A Model Distinguishing Production and Consumption Bundles

Sharon Schalk

Abstract

In contrast with the classical theory of Arrow and Debreu, a model of a private ownership economy is presented in which production and consumption bundles are treated separately. Each of the two types of bundles is assumed to establish a convex cone. This also offers a point of contrast in comparison with the classical theory. The main part in the modelling is the introduction of production technologies which can be thought of as replacing the notion of production sets in Arrow and Debreu's model. It is shown that under mild economically interpretable conditions, a Walrasian equilibrium exists.

Correspondence to:
S. Schalk
Department of Econometrics and CentER
Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands

-The author would like to thank Stef van Eindhoven and Dolf Talman for their valuable comments.
-This research is part of the VF-program "Competition and Cooperation".
-The author is financially supported by the Cooperation Centre Tilburg and Eindhoven Universities, The Netherlands.

Introduction

In this paper, we present a mathematical model of a private ownership economy, and prove that under certain assumptions this model allows for existence of Walrasian equilibria. There are two main differences between the model presented here and the classical models (cf. [ArDe54]), differences, which are outlined in the following two statements.

- Commodities are not assumed to occur separately.
- Production and consumption are not treated on the same level.

Firstly, in the classical model one starts from the assumption that commodities are separately tradable. We shall not distinguish separate commodities and, in fact, not consider the concept of commodity at all, but replace this concept by the concept of “economy bundle". In the classical terminology, an economy bundle would be called a commodity bundle, but we avoid the use of the term commodity bundle since it can lead to confusion, having a fixed meaning in classical equilibrium theory as being a collection of several separately available goods. We are aware of the fact that it is hard to think of a real-world example, which fits in our model description and in which commodities do not enter the discussion. Therefore, one might think of our model as describing the non-classical situation in which fixed links between different commodities may be assumed present. For instance, we can model an economy in which only fixed, prescribed combinations of commodities can be traded.
In our model, we consider the set of economy bundles to be modelled by a convex cone in a real vector space; classically, the set of commodity bundles of an economy, where n different commodities are present, is modelled by the positive orthant $(\mathbb{R}^n)^+$ of the Euclidean space $\mathbb{R}^n$. Our model of a private ownership economy is only in terms of convex cones and its properties, and does not involve any vector space terminology. We emphasize this, by introducing eight axioms on a set, in which addition and scalar multiplication over the positive reals are defined, and which resemble the vector space axioms. A set satisfying these axioms is called a salient half-space. Each pointed convex cone in which addition and scalar multiplication are defined through the vector space operations, is a salient half-space. Furthermore, each salient half-space induces an ordered vector space for which the salient half-space is the positive cone. Summarising, in the model introduced in this paper, each element of the salient half-space under consideration, represents an economy bundle.

Although the model is presented in the general terms of salient half-spaces, existence of Walrasian equilibria can be guaranteed only if some assumptions are made, of which the assumption that the vector space for which the salient half-space is the positive cone, is finite dimensional, is the strongest. Despite this, we feel that the essential idea of this model is the use of the concept of salient half-space and concepts related to it. Of course, in proving the Existence Theorem, we make use of several properties of finite dimensional vector spaces, but they are part of the technical mathematical tools and not of the structure of the model. Forcing ourselves to cope with this general model structure, we have to apply an analysis and techniques which may be of use when tackling models for private ownership economies where the finite dimensionality restriction is not satisfied.

Secondly, production and consumption are not treated on the same level. We assume that an economy bundle in a private ownership economy is a unique concatenation of a consumption (economy) bundle and a production (economy) bundle. So, we distinguish between production bundles and consumption bundles. Only production bundles can be used as input for a production process and the output of such a process is always a consumption bundle. We model this by introducing a collection of production bundles, and a collection of consumption bundles, described by the salient half-space C_{prod} and C_{cons}, respectively, with the set C of economy bundles being equal to the Carthesian product set $C_{\text{prod}} \times C_{\text{cons}}$. Each economy bundle $(x, y) \in C$ represents a production process, where it is possible to produce consumption bundle $y \in C_{\text{cons}}$ from production bundle $x \in C_{\text{prod}}$. If a collection T of production processes satisfies certain conditions, which will be specified later, it is called a production technology. As far as we know, the classical models do not distinguish between consumption (commodity) bundles and production (commodity) bundles: instead of introducing a production technology T as a subset of $C_{\text{prod}} \times C_{\text{cons}}$ (in the classical situation with k production goods and l consumption goods, this would be $(\mathbb{R}^k)^+ \times (\mathbb{R}^l)^+$), the classical models recognise a production technology (production set) as a subset Y of the Euclidean vector space $\mathbb{R}^n$. Globally speaking, the product set $C_{\text{prod}} \times C_{\text{cons}}$ is replaced by the vector lattice $\mathbb{R}^n$. Indeed, $\mathbb{R}^n$ is regarded as the sum of the positive cone $(\mathbb{R}^n)^+$ and the negative cone $(\mathbb{R}^n)^-$ by writing each input-output vector $x \in \mathbb{R}^n$ as $x^+ - x^-$, with output vector x^+ and input vecor x^- defined by $x^+ := 0 \vee x$ and $x^- := (-x) \vee 0$. So to each $x \in Y$ there is associated a unique pair $(x^+, x^-) \in (\mathbb{R}^n)^+ \times (\mathbb{R}^n)^+$, and thus Y can be seen as a subset of $(\mathbb{R}^n)^+ \times (\mathbb{R}^n)^+$. In this paper, we shall not discuss whether the classical notion of production technology (Y) is generalised by our notion of production technology (T).

The introduction of the concept of production and consumption bundles also gives rise to a slightly altered definition of Walrasian equilibrium. Disregarding the concept of commodity, we cannot speak of the price of a commodity, and so, we use the notion of “value functional". All this results in a study in which new mathematical techniques are provided to prove existence of equilibria. For instance, the classical approach of examining excess demand and supply in each individual market cannot be adopted. We prove existence of a Walrasian equilibrium by constructing a so called equilibrium function. This function is defined on the set of value functionals, and its zeroes correspond to equilibrium value functionals.

We conclude this introduction by describing the contents of the different sections. Section 1 contains the introduction of the mathematical concepts and theorems which are used to construct the model and to prove the existence theorem. Its main item is the introduction of the concept of salient half-space and its relationship with vector spaces. The presentation in this section is almost self containing. In Section 2 we describe the mathematical model introducing the features of the economic agents, and of the production technologies. Furthermore, the Existence Theorem is stated and the mathematical assumptions, needed in its proof, are introduced. Section 3 is devoted to some properties for individual agents and production technolgies, the introduction of the concept of equilibrium function and proof of the Existence Theorem.

Contents

1 Mathematical concepts

The purpose of this section is the description of the mathematical concepts involved in our model (cf. Section 2) of a private ownership economy as well as in the existence proof (cf. Section 3).

1.1 Salient half-space

We start with the concept of salient half-space since we shall use this notion to model the set of economy bundles. Thereafter, we describe some similarities and differences between salient half-spaces, vector spaces, and convex cones.

Definition 1.1.1 *A salient half-space is a set C with the following properties:*

- *An addition is defined on C, which is commutative, associative and satisfies*

  1.1.1.a) *there exists an element $v \in C$, called the vertex of C, such that $x + y = v \iff x = y = v$,*

  1.1.1.b) *for every $x \in C$ the mapping add$_x : C \to C$, defined by add$_x(y) := y + x$, is injective.*

- *To every pair $x \in C$ and $\alpha \geq 0$, there corresponds an element $\alpha x \in C$, called the (scalar) product of α and x. Scalar multiplication over $I\!R^+$ thus defined, is associative and satisfies the distributive laws. Furthermore $1x = x$ holds for every $x \in C$.*

Condition 1.1.1.b states that the mapping add$_x$ is injective for all $x \in C$. Note that Condition 1.1.1.a implies that for all $x \neq v$ this mapping is not surjective. Also note that if $I\!R^+$ were replaced by $I\!R$ in Definition 1.1.1, the set C verifies the axioms dealing with scalar multiplication satisfied by a vector space (cf. [Halm87]).

Example
Let C be a pointed convex cone in a vector space V, then C is a salient half-space with the zero-element of V as vertex, and addition and multiplication defined in the natural way. Recall that a subset C of a vector space V is called a cone if $\alpha x \in C$ for all $x \in C$ and $\alpha \geq 0$. A cone is called pointed if the zero-element of V is the only extreme point of C. A subset D of a vector space is called convex if $\tau x + (1 - \tau)y \in D$ for all $x, y \in D$ and $\tau \in [0, 1]$. Thus, a cone in a vector space is convex if and only if it is closed under addition.

We shall see that the converse also holds: for every salient half-space C, there is a vector space $V[C]$ such that C is a pointed convex cone in $V[C]$. But first we derive some properties of salient half-spaces.

Lemma 1.1.2 *The vertex of a salient half-space C is unique.*

Proof
Suppose both v and w are vertices of C, then from $w + w = w$ it immediately follows that $v + w + w = v + w$. Applying Condition 1.1.1.b, we get $v + w = v$ and, because v is a vertex of C, $w = v$ follows from Condition 1.1.1.a. $\square$

Lemma 1.1.3 *For every salient half-space C, its vertex v satisfies the following three properties:*

$$\begin{array}{lll} 1.1.3.a) & \forall \alpha > 0 & : \alpha v = v, \\ 1.1.3.b) & \forall x \in C & : x + v = x, \\ 1.1.3.c) & \forall x \in C & : 0x = v. \end{array}$$

Proof
a) We prove that αv is a vertex of C for all $\alpha > 0$, then by the preceding lemma $\alpha v = v$. Consider the following equivalent assertions:
$x + y = \alpha v \iff \frac{1}{\alpha}x + \frac{1}{\alpha}y = v \iff (\frac{1}{\alpha}x = v) \wedge (\frac{1}{\alpha}y = v) \iff (x = \alpha v) \wedge (y = \alpha v)$.
b) Let $x \in C$ and define $y := x + v$. Then $y + y = 2y = 2(x + v) = 2x + v = x + (x + v) = x + y$. Applying Condition 1.1.1.b yields $y = x$.
c) Let $x \in C$, then by Property 1.1.3.b and the distributiveness of scalar multiplication over $I\!R^+$, we get $0x + 0x = (0 + 0)x = 0x = 0x + v$. So, Condition 1.1.1.b yields $0x = v$. $\square$

From Property 1.1.3.b together with Condition 1.1.1.a and 1.1.1.b, we conclude that $(C, +)$ is a semi-group with zero-element v. Since in a salient half-space, scalar multiplication is defined only over $I\!R^+$ and due to Condition 1.1.1.a, $(C, +)$ is not a group, but a semi-group. However, we can extend $(C, +)$ to a group in a similar way as $I\!N \cup \{0\}$ extends to $Z\!\!\!Z$. We shall present this extension in short. Define the equivalence relation $\sim$ on the product set $C \times C$ by:

$$(x_1, x_2) \sim (y_1, y_2) :\iff x_1 + y_2 = y_1 + x_2.$$

Let $V[C]$ be the collection of all equivalent classes $[(y_1, y_2)] := \{(z_1, z_2) \in C \times C \mid (z_1, z_2) \sim (y_1, y_2)\}$, so $V[C] := (C \times C)/_{\sim}$. Unambiguously, we can define the following addition and scalar multiplication on $V[C]$:

$$
\begin{array}{lcl}
[(y_1, y_2)] + [(z_1, z_2)] & := & [(y_1 + z_1, y_2 + z_2)] \\
\alpha[(y_1, y_2)] & := & \left\{ \begin{array}{ll} [(\alpha y_1, \alpha y_2)] & \text{if } \alpha \geq 0 \\ [((-\alpha) y_2, (-\alpha) y_1)] & \text{if } \alpha < 0. \end{array} \right.
\end{array}
$$

We shall make plausible that with these definitions, the set $V[C]$ becomes a real vector space. We call $V[C]$ the vector space generated by the salient half-space C. In general, if $(A, +)$ is a semigroup with a zero-element, then the above construction can be applied to construct a group. So the proof that $V[C]$ is indeed a vector space can concentrate on the introduction of the scalar product over negative α. The construction yields that $[(v, v)]$ is the origin of $V[C]$ and $-[(y_1, y_2)] = [(y_2, y_1)]$. Note that multiplication by negative scalars is defined properly. Let $\alpha > 0$ then

$$(-\alpha)[(y_1, y_2)] = \alpha(-1)[(y_1, y_2)] = \alpha[(y_2, y_1)] = \alpha(-[(y_1, y_2)]).$$

Furthermore, the salient half-space C is a total subset of the vector space $V[C]$, i.e., the linear span of C equals $V[C]$. The vertex v of C coincides with the origin of the vector space $V[C]$, and henceforward we shall denote the vertex of a salient half-space by 0.

Definition 1.1.4 *On a salient half-space C the partial ordering $\geq_C$ is defined by*

$$
\begin{array}{lll}
x \geq_C y & \textit{if and only if} & \exists z \in C : x = y + z. \\
x >_C y & \textit{if and only if} & \exists z \in C \setminus \{0\} : x = y + z.
\end{array}
$$

The salient half-space C, when identified with $\{[(y_1, y_2)] \in V[C] \mid \exists x \in C : [(y_1, y_2)] \sim [(x, 0)]\}$, can be regarded as a subset of $V[C]$. The partial order relation $\geq_C$, defined on C, can be extended to a partial order relation on $V[C]$ by defining for all $[(y_1, y_2)], [(z_1, z_2)] \in V[C]$:

$$[(y_1, y_2)] \geq [(z_1, z_2)] \text{ if } \exists [(x_1, x_2)] \in C : [(y_1, y_2)] = [(z_1, z_2)] + [(x_1, x_2)].$$

Note that this is equivalent with $y_1 + z_2 + x_2 = y_2 + z_1 + x_1$, or

$$y_1 + z_2 \geq_C y_2 + z_1.$$

Also, note that $C := \{[(y_1, y_2)] \in V[C] \mid [(y_1, y_2)] \geq [(0,0)]\}$.

