j} \leq k + Z_{i,j}, \quad Z_{i,i} + Z_{j,j} \leq k + X_{i,j}, \tag{22}$$
$$\begin{aligned} &X_{i,p} + X_{j,p} \leq Z_{p,p} + X_{i,j}, \quad Z_{i,p} + Z_{j,p} \leq Z_{p,p} + Z_{i,j}, \\ &X_{i,p} + X_{j,p} \leq Z_{p,p} + Z_{i,j}, \\ &X_{i,p} + Z_{j,p} \leq Z_{p,p} + X_{i,j}, \quad Z_{i,p} + X_{j,p} \leq Z_{p,p} + X_{i,j}, \end{aligned} \tag{23}$$

$$\begin{aligned} Z_{i,i} + Z_{j,j} + Z_{p,p} &\leq X_{i,j} + X_{i,p} + X_{j,p} + k, \\ Z_{i,i} + Z_{j,j} + Z_{p,p} &\leq Z_{i,j} + Z_{i,p} + Z_{j,p} + k, \\ Z_{i,i} + Z_{j,j} + Z_{p,p} &\leq Z_{i,j} + X_{i,p} + X_{j,p} + k, \\ Z_{i,i} + Z_{j,j} + Z_{p,p} &\leq X_{i,j} + X_{i,p} + Z_{j,p} + k, \\ Z_{i,i} + Z_{j,j} + Z_{p,p} &\leq X_{i,j} + Z_{i,p} + X_{j,p} + k, \end{aligned} \tag{24}$$

where $i, j, p \in [n], i \neq j \neq p$. Inequalities (21) correspond to (17), (22) correspond to (18), and so on.

Some of the BQP constraints (21)–(24) are redundant for the relaxation (15). In particular, (21) follows from (15g). Also, constraints (22) are redundant, since the 2-clique constraints are redundant for the SDP relaxations (11) and (12). Therefore, in numerical experiments, we use inequalities of the type (23) and (24). Notice that the first inequality in each of the two sets (23) and (24) is only valid for $k \geq 3$.

One could also consider the triangle inequalities for Y in (12); that is, $Y_{ij} + Y_{jp} - Y_{ip} \leq 1$ for any $i, j, p \in [n]$. However, those inequalities follow from (19) and the nonnegativity of Y.

4.3. The M*k*CS as the Maximum Stable Set Problem

The M*k*CS problem on a graph G can be considered as a stable set problem on the Cartesian product of the complete graph on k vertices and G. This result was proved by Narasimhan (1989). For the sake of completeness, we present a short proof. We also prove that the Schrijver's ϑ'-number on the Cartesian product of the mentioned two graphs equals the optimal value of the vector-lifting relaxation (12).

We denote by $K_k = (V_k, E_k)$, where $V_k = [k]$, the complete graph on k vertices. The Cartesian product $K_k \square G$ of graphs K_k and $G = (V, E)$ is a graph with the vertex set $V_k \times V$ and the edge set $E_\square$, where two vertices (u, i) and (v, j) are adjacent if $u = v$ and $(i,j) \in E$ or $i = j$ and $(u,v) \in E_k$. The following result shows that the MkCS problem on G corresponds to the stable set problem on $K_k \square G$.

Theorem 4 (Narasimhan 1989). *Let $G(V,E)$, and let K_k be the complete graph on k vertices. Then, $\alpha_k(G)\alpha(K_k \square G)$.*

Proof. First, if $S_1, \ldots, S_k$ are disjoint stable sets in G, then $\{1\} \times S_1, \ldots, \{k\} \times S_k$ is a stable set in $K_k \square G$. On the other hand, let S be a stable set in $K_k \square G$ of the largest cardinality. Then S can be partitioned into $S_1, \ldots, S_k$ such that $S_1 = \{1\} \times \hat{S}_1, \ldots, S_k = \{k\} \times \hat{S}_k$, $\hat{S}_1 \subseteq V, \ldots, \hat{S}_k \subseteq V$. Moreover, $\hat{S}_1, \ldots, \hat{S}_k$ are disjoint, since $u \in \hat{S}_l \cap \hat{S}_p$ for some $l, p \in [k]$ with $l \neq p$ implies that there is an edge between (l, u) and (p, u) that is also in the stable set S. Hence, $\hat{S}_1, \ldots, \hat{S}_k$ are disjoint stable sets in G. □

The Schrijver's ϑ'-number on $K_k \square G$ is as follows:

$$\vartheta'(K_k \square G) = \max_{Y \in \mathbb{S}^{nk}} \ \langle J, Y \rangle \tag{25}$$
$$\text{s.t.} \quad Y_{ij}^{rr} = 0, \ \text{for all } \{ij\} \in E, \ r \in [k]$$
$$Y_{ii}^{rl} = 0, \ \text{for all } i \in [n], \ r, l \in [k], \ r \neq l$$
$$\langle I, Y \rangle = 1, \ Y \succcurlyeq 0, \ Y \geq 0,$$

where Y is of the size $nk \times nk$. The SDP relaxation (25) follows directly from (9) and the definition of $K_k \square G$. Next, we relate (25) and (12).

Lemma 4. *The Schrijver's ϑ'-number on $K_k \square G$ equals the optimal value of the vector-lifting relaxation* (12); that is, $\vartheta'(K_k \square G) = \theta_k^2(G) \geq \alpha_k(G)$.

Proof. See the online supplement.

Remark 2. It is well known that the clique constraints are redundant for the Lovász ϑ-number (see, e.g., chapter 9 of Grötschel et al. 1988). Therefore, the clique constraints are redundant for the SDP relaxation (25), and consequently also for (12) and (11).