In literature, it is common to introduce a pointed convex cone in a vector space, therewith introducing a partial order on this vector space. Here, we introduce these notions the other way around, since we consider the salient half-space, rather than the vector space, to be the essential element of the model.

Definition 1.1.5 *An element u of C is called an order unit for C if*

$$\forall x \in C \ \exists \lambda \geq 0 : x \leq_C \lambda u.$$

Lemma 1.1.6 *Let u be an order unit for C, and let $[(y_1, y_2)] \in V[C]$. Then*

$$\exists \lambda \geq 0 : -\lambda[(u,0)] \leq [(y_1, y_2)] \leq \lambda[(u,0)].$$

Proof
Since u is an order unit for C, we find $\begin{cases} \exists \lambda_1 \geq 0 : y_1 \leq_C \lambda_1 u \\ \exists \lambda_2 \geq 0 : y_2 \leq_C \lambda_2 u. \end{cases}$
Define $\lambda := \max\{\lambda_1, \lambda_2\}$, then $\begin{cases} y_1 \leq_C y_2 + \lambda u \\ y_2 \leq_C y_1 + \lambda u. \end{cases}$ □

1.2 Salient half-dual space

Let C^* be the set of all half-linear functionals $p : C \to I\!R^+$, i.e., the set of all functions p defined on C satisfying

$$\begin{cases} p(x+y) = p(x) + p(y) & \forall x, y \in C \\ p(\alpha x) = \alpha p(x) & \forall x \in C \ \forall \alpha \geq 0, \end{cases}$$

then C^* is a salient half-space also, where the zero-functional is its vertex and addition and positive scalar multiplication are defined pointwise; for $p, q \in C^*$ and $\alpha \geq 0$:

$$\begin{cases} (p+q)(x) := p(x) + q(x) & \forall x \in C \\ (\alpha p)(x) := \alpha p(x) & \forall x \in C. \end{cases}$$

We call C^* the salient half-dual space (or in short: the half-dual) of C. It turns out that existence of one order unit in C is sufficient to guarantee that C^* is non-trivial, i.e., $C^* \neq \{0\}$.

Proposition 1.2.1 *If C has an order unit, then $C^* \neq \{0\}$.*

Proof
Let u be an order unit for C. Define the set $U \subset V[C]$ by $U := \{\lambda[(u,0)] \mid \lambda \in I\!R\}$, then U is a subspace of $V[C]$. By Lemma 1.1.6, we find

$$\forall[(y_1,y_2)] \in V[C]\ \exists\lambda \geq 0 : -\lambda[(u,0)] \leq [(y_1,y_2)] \leq \lambda[(u,0)].$$

Thus, we can define the sublinear functional $q : V[C] \to I\!R$ by

$$q([(y_1,y_2)]) := \inf\{\lambda \mid [(y_1,y_2)] \leq \lambda[(u,0)]\}.$$

Define $f(\lambda[(u,0)]) := \lambda$, for every $\lambda \in I\!R$. With this definition, $f : U \to I\!R$ is a positive linear functional on U satisfying $\forall\lambda \in I\!R : f(\lambda[(u,0)]) = q(\lambda[(u,0)])$. By the Hahn-Banach Theorem, there exists a linear functional $\tilde{f} : V[C] \to I\!R$ such that on the set U, $\tilde{f}$ is equal to f, and $\forall[(y_1,y_2)] \in V[C] : \tilde{f}([(y_1,y_2)]) \leq q([(y_1,y_2)])$. For every $[(x_1,x_2)] \in C$ it holds that $q([(x_1,x_2)]) \geq 0$. We conclude that the functional $\tilde{f}$ acts positively on C since for all $[(x_1,x_2)] \in C : \tilde{f}(-[(x_1,x_2)]) \leq q(-[(x_1,x_2)]) \leq 0$. $\square$

Applying Definition 1.1.4 on the salient half-dual space, we find the pre-ordering $\geq_{C^*}$ on C^*, which is given by

$$\begin{array}{lll} p \geq_{C^*} q & \text{if and only if} & \exists r \in C^* : p = q + r. \\ p >_{C^*} q & \text{if and only if} & \exists r \in C^* \setminus \{0\} : p = q + r. \end{array}$$

Note that this partial order relation is equivalent with the standard partial order relation on functionals in $(V[C])^*$:

$$\begin{array}{lll} p \geq_{C^*} q & \iff & \forall x \in C : p(x) \geq q(x). \\ p >_{C^*} q & \iff & (\forall x \in C : p(x) \geq q(x)) \wedge (\exists x \in C : p(x) > q(x)). \end{array}$$

First we examine the relationship between the vector space $V[C^*]$, generated by the half-dual C^* of C, and the dual space $(V[C])^*$ of $V[C]$.

Proposition 1.2.2 *$V[C^*]$ is canonically injected in $(V[C])^*$ and therefore can be considered a subspace of $(V[C])^*$. Furthermore, $C^* = \{p \in (V[C])^* \mid \forall x \in C : p(x) \geq 0\}$.*

Proof
Let $[(p_1, p_2)] \in V[C^*]$ and define for every $[(y_1, y_2)] \in V[C]$:

$$[(p_1, p_2)]\,([(y_1, y_2)]) := p_1(y_1) - p_1(y_2) - p_2(y_1) + p_2(y_2).$$

It is easy to check that this definition is independent of the choice of the representatives (y_1, y_2) and (p_1, p_2), and that with this definition $[(p_1, p_2)]$ acts as a linear functional on $V[C]$. Secondly, it is easy to check that the mapping, described above, which adds a linear functional to every pair $[(p_1, p_2)] \in V[C^*]$ is linear. Furthermore, if $\forall[(x_1, x_2)] \in C$ it holds that $[(p_1, p_2)]([(x_1, x_2)]) = 0$, then $\forall x \in C : [(p_1, p_2)]([(x, 0)]) = p_1(x) - p_2(x) = 0$, and we conclude $p_1 = p_2$, or, in other words, $[(p_1, p_2)] = [(0, 0)]$. $\square$

In the sequel we shall regard C^* as a subset of $(V[C])^*$.
Let W be a vector space. Then $S \subset W^*$ is said to be separating the elements of W if $\forall x \in W \setminus \{0\}\ \exists p \in S : p(x) \neq 0$. Note, that in this connection, a subset $S \subset C^*$ is said to be separating the elements of C if $\forall x, y \in C, x \neq y\ \exists p \in S : p(x) \neq p(y)$.

Lemma 1.2.3 *A set $S_0 \subset C^*$ separates the elements of C if and only if the collection $S := \{[(p_1, p_2)] \mid p_1, p_2 \in S_0\} \subset V[C^*]$ separates the elements of $V[C]$.*

Proof
Let $x, y \in C$ satisfy $x \neq y$. Consider the following sequence of equivalent statements

$$\begin{aligned}
&\forall p \in S_0 : p(x) = p(y),\\
&\forall p_1, p_2 \in S_0 : p_1(x) + p_2(y) = p_1(y) + p_2(x),\\
&\forall [(p_1, p_2)] \in S : p_1(x) + p_2(y) - p_1(y) - p_2(x) = 0,\\
&\forall [(p_1, p_2)] \in S : [(p_1, p_2)]\,([(x, y)]) = 0.
\end{aligned}$$

$\square$

From now on, we assume that $V[C]$ is finite-dimensional. As usual in this situation, we identify $V[C]$ and its bidual $(V[C])^{**}$, i.e., we identify each $x \in V[C]$ with its action $p \mapsto p(x)$ on $(V[C])^*$. To show this duality to full advantage, instead of $p(x)$, we write $[x, p]$ for every $p \in (V[C])^*$ and $x \in V[C]$. Note that with this identification, we have $C \subseteq C^{**}$. Since in this paper, we are particularly interested in salient half-spaces, and since we regard the vector space generated by a salient half-space merely as a mathematical tool, we shall often adopt the notation $[x,p]_C$ to denote $p(x)$ where $x \in C$ and $p \in C^*$.

Because $C \subseteq C^{**}$, we can consider the partial ordering $\geq_{C^{**}}$ on C as follows. Let $x, y \in C$, then

$$\begin{aligned} x \geq_{C^{**}} y &\iff \exists z \in C^{**} : x = y + z \\ &\iff \forall p \in C^* : [p,x]_{C^*} \geq [p,y]_{C^*} \\ &\iff \forall p \in C^* : [x,p]_C \geq [y,p]_C. \end{aligned}$$

So, if $C^{**} = C$, then $x \geq_C y$ is equivalent with $\forall p \in C^* : [x,p]_C \geq [y,p]_C$.

Proposition 1.2.4 *Let $C^{**} = C$. Then C^* separates the elements of C.*

Proof
Let $x, y \in C$, and suppose $\forall p \in C^* : [x,p]_C = [y,p]_C$. Of course, since $C^{**} = C$, this means $x \geq_C y$ and $y \geq_C x$. The order relation $\geq_C$ being anti-symmetric, this implies $x = y$. □

By Lemma 1.2.3 we find that $V[C^*]$ is a subspace of $(V[C])^*$, separating the elements of the finite dimensional vector space $V[C]$. This yields

$$V[C^*] = (V[C])^*. \tag{1}$$

It is in general not true, that (1) implies $C^{**} = C$, since the latter equality is related to a non-algebraic condition on C.
Finally, we mention the consequences of the condition $C^{**} = C$ for the partial ordering on C:

$$\begin{aligned} x \geq_C y &:\iff \exists z \in C : x = y + z \\ &\iff \forall p \in C^* : [x,p]_C \geq [y,p]_C, \\ x >_C y &:\iff \exists z \in C \setminus \{0\} : x = y + z \\ &\iff (\forall p \in C^* : [x,p]_C \geq [y,p]_C) \wedge (\exists p \in C^* : [x,p]_C > [y,p]_C). \end{aligned}$$

1.3 Topology and order units

In the following, we shall assume C to be a salient half-space satisfying the conditions presented at the end of Subsection 1.2, i.e. $C \neq \{0\}$, $\dim(V[C]) < \infty$, and $C^{**} = C$. Note that if a salient half-space C satisfies these conditions, so does its dual C^*, since $(V[C^*])^* = (V[C^*])^{***}$. Therefore, every lemma or proposition derived for C with dual space C^* has a dual lemma or proposition for C^* and its dual C.

Definition 1.3.1 *Let $(x_n)_{n \in I\!N}$ be a sequence in C, then we say that $(x_n)_{n \in I\!N}$ converges to x (notation: $x_n \to x$), if $\forall f \in C^* : \lim_{n \to \infty} f(x_n) = f(x)$.*

Definition 1.3.2 *A set $S \subset C$ is $\mathcal{T}(C, C^*)$-closed in C, if for all sequences $(x_n)_{n \in I\!N}$ in S, satisfying $x_n \to x \in C$, it holds that $x \in S$.*

Thus, a topology is defined on C, where $O \subset C$ is an open set if and only if $C \setminus O$ is $\mathcal{T}(C, C^*)$-closed. The proof that the collection of al such open sets satisfies the conditions of a topology for C is straightforward. Of course, we shall denote this topology by $\mathcal{T}(C, C^*)$.

Similarly, we find the following definition for the vector space $V[C]$:

Definition 1.3.3 *Let $(y_n)_{n \in I\!N}$ be a sequence in $V[C]$, then we say that $(y_n)_{n \in I\!N}$ converges to y (notation: $y_n \to y$), if $\forall f \in C^* : \lim_{n \to \infty} [(f, 0)](y_n) = [(f, 0)](y)$. A set $S \subset V[C]$ is $\mathcal{T}(V[C], C^*)$-closed in $V[C]$ if for all sequences $(y_n)_{n \in I\!N}$ in S, satisfying $y_n \to y \in V[C]$, it holds that $y \in S$.*

Thus, the topology $\mathcal{T}(V[C], C^*)$ is defined on $V[C]$. This topology is Hausdorf, since C^* separates the elements of $V[C]$. Considering C as a subset of $V[C]$, the toplology $\mathcal{T}(C, C^*)$ is the relative toplogy on C, induced by $\mathcal{T}(V[C), C^*)$. The construction of $V[C]$ from C implies that C is solid in $V[C]$. Note that C is a $\mathcal{T}(V[C], C^*)$-closed set due to $C^{**} = C$, while it is both open and closed in $\mathcal{T}(C, C^*)$.
We shall denote the $\mathcal{T}(V[C], C^*)$-interior of a set $A \subset V[C]$ by $\text{int}(A)$ and the boundary of A by ∂A. In the following, we shall use the notation $\text{int}(C)$ to denote the $\mathcal{T}(V[C], C^*)$-interior of C, where C is regarded as a subset of $V[C]$. Since C is solid in $V[C]$, $\text{int}(C) \neq \emptyset$. With the notation ∂C, we denote $C \setminus \text{int}(C)$. Since, in this paper, we regard the salient half-space C, rather than the vector space $V[C]$, to be the essential concept, we would like to have a salient half-space-related characterisation of $\text{int}(C)$.

Lemma 1.3.4 *Let $x_0 \in C$. Then $x_0 \in \text{int}(C)$ if and only if $\forall p \in C^* \setminus \{0\}$: $[x_0, p]_C > 0$.*

Proof
Suppose there exists $p \in C^*$ such that $[x_0, p]_C = 0$. Since $x_0 \in \text{int}(C)$ there is an open set $O \in \mathcal{T}(V[C], C^*)$ satisfying $\{x_0\} + O \subset C$. For all $y \in O$, $[y, p]_C = [x_0 + y, p]_C \geq 0$, from which we conclude that $p = 0$.
For the converse, suppose $x_0 \in \partial C \setminus \{0\}$. Since C is convex, int(C) is convex, so by the Weak Separation Theorem of Minkowski ([Pani93, p.60])

$$\exists\, p_0 \in (V[C])^* \setminus \{0\}\ \exists\, \alpha \in I\!R : \begin{cases} \forall\, \lambda \geq 0 : & [\lambda x_0, p_0] \leq \alpha \\ \forall\, x \in \text{int}(C) : & [x, p_0] \geq \alpha. \end{cases}$$

On the one hand we can choose λ equal to 0, and on the other hand int(C) contains a sequence of elements converging to 0. So, we find $\alpha = 0$, and as a consequence $p_0 \in C^* \setminus \{0\}$. By choosing λ equal to 1, we find $[x_0, p_0]_C \leq 0$. □

Note that as a consequence of this lemma, every element $x \in \partial C$ satisfies $\exists p \in C^* \setminus \{0\} : [x, p]_C = 0$.

Every $p \in C^*$ induces a seminorm $q_p : V[C] \to I\!R^+$ by $q_p(y) := |p(y)|$. This separating collection of seminorms, $\{q_p \mid p \in C^*\}$, generates the topology $\sigma(V[C], C^*)$, and since both topologies $\mathcal{T}(V[C], C^*)$ and $\sigma(V[C], C^*)$ are locally convex and use the sequence-based definition of convergence, they are equivalent. Since $V[C]$ is finite-dimensional, topology $\sigma(V[C], C^*)$ is induced by any norm on $V[C]$. Given any element $p_0 \in \text{int}(C)$, we shall construct a norm on $V[C]$.

Proposition 1.3.5 *Let $p_0 \in \text{int}(C)^*$. Then there exists a norm $\parallel . \parallel_{p_0}$ on $V[C]$, such that $\forall x \in C : \parallel x \parallel_{p_0} = [x, p_0]_C$.*

Proof
For every $y \in V[C]$ define $\parallel y \parallel_{p_0} := \inf\{[x_1 + x_2, p_0]_C \mid x_1, x_2 \in C$ with $y + x_2 = x_1\}$. It is not difficult to check that $\parallel . \parallel_{p_0}$ indeed is a norm on $V[C]$. To prove that $\forall x \in C : \parallel x \parallel_{p_0} = [x, p_0]_C$, we remark that $\forall x \in C : [x, p_0]_C \leq \parallel x \parallel_{p_0}$, since for all $x, x_1, x_2 \in C$ satisfying $x + x_2 = x_1$ it holds that $x_1 + x_2 = x + 2x_2 \geq_C x$. Furthermore, we can choose $x_1 = x$ and $x_2 = 0$ to obtain that $\parallel x \parallel_{p_0} \leq [x, p_0]_C$. □

Since $C^{**} = C$, each $x_0 \in \text{int}(C)$ induces a norm $\parallel . \parallel_{x_0}$ on C^*.

Corollary 1.3.6 *Let S be a subset of C and let $p_0 \in \mathrm{int}(C^*)$. Then S is bounded if and only if the set $\{[x, p_0]_C \mid x \in S\}$ is bounded.*

Corollary 1.3.7 *For all $p_0 \in int(C^*)$, the sets $K_1(p_0) := \{x \in C \mid [x, p_0]_C \leq 1\}$ and $L_1(p_0) := \{x \in C \mid [x, p_0]_C = 1\}$ are compact.*

Proof
Let $p_0 \in \mathrm{int}(C^*)$ be given. The sets $K_1(p_0)$ and $L_1(p_0)$ are closed subsets of the unit sphere $\{x \in C \mid \| x \|_{p_0} \leq 1\}$. □

Proposition 1.3.8 *Every $x_0 \in \mathrm{int}(C)$ is an order unit for C. Moreover, if $x_0 \in \mathrm{int}(C)$ then for every $x \in C$ there exists $\varphi(x) \geq 0$ such that $x \leq_C \varphi(x)x_0$, and if in addition $x \in \mathrm{int}(C)$ then there exists $\psi(x) > 0$ such that $\psi(x)x_0 \leq_C x$.*

Proof
Let $x_0 \in \mathrm{int}(C)$. Now, the statement

$$\forall x \in C \; \exists \psi(x), \varphi(x) \geq 0 : \psi(x)x_0 \leq_C x \leq_C \varphi(x)x_0 \tag{2}$$

is equivalent with

$$\forall x \in C \; \exists \psi(x), \varphi(x) \geq 0 \; \forall p \in C^* : \psi(x)[x_0, p]_C \leq [x, p]_C \leq \varphi(x)[x_0, p]_C.$$

Define the compact set $L_1(x_0) := \{p \in C^* \mid [x_0, p]_C = 1\}$. Then $C^* = \{\alpha p \mid p \in L_1(x_0), \alpha \geq 0\}$. Now, statement (2) is equivalent with

$$\forall \; x \in C \; \exists \; \psi(x), \varphi(x), \geq 0 \;\; \forall \; p \in L_1(x_0) : \psi(x) \leq [x, p]_C \leq \varphi(x).$$

For every $x \in C$ define

$$\begin{aligned} \varphi(x) &:= \max\{[x, p]_C \mid p \in L_1(x_0)\} \\ \psi(x) &:= \min\{[x, p]_C \mid p \in L_1(x_0)\}. \end{aligned}$$

Then $\psi(x) \leq [x, p]_C \leq \varphi(x)$ for all $p \in L_1(x_0)$. □

Corollary 1.3.9 *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathrm{int}(C)$, with limit $x_0 \in \mathrm{int}(C)$. Then there are sequences $(\psi_n)_{n \in \mathbb{N}}$ and $(\varphi_n)_{n \in \mathbb{N}}$ such that*

$$\psi_n x_0 \leq_C x_n \leq_C \varphi_n x_0 \text{ and } \lim_{n \to \infty} \psi_n = \lim_{n \to \infty} \varphi_n = 1.$$

Proof
Using the notation of the previous proof, let p satisfy $[x_0, p]_C = \varphi(x_0)$ and, similarly, let for all $n \in I\!N$, p_n satisfy $[x_n, p_n]_C = \varphi(x_n)$. Since, for all $n \in I\!N : \varphi(x_n) \geq [x_n, p]_C$, we find that $\liminf_{n\to\infty} \varphi(x_n) \geq [x_0, p]_C$. Let $(x_{n_k})_{k\in I\!N}$ be a subsequence of $(x_n)_{n\in I\!N}$, satisfying $\limsup_{n\to\infty} \varphi(x_n) = \lim_{k\to\infty} \varphi(x_{nk})$. The sequence $(p_{nk})_{k\in I\!N}$ lies in the compact set $L_1(x_0)$, so $(p_{nk})_{k\in I\!N}$ can be assumed convergent with limit $q \in L_1(x_0)$. Now, we find

$$\begin{aligned}\limsup_{n\to\infty} \varphi(x_n) &= \lim_{k\to\infty} \varphi(x_{nk}) = \lim_{k\to\infty} [x_{nk}, p_{nk}]_C \\ &= [x_0, q]_C \leq [x_0, p]_C \leq \liminf_{n\to\infty} \varphi(x_n).\end{aligned}$$

A similar argument can be used to prove $\lim_{n\to\infty} \psi(x_n) = \psi(x_0)$. □

Brouwer's Fixed Point Theorem [Conw90, p.149]
Let K be a non-empty compact convex subset of a finite-dimensional normed vector space X and let $\mathcal{F} : K \to K$ be a continuous function, then there exists $x \in K$ such that $\mathcal{F}(x) = x$, i.e., $\mathcal{F}$ has a fixed point in K.

Brouwer's Fixed Point Theorem has the following consequence for continuous functions on a salient half-space C satisfying $C^{**} = C$.

Proposition 1.3.10 *Let C be a salient half-space satisfying $C^{**} = C$ and $\dim(V[C]) < \infty$. Let $\mathcal{G} : C\backslash\{0\} \to C$ be a continuous function. Then there exists an $x \in C\backslash\{0\}$ such that $\mathcal{G}(x) = \alpha x$ for some $\alpha \geq 0$. In fact, for all $p_0 \in \text{int}(C^*)$ there is $x \in C$ such that $\mathcal{G}(x) = [\mathcal{G}(x), p_0]x$.*

Proof
Let $p_0 \in \text{int}(C^*)$. The set $K_1(p_0) := \{x \in C \mid [x, p_0]_C = 1\}$ is non-empty, convex and compact by Corollary 1.3.7. Define $\mathcal{F}(x) := \frac{x+\mathcal{G}(x)}{1+[\mathcal{G}(x),p_0]_C}$, $x \in K_1(p_0)$, then $\mathcal{F} : K_1(p_0) \to K_1(p_0)$ is a continuous function. By the preceding theorem the function $\mathcal{F}$ has a fixed point x in $K_1(p_0)$, so

$$x = \mathcal{F}(x) = \frac{x + \mathcal{G}(x)}{1 + [\mathcal{G}(x), p_0]_C},$$

hence $\mathcal{G}(x) = [\mathcal{G}(x), p_0]_C x$. □

We finish this subsection with the introduction of a Lebesque measure. Let C be a salient half-space and n the dimension of $V[C]$. Let $x_0 \in \text{int}(C)$ and consider the hyperplane $H_1(x_0) := \{p \in (V[C])^* \mid [x_0, p] = 1\}$ of the dual space $(V[C])^*$. Let $\Phi : I\!R^{n-1} \to H_1(x_0)$ be an affine parametrisation of $H_1(x_0)$ and endow $H_1(x_0)$ with the topology such that Φ is a homeomorphism. Take the standard Lebesque measure λ on $I\!R^{n-1}$ and define μ to be the measure on $H_1(x_0)$ induced by Φ and λ. Hence, for every subset A of $H_1(x_0)$ we have $\mu(A) = \lambda(\Phi^{\leftarrow}(A))$ and for a real-valued function f on (a subset of) $H_1(x_0)$, for which $f \circ \Phi$ is continuous, f is integrable with respect to μ, and

$$\int_A f d\mu = \int_{\Phi^{\leftarrow}(A)} (f \circ \Phi) d\lambda.$$

This measure μ is a regular Borel measure. Therefore, if f is continuous on a subset A of $H_1(x_0)$ with a dense interior, and if the set $L := \{x \in A \mid f(x) < 0\}$ satisfies $\mu(L) = 0$, then $L = \emptyset$, i.e. $\forall x \in A : f(x) \geq 0$.
Let W denote a finite-dimensional real vector space with $\{g_1, \ldots g_m\}$ a basis in the dual space W^*, and let $f : H_1(x_0) \to W$ be continuous. Then $\forall i \in \{1, \ldots, m\}$: $g_i \circ f$ is continuous from $H_1(x_0)$ into $I\!R$, and by $\int_A f d\mu$ we denote the unique element in W which satisfies

$$\forall i \in \{1, \ldots, m\} : \int_A (g_i \circ f) d\mu = g_i \circ \left(\int_A f d\mu \right).$$

For a norm $\parallel . \parallel$ on the vector space W, we have

$$\parallel \int_A f d\mu \parallel \leq \int_A \parallel f \parallel \; d\mu.$$

1.4 Direct sums and extremal sets

In our model (cf. Section 2) we shall define a production technology set which will be a subset of a direct sum of two salient half-spaces. In this subsection, we introduce such direct sums and derive some of their properties, which will be used in Section 3.

Definition 1.4.1 *Let C_a and C_b be two salient half-spaces. Their direct sum is the salient half-space $C_a \oplus C_b$, consisting of all ordered pairs $x = (x^a, x^b)$ with $x^a \in C_a$ and $x^b \in C_b$. The salient half-space operations are for all $x, y \in C_a \oplus C_b$ and for all $\alpha \geq 0$ given by:*

$$\left\{ \begin{array}{lcl} (x+y)^a & := & x^a + y^a \\ (\alpha x)^a & := & \alpha x^a \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{lcl} (x+y)^b & := & x^b + y^b \\ (\alpha x)^b & := & \alpha x^b. \end{array} \right.$$

For every $x \in C_a \oplus C_b$, there are unique $x^a \in C_a$ and $x^b \in C_b$ such that $x = (x^a, x^b)$. Since $C_a \oplus C_b$ is a salient half-space, every property thusfar derived for salient half-spaces is also applicable to $C_a \oplus C_b$.
On the direct sum $C_a \oplus C_b$ the pre-ordering $\geq_{(C_a \oplus C_b)}$ is given by:

$$x \ \geq_{(C_a \oplus C_b)} \ y :\Longleftrightarrow \begin{cases} x^a \geq_{C_a} y^a \\ x^b \geq_{C_b} y^b. \end{cases}$$

We continue this subsection on direct sums by remarking that

$$V[C_a \oplus C_b] = V[C_a] \oplus V[C_b],$$

where the second $\oplus$ denotes the usual direct sum defined for two vector spaces (cf. [Halm87]), and that

$$(C_a \oplus C_b)^* = C_a^* \oplus C_b^*,$$

where the action of $p \in C_a^* \oplus C_b^*$ on an element $x \in C_a \oplus C_b$ is defined by

$$[x, p]_{(C_a \oplus C_b)} \quad = \quad [x^a, p^a]_{C_a} + [x^b, p^b]_{C_b}.$$

To simplify notation we shall use C to denote $C_a \oplus C_b$. Furthermore, we shall write $[.,.]_a$ and $[.,.]_b$ instead of $[.,.]_{C_a}$ and $[.,.]_{C_b}$, respectively. Hence, for every $x \in C, p \in C^*$ we write $[x,p]_C = [x^a, p^a]_a + [x^b, p^b]_b$. Also, we shall write $\geq_a$ and $\geq_b$ instead of $\geq_{C_a}$ and $\geq_{C_b}$.

Definition 1.4.2 *For all $x \in C$ we define the set F_x by*

$$F_x := \{z \in C \mid z^a \geq_a x^a \textit{ and } z^b \leq_b x^b\}. \tag{3}$$

Let $T \subset C$. For all $x \in T$ we define the set $R_x(T)$ by

$$R_x(T) := \{z \in T \mid x \in F_z \textit{ and } F_z \subset T\}. \tag{4}$$

Furthermore, the set $E(T)$ is defined by

$$E(T) := \{x \in T \mid R_x(T) = \{x\}\}. \tag{5}$$

Without proof we state the following two consequences of these definitions.

Lemma 1.4.3 *Let $x \in C$. Then*

- $\forall y \in F_x : F_y \subset F_x$.
- *If* $y \in F_x$ *and* $x \neq y$, *then* $x \notin F_y$.