5. Matrix-Lifting SDP Relaxations

In this section, we derive a matrix-lifting SDP relaxation for the MkCS problem. Relaxations obtained by the matrix-lifting approach are known to have less variables and constraints than the corresponding relaxations obtained by the vector-lifting approach. However, relaxations obtained by those two approaches may be equal (see Ding et al. 2011). We show here that our matrix-lifting SDP relaxation is dominated by the vector-lifting relaxations from the previous section. However, numerical results show that the relaxation derived here is preferable among other bounding approaches for large graphs. Namely, the matrix-lifting relaxation provides often the same bound as the strongest vector-lifting bound while requiring significantly less computational effort (see the online supplement for more details). We also apply symmetry reduction on colors to further reduce the relaxation introduced here.

Let $X \in \{0,1\}^{n \times k}$ be a solution to the IP problem (1), and consider

$$Y = \begin{bmatrix} I_k \\ X \end{bmatrix} [\, I_k \quad X^\top \,] = \begin{bmatrix} I_k & X^\top \\ X & XX^\top \end{bmatrix}.$$

Linearizing the block $XX^\top$, we obtain the following matrix-lifting SDP relaxation for (1):

$$\max_{Z \in \mathbb{S}^n, \, X \in \mathbb{R}^{n \times k}} \ \langle I, Z \rangle \tag{26}$$
$$\text{s.t.} \quad Z_{ij} = 0 \ \text{ for } \{ij\} \in E$$
$$Z_{ii} \leq 1 \ \text{ for } i \in [n]$$
$$Z_{ii} = \sum_{r \in [k]} X_{ir} \ \text{ for } i \in [n]$$
$$\begin{bmatrix} I_k & X^\top \\ X & Z \end{bmatrix} \succcurlyeq 0, \ Z \geq 0, \ X \geq 0.$$

Here, the positive semidefinite constraint is imposed on a matrix variable of the size $(k+n) \times (k+n)$. The zero pattern and constraints on the diagonal follow directly from the construction. The relaxation (26) has no constraints that ensure that a vertex can be colored by only one color, while vector-lifting relaxations have such constraints.

Notice that the constraint $\operatorname{diag}(Y) \leq e$ is redundant when $k = 1$ and that the resulting SDP relaxation corresponds to one of the SDP relaxations for the Schrijver number (see Grötschel et al. 1988).

5.1. Symmetry Reduction on Colors

In this section, we exploit the invariance of the MkCS problem with respect to color permutations to reduce the number of variables in the SDP relaxation (26).

Let (Z, X) be a feasible solution to (26), and let $\bar{X}$ be the average over all column permutations of X. Since the SDP relaxation (26) is convex and invariant under permutations of the colors, $(Z, \bar{X})$ is feasible for (26). By construction, all columns of $\bar{X}$ are equal to each other. Therefore, it is sufficient to consider solutions (Z, X) for (26), where the columns of X are equal to each other. If we denote a column of X by x, then the constraint $Z_{ii} = \sum_{r \in [k]} X_{ir}$ for $i \in [n]$ reduces to

$$\operatorname{diag}(Z) = kx, \tag{27}$$

and the SDP constraint to $\left[\begin{smallmatrix} I_k & ex^\top \\ xe^\top & Z \end{smallmatrix}\right] \succcurlyeq 0$, where $e \in \mathbb{R}^k$. Now, we use the Schur complement and (27) to rewrite the SDP constraint: $Z \succcurlyeq 0, Z - xe^\top ex^\top =$

$Z - kxx^\top = Z - \frac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top \succcurlyeq 0$. The next result follows from the aforementioned discussion.

Theorem 5. *The matrix-lifting SDP relaxation* (26) *is equivalent to the following relaxation*:

$$\begin{aligned}\theta_k^3(G) = \max_{Z\in\mathbb{S}^n}\ & \langle I, Z\rangle \\ \text{s.t. } & Z_{ij} = 0 \text{ for } \{ij\} \in E \\ & Z_{ii} \le 1 \text{ for } i \in [n] \\ & \begin{bmatrix} k & \mathrm{diag}(Z)^\top \\ \mathrm{diag}(Z) & Z \end{bmatrix} \succcurlyeq 0,\ Z \ge 0,\end{aligned} \tag{28}$$

and the latter problem is strictly feasible.

Proof. The first part follows from the construction. To show strict feasibility, consider $M \succ 0$ from Lemma 2. Let $A_{\bar{G}}$ be the adjacency matrix of the complement of G. Then there exists $\varepsilon > 0$ such that $M + \varepsilon\begin{bmatrix} 0 & 0^\top \\ 0 & A_{\bar{G}}\end{bmatrix} \succ 0$. Therefore, matrix $Z = \frac{1}{(n+1)}I + \varepsilon A_{\bar{G}}$ is a strictly feasible solution of (28) by construction. □

Let us relate our matrix- and vector-lifting relaxations for the M*k*CS problem. Note that the matrix-lifting relaxation does not impose orthogonality constraints that correspond to the incidence vectors of different colors. Further, note that $\mathrm{diag}(Z) \le e$ is redundant for (15) when the orthogonality constraints are imposed (see Lemma 1). Now, when we put those observations together, we arrive at the following result.

Theorem 6. *The SDP relaxation* (28) *is equivalent to the following vector-lifting relaxation*:

$$\begin{aligned}\max_{Z,X\in\mathbb{S}^n}\ & \langle I, Z\rangle \\ \text{s.t. } & Z_{ij} = 0, \quad \text{for } \{ij\} \in E \\ & Z_{ii} \le 1, \quad \text{for } i \in [n] \\ & Z \ge 0,\ X \ge 0 \\ & Z - X \succcurlyeq 0,\ \begin{bmatrix} 1 & \mathrm{diag}(Z)^\top \\ \mathrm{diag}(Z) & Z + (k-1)X \end{bmatrix} \succcurlyeq 0.\end{aligned} \tag{29}$$

Proof. First, let (Z, X) be a feasible solution for the SDP relaxation from the theorem. We claim that Z is feasible for problem (28), and the corresponding objective values are equal. The linear constraints in (28) are readily satisfied. To verify that the SDP constraint in (28) is also satisfied, we proceed as follows: $Z - \frac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top \succcurlyeq Z - \frac{1}{k}Z - \frac{k-1}{k}X = \frac{k-1}{k}(Z - X) \succcurlyeq 0$. Here, we exploit two positive semidefinite constraints from (29).

Now, let Z be a feasible solution to (28). We claim that $(Z, X) := (Z, \frac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top)$ is feasible for the SDP relaxation (29), and the corresponding objective values are equal. The linear constraints of the SDP relaxation (29) are clearly satisfied for previously defined (Z, X). The SDP constraint of (28) implies $Z \succcurlyeq 0$ and $Z - X = Z - \frac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top \succcurlyeq 0$, by the Schur complement. Therefore, the first SDP constraint in (29) is satisfied. Finally, $Z + (k-1)X = Z + \frac{(k-1)}{k}\mathrm{diag}(Z)\ \mathrm{diag}(Z)^\top \succcurlyeq 0$ and $Z + (k-1)X - \mathrm{diag}(Z)\ \mathrm{diag}(Z)^\top = Z + (k-1)X - kX = Z - X \succcurlyeq 0$, which implies the second SDP constraint in (29). □

Note that the feasible region of the SDP relaxation from Theorem 6 is a superset of the feasible region of the SDP relaxation for $\theta_k^2(G)$. The following result follows directly from the previous theorem and Theorem 2:

$$\theta_k^1(G) \le \theta_k^2(G) \le \theta_k^3(G),$$

where $\theta_k^1(G)$ is the optimal solution of the SDP relaxation (15), $\theta_k^2(G)$ is the optimal solution of (15) without constraints (15f) and (15g), and $\theta_k^3(G)$ is the optimal solution of (28).

In the previous section, we concluded that $\theta_k^3(G) = \vartheta'(G)$ when $k = 1$, where $\vartheta'(G)$ is the Schrijver number. We were not able to establish a relation between $\theta_k^3(G)$ and $\vartheta'(G)$ when $k > 1$. However, our numerical results (see the online supplement) suggest the following result.

Conjecture 1. *For $k \ge 2$, the upper bound $\theta_k^3(G)$ (see* (28)*) is at least as good as the upper bound $\vartheta_k'(G)$ (see* (9)*).*

We conclude this section listing some cases when our bounds are tight.

Lemma 5. *For a given k, let G be a graph such that $\alpha_k(G) = k\vartheta(G)$. Then,*

$$\vartheta_k(G) = \vartheta_k'(G) = \theta_k^1(G) = \theta_k^2(G) = \theta_k^3(G) = k\alpha(G) = \alpha_k(G).$$

Proof. Let G be any graph. We have $\alpha_k(G) \le k\alpha(G) \le k\vartheta(G)$. We have already shown that $\alpha_k(G) \le \theta_k^1(G) \le \theta_k^2(G) \le \theta_k^3(G)$. We claim that $\theta_k^3(G) \le k\vartheta(G)$. To prove this, given Z a feasible solution to (28), let $\hat{Z} = \frac{1}{\langle I, Z\rangle}Z$. Then $\hat{Z}$ is feasible for (8). It is enough to show then that $k\langle J, \hat{Z}\rangle \ge \langle I, Z\rangle$; that is, $k\langle J, Z\rangle \ge \langle I, Z\rangle^2$. But, using the Schur complement, the last constraint from (28) is equivalent to $kZ - \mathrm{diag}(Z)\mathrm{diag}(Z)^\top \succcurlyeq 0$, which implies that $k\langle J, Z\rangle - \langle I, Z\rangle^2 = ke^\top Ze - (e^\top \mathrm{diag} Z)^2 = e^\top(kZ - \mathrm{diag}(Z)\mathrm{diag}(Z)^\top)e \ge 0$.

Also, $\alpha_k(G) \le \vartheta_k'(G) \le \vartheta_k(G)$. We claim that $\vartheta_k(G) \le k\vartheta(G)$. To prove this, notice that if Z is a feasible solution to (7), then $\frac{1}{k}Z$ is feasible for (8).

Now, if G is such that $\alpha_k(G) = k\vartheta(G)$, all previous inequalities become equalities. □

Notice that $\alpha_k(G) \le k\alpha(G) \le k\vartheta(G)$, and thus the assumption $\alpha_k(G) = k\vartheta(G)$ in Lemma 5 is equivalent to $\alpha_k(G) = k\alpha(G)$ and $\alpha(G) = \vartheta(G)$. Several families of graphs satisfy these conditions. For instance, if G is a perfect graph with at least K nonintersecting independent sets of size $\alpha(G)$, then $\alpha_k(G) = k\alpha(G) = k\vartheta(G)$ for all $k \le K$. In Proposition 1, we characterize a family of (not necessarily perfect) graphs satisfying the conditions of Lemma 5. The condition is given in terms of the chromatic number of the complement graph of G, $\chi(\bar{G})$. Notice that $\chi(\bar{G})$ is equal to the clique cover number of G.