Lemma 1.4.4 *Let* $T \subset C$. *Assume* $T = \bigcup_{e \in E(T)} F_e$ *and assume* $\forall e, f \in E(T)\ \forall \tau \in [0,1] : \tau e + (1-\tau) f \in T$. *Then the set* T *is convex.*

Proof
Let $x, y \in T$ and $\tau \in [0,1]$. By the first property of T, there exist $e, f \in E(T)$ such that $x \in F_e$ and $y \in F_f$. Thus,

$$\begin{cases} \exists \tilde{x}^a \in C_a : x^a = e^a + \tilde{x}^a \\ \exists \tilde{x}^b \in C_b : e^b = x^b + \tilde{x}^b \end{cases} \quad \text{and} \quad \begin{cases} \exists \tilde{y}^a \in C_a : y^a = f^a + \tilde{y}^a \\ \exists \tilde{y}^b \in C_b : f^b = y^b + \tilde{y}^b. \end{cases}$$

To prove convexity of T we shall show that $\tau x + (1-\tau) y \in F_{(\tau e + (1-\tau) f)}$. Indeed, this proves the assertion since both properties of T, combined with the first property of Lemma 1.4.3, yield $F_{(\tau e + (1-\tau) f)} \subset T$.
Firstly, note that

$$\begin{aligned} \tau x^a + (1-\tau) y^a &= \tau(e^a + \tilde{x}^a) + (1-\tau)(f^a + \tilde{y}^a) \\ &= (\tau e^a + (1-\tau) f^a) + (\tau \tilde{x}^a + (1-\tau)\tilde{y}^a), \end{aligned}$$

and secondly,

$$(\tau x^b + (1-\tau) y^b) + (\tau \tilde{x}^b + (1-\tau)\tilde{y}^b) = \tau e^b + (1-\tau) f^b.$$

Since $\tau \tilde{x}^a + (1-\tau)\tilde{y}^a \in C_a$ and $\tau \tilde{x}^b + (1-\tau)\tilde{y}^b \in C_b$, we conclude that $\tau x + (1-\tau) y \in F_{(\tau e + (1-\tau) f)}$. $\square$

Define the function $\mathcal{G} : C \times C^* \to I\!R$ by

$$\mathcal{G}(x,p) := [x^b, p^b]_b - [x^a, p^a]_a. \tag{6}$$

In the next section, this function $\mathcal{G}$ will be used to model the profit, or gain, of a production process.
Note that the following two properties are a direct consequence of the definition of $\mathcal{G}$ and F_x.

- Let $x \in C$, $p \in C^*$ and $y \in F_x$, then $\mathcal{G}(x,p) \geq \mathcal{G}(y,p)$.

- Let $x \in C$, $p \in \text{int}(C^*)$ and let $y \in F_x$ sarisfy $y \neq x$, then $\mathcal{G}(x,p) > \mathcal{G}(y,p)$.

Definition 1.4.5 *For $T \subset C$, we define the extended real function $\chi : C^* \to [-\infty, \infty]$ by*

$$\chi(p) := \sup_{x \in T} \mathcal{G}(x,p).$$

Lemma 1.4.6 *Let T be a subset of C satisfying $\forall x \in T : F_x \subset T$, let $p \in \text{int}(C_a^*) \times C_b^*$ satisfy $\chi(p) = \infty$, and let $\alpha \in I\!R$. Then $L_\alpha^T(p) := \{x \in T \mid \mathcal{G}(x,p) = \alpha\}$ is an unbounded set.*

Proof
Suppose the set $L_\alpha^T(p)$ is bounded. We shall prove that in this case the set $K_\alpha^T(p) := \{x \in T \mid \mathcal{G}(x,p) \geq \alpha\}$ is also bounded, which is in contradiction with $\chi(p) = \infty$. Since $L_\alpha^T(p)$ is assumed to be bounded, there exists $y_0 \in \text{int}(C)$ such that $\forall x \in L_\alpha^T(p) : x \leq_C y_0$. This yields $L_\alpha^T(p) \subset F_{(0,y_0^b)}$. If $K_\alpha^T(p)$ were unbounded, then there would be an unbounded sequence $(x_n)_{n \in I\!N}$ in $K_\alpha^T(p)$. Since $[x_n^b, p^b]_b \geq \mathcal{G}(x_n,p) + [x_n^a, p^a]_a$ and $p^a \in \text{int}(C_a^*)$, the sequence $\left(x_n^b\right)_{n \in I\!N}$ would be unbounded in C_b. So, there would be $n_0 \in I\!N$ such that $(x_{n_0}^a, x_{n_0}^b) \notin F_{(0,y_0^b)}$. Now, take any $z_0^a \in \text{int}(C_a)$. Since $x_{n_0} \in K_\alpha^T(p) \subset T$ and since $\forall x \in T : F_x \subset T$, there exists $\lambda \geq 0$ such that $(x_{n_0}^a + \lambda z_0^a, x_{n_0}^b) \in L_\alpha^T(p)$, which is in contradiction with $\forall x \in L_\alpha^T(p) : x^b \leq_b y_0^b$. □

Lemma 1.4.7 *Let T be a closed set in C, satisfying $\forall x \in T : F_x \subset T$. Assume there is $p \in \text{int}(C_a^*) \times C_b^*$ such that $\chi(p) < \infty$ and there is at most one element $x_p \in T$ such that $\mathcal{G}(x_p,p) = \chi(p)$. Then the following assertions are equivalent.*

1. *There is $x_p \in T$ such that $\mathcal{G}(x_p,p) = \chi(p)$.*
2. *There is an $\mathcal{T}(C^*,C)$-open neighbourhood O of p such that every $q \in O$ satisfies $\chi(q) < \infty$.*

Proof
Assume assertion 1 holds, so there is precisely one $x_p \in T$ such that $\mathcal{G}(x_p,p) = \chi(p)$. The proof of assertion 2 is by contradiction. So, suppose that for every $\mathcal{T}(C^*,C)-$open neighbourhood O of p there exists $q \in O$ such that $\chi(q) = \infty$.

Then there is a sequence $(q_n)_{n\in\mathbb{N}}$ in $\text{int}(C_a^*) \times C_b^*$ converging to p, that satisfies $\forall n \in \mathbb{N} : \chi(q_n) = \infty$. Define the set $S := \{x \in C \mid [x, p_0]_C = 1 + [x_p, p_0]_C\}$, for certain fixed $p_0 \in \text{int}(C^*)$. By Lemma 1.4.6, for all $n \in \mathbb{N}$ the set $L_n := \{x \in T \mid \mathcal{G}(x, q_n) = \mathcal{G}(x_p, q_n)\}$ is unbounded. The set L_n is also convex, and since $\forall n \in \mathbb{N} : x_p \in L_n$, we find that $\forall n \in \mathbb{N} : S \cap L_n \neq \emptyset$. Indeed, since each L_n is unbounded, there is an $y_n \in L_n$ such that $y_n \notin S$, and since L_n is convex, there is $\tau \in (0, 1)$ such that $\tau x_p + (1 - \tau)y_n \in S$. For every $n \in \mathbb{N}$ choose $x_n \in S \cap L_n$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, so there is a convergent subsequence with limit $x \in S \cap T$. By the continuity of $\mathcal{G}$, we find $\mathcal{G}(x, p) = \mathcal{G}(x_p, p)$. Because $[x, p_0]_C \neq [x_p, p_0]_C$, we come to a contradiction, since we have found two different elements of T optimizing $\mathcal{G}(., p)$ in T.
For the converse, suppose assertion 2 is valid. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in T satisfying

$$\lim_{n\to\infty} \mathcal{G}(x_n, p) = \chi(p) \geq 0.$$

Since $p^a \in \text{int}(C_a^*)$, the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded if and only if the sequence $([x_n^a, p^a]_a)_{n\in\mathbb{N}}$ is. So, we conclude that the sequences $(x_n^a)_{n\in\mathbb{N}}$ and $\left(x_n^b\right)_{n\in\mathbb{N}}$ are either both bounded or both unbounded. We shall prove that the sequence $(x_n^a)_{n\in\mathbb{N}}$ is bounded in C_a. Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in C, and so admits a convergent subsequence. Closedness of T then yields the desired result. By assertion 2, there exists $\varepsilon > 0$ such that $\sup\{\mathcal{G}(x, p_\varepsilon) \mid x \in T\} = \chi(p_\varepsilon) < \infty$, where $p_\varepsilon := ((1-\varepsilon)p^a, p^b)$. Since $\chi(p_\varepsilon) \geq \mathcal{G}(x_n, p_\varepsilon) = \mathcal{G}(x_n, p) + \varepsilon[x_n^a, p^a]_a$ for all $n \in \mathbb{N}$, since $\lim_{n\to\infty} \mathcal{G}(x_n, p) = \chi(p) \geq 0$, and since $p^a \in \text{int}(C_a^*)$ we conclude that $(x_n^a)_{n\in\mathbb{N}}$ is a bounded sequence in C_a. □

Using the previous lemma twice in a row, we obtain the following.

Corollary 1.4.8 *Let T be a closed set in C, satisfying $\forall x \in T : F_x \subset T$. Assume that for every $q \in \text{int}(C^*)$ satisfying $\chi(q) < \infty$ there is at most one $x_q \in T$ such that $[x_q, q]_C = \chi(q)$. Assume there is $p \in \text{int}(C^*)$ such that $\chi(p) < \infty$ and there is precisely one element $x_p \in T$ such that $\mathcal{G}(x_p, p) = \chi(p)$. Then, there is an $\mathcal{T}(C^*, C)$-open neighbourhood O of p, $O \subset \text{int}(C^*)$, such that $\forall q \in O\ \exists x_q \in T : \mathcal{G}(x_q, q) = \chi(q)$.*

Corollary 1.4.9 *Let T be a closed set in C, satisfying $\forall x \in T : F_x \subset T$, let $p \in \text{int}(C^*)$ satisfy $\mathcal{G}(x_0, p) = \chi(p)$ for a unique $x_0 \in T$. Let $\alpha \in \mathbb{R}$. Then $K_\alpha^T(p) := \{x \in T \mid \mathcal{G}(x, p) \geq \alpha\}$ is compact.*

Proof
Since $K_\alpha^T(p)$ is closed, we only have to prove that it is bounded. Suppose $(x_n)_{n\in I\!N}$ is an unbounded sequence in $K_\alpha^T(p)$. Since $\forall n \in I\!N : \alpha \leq \mathcal{G}(x_n, p) \leq \chi(p)$, both the sequences $(x_n^a)_{a\in I\!N}$ and $\left(x_n^b\right)_{b\in I\!N}$ are unbounded. By Lemma 1.4.7 there is an $\mathcal{T}(C, C^*)$-open neighbourhood O of p such that every $q \in O$ satisfies $\chi(q) < \infty$. In order to obtain a contradiction, we shall prove that $\exists \tilde{p} \in O : \chi(\tilde{p}) = \infty$. Since O is open, there is $\varepsilon > 0$ such that $p_\varepsilon := ((1-\varepsilon)p^a, p^b) \in O$. Note that $\mathcal{G}(x_n, p_\varepsilon) = \mathcal{G}(x_n, p) + \varepsilon[x_n^a, p^a]_a$. Since $\forall n \in I\!N : \mathcal{G}(x_n, p) \geq \alpha$ and since $(x_n^a)_{n\in I\!N}$ is unbounded, $\sup \mathcal{G}(x_n, p) = \infty$. □

2 The economic model

2.1 Economy bundles and economy value functionals

As mentioned in the introduction, the main goal of this paper is the introduction of a model of a private ownership economy, which differs from the Classical models in the following two aspects.

- Commodities are not assumed to occur separately. Instead of introducing the commodity space $(\mathbb{R}^n)^+$ describing n different commodities, we shall only assume appearance of so called economy bundles. Here, we use the term “economy bundle" to describe exchangable objects in the economy. Thus, economy bundles can represent a single commodity, a bundle of commodities or a fixed combination of commodities, of which one of the elements can only be obtained by buying this specific fixed combination, i.e., of which one element is not sold separately. The latter case describes a situation in which our model allows for links between commodities.

- Production and consumption are not treated on the same level. In the model, two different types of economy bundles occur: production bundles which can be used as input to production processes, and consumption bundles which can be output of these processes. Despite the terminology, bundles of both types can be consumed by economic agents and bundles of both types will be present in the initial endowment. However, the production processes can convert only production bundles into consumption bundles and not the other way around.

In our model, we incorporate the above described situation as follows.
Firstly, considering economy bundles instead of separate commodities, we model the set of all economy bundles in the economy by a salient half-space C (cf. Definition 1.1.1), reflecting that the only possible manipulations with economy bundles are adding and scaling over $\mathbb{R}^+$. If $x, y \in C$ represent two economy bundles then we can speak of the sum $x + y$ of x and y, and if $\alpha \geq 0$ we can speak of the scaled version αx of x. Both $x + y$ and αx are economy bundles in C. Requiring the economy bundle set C to be salient (Condition 1.1.1.a) describes the fact that it is impossible for two economy bundles to cancel each other out after addition.
Secondly, considering two types of economy bundles, we assume that C is the direct sum of two salient half-spaces C_{prod} and C_{cons}, where C_{prod} and C_{cons} consists

of all production bundles and all consumption bundles, respectively. Both C_{prod} and C_{cons} are assumed to be non-trivial, i.e., assumed to be unequal to $\{0\}$. So, C is also non-trivial. In every economy bundle $x \in C$, each of the two types is uniquely represented: $x = (x^{\text{prod}}, x^{\text{cons}})$ with $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$.

Since, in our model commodities are not assumed to occur separately, the price of a single commodity is not a meaningful concept. Instead, we speak of the value of an economy bundle, which will be determined on the basis of “value functionals". These value functionals are described by subadditive positive functionals on C. The set of all such functionals has been introduced in Section 1 as the salient half-dual space C^* and we have seen that $C^* = (C_{\text{prod}})^* \oplus (C_{\text{cons}})^*$. Let $x \in C$ and $p \in C^*$, then the value of economy bundle x with respect to the value functional p equals

$$[x, p]_C := [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}} + [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}}.$$

Instead of the notation $[x, p]_C$ we shall mostly write $\mathcal{V}(x, p)$ for the value of the pair (x, p) with $x \in C$ and $p \in C^*$.