Proposition 1. *Let G be a graph such that $|V(G)| = \chi(\bar{G})\chi(G)$. Then Lemma 5 holds for all $k \le \chi(\bar{G})$.*

Proof. For any graph, $\chi(\bar{G}) \ge \alpha(G) \ge |V(G)|/\chi(G)$. Thus, $|V(G)| = \chi(\bar{G})\chi(G)$ implies that $\chi(\bar{G}) = \alpha(G)$ and $\alpha_{\chi(G)}(G) = \chi(G)\alpha(G)$. Using the Lovász sandwich theorem (Lovász 1979) $\alpha(G) \le \vartheta(G) \le \chi(\bar{G})$, we obtain that $\alpha(G) = \vartheta(G)$, from which the statement follows. □

The set of vertex-transitive graphs contains a number of nontrivial examples for Proposition 1 (see, e.g., Table 1). A graph is vertex-transitive if its automorphism group acts transitively on vertices, that is, if for every two vertices there is an automorphism that maps one to the other. In the table, $H(v,d) := H(v,d,1)$ denotes the Hamming graph, and $J(v,d) := J(v,d,d-1)$ denotes the Johnson graph. For definitions of these graphs, see Section 6.

6. Reductions Using Graph Symmetry

In this section, we first prove that several inequalities in the strongest vector-lifting relaxation (15) are redundant for vertex-transitive graphs. Then, we present reduced SDP relaxations for different classes of highly symmetric graphs. We say that a graph is *highly symmetric* if its adjacency matrix belongs to an association scheme (see, e.g., Delsarte 1973, Schrijver 1979, van Dam and Sotirov 2015).

Proposition 2. *Let G be a vertex-transitive graph. Then constraints (15g) are redundant for the SDP relaxation (15a)–(15e).*

Proof. Let (Z, X) be an optimal solution for the SDP relaxation (15a)–(15e). By averaging and vertex transitivity of the graph, we obtain an optimal solution $(\bar{Z}, \bar{X})$ such that $\bar{Z}_{ii} = \bar{Z}_{jj}$ for all i, j. Given i and j, let $z = \bar{Z}_{ii} = \bar{Z}_{jj}$ and $y = \bar{Z}_{ij} + (k-1)\bar{X}_{ij}$. Looking at the principal submatrix of constraint (15e) indexed by i and j, we obtain $\begin{bmatrix} z & y \\ y & z \end{bmatrix} \succcurlyeq 0$. This implies that $z^2 \ge y^2$, which is equivalent to (15g) from the nonnegativity of z and y. □

In general, constraints (15f) are not redundant for the SDP relaxation (15) and vertex-transitive graphs. For example, for the Petersen graph and $k = 2$, we have that $\theta_k^1(G)$ equals 7.5, whereas $\theta_k^2(G)$ is equal to 8.

We will consider symmetry reduction for the Hamming and Johnson graphs. We apply the general theory of symmetry reduction to our SDP relaxations (see, e.g., Schrijver 1979, Gatermann and Parrilo 2004, de Klerk and Sotirov 2008, van Dam and Sotirov 2015) and therefore omit the details. Whereas in the aforementioned papers and others in the literature symmetry-reduced relaxations are linear programming relaxations, our simplified relaxations have also second-order cone constraints.

Let us now define the Hamming graphs. The vertex set V is the set of d-tuples of letters from an alphabet of size q, so $n := |V| = q^d$. The adjacency matrices $H(d,q,j)$ $(j = 0,\ldots,d)$ of the Hamming association scheme are defined by the number of positions in which two d-tuples differ. In particular, $H(d,q,j)_{x,y} = 1$ if x and y differ in j positions, for $x, y \in V$ $(j = 0,\ldots,d)$, that is, if their Hamming distance $d(x,y) = j$; $H(d,q,1)$ is the adjacency matrix of the well-known Hamming graph, which can also be obtained as the Cartesian product of d copies of the complete graph K_q. Further, we denote by $H^-(d,q,j)$ the graph whose Hamming distance $d(x,y) \le j$. The matrices of the Hamming association scheme can be simultaneously diagonalized. The eigenvalues (character table) of the Hamming scheme can be expressed in terms of Krawtchouk polynomials:

$$K_i(u) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{u}{j}\binom{d-u}{i-j}, \quad i, u = 0, \ldots, d.$$

In particular, eigenvalues of $B_i := H(d,q,i)$ $(i = 0,1,\ldots,d)$ are $K_i(j)$ for $j = 0,1,\ldots,d$.

Now, let us consider the SDP relaxation (9). Since the relaxation is invariant under the permutation group of the Hamming graph, we can restrict optimization of the SDP relaxation to feasible points in the Bose-Mesner algebra (see, e.g., Gatermann and Parrilo 2004, de Klerk and Sotirov 2008). Therefore, we assume that $Z = \sum_{i=0}^{d} z_i B_i$ in (9). Further, we consider the case in which the adjacency matrix corresponds to

Table 1. Graphs for Which All of Our SDP Upper Bounds Are Tight

Graph	$\lvert V(G)\rvert$	$\chi(G)$	$\chi(\bar{G})$	
$H(v, d)$	d^v	d	d^{v-1}	Sabidussi (1957)
$J(v,2)$, v even	$\binom{v}{2}$	$v-1$	$\frac{v}{2}$	Etzion and Bitan (1996)
$J(v,3)$, $v \equiv 1$ or 3 mod 6	$\binom{v}{3}$	$v-2$	$\frac{v(v-1)}{6}$	Brouwer et al. (1990), Etzion and Bitan (1996)

$H(d,q,1)$. Relaxations are similar for any other $H(d,q,j)$ or $H^-(d,q,j)$ $(j=2\dots,d)$. After the substitution, the SDP relaxation (9) reduces to

$$\begin{aligned}\vartheta'_k(G) = \max_{z\in\mathbb{R}^{d+1}} \quad & k+\sum_{i=2}^{d} z_i\langle J,B_i\rangle \qquad (30)\\ \text{s.t.} \quad & \frac{k}{n}+\sum_{i=2}^{d} z_iK_i(j)\ge 0, \text{ for } j\in\{0,1,\dots,d\}\\ & 1-\frac{k}{n}-\sum_{i=2}^{d} z_iK_i(j)\ge 0, \text{ for } j\in\{0,1,\dots,d\}\\ & z_0=\frac{k}{n},\ z_1=0,\ z_i\ge 0, \text{ for } i\in\{2,\dots,d\}.\end{aligned}$$

Note that (30) is a linear program. Next, we reduce our matrix-lifting SDP relaxation (28) by using similar arguments as before. The resulting θ^3-bound is as follows:

$$\begin{aligned}\theta^3(G) = \max_{z\in\mathbb{R}^{d+1}} \quad & n\cdot z_0 \qquad (31)\\ \text{s.t.} \quad & \sum_{i=0}^{d} z_iK_i(0)-\frac{n}{k}z_0^2\ge 0\\ & \sum_{i=0}^{d} z_iK_i(j)\ge 0, \text{ for } j\in\{1,\dots,d\}\\ & z_0\le 1,\ z_1=0,\ z_i\ge 0, \text{ for } i\in\{0,2,3,\dots,d\}.\end{aligned}$$

Note that (31) is an optimization problem with a linear objective, $2d+1$ linear inequalities, and one convex quadratic constraint. Finally, we simplify the SDP relaxation whose optimal value is denoted by $\theta^2(G)$, that is, the vector-lifting SDP relaxation (15a) without (15f) and (15g). Here, we also may restrict $X=\sum_{i=0}^{d} x_iB_i$:

$$\begin{aligned}\theta^2(G) = \max_{z,x\in\mathbb{R}^{d+1}} \quad & n\cdot z_0 \qquad (32)\\ \text{s.t.} \quad & \sum_{i=0}^{d} (z_i-x_i)K_i(j)\ge 0, \text{ for } j\in\{0,1,\dots,d\}\\ & \sum_{i=0}^{d} z_iK_i(0)+(k-1)\sum_{i=0}^{d} x_iK_i(0)-nz_0^2\ge 0\\ & \sum_{i=0}^{d} z_iK_i(j)+(k-1)\sum_{i=0}^{d} x_iK_i(j)\ge 0,\\ & \quad \text{for } j\in\{1,\dots,d\}\\ & x_0=0,\ x_i\ge 0, \text{ for } i\in\{1,2,3,\dots,d\}\\ & z_0\le 1,\ z_1=0,\ z_i\ge 0, \text{ for } i\in\{0,2,3,\dots,d\}.\end{aligned}$$

Note that the optimization problem (32) has a linear objective, one second-order cone constraint, and several linear constraints.

Finally, to compute $\theta^1(G)$, we add to (32) the symmetry-reduced inequalities (15f), that is, $1-2z_0+z_i+(k-1)x_i\ge 0$, $i=1,\dots,d$.

One can similarly derive simplified SDP relaxations for graphs whose corresponding algebra is diagonalizable, such as for the Johnson graph $J(v,d,q)$. The Johnson graph is defined as follows. Let Ω be a fixed set of size v, and let d be an integer such that $1\le d\le v/2$. The vertices of the Johnson graph $J(v,d,q)$ are the subsets of Ω with size d. Two vertices are connected if the corresponding sets have q elements in common. In the literature, the graph $J(v,d,d-1)$ is known as the *Johnson graph* $J(v,\ d)$, and $J(v,d,0)$ is known as the *Kneser graph* $K(v,\ d)$. Matrices corresponding to $J(v,d,q)$, $q=0,1,\dots,d$ can be simultaneously diagonalized. The eigenvalues (character table) of the Johnson scheme can be expressed in terms of Eberlein polynomials:

$$E_i(u):=\sum_{j=0}^{i}(-1)^j\binom{u}{j}\binom{d-u}{i-j}\binom{v-d-u}{i-j}, \quad i,u=0,\dots,d.$$

Eigenvalues of $J(v,d,i)$ $(i=0,1,\dots,d)$ are $E_i(j)$ for $j=0,1,\dots,d$. Now, one can proceed similarly as with the Hamming graphs in order to obtain simplified relaxations for the Johnson graphs. The resulting relaxations for the Johnson graphs differ from the relaxations for the Hamming graphs in the type of polynomials.

7. Symmetry Reductions for Other Partition Problems

Notice that a k-colorable subgraph of a graph corresponds to a partition of the graph's vertices into $k+1$ subsets, that is, k independent sets and the rest of the vertices. Therefore, one can consider the k-colorable subgraph problem as a graph partition problem, which is invariant under permutations of the subsets. It is not difficult to verify that other graph partition problems such as the max-k-cut problem and the k-equipartition problem are also invariant under permutations of the subsets. The max-k-cut problem is the problem of partitioning the vertex set of a graph into k sets such that the total weight of edges joining different sets is maximized. For the problem formulation and related SDP relaxations, see, for example, Delorme and Poljak (1993), de Klerk et al. (2004), and Rendl (2016). The k-equipartition problem is the problem of partitioning the vertex set of a graph into k sets of equal cardinality such that the total weight of edges joining different sets is minimized. For the problem formulation and related SDP relaxations, for example, Karisch and Rendl (1998), Wolkowicz and Zhao (1999), Sotirov (2012), and van Dam and Sotirov (2015).