2.2 Economic agents

The features of an economic agent are an economy bundle $w = (w^{\text{prod}}, w^{\text{cons}}) \in C$, called initial endowment, and a preference relation $\succeq$ defined on C, on the basis of which the agent is supposed to make choices. By $x \succeq y$ we denote that the agent considers economy bundle x to be at least as preferable as bundle y. By $x \succ y$ we mean $x \succeq y$ and $\neg(y \succeq x)$. Finally, by $x \sim y$ we denote that the agent is indifferent in his choice between x and y. This preference relation $\succeq$ on C satisfies reflexivity, transitivity and completeness.
For a given value $\kappa \geq 0$ and a value functional $p \in C^*$, the budget set $B(p, \kappa) := \{x \in C \mid \mathcal{V}(x, p) \leq \kappa\}$ consists of all economy bundles that can be afforded given value κ and value functional p. The set $D(p, \kappa) := \{x \in B(p, \kappa) \mid \forall y \in B(p, \kappa) : x \succeq y\}$ of all best (most preferable) elements of the budget set $B(p, \kappa)$, is called the demand set. In the final model, κ will be specified as being the value $\mathcal{V}(w, p)$ of the initial endowment plus the values of the shares in the profit of production.

2.3 Production processes and technologies

Since we deal with an exchange economy with production, we have to model so called production processes, i.e., processes that incorporate the possibility to convert production bundles into consumption bundles. For our model this means that we say that an economy bundle $x \in C$ is a production process if consumption bundle $x^{\text{cons}} \in C_{\text{cons}}$ can be obtained from production bundle $x^{\text{prod}} \in C_{\text{prod}}$ as input. A collection of production processes being technologically feasible (i.e. satisfying conditions a, b and c of Definition 2.3.1, which will be defined later on) is said to be a production technology. A production technology is modelled by a subset T of C. One may think of a production technology as being the set of all production processes that can be executed due to the presence of a specific group of machinery. So, each production technology T will satisfy the following natural assumptions from an economic point of view:

a) The production process “no production" belongs to T;

b) A production process in T with zero input has zero output;

c1) Free disposal of input;

c2) Free disposal of output.

Free disposal of input states that if $x = (x^{\text{prod}}, x^{\text{cons}})$ is an executable production process and $\tilde{x}^{\text{prod}} = x^{\text{prod}} + y^{\text{prod}}$ for some $y^{\text{prod}} \in C_{\text{prod}}$, then $(\tilde{x}^{\text{prod}}, x^{\text{cons}})$ is also a feasible production process since after disposal of y^{prod}, production process x can be exectuted. Put differently, if $x \in T$ and $\tilde{x}^{\text{prod}} \in C_{\text{prod}}$ with $\tilde{x}^{\text{prod}} \geq_{\text{prod}} x^{\text{prod}}$ then $(\tilde{x}^{\text{prod}}, x^{\text{cons}}) \in T$. Similarly, free disposal of output states that if $x = (x^{\text{prod}}, x^{\text{cons}})$ is a feasible production process and $x^{\text{cons}} = y^{\text{cons}} + \tilde{x}^{\text{cons}}$ for some $y^{\text{cons}}, \tilde{x}^{\text{cons}} \in C_{\text{cons}}$, then $(x^{\text{prod}}, \tilde{x}^{\text{cons}})$ is also a feasible production process since after production of x^{cons} out of x^{prod}, y^{cons} can be disposed of, leaving $\tilde{x}^{\text{cons}}$ as output. So, if $x \in T$ and $\tilde{x}^{\text{cons}} \in C_{\text{cons}}$ with $\tilde{x}^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}$ then $(x^{\text{prod}}, \tilde{x}^{\text{cons}}) \in T$.
In fact, for every $x \in T$, the set F_x (as defined in Definition 1.4.2) is a subset of T, since F_x consists of precisely all the production processes in C which are executable due to the fact that x is executable and the two free disposal properties c1 and c2. Moreover, the statement $\forall x \in T : F_x \subset T$ is equivalent with $\forall e \in E(T) : F_e \subset T$. Indeed, if $x \in T$, then $\exists e \in E(T) : x \in F_e$. By the first property of Lemma 1.4.3, we find $F_x \subset F_e \subset T$.

Now, we come to the definition of the concept of production technology.

Definition 2.3.1 *A set $T \subset C$ is a production technology if the set T has the following properties:*

a) $(0,0) \in T$,

b) *If* $(0, x^{\text{cons}}) \in T$ *then* $x^{\text{cons}} = 0$,

c) $T = \bigcup_{e \in E(T)} F_e$.

We call a production process $(x^{\text{prod}}, x^{\text{cons}})$ of a technology T efficient, if at least x^{prod} is needed to produce x^{cons}, and if it is not possible to produce more than x^{cons} out of x^{prod}. Mathematically speaking, this boils down to the following definition.

Definition 2.3.2 *For a production technology T, a production process $x \in T$ is efficient if $\forall y \in C$:*

- $((y^{\text{prod}}, x^{\text{cons}}) \in T$ and $y^{\text{prod}} \leq_{\textit{prod}} x^{\text{prod}}) \Longrightarrow y^{\text{prod}} = x^{\text{prod}}$;
- $((x^{\text{prod}}, y^{\text{cons}}) \in T$ and $y^{\text{cons}} \geq_{\textit{cons}} x^{\text{cons}}) \Longrightarrow y^{\text{cons}} = x^{\text{cons}}$.

By Definition 1.4.2, the set $E(T)$ consists of precisely all efficient production processes in T. Note that $(0,0) \in E(T)$.

Given a value functional $p \in C^*$ and a production process $x \in T$, the gain $\mathcal{G}(x,p)$ of the pair (x,p) equals the value of the produced economy bundle x^{cons} minus the value of the production bundle x^{prod}, used as input. So,

$$\mathcal{G}(x,p) := [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} - [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}}. \tag{7}$$

Recall from Subsection 1.4 that $\forall p \in C^* \ \forall x \in T \ \forall y \in F_x : \mathcal{G}(y,p) \leq \mathcal{G}(x,p)$. This inequality is strict if $p \in \text{int}(C^*)$ and $y \neq x$. Since for every pair $(x,p) \in C \times C^*$ we can speak of both its value, were x is considered as an economy bundle, and its gain, where x is considered as a production process, we have introduced the distinguised notation $\mathcal{V}(x,p)$ and $\mathcal{G}(x,p)$. Note that $\mathcal{V}$ is a mapping from $C \times C^*$ into $\mathbb{R}^+$, while $\mathcal{G}$ is a mapping into $\mathbb{R}$.

Given $p \in C^*$, the (possibly empty) set of all gain maximizing production processes in T is called the supply set $S(p)$ of T, i.e.,

$$S(p) = \{x \in T \mid \forall y \in T : \mathcal{G}(x,p) \geq \mathcal{G}(y,p)\}. \tag{8}$$

The conditions on T and the definition of $E(T)$ imply that $\forall p \in C^* : S(p) \subset E(T)$.

2.4 Agents, production and equilibrium

Let I denote the number of economic agents and J the number of production technologies present in our model of a private ownership economy. The set of agents and the set of production processes is labelled by $i \in \{1, \ldots, I\}$ and $j \in \{1, \ldots, J\}$, respectively. For each $i \in \{1, \ldots, I\}, j \in \{1, \ldots, J\}$, agent i has share $\theta_{ij} \in [0, 1]$ in the gain of production technology T_j, i.e., if production process $x_j \in T_j$ is executed at value functional p, the gain $\mathcal{G}(x_j, p)$ of this production process is divided amongst the agents, such that agent i receives $\theta_{ij}\mathcal{G}(x_j, p)$. For all $j \in \{1, \ldots, J\}$ these shares satisfy $\sum_{i=1}^{I} \theta_{ij} = 1$.

At value functional $p \in C^*$ and executed production processes $x_j \in T_j$, $j \in \{1, \ldots, J\}$, the value $\kappa_i(p; x_1, \ldots, x_J)$ of agent i is defined by

$$\kappa_i(p; x_1, \ldots, x_J) := \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij}\mathcal{G}(x_j, p),$$

where the first term denotes the value of the initial endowment of agent i and the second term denotes the total value received from shares in the gain of the production technologies. At given value functional $p \in C^*$, each agent will have maximal value $\kappa_i(p; x_1, \ldots, x_J)$, if for all $j \in \{1, \ldots, J\}$, production process x_j is an element of the supply set $S_j(p)$ of technology T_j. Since, eventually, we want to go from supply sets to supply functions, an extra assumption on the production technologies is needed, guaranteeing that

$$\{p \in \text{int}(C^*) \mid \forall j \in \{1, \ldots, J\} : S_j(p) \neq \emptyset\} \neq \emptyset. \tag{9}$$

This condition will be presented in Subsection 3.1.

In case (9) is satisfied, each $s_j \in S_j(p)$ yields the maximal gain $\mathcal{G}(s_j, p)$ of technology T_j at value functional $p \in A$. Thus, we may define the value

$$\mathcal{K}_i(p) := \kappa_i(p; s_1, \ldots, s_J).$$

With the value $\kappa = \mathcal{K}_i(p)$, the budget set $B_i(p, \kappa)$ of agent i is given by

$$B_i(p) := B_i(p, \mathcal{K}_i(p)) = \{x \in C \mid \mathcal{V}(x, p) \leq \mathcal{K}_i(p)\}$$

and similarly, the corresponding demand set is given by

$$D_i(p) := \{x \in B_i(p) \mid \forall y \in B_i(p) : x \succeq_i y\}.$$

Given this model of an exchange economy, the relevant question is whether or not there exists a Walrasian equilibrium. We shall not answer this question completely, but we shall present additional assumptions for this model, such that existence of such equilibria is guaranteed.

Definition 2.4.1 *A Walrasian equilibrium (or in short: equilibrium) is an* $(I + J + 1)$*-tuple* $((d_i)_{i=1}^{I}, (s_j)_{j=1}^{J}, p_{eq})$ *consisting of*

- $p_{eq} \in C^*$,
- $d_i \in D_i(p_{eq})$ *for all* $i \in \{1, \ldots, I\}$;
- $s_j \in S_j(p_{eq})$ *for all* $j \in \{1, \ldots, J\}$;
- $\sum_{i=1}^{I} d_i + \sum_{j=1}^{J} ({s_j}^{\text{prod}}, 0) = \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} (0, {s_j}^{\text{cons}})$.

We call p_{eq} a (Walrasian) equilibrium value functional.

In the following section, we present additional assumptions on our model, which guarantee existence of Walrasian equilibria. In fact, in that section, we shall prove the following.

Existence Theorem
The model of a private ownership economy, described above, admits a Walrasian equilibrium, under the following assumptions:

A1 $C^{**} = C$.

A2 $V[C]$ *is finite-dimensional.*

A3 *For every* $j \in \{1, \ldots, J\}$*, production technology* T_j *satisfies*

- **a)** T_j *is closed with respect to topology* $\mathcal{T}(C, C^*)$,
- **b)** *if* $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0, 1)$ *then* $\tau e_1 + (1 - \tau)e_2 \in T_j$ *and* $\tau e_1 + (1 - \tau)e_2 \notin E(T_j)$.

A4 *For every* $i \in \{1, \ldots, I\}$*, preference relation* $\succeq_i$ *is*

- **a)** *monotone:* $\forall\, x, y \in C : x \geq_C y$ *implies* $x \succeq_i y$,

b) *strictly convex:* $\forall x, y \in C$, $\tau \in (0,1) : x \succeq_i y$ *and* $x \neq y$ *imply* $\tau x + (1-\tau)y \succ_i y$,

c) *continuous:* $\forall y \in C$ *the sets* $\{x \in C \mid x \succeq_i y\}$ *and* $\{x \in C \mid y \succeq_i x\}$ *are closed in* C.

A5 *Furthermore,*

a) $\exists p \in \operatorname{int}(C^*) : p \in \bigcap_{j=1}^{J} Dom(\mathcal{S}_j)$.

b) *For every sequence* $(p_n)_{n\in\mathbb{N}}$ *in* $\operatorname{int}(C^*)$ *with non-zero limit, there is* $i_0 \in \{1, \ldots, I\}$ *such that* $\liminf_{n\to\infty}\{\mathcal{K}_{i_0}(p_n) \mid n \in \mathbb{N}\} > 0$.

3 The mathematical model, existence of equilibrium

In this section, we shall prove the Existence Theorem presented in Subsection 2.4. In this theorem we prove existence of Walrasian equilibria in the model of a private ownership economy, presented in Section 2, given the five additional Mathematical Assumptions A1 - A5, as mentioned at the end of the previous section.
In Section 1, we suggested Assumption A1 already, to guarantee that C is a closed subset of $V[C]$ with respect to $\mathcal{T}(V[C], C^*)$. Furthermore, we assumed that the vector space $V[C]$, generated by economy bundle set C, is finite-dimensional (Assumption A2), to ensure that every bounded set in C is pre-compact. Assumption A3 implies that instead of supply sets, we can deal with supply functions. Given a production technology T, elements of the corresponding supply set belong to $E(T)$. So, in order to guarantee that we can use supply functions, we introduce conditions on $E(T)$ which resemble decreasing returns to scale or strict convexity conditions. The assumption that T is closed will guarantee the continuity of the supply function. Similarly, Assumption A4 implies that we can deal with demand functions and that these functions are continuous. All this will be shown in Subsection 3.1. Assumption A5.a implies that the total supply function has a non-trivial domain. We have not yet reached the point that the other assumptions (in particular Assumption A3) lead to redundancy of A5.a. In Subsection 3.3 we shall construct an equilibrium function as defined in Definition 3.3.1. Finally, in Subsection 3.4, we shall prove the Existence Theorem, using the constructed function. Here, we shall use Asumption A5.b, which looks rather technical. It is a condition weaker than the usual one which requires that the total initial endowment is strictly positive. In fact, Condition A5.b is satisfied if $w^{\text{prod}}_{\text{total}}$ is strictly positive.