It is known that vector- and matrix-lifting SDP relaxations for the max-k-cut and k-equipartition problems are equivalent. In particular, de Klerk et al. (2004) prove the equivalence of the relaxations for the max-k-cut problem by exploiting the invariance of the max-k-cut problem under permutations of the subsets. Sotirov (2012) proves the equivalence of three different SDP relaxations for the k-equipartition problem: a matrix-lifting relaxation, a vector-lifting relaxation, and an SDP relaxation for the k-

equipartition problem derived as a special case of the quadratic assignment problem.

Here, we prove the same results by using the approach from Sections 4.1 and 5.1. We remark that the proof here is more elegant than the one from Sotirov (2012).

We denote the optimal value of the vector-lifting SDP relaxation for the max-k-cut (resp., k-equipartition) problem on graph G by $MkC_v(G)$ (resp., $Ek_v(G)$). The vector-lifting relaxations of both problems are particular cases of the relaxation for the general graph partition problem by Wolkowicz and Zhao (1999). Let L be the Laplacian matrix of G; then the symmetry-reduced versions of vector-lifting relaxations are given as follows:

$$MkC_v(G) = \max_{Z,X\in\mathbb{S}^n} \Big\{ \tfrac{1}{2}\langle L, Z\rangle : X_{ii} = 0 \ \ \forall i \in [n], \\ Z_{ii} = 1 \ \ \forall i \in [n], \\ Z \geq 0, X \geq 0, Z - X \succcurlyeq 0, Z \\ +(k-1)X - J \succcurlyeq 0 \Big\}.$$

$$Ek_v(G) = \min_{Z,X\in\mathbb{S}^n} \Big\{ \tfrac{1}{2}\langle L, Z\rangle : X_{ii} = 0 \ \ \forall i \in [n], \\ Z_{ii} = 1 \ \ \forall i \in [n], \\ Z \geq 0, X \geq 0, Z - X \succcurlyeq 0, \\ Z + (k-1)X - J \succcurlyeq 0, Ze = \frac{n}{k}e \Big\}.$$

Next, we look at the matrix-lifting relaxations for the two problems. For the max-k-cut problem, reducing the matrix-lifting SDP relaxation results in the relaxation by van Dam and Sotirov (2016), which is equivalent to the relaxation by Frieze and Jerrum (1995). Similarly, for the k-equipartition problem, reducing the matrix-lifting SDP relaxation results in a well-known SDP relaxation by Karisch and Rendl (1998), which is equivalent to the relaxation by Sotirov (2014). The relaxations are as follows, with the notation analogous to the notation in the vector-lifting case:

$$MkC_m(G) = \max_{Z\in\mathbb{S}^n} \Big\{ \frac{1}{2}\langle L, Z\rangle : Z_{ii} = 1 \ \ \forall i \in [n], \ Z \geq 0, \\ Z - \frac{1}{k}J \succcurlyeq 0 \Big\}.$$

$$Ek_m(G) = \min_{Z\in\mathbb{S}^n} \Big\{ \frac{1}{2}\langle L, Z\rangle : Z_{ii} = 1 \ \ \forall i \in [n], \ Z \geq 0, \\ Z - \frac{1}{k}J \succcurlyeq 0, \ Ze = \frac{n}{k}e \Big\}.$$

To show that the vector- and matrix-lifting relaxations are equivalent, observe that, for both the k-equipartition and max-k-cut problem, one can construct feasible solutions to the matrix-lifting relaxation from the vector-lifting relaxation with the same objective value, and the other way round. First, from a feasible solution Z for each of the symmetry-reduced matrix-lifting relaxations, we obtain a feasible solution $(Z, \frac{1}{k-1}(J - Z))$ for the corresponding symmetry-reduced vector-lifting relaxation. For the opposite direction, a feasible solution (Z, X) to each of the symmetry-reduced vector-lifting relaxations provides a feasible solution Z to the corresponding matrix-lifting relaxation, as in the case of the MkCS problem (see also the proof of Theorem 6). Hence, the vector- and matrix-lifting relaxations are equivalent.

8. Conclusion

This paper combines several modeling approaches to derive strong bounds for the maximum k-colorable subgraph problem and related problems.

We first analyze the existing upper bound for the MkCS problem known as the *generalized* ϑ*-number* (see (7)). Then, we strengthen it by adding nonnegativity constraints to the corresponding SDP relaxation. We call the resulting upper bound the *generalized* ϑ'*-number* (see (9)). Then, we propose several new SDP relaxations for the MkCS problem with increasing complexities. The sizes of our new SDP relaxations initially depend on the number of colors k and the number of the vertices in the graph (see (11), (12), and (26)). To reduce the sizes of those three SDP relaxations, we exploit the fact that the MkCS problem is invariant with respect to color permutations. The reduction results in the SDP relaxations with at most two SDP constraints of order at most $(n+1)$ for any k and any graph type (see Theorem 3, Corollary 1 in the online supplement, and Theorem 5). The resulting relaxations provide the following upper bounds for the MkCS problem: θ_k^1 (see (15)), θ_k^2 (see (15) without constraints (15f) and (15g)), and θ_k^3 (see (28)). In Proposition 1, we characterize a family of graphs for which those bounds are tight. To improve our strongest relaxation, we add nonredundant, symmetry-reduced, Boolean quadric polytope inequalities (see (23)–(24)).

We further reduce relaxations for several classes of highly symmetric graphs, including the Johnson and Hamming graphs (see Section 6). The resulting relaxations are linear programs or linear programs with one convex quadratic constraint. Finally, we show that the vector- and matrix-lifting relaxations for the max-k-cut problem and the k-equipartition problem are equivalent by exploiting the invariance of the problems under permutations of the subsets (see Section 7).

We compute upper and lower bounds for graphs considered in Campêlo and Corrêa (2010), and Januschowski and Pfetsch (2011a) with up to 500 vertices. We also compute bounds for the MkCS problem for highly symmetric graphs with up to 6, 760, and 448 edges. We solve the problem for several graphs to optimality and obtain stronger bounds than in Campêlo and Corrêa (2010) for all but one tested graph. See the online supplement for numerical results. Our lower bounds on the chromatic number of a graph are competitive with bounds from the literature.

Acknowledgments
The authors are grateful to Monique Laurent for raising a question about the relation between the maximum k-colorable subgraph and the maximum stable set problem. The second author thanks Renate van der Knaap for implementing a tabu search algorithm and for computing lower bounds. The authors also thank two anonymous referees for suggestions that led to an improvement of this paper.

References
Addario-Berry L, Kennedy W, King AD, Li Z, Reed B (2010) Finding a maximum-weight induced k-partite subgraph of an i-triangulated graph. *Discrete Appl. Math.* 158(7):765–770.

Alizadeh F (1995) Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.* 5(1):13–51.

Bentert M, van Bevern R, Niedermeier R (2019) Inductive k-independent graphs and c-colorable subgraphs in scheduling: A review. *J. Scheduling* 22:3–20.

Bresar B, Valencia-Pabon M (2019) Independence number of products of Kneser graphs. *Discrete Math.* 342(4):1017–1027.

Brouwer A, Shearer J, Sloane N, Smith W (1990) A new table of constant weight codes. *IEEE Trans. Inform. Theory* 36(6): 1334–1380.

Burer S (2009) On the copositive representation of binary and continuous nonconvex quadratic programs. *Math. Programming* 120(2):479–495.

Campêlo M, Corrêa RC (2010) A combined parallel Lagrangian decomposition and cutting-plane generation for maximum stable set problems. *Electronic Notes Discrete Math.* 36:503–510.

de Klerk E, Sotirov R (2008) Exploiting group symmetry in semidefinite programming relaxations of the quadratic assignment problem. *Math. Programming* 122(2):225–246.

de Klerk E, Pasechnik D, Warners J (2004) On approximate graph colouring and MAX-k-CUT algorithms based on the θ-function. *J. Combin. Optim.* 8(3):267–294.

Delorme C, Poljak S (1993) Laplacian eigenvalues and the maximum cut problem. *Math. Programming* 62(1):557–574.

Delsarte P (1973) An algebraic approach to the association schemes of coding theory. Philips Research Reports Supplements No. 10, Phillips Research Laboratories, Eindhoven, Netherlands.

Ding Y, Ge D, Wolkowicz H (2011) On equivalence of semidefinite relaxations for quadratic matrix programming. *Math. Oper. Res.* 36(1):88–104.

Erdős P, Ko C, Rado R (1961) Intersection theorem for system of finite sets. *Quart. J. Math.* 12(1):313–318.

Etzion T, Bitan S (1996) On the chromatic number, colorings, and codes of the Johnson graph. *Discrete Appl. Math.* 70(2):163–175.

Fan K (1949) On a theorem of Weyl concerning eigenvalues of linear transformations I. *Proc. Natl. Acad. Sci. USA* 35(11):652–655.

Fouilhoux P, Mahjoub AR (2006) Polyhedral results for the bipartite induced subgraph problem. *Discrete Appl. Math.* 154(15):2128–2149.

Fouilhoux P, Mahjoub AR (2012) Solving VLSI design and DNA sequencing problems using bipartization of graphs. *Comput. Optim. Appl.* 51(2):749–781.

Frieze A, Jerrum M (1995) Improved approximation algorithms for MAX k-CUT and MAX BISECTION. Balas E, Clausen J, eds., *Integer Programming and Combinatorial Optimization* (Springer, Berlin), 1–13.

Füredi Z (1983) An extremal problem concerning Kneser's conjecture. *Studia Sci. Math. Hungarica* 18:335–341.

Gatermann K, Parrilo P (2004) Symmetry groups, semidefinite programs, and sums of squares. *J. Pure Appl. Algebra* 192(1–3): 95–128.

Godsil C, Royle G (2001) *Algebraic Graph Theory* (Springer, Berlin).

Goemans MX, Williamson DP (1994) .879 approximation algorithms for MAX CUT and MAX 2SAT. *Proc. 26th Annual ACM Sympos. Theory Comput.* (ACM, New York), 422–431.

Grötschel M, Lovász L, Schrijver A (1988) *Geometric Algorithms and Combinatorial Optimization* (Springer, Berlin).

Gvozdenović N, Laurent M (2008) The operator ψ for the chromatic number of a graph. *SIAM J. Optim.* 19(2):572–591.

Halldórsson MM, Halpern JY, Li LE, Mirrokni VS (2004) On spectrum sharing games. *Proc. 23rd Annual ACM Sympos. Principles Distributed Comput.* (ACM, New York), 107–114.

Hertz A, Montagné R, Gagnon F (2016) Constructive algorithms for the partial directed weighted improper coloring problem. *J. Graph Algorithms Appl.* 20(2):159–188.