3.1 Supply and demand functions

In this section we show that Assumption A3 guarantees that for all $j \in \{1, \ldots, J\}$ and for every value functional p taken from some specific set $\text{Dom}(\mathcal{S}_{\text{total}}) \subset C^*$, every supply set $S_j(p) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x, p) \geq \mathcal{G}(y, p)\}$ consists of exactly one element. Furthermore, we show that Assumption A4 guarantees that for all $i \in \{1, \ldots, I\}$ and for every value functional p taken from some specific set $\text{Dom}(\mathcal{S}_{\text{total}}) \subset \text{Dom}(\mathcal{S}_{\text{total}})$, every demand set $D_i(p) = \{x \in B_i(p) \mid \forall y \in B_i(p) : x \succeq_i y\}$ consists of exactly one element. Thus, we are able to define the supply functions $\mathcal{S}_j : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$ and the demand functions $\mathcal{D}_i : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$. Furthermore, we shall derive some properties of these functions. Since these properties do not depend on the specific agent or production technology, we shall,

for a moment, drop the index i and j in this subsection.
First, consider a production technology T with efficiency set $E(T)$. Note that by Assumption A3.b and Lemma 1.4.4, the production technology T is a convex set in C.

Lemma 3.1.1 *Let $p \in \text{int}(C^*)$. Then the supply set $S(p)$ contains at most one element.*

Proof
Suppose both s_1 and $s_2 \in S(p)$ and $s_1 \neq s_2$. By Assumption A3.b, $s := \frac{1}{2}(s_1 + s_2) \in T \setminus E(T)$. Since $T \setminus E(T) = \{x \in T \mid \exists y \in E(T), y \neq x : x \in F_y\}$, there exists $y \in T, y \neq s : s \in F_y$. Now, $\mathcal{G}(y,p) > \mathcal{G}(s,p) = \mathcal{G}(s_1,p)$, which is in contradiction with s_1 being an element of the supply set $S(p)$. □

The previous lemma enables us to define the supply function $\mathcal{S} : \text{Dom}(\mathcal{S}) \to E(T)$, where

$$\text{Dom}(\mathcal{S}) := \{p \in \text{int}(C^*) \mid \exists x_0 \in E(T)\ \forall x \in T : \mathcal{G}(x_0,p) > \mathcal{G}(x,p)\}.$$

Note that by Corollary 1.4.8, $\text{Dom}(\mathcal{S})$ is a $\mathcal{T}(C^*, C)$-open salient half-space in $\text{int}(C^*)$.

Proposition 3.1.2 *The supply function $\mathcal{S}$: Dom$(\mathcal{S}) \to E(T)$ is continuous.*

Proof
Let $(p_n)_{n \in I\!N}$ be a sequence in $\text{Dom}(\mathcal{S})$ with limit $p \in \text{Dom}(\mathcal{S})$. Suppose the sequence $(\mathcal{S}(p_n))_{n \in I\!N}$ does not converge to $\mathcal{S}(p)$. Taking a subsequence if necessary, we may assume that

$$\exists \varepsilon > 0\ \forall n \in I\!N : \| \mathcal{S}(p_n) - \mathcal{S}(p) \| \geq \varepsilon.$$

Define $x_n := \lambda_n \mathcal{S}(p_n) + (1 - \lambda_n)\mathcal{S}(p)$ with $\lambda_n := \frac{\varepsilon}{\|\mathcal{S}(p_n) - \mathcal{S}(p)\|} \in (0,1]$, then, by Assumption A3.b, $x_n \in T \setminus E(T)$ and $\| x_n - \mathcal{S}(p) \| = \varepsilon$. The sequence $(x_n)_{n \in I\!N}$ is bounded, so there is a convergent subsequence $(x_{nk})_{k \in I\!N}$ with limit $x \in T$ (Assumption A3.a), satisfying $\| x - \mathcal{S}(p) \| = \varepsilon$. Since $x_n = \lambda_n \mathcal{S}(p_n) + (1 - \lambda_n)\mathcal{S}(p)$ with $\lambda \in (0,1]$, we find $\mathcal{G}(x_n, p_n) \geq \min\{\mathcal{G}(\mathcal{S}(p_n), p_n), \mathcal{G}(\mathcal{S}(p), p_n)\} = \mathcal{G}(\mathcal{S}(p), p_n)$. The function $\mathcal{G} : C \times C^* \to I\!R$ is continuous, so $\mathcal{G}(x,p) \geq \mathcal{G}(\mathcal{S}(p), p)$. Since $x \in T,\ x \neq \mathcal{S}(p)$, this is in contradiction with the properties of $\mathcal{S}(p)$. □

Corollary 3.1.3 *Let $(p_n)_{n \in I\!N}$ be a sequence in Dom$(\mathcal{S})$, with limit $p \in \text{int}(C^*)$. If the sequence $(\mathcal{S}(p_n))_{n \in I\!N}$ is convergent with limit $s \in C$, then $p \in$ Dom$(\mathcal{S})$ and $s = \mathcal{S}(p)$.*

Proof
Since $\forall n \in I\!N \; \forall x \in T : \mathcal{G}(\mathcal{S}(p_n), p_n) \geq \mathcal{G}(x, p_n)$, the continuity of the function $\mathcal{G} : C \times C^* \to I\!R$ guarantees that $\forall x \in T : \mathcal{G}(s, p) \geq \mathcal{G}(x, p)$. By Assumption A3.a, the set T is closed, so Lemma 3.1.1 yields $s = \mathcal{S}(p)$. □

Corollary 3.1.4 *Let $(p_n)_{n \in I\!N}$ be a sequence in Dom$(\mathcal{S})$ convergent to $p \in \text{int}(C^*) \setminus$ Dom$(\mathcal{S})$. Let $p_0 \in \text{int}(C^*)$, then $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$.*

Proof
If the sequence $(\mathcal{S}(p_n))_{n \in I\!N}$ were bounded, then there would be a convergent subsequence $(\mathcal{S}(p_{n_k}))_{k \in I\!N}$ with limit $s \in C$. This would be in contradiction with the previous corollary. So, the sequence $(\mathcal{S}(p_n))_{n \in I\!N}$ is unbounded.
For all $\alpha \in I\!R$ the set $L_{p_0}(\alpha) = \{x \in T \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact (Corollary 1.4.9), and so we find that $\forall \alpha \in I\!R \; \exists N \in I\!N \; \forall n > N : \mathcal{G}(\mathcal{S}(p_n), p_0) \leq \alpha$, and we conclude $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$. □

The previous statements hold for every production technology T satisfying

a) T is closed with respect to topology $\mathcal{T}(C, C^*)$,

b) if $e_1, e_2 \in E(T)$, $e_1 \neq e_2$, $\tau \in (0,1)$ then $\tau e_1 + (1-\tau)e_2 \in T$ and $\tau e_1 + (1-\tau)e_2 \notin E(T)$.

So, when we define

$$\text{Dom}(\mathcal{S}_{\text{total}}) := \bigcap_{j=1}^{J} \text{Dom}(\mathcal{S}_j),$$

we can construct the total supply function $\mathcal{S}_{\text{total}} : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$, by defining for every $p \in \text{Dom}(\mathcal{S}_{\text{total}})$

$$\mathcal{S}_{\text{total}}(p) := \sum_{j=1}^{J} \mathcal{S}_j(p).$$

Here, we need Assumption A5.a, stating that $\text{Dom}(\mathcal{S}_{\text{total}}) \neq \emptyset$.

We shall now consider one agent with the following characteristics: initial endowment $w \in C$, preference relation $\succeq$ defined on C, value $\kappa \geq 0$, budget set $B(p, \kappa)$ and, demand set $D(p, \kappa)$. Recall that the budget set $B(p, \kappa) = \{x \in C \mid \mathcal{V}(x, p) \leq \kappa\}$ and demand set $D(p, \kappa) = \{x \in B(p, \kappa) \mid \forall y \in B(p, \kappa) : x \succeq y\}$ are defined for every $p \in C^*$ and every $\kappa \geq 0$. Using Assumption A4, we derive some properties for both the budget and demand set, and we prove that for every $p \in \text{int}(C^*)$, the demand set $D(p, \kappa)$ consists of precisely one element.

Lemma 3.1.5 *Let $\kappa \geq 0$ and $p \in \text{int}(C^*)$. Then the demand set $D(p, \kappa)$ at value functional p and value κ, is non-empty.*

Proof
By Corollary 1.3.7, the budget set $B(p, \kappa)$ is compact in C. For every $b \in B(p, \kappa)$, define the set $G(b) := \{x \in B(p, \kappa) \mid b \succ x\}$. The preference relation $\succeq$ is continuous (Assumption A4.c), so every set $G(b)$ is open. Suppose the demand set at value functional p and value κ were empty, then every $b_0 \in B(p, \kappa)$ is an element of at least one $G(b)$. The collection $\{G(b) \mid b \in B(p, \kappa)\}$ is an open cover of the compact set $B(p, \kappa)$, so there is a finite subset $F \subset B(p, \kappa)$ such that $B(p, \kappa) = \bigcup_{f \in F} G(f)$. The preference relation $\succeq$ being transitive, F has a maximal element $f_1 \in F$. Since, $f_1 \in G(f_2)$ for some $f_2 \in F$, $f_2 \neq f_1$, we arrive at a contradiction. $\square$

Lemma 3.1.6 *Let $\kappa \geq 0$ and $p \in \text{int}(C^*)$ Then the demand set $D(p, \kappa)$ contains precisely one element.*

Proof
Suppose both d_1 and d_2 belong to $D(p, \kappa)$ and $d_1 \neq d_2$. On the one hand, using Assumption A4.b we find $\tau d_1 + (1 - \tau)d_2 \succ d_1$ for all $\tau \in (0, 1)$. And, on the other hand, using convexity of the budget set, we find $\tau d_1 + (1 - \tau)d_2 \in B(p, \kappa)$ for all $\tau \in (0, 1)$. $\square$

For every $i \in \{1, \ldots, I\}$ and for every $p \in \text{Dom}(\mathcal{S}_{\text{total}})$, we can define value $\mathcal{K}_i(p)$ at p, as introduced in Subsection 2.4, by

$$\mathcal{K}_i(p) := \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p),$$

where w_i is the inital endowment of agent i and θ_{ij} is his share in the gain of production technology T_j. Note that $\mathcal{K}_i(p) \geq 0$, for all $p \in \text{Dom}(\mathcal{S}_{\text{total}})$. Since for every $j \in \{1, \ldots, J\}$ the supply function $\mathcal{S}_j : \text{Dom}(\mathcal{S}_j) \to C$ is continuous, and since $\mathcal{G}$ and $\mathcal{V}$ are bicontinuous on $C \times C^*$, the value function $\mathcal{K}_i : \text{Dom}(\mathcal{S}_{\text{total}}) \to I\!R^+$ is continuous for all $i \in \{1, \ldots, I\}$. Using $\mathcal{K}_i(p)$, the budget set $B_i(p) := B_i(p, \mathcal{K}_i(p))$ can be defined for every $p \in \text{Dom}(\mathcal{S}_{\text{total}})$, and therewith the demand set $D_i(p) := D_i(p, \mathcal{K}_i(p))$ consisting of all best elements of $B_i(p)$. Each budget set $B_i(p)$ and demand set $D_i(p)$ is only defined for $p \in \text{Dom}(\mathcal{S}_{\text{total}})$, since only for these value functionals, $\mathcal{K}_i(p)$ is defined. Note that for every fixed $p_0 \in \text{Dom}(\mathcal{S}_{\text{total}})$, the statements for $\kappa \geq 0$, $B(p_0, \kappa)$ and $D(p_0, \kappa)$, also apply for $\mathcal{K}(p_0)$ and the sets $B(p_0)$ and $D(p_0)$. Since $\text{Dom}(\mathcal{S}_{\text{total}}) \subset \text{int}(C^*)$, we find, using Lemma 3.1.6, that for every $p \in \text{Dom}(\mathcal{S}_{\text{total}})$, the demand set $D_i(p)$ consists of precisely one element. So, we are able to define the demand function $\mathcal{D}_i : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$.

Next, we shall derive some properties for these demand functions, concerning their continuity. Since these properties do not depend upon the index i, we shall again drop this index for a moment and consider an agent with initial endowment $w \in C$, preference relation $\succeq$ on C, and demand function $\mathcal{D} : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$. Let us state some preliminary lemmas concerning the budget set and the demand set of this agent.

Lemma 3.1.7 *Let $p \in C^*$, $\kappa > 0$, $x \in C$, and suppose $x \succeq b$ for all $b \in B(p, \kappa)$ satisfying $\mathcal{V}(b, p) < \kappa$. Then $x \succeq b$ for all $b \in B(p, \kappa)$.*

Proof

Let $b \in B(p, \kappa)$ satisfy $\mathcal{V}(b, p) = \kappa$. We shall prove that $x \succeq b$. Clearly, $b \neq 0$. So, for all $\tau \in [0, 1)$ we have $\mathcal{V}(\tau b, p) < \kappa$ and thus $x \succeq \tau b$. By Assumption A4.c, the preference relation $\succeq$ is continuous, so $x \succeq b$. □

Lemma 3.1.8 *Let $\kappa \geq 0$, $p \in C^*$, $x \in C$ and suppose $\exists d \in D(p, \kappa) : x >_C d$. Then $x \notin B(p, \kappa)$.*

Proof

Due to the monotony of the preference relation (Assumption A4.a), $x >_C d$ implies $x \succeq d$. By the strict convexity of the preference relation (Assumption A4.b), we find that $\frac{1}{2}(x + d) \succ d$. Now, suppose $x \in B(p, \kappa)$, then due to the convexity of $B(p,\kappa)$ we would find $\frac{1}{2}(x+d) \in B(p,\kappa)$, which is in contradiction with the optimality of d. $\square$

Lemma 3.1.9 *Let $\kappa \geq 0$ and $p \in \text{int}(C^*)$. Then $\mathcal{V}(\mathcal{D}(p,\kappa),p) = \kappa$.*

Proof
In case $\kappa = 0$, the budget set $B(p,\kappa)$ equals $\{0\}$, and thus $\mathcal{V}(\mathcal{D}(p,\kappa),p) = \mathcal{V}(0,p) = 0$. Now, suppose $\kappa > 0$ and $\mathcal{V}(\mathcal{D}(p,\kappa),p) < \kappa$. There is $x_0 \in \text{int}(C)$ such that $x_0 >_C \mathcal{D}(p,\kappa)$ and $\mathcal{V}(x_0,p) > \kappa$. Consider the convex combination $\tau x_0 + (1-\tau)\mathcal{D}(p,\kappa)$ with $\tau \in (0,1)$ so small that $\mathcal{V}(\tau x_0 + (1-\tau)\mathcal{D}(p,\kappa),p) \leq \kappa$. Then $\tau x_0 + (1-\tau)\mathcal{D}(p,\kappa) \in B(p,\kappa)$ and $\tau x_0 + (1-\tau)\mathcal{D}(p,\kappa) >_C \mathcal{D}(p,\kappa)$. By Lemma 3.1.8, we come to a contradiction. $\square$

Lemma 3.1.10 *Let $(p_n)_{n\in I\!N}$ be a convergent sequence in Dom($\mathcal{S}_{\text{total}}$) with limit $p \in C^*$, and assume the sequence $\left(\mathcal{K}_((p)_n)\right)_{n\in I\!N}$ is convergent with limit κ. If $\kappa > 0$ and the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ is convergent, then $p \in \text{int}(C^*)$.*

Proof
If p would be an element of $\partial(C^*)$, then there would be an element $x \in C \setminus \{0\}$, such that $\mathcal{V}(x,p) = 0$. Since $\forall y \in B(p,\kappa) : y+x \in B(p,\kappa)$, and since $y+x >_C y$, Corollary 3.1.8 would yield that $B(p,\kappa)$ does not contain a maximal element with respect to $\succeq$. To prove the assertion, we shall show existence of a maximal element in $B(p,\kappa)$, assuming that the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ is convergent and that $\kappa > 0$. In fact, using Lemma 3.1.7, we shall prove that the limit d of the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ is maximal in $B(p,\kappa)$. Indeed, let $b \in B(p,\kappa)$ satisfy $\mathcal{V}(b,p) < \kappa$. Then there is $N \in I\!N$ such that $\forall\ n > N : \mathcal{V}(b,p_n) < \mathcal{K}(p_n)$. So, $\mathcal{D}(p_n) \succeq b$ for all $n > N$. Continuity of the preference relation (Assumption A4.c) yields $d \succeq b$, and by Lemma 3.1.7 we conclude that d is maximal in $B(p,\kappa)$. $\square$

The preceding lemma will be applied (taking a subsequence of $(\mathcal{D}(p_n))_{n\in I\!N}$ if necessary) in the following way.

Corollary 3.1.11 *If $(p_n)_{n\in I\!N}$ is a convergent sequence in Dom($\mathcal{S}_{\text{total}}$) with limit $p \in \partial C^*$, and if the sequence $(\mathcal{K}(p_n))_{n\in I\!N}$ is convergent with limit $\kappa > 0$ then $(\mathcal{D}(p_n))_{n\in I\!N}$ is unbounded.*

To conclude this subsection on properties of individual agents, we prove that the demand function $\mathcal{D} : \text{Dom}(\mathcal{S}_{\text{total}}) \to C$ is continuous. For this we need the following lemma.

Lemma 3.1.12 *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Dom($\mathcal{S}_{\text{total}}$) convergent to $p \in$ Dom($\mathcal{S}_{\text{total}}$). Then the following two properties hold.*

1) *If $b_n \in B(p_n)$ for each $n \in \mathbb{N}$, then there is a subsequence $(b_{nk})_{k\in\mathbb{N}}$ that converges to some $b \in B(p)$.*

2) *For each $b \in B(p)$ there exists a convergent sequence $(b_n)_{n\in\mathbb{N}}$ with limit b, such that $b_n \in B(p_n)$ for all $n \in \mathbb{N}$.*

Proof

1) Since $p \in \text{int}(C^*)$ is an order unit, there is, by Corollary 1.3.9, a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfying $\forall n \in \mathbb{N} : \psi_n > 0$ and $\lim_{n\to\infty} \psi_n = 1$, such that

$$\forall\, n \in \mathbb{N} :\ \psi_n p \leq_{C^*} p_n.$$

Because $b_n \in B(p_n)$ for all $n \in \mathbb{N}$, we find $\psi_n[b_n, p]_C \leq [b_n, p_n]_C \leq \mathcal{K}(p_n)$. Since the function $\mathcal{K} : \text{Dom}(\mathcal{S}_{\text{total}}) \to \mathbb{R}^+$ is continuous, the sequence $(\mathcal{K}(p_n))_{n\in\mathbb{N}}$ is bounded. And since $p \in \text{int}(C^*)$, boundedness of $[b_n, p]_C$ implies that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded (Lemma 1.3.6). So, $(b_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(b_{nk})_{k\in\mathbb{N}}$ with limit $b \in C$. Since $\forall k \in \mathbb{N}$: $\mathcal{V}(b_{nk}, p_{nk}) \leq \mathcal{K}(p_{nk})$, the limit b belongs to $B(p)$.

2) Let $b \in B(p)$. If $\mathcal{V}(b, p) < \mathcal{K}(p)$ then $\exists N \in \mathbb{N}\ \forall n > N : \mathcal{V}(b, p_n) < \mathcal{K}(p_n)$, and so, if we choose $b_n := b$ for all $n > N$, we are done. Therefore, we may as well assume $\mathcal{V}(b, p) = \mathcal{K}(p)$. For every $n \in \mathbb{N}$, define $\tau_n := \frac{\mathcal{K}(p_n)}{\mathcal{V}(b,p_n)}$. Note that $\lim_{n\to\infty} \tau_n = 1$. Now put $b_n := \tau_n b$, then $\forall n \in \mathbb{N} : \mathcal{V}(b_n, p_n) = \mathcal{K}(p_n)$ and $\lim_{n\to\infty} b_n = b$.

□

Lemma 3.1.12 expresses the type of continuity that we need in order to prove the continuity of the individual demand functions $\mathcal{D}$.

Lemma 3.1.13 *The demand function $\mathcal{D}$ is continuous on Dom($\mathcal{S}_{\text{total}}$).*

Proof
Let $(p_n)_{n\in I\!N}$ be a sequence in Dom$(\mathcal{S}_{\text{total}})$ converging to some $p \in$ Dom$(\mathcal{S}_{\text{total}})$. Suppose the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ does not converge to $\mathcal{D}(p)$, then without loss of generality any subsequence of $(\mathcal{D}(p_n))_{n\in I\!N}$ does not converge to $\mathcal{D}(p)$. By 1) of the preceding lemma, the sequence $(\mathcal{D}(p_n))_{n\in I\!N}$ has a subsequence $(\mathcal{D}(p_{nk}))_{k\in I\!N}$ that converges to some $b \in B(p)$. Now, the proof is done if we can show that $b = \mathcal{D}(p)$. Let $x \in B(p)$. By 2) of the preceding lemma, for all $n \in I\!N$ there is $x_n \in B(p_n)$ satisfying $x_n \to x$. Since the preference relation $\succeq$ is continuous (Assumption A4.c), we find that if $\forall n \in I\!N : \mathcal{D}(p_n) \succeq x_n$, then $b \succeq x$. So, $b = \mathcal{D}(p)$. □

Analogous to the construction of the total supply function $\mathcal{S}_{\text{total}}$, we now are able to construct the total demand function $\mathcal{D}_{\text{total}}$: Dom$(\mathcal{S}_{\text{total}}) \to C$, by defining for every $p \in$ Dom$(\mathcal{S}_{\text{total}})$

$$\mathcal{D}_{\text{total}}(p) := \sum_{i=1}^{I} \mathcal{D}_i(p).$$

3.2 The mathematical model

Consider the presented model of an exchange economy with J production technologies, each with production technology set $T_j \subset C$ satisfying Assumption A3, and with I economic agents, each with initial endowment $w_i \in C$, preference relation $\succeq_i$ defined on C, satisfying Assumption A4, and shares $\theta_{ij} \geq 0$, satisfying $\sum_{i=1}^{I} \theta_{ij} = 1$. We have seen that there can be defined corresponding supply functions $\mathcal{S}_j$: Dom$(\mathcal{S}_j) \to C$, $j \in \{1, \ldots, J\}$, and demand functions $\mathcal{D}_i$: Dom$(\mathcal{S}_{\text{total}}) \to C$, $i \in \{1, \ldots, I\}$. These functions are continuous. Furthermore, we have defined the total supply function $\mathcal{S}_{\text{total}}$ and the total demand function $\mathcal{D}_{\text{total}}$ on Dom$(\mathcal{S}_{\text{total}})$.

In Lemma 3.1.9, we have seen that $\forall i \in \{1, \ldots, I\}\ \forall p \in$ Dom$(\mathcal{S}_{\text{total}})$:

$$\mathcal{V}(\mathcal{D}_i(p), p) = \mathcal{K}_i(p) = \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p).$$

So, as a consequence of this lemma, we find an adapted version of Walras' law, namely that for all $p \in$ Dom$(\mathcal{S}_{\text{total}})$:

$$\mathcal{V}(\mathcal{D}_{\text{total}}(p), p) = \mathcal{V}(w_{\text{total}}, p) + \mathcal{G}(\mathcal{S}_{\text{total}}(p), p), \tag{10}$$

where the total initial endowment w_{total} is defined by $w_{\text{total}} := \sum_{i=1}^{I} w_i$.

We introduce the convenient notation

$$\mathcal{Z}(p,q) := \mathcal{V}(\mathcal{D}_{\text{total}}(p), q) - \mathcal{G}(\mathcal{S}_{\text{total}}(p), q) - \mathcal{V}(w_{\text{total}}, q), \tag{11}$$

where $p \in \text{Dom}(\mathcal{S}_{\text{total}})$ and $q \in C^*$. The function $\mathcal{Z} : \text{Dom}(\mathcal{S}_{\text{total}}) \times C^* \to I\!R$ thus defined is bicontinuous. Walras' law (10) reads

$$\forall p \in \text{Dom}(\mathcal{S}_{\text{total}}) : \mathcal{Z}(p,p) = 0. \tag{12}$$

The convenience of this notation is also shown in the next lemma, where a characterisation of equilibrium value functionals (cf. Definition 2.4.1) is given. Note, that each equilibrium value functional is an element of $\text{Dom}(\mathcal{S}_{\text{total}})$.

Lemma 3.2.1 *Let $p \in$ Dom$(\mathcal{S}_{\text{total}})$. Then p is an equilibrium value functional if and only if $\mathcal{Z}(p,q) \leq 0$ for all $q \in C^*$.*

Proof
If $\mathcal{Z}(p,q) \leq 0$ for all $q \in C^*$, then

$$\mathcal{D}_{\text{total}}(p) + (\mathcal{S}^{\text{prod}}_{\text{total}}(p), 0) \leq_C w_{\text{total}} + (0, \mathcal{S}^{\text{cons}}_{\text{total}}(p)).$$

Now, apply $\mathcal{V}(.,p)$ on both sides of this inequality. Since $p \in \text{int}(C^*)$, Walras' law implies there is equality. □

3.3 Construction of an equilibrium function

In this subsection we introduce the concept of equilibrium function, on the basis of which we shall prove existence of an equilibrium value functional.

Definition 3.3.1 *Let there be given a model of a private ownership economy as described in Subsection 3.2. Let X be a subset of C^*. A function $\mathcal{F}^{eq} : X \to C^*$ is an equilibrium function if for every $p \in X \cap$ Dom$(\mathcal{S}_{\text{total}})$:*

$$\mathcal{F}^{eq}(p) = 0 \textit{ if and only if } \mathcal{Z}(p,q) \leq 0 \textit{ for all } q \in C^*,$$

The problem of proving existence of a Walrasian equilibrium can now be replaced by constructing an equilibrium function with zeroes in $\mathrm{Dom}(\mathcal{S}_{\text{total}})$. This section will deal mainly with the construction of an equilibrium function.

By Corollary 1.3.7, the section $L_1(x_0) := \{p \in C^* \mid \mathcal{V}(x_0, p) = 1\}$ is compact for every $x_0 \in \mathrm{int}(C)$. In the mathematical introduction we have constructed the Lebesque measure μ on such a section.

Definition 3.3.2 *Let $x_0 \in \mathrm{int}(C)$, and let $L_1(x_0) := \{p \in C^* \mid \mathcal{V}(x_0, p) = 1\}$. On Dom$(\mathcal{S}_{\text{total}})$, the function $\mathcal{F}_0^{eq}$: Dom$(\mathcal{S}_{\text{total}}) \to C^*$ is defined by*

$$\mathcal{F}_0^{eq}(p) := \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p,q)\} q d\mu(q).$$

Lemma 3.3.3 *The function $\mathcal{F}_0^{eq}$: Dom$(\mathcal{S}_{\text{total}}) \to C^*$ is an equilibrium function.*

Proof
Let $p \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$. Clearly, if $\forall q \in C^* : \mathcal{Z}(p,q) \leq 0$, then $\mathcal{F}_0^{eq}(p) = 0$. Now, suppose for some $p \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$ we have $\mathcal{F}_0^{eq}(p) = 0$. Then

$$\begin{aligned} 0 &= \mathcal{Z}(p, \mathcal{F}_0^{eq}(p)) \\ &= \mathcal{Z}(p, \int_{L(p)} \mathcal{Z}(p,q) q d\mu(q)) \\ &= \int_{L(p)} (\mathcal{Z}(p,q))^2 d\mu(q), \end{aligned}$$

where $L(p) := \{q \in\ L_1(x_0) \mid \mathcal{Z}(p,q) > 0\}$. It follows that $\mu(L(p)) = 0$ and therefore $L(p) = \emptyset$ due to the continuity of $q \mapsto \mathcal{Z}(p,q)$ (cf. Subsection 1.3). Hence, for all $q \in L_1(x_0)$ it holds that $\mathcal{Z}(p,q) \leq 0$. □

In order to prove existence of a Walrasian equilibrium, we are going to adapt $\mathcal{F}_0^{eq}$: $\mathrm{Dom}(\mathcal{S}_{\text{total}}) \to C^*$ to an equilibrium function $\mathcal{F}^{eq} : C^* \to C^*$, and show that this adaption is continuous. Thereafter, we are able to use Proposition 1.3.10 to prove that the equilibrium function $\mathcal{F}^{eq}$ has zeroes.