Hertz A, Montagné R, Gagnon F (2018) Online algorithms for the maximum k-colorable subgraph problem. *Comput. Oper. Res.* 91:209–224.

Hüffner F (2005) Algorithm engineering for optimal graph bipartization. Nikoletseas SE, ed. *Experimental and Efficient Algorithms* (Springer, Berlin), 240–252.

Januschowski T, Pfetsch ME (2011a) Branch-cut-and-propagate for the maximum k-colorable subgraph problem with symmetry. Achterberg T, Beck JC, eds. *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems* (Springer, Berlin), 99–116.

Januschowski T, Pfetsch ME (2011b) The maximum k-colorable subgraph problem and orbitopes. *Discrete Optim.* 8(3):478–494.

Karisch SE, Rendl F (1998) Semidefinite programming and graph equipartition. Pardalos PM, Wolkowicz H, eds. *Topics in Semidefinite and Interior-Point Methods* (American Mathematical Society, Providence, RI), 77–95.

Karp RM (1972) Reducibility among combinatorial problems. Miller RE, Thatcher JW, Bohlinger JD, eds. *Complexity of Computer Computations* (Springer, Boston), 85–103.

Koster A, Scheffel M (2007) A routing and network dimensioning strategy to reduce wavelength continuity conflicts in all-optical networks. *Proc. INOC 2007, Internat. Network Optim. Conf., Spa, Belgium, April 22–25*.

Lee K, Funabiki N, Takefuji Y (1992) A parallel improvement algorithm for the bipartite subgraph problem. *IEEE Trans. Neural Networks* 3(1):139–145.

Lewis J, Yannakakis M (1980) The node-deletion problem for hereditary properties is NP-complete. *J. Comput. System Sci.* 20(2):219–230.

Lippert R, Schwartz R, Lancia G, Istrail S (2002) Algorithmic strategies for the single nucleotide polymorphism haplotype assembly problem. *Brief Bioinform.* 3(1):23–31.

Lovász L (1979) On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory* 25(1):1–7.

Lund C, Yannakakis M (1993) The approximation of maximum subgraph problems. Lingas A, Karlsson R, Carlsson S, eds. *Automata, Languages and Programming* (Springer, Berlin), 40–51.

Marek-Sadowska M (1984) An unconstrained topological via minimization problem for two-layer routing. *IEEE Trans. Comput. Aided Des. Integrated Circuits Systems* 3(3):184–190.

Mohar B, Poljak S (1993) Eigenvalues in combinatorial optimization. Brualdi RA, Friedland S, Klee V, eds. *Combinatorial and Graph-Theoretical Problems in Linear Algebra* (Springer, New York), 107–151.

Narasimhan G (1989) The maximum k-colorable subgraph problem. Unpublished doctoral dissertation, University of Wisconsin–Madison.

Narasimhan G, Manber R (1990) A generalization of Lovász's ϑ function. *Polyhedral Combinatorics* (American Mathematical Society, Providence, RI), 19–27.

Padberg M (1989) The Boolean quadric polytope: Some characteristics, facets and relatives. *Math. Programming* 45(1):139–172.

Papadimitriou CH, Yannakakis M (1991) Optimization, approximation, and complexity classes. *J. Comput. System Sci.* 43(3):425–440.

Rendl F (2016) Semidefinite relaxations for partitioning, assignment and ordering problems. *Ann. Oper. Res.* 240:119–140.

Rendl F, Sotirov R (2018) The min-cut and vertex separator problem. *Comput. Optim. Appl.* 69(1):159–187.

Rendl F, Sotirov R, Truden C (2019) Lower bounds for the bandwidth problem. Preprint, submitted April 14, https://arxiv.org/abs/1904.06715.

Sabidussi G (1957) Graphs with given group and given graph-theoretical properties. *Canadian J. Math.* 9:515–525.

Schrijver A (1979) A comparison of the Delsarte and Lovász bounds. *IEEE Trans. Inform. Theory* 25(4):425–429.

Sherali HD, Adams WP (1994) A hierarchy of relaxations and convex hull characterizations for mixed-integer zero-one programming problems. *Discrete Appl. Math.* 52(1):83–106.

Sotirov R (2012) SDP relaxations for some combinatorial optimization problems. Anjos M, Lasserre J, eds. *Handbook of Semidefinite, Conic and Polynomial Optimization: Theory, Algorithms, Software and Applications* (Springer, New York), 795–820.

Sotirov R (2014) An efficient semidefinite programming relaxation for the graph partition problem. *INFORMS J. Comput.* 26(1):16–30.

Subramanian AP, Gupta H, Das SR, Buddhikot MM (2007) Fast spectrum allocation in coordinated dynamic spectrum access based cellular networks. *Proc. 2nd IEEE Internat. Sympos. New Frontiers Dynam. Spectrum Access Networks* (IEEE, Piscataway, NJ), 320–330.

van Dam E, Sotirov R (2016) New bounds for the max-k-cut and chromatic number of a graph. *Linear Algebra Appl*. 488:216–234.

van Dam ER, Sotirov R (2015) Semidefinite programming and eigenvalue bounds for the graph partition problem. *Math. Programming* 151(2):379–404.

Wolkowicz H, Zhao Q (1999) Semidefinite programming relaxations for the graph partitioning problem. *Discrete Appl. Math.* 96–97:461–479.

Yannakakis M, Gavril F (1987) The maximum *k*-colorable subgraph problem for chordal graphs. *Inform. Processing Lett.* 24(2):133–137.