Definition 3.3.4 *Let $p_0 \in$ Dom$(\mathcal{S}_{\text{total}})$. The function $\mathcal{F}^{eq} : C^* \to C^*$ is defined by*

$$\mathcal{F}^{eq}(p) := \begin{cases} (1 - \eta(\mathcal{Z}(p,p_0)))\mathcal{F}_0^{eq}(p) + \eta(\mathcal{Z}(p,p_0))p_0 & p \in \textit{Dom}(\mathcal{S}_{\text{total}}) \\ p_0 & p \notin \textit{Dom}(\mathcal{S}_{\text{total}}). \end{cases}$$

Here the sigma-oidal function $\eta : I\!R \to [0,1]$ *is defined by*

$$\eta(\alpha) := \begin{cases} 0 & \textit{if } \alpha \leq 0 \\ \alpha & \textit{if } 0 < \alpha < 1 \\ 1 & \textit{if } 1 \leq \alpha. \end{cases}$$

Note that

$$\forall \alpha \in I\!R : \alpha\eta(\alpha) \geq 0. \tag{13}$$

Lemma 3.3.5 *The function* $\mathcal{F}^{eq} : C^* \to C^*$ *is an equilibrium function.*

Proof
Suppose $\mathcal{F}^{eq}(p) = 0$ for some $p \in C^*$, then from the definition of $\mathcal{F}^{eq}$ it follows that $p \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$. Because $\mathcal{F}_0^{eq} : \mathrm{Dom}(\mathcal{S}_{\text{total}}) \to C^*$ is an equilibrium function, and because the set C^* is salient, the following assertions are equivalent,

$$\begin{aligned} \mathcal{F}^{eq}(p) = 0 \iff & (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{F}_0^{eq}(p) + \eta(\mathcal{Z}(p, p_0))p_0 = 0 \\ \iff & \eta(\mathcal{Z}(p, p_0))p_0 = 0 \text{ and } (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{F}_0^{eq}(p) = 0 \\ \iff & \eta(\mathcal{Z}(p, p_0)) = 0 \text{ and } \mathcal{F}_0^{eq}(p) = 0 \\ \iff & \mathcal{Z}(p, q) \leq 0 \text{ for all } q \in C^*. \end{aligned}$$

□

The following lemma shows that if the equilibrium function $\mathcal{F}^{eq} : C^* \to C^*$ is continuous, we can, indeed, use Proposition 1.3.10 to prove that $\mathcal{F}^{eq}$ has zeroes, i.e., to prove that Walrasian equilibria exist in this model of a private ownership economy.

Lemma 3.3.6 *Let* $p \in C^*$ *and let* $\mathcal{F}^{eq}$ *be defined as in Definition 3.3.4. Then* $(\mathcal{F}^{eq}(p) = 0) \iff (\exists\, \alpha \geq 0 : \mathcal{F}^{eq}(p) = \alpha p)$.

Proof
Suppose $\mathcal{F}^{eq}(p) = \alpha p$ for some $\alpha \geq 0$. From the definition of $\mathcal{F}^{eq}$ it immediately follows that $p \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$. Walras' law (equation (12)) yields

$$\mathcal{Z}(p, \mathcal{F}^{eq}(p)) = \alpha\mathcal{Z}(p, p) = 0.$$

Since $\mathcal{Z}(p, \mathcal{F}_0^{eq}(p)) = \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p, q)\}\mathcal{Z}(p, q)d\mu(q) \geq 0$, using equation (13) and using the definition of $\mathcal{F}^{eq}(p)$ for $p \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$ we find

$$\mathcal{Z}(p, \mathcal{F}^{eq}(p)) = \underbrace{(1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{Z}(p, \mathcal{F}_0^{eq}(p))}_{\geq 0} + \underbrace{\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0)}_{\geq 0} = 0.$$

Clearly

$$(1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{Z}(p, \mathcal{F}_0^{eq}(p)) = 0 \tag{14}$$

and

$$\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0) = 0. \tag{15}$$

Now suppose $\mathcal{Z}(p, \mathcal{F}_0^{eq}(p)) \neq 0$, then by equation (14), $\eta(\mathcal{Z}(p, p_0)) = 1$ and by the definition of η we find $\mathcal{Z}(p, p_0) \geq 1$. Since this is in contradiction with equation (15), we conclude

$$\mathcal{Z}(p, \mathcal{F}_0^{eq}(p)) = \int_{L_1(x_0)} \max\{0, \mathcal{Z}(p, q)\}\mathcal{Z}(p, q) d\mu(q) = 0.$$

So, for all $q \in L_1(x_0) : \mathcal{Z}(p, q) \leq 0$. Since $\mathcal{F}^{eq} : C^* \to C^*$ is an equilibrium function, $\mathcal{F}^{eq}(p) = 0$. □

3.4 Equilibrium function, existence of zeroes

In the previous section, we constructed the function $\mathcal{F}^{eq} : C^* \to C^*$, and proved that this function is an equilibrium function. In order to prove existence of Walrasian equilibria, we only have to prove continuity of $\mathcal{F}^{eq}$ on $C^* \setminus \{0\}$, since in this case Theorem 1.3.10 and Lemma 3.3.6 yield the desired result. We start with proving that the function $\mathcal{F}_0^{eq}$ is continuous on Dom($\mathcal{S}_{\text{total}}$).

Lemma 3.4.1 *The function $\mathcal{F}_0^{eq}$ is continuous on Dom($\mathcal{S}_{\text{total}}$).*

Proof
Recall the definition of x_0 and $L_1(x_0)$ in Definition 3.3.2. Impose on C^* the norm $\| \cdot \|_{x_0}$, and let $\| \cdot \|$ be the norm on C, dual to the norm $\| \cdot \|_{x_0}$. We recall that, by definition, for all $q \in L_1(x_0)$ we have $\| q \|_{x_0} = 1$.
Since, for $\alpha \in I\!R : \max\{0, \alpha\} = \frac{1}{2}(|\alpha| + \alpha)$, we find for $\alpha, \beta \in I\!R$:

$$|\max\{0, \alpha\} - \max\{0, \beta\}| \leq |\alpha - \beta|.$$

From this, we conclude that for all $p_1, p_2 \in$ Dom($\mathcal{S}_{\text{total}}$) and $q \in C^*$:

$$\begin{aligned}
& | \max\{0, \mathcal{Z}(p_1, q)\} - \max\{0, \mathcal{Z}(p_2, q)\} | \\
\leq\ & | \mathcal{Z}(p_1, q) - \mathcal{Z}(p_2, q) | \\
=\ & |\mathcal{V}(\mathcal{D}_{\text{total}}(p_1), q) - \mathcal{G}(\mathcal{S}_{\text{total}}(p_1), q) - \mathcal{V}(\mathcal{D}_{\text{total}}(p_2), q) + \mathcal{G}(\mathcal{S}_{\text{total}}(p_2), q)| \\
\leq\ & \| q \|_{x_0} \ (\| \mathcal{D}_{\text{total}}(p_1) - \mathcal{D}_{\text{total}}(p_2) \| + \| (0, \mathcal{S}^{\text{cons}}_{\text{total}}(p_1)) - (0, \mathcal{S}^{\text{cons}}_{\text{total}}(p_2)) \| \\
& + \| (\mathcal{S}^{\text{prod}}_{\text{total}}(p_1), 0) - (\mathcal{S}^{\text{prod}}_{\text{total}}(p_2), 0) \| \ ).
\end{aligned}$$

From the above, we find for $p_1, p_2 \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$:

$$\begin{aligned} & \| \mathcal{F}_0^{eq}(p_1) - \mathcal{F}_0^{eq}(p_2) \|_{x_0} \\ \leq & \int_{L_1(x_0)} | \max\{0, \mathcal{Z}(p_1, q)\} - \max\{0, \mathcal{Z}(p_2, q)\} | \ \| q \|_{x_0} \ d\mu(q). \\ = & (\| \mathcal{D}_{\text{total}}(p_1) - \mathcal{D}_{\text{total}}(p_2) \| + \| (0, \mathcal{S}^{\text{cons}}_{\text{total}}(p_1)) - (0, \mathcal{S}^{\text{cons}}_{\text{total}}(p_2)) \| \\ & + \| (\mathcal{S}^{\text{prod}}_{\text{total}}(p_1), 0) - (\mathcal{S}^{\text{prod}}_{\text{total}}(p_2), 0) \|) \, \mu(L_1(x_0)). \end{aligned}$$

Since $\mathcal{D}_{\text{total}}$ and $\mathcal{S}_{\text{total}}$ are continuous on $\mathrm{Dom}(\mathcal{S}_{\text{total}})$, it follows that $\mathcal{F}_0^{eq}$ is continuous on $\mathrm{Dom}(\mathcal{S}_{\text{total}})$. □

Proposition 3.4.2 *The function $\mathcal{F}^{eq} : C^* \setminus \{0\} \to C^*$ is continuous.*

Proof
The function $q \mapsto \eta(\mathcal{Z}(q, p_0))$ is continuous on $\mathrm{Dom}(\mathcal{S}_{\text{total}})$, and $\mathcal{F}_0^{eq}$ is continuous on $\mathrm{Dom}(\mathcal{S}_{\text{total}})$, so the function $\mathcal{F}^{eq}$ is continuous on $\mathrm{Dom}(\mathcal{S}_{\text{total}})$. Remains to prove the continuity of $\mathcal{F}^{eq}$ on $C^* \setminus (\mathrm{Dom}(\mathcal{S}_{\text{total}}) \cup \{0\})$. By definition, $\mathcal{F}^{eq}(p) = p_0$ for all $p \in C^* \setminus \mathrm{Dom}(\mathcal{S}_{\text{total}})$, so we only have to consider a sequence $(p_n)_{n \in \mathbb{N}}$ in $\mathrm{Dom}(\mathcal{S}_{\text{total}})$ with limit $p \notin \mathrm{Dom}(\mathcal{S}_{\text{total}})$. Now, suppose the sequence $(\mathcal{F}^{eq}(p_n))_{n \in \mathbb{N}}$ does not converge to $\mathcal{F}^{eq}(p) = p_0$. Taking a subsequence if necessary, we may assume $\mathcal{F}^{eq}(p_n) \neq p_0$, for all $n \in \mathbb{N}$, i.e. $\forall n \in \mathbb{N} : p_n \in \mathrm{Dom}(\mathcal{S}_{\text{total}})$. Since $p \notin \mathrm{Dom}(\mathcal{S}_{\text{total}})$ means either $p \in \partial C^*$ or $p \in \mathrm{int}(C^*) \setminus \mathrm{Dom}(\mathcal{S}_{\text{total}})$.
In the first situation, by Assumption A5.b and Corollary 3.1.11, we find that the sequence $(\mathcal{D}_{\text{total}}(p_n))_{n \in \mathbb{N}}$ is unbounded.
In the second situation, $\exists j \in \{1, \ldots, J\} : p \notin \mathrm{Dom}(\mathcal{S}_j)$, and by Corollary 3.1.4 for all $j \in \{1, \ldots, J\}$, satisfying $p \notin \mathrm{Dom}(\mathcal{S}_j)$ it holds that $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$. Hence, $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_{\text{total}}(p_n), p_0) = -\infty$.
Either way, in the first situation with the help of Corollary 1.4.9, we conclude

$$\lim_{n\to\infty} \mathcal{Z}(p_n, p_0) = \lim_{n\to\infty} (\mathcal{V}(\mathcal{D}_{\text{total}}(p_n), p_0) - \mathcal{G}(\mathcal{S}_{\text{total}}(p_n), p_0) - \mathcal{V}(w_{\text{total}}, p_0)) = \infty.$$

Hence, $\exists n_0 \in \mathbb{N} : \mathcal{Z}(p_{n_0}, p_0) \geq 1$ (cf. Corollary1.3.6). So $\mathcal{F}^{eq}(p_{n_0}) = p_0$. This is in contradiction with the assumption that $\mathcal{F}^{eq}(p_n) \neq p_0$ for all $n \in \mathbb{N}$. □

Finally we come to the proof of the main theorem of this paper as presented in Subsection 2.4.

Proof of Existence Theorem

Since the equilibrium function $\mathcal{F}^{eq}$ is continuous on $C^* \setminus \{0\}$, applying Proposition 1.3.10 yields that there is some $p \in C^* \setminus \{0\}$ such that $\mathcal{F}^{eq}(p) = \alpha p$ for some $\alpha \geq 0$. Lemma 3.3.6 yields $\mathcal{Z}(p, q) \leq 0$ for all $q \in C^*$. $\square$

References

[ArDe54] K.J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica 22, 1954, pp.265-290.

[ArHa71] K.J. Arrow and F.H. Hahn, *General Competitive Analysis*, Mathematical Economic Texts vol. 6, 1971.

[Conw90] J.B. Conway, *A Course in Functional Analysis*, second edition, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, Berlin, 1990.

[Debr59] G. Debreu, *Theory of Value*, *An Axiomatic Analysis of Economic Equilibrium*, Yale University Press, New Haven, 1959.

[Halm87] P.R. Halmos, *Finite Dimensional Vector Spaces*, Undergraduate Texts in Mathematics, Springer-Verlag, Berlin, 1987.

[Pani93] M.J. Panik, *Fundamentals of Convex Analysis*, *Duality, Seperation, Representation and Resolution*, Theory and Decision Library, Series B: Mathematical and Statistical Methods, vol. 24, Kluwer Academic Publishers, Dordrecht, 1993.