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**SOME IMPLICATIONS ON AMORPHIC ASSOCIATION
SCHEMES**

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Some implications on amorphic association schemes

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Abstract

We give an overview of results on amorphic association schemes. We give the known constructions of such association schemes, and enumerate most such association schemes on up to 49 vertices. Special attention is paid to cyclotomic association schemes. We give several results on when a strongly regular decomposition of the complete graph is an amorphic association scheme. This includes a new proof of the result that a decomposition of the complete graph into three strongly regular graphs is an amorphic association scheme, and the new result that a strongly regular decomposition of the complete graph for which the union of any two relations is again strongly regular must be an amorphic association scheme.

1 Introduction

In this paper, we give an overview of results (including some new ones) on amorphic association schemes. These are association schemes with the (exceptional) property that any fusion is again an association scheme. Amorphic association schemes are in some sense at the opposite end of distance-regular graphs in the spectrum of all association schemes. A distance-regular graph has the property that its adjacency algebra (the algebra of all polynomials in its adjacency matrix) is the Bose-Mesner algebra of the corresponding association scheme (consisting of the distance-graphs of the distance-regular graph), that is, we say that the distance-regular graph generates the whole association scheme. More generally, any d -class association scheme which has as one of its relations a graph with $d + 1$ distinct eigenvalues is generated by this graph. The following question pops up: which association schemes are generated by one of its relations, or by a union of some of its relations? It is clear that if each relation of an association scheme has few distinct eigenvalues, then this is not likely to happen. In a “bad” case, all relations are strongly regular graphs. Or even “worse”, any union of relations is a strongly regular graph. This extreme situation will turn out to be precisely the case of an amorphic association scheme.

Instead of the adjacency algebra, one could also consider the coherent closure of a relation, or perhaps of a union of relations. By coherent closure of a relation we mean the smallest (in terms of number of classes) association scheme that contains this particular relation, if such an association scheme exists. If we consider the case of distance-regular graphs again, then it is clear that the coherent closure of the graph is the entire association scheme corresponding to this distance-regular

graph. Given any association scheme, one might ask whether one of its relations (or perhaps a union of some of its relations) has coherent closure equal to the entire association scheme. Put differently: is there a relation for which there is no (nontrivial) fusion scheme that contains this relation. Here the “worst” thing that can happen is that any fusion of the association scheme is an association scheme itself. This is precisely how amorphic association schemes are defined.

The further set-up of this paper is as follows. In Section 2, we shall give basic definitions and information on association schemes and strongly regular graphs. In Section 3 we introduce amorphic association schemes, and in Section 4 we prove the basic results on the parameters of such schemes. This includes A.V. Ivanov’s classification result that all relations in an amorphic association scheme with at least three classes are strongly regular graphs of (negative) Latin square type. In Section 5, we give several basic constructions of amorphic association schemes, using mutually orthogonal Latin squares, Hadamard matrices, particular points sets in projective planes, cyclotomic association schemes, and partial difference sets. In Section 5.4 we give a new, self-contained, proof that if p is semiprimitive modulo d , then the cyclotomic d -class association scheme on a finite field of characteristic p is amorphic. Most small amorphic association schemes on up to 49 vertices are enumerated in Section 6. In Section 7, we consider strongly regular decompositions of the complete graph, and tackle the question of when such decompositions are amorphic association schemes. This includes a new proof in Section 7.1 of the result that a decomposition of the complete graph into three strongly regular graphs is an amorphic association scheme, and the new result in Section 7.3 that a strongly regular decomposition of the complete graph for which the union of any two relations is again strongly regular is also an amorphic association scheme. We finish with some miscellaneous results in Section 8.

2 Preliminaries

A d -class association scheme (cf. [3, 7, 27]) is a set of relations $R_i, i = 0, 1, \dots, d$ on a set V , such that

- 1) the relation R_0 is the identity relation $\{(x, x) | x \in V\}$ on V ;
- 2) the relations $R_i, i = 0, 1, \dots, d$ partition the complete relation $V \times V$;
- 3) the reverse $R_i^\top = \{(x, y) | (y, x) \in R_i\}$ of the relation R_i is also one of the relations, say $R_{i'}$, for some i' ;
- 4) for every h, i, j there is a constant p_{ij}^h (called an intersection number) such that for every $(x, y) \in R_h$, the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^h .

As a warning to the reader, we mention the following. Each of the relations $R_i, i \neq 0$ can be seen as a (possibly directed) graph without loops. In this paper we shall usually talk about graphs instead of relations. By Property 2), the graphs $R_i, i \neq 0$ in an association scheme form an (arc/edge) decomposition of the complete graph. Sometimes we shall call this decomposition an association scheme (without mentioning the trivial relation R_0).

An association scheme is called *primitive* if all graphs $R_i, i \neq 0$ are connected. The association scheme is called *commutative* if $p_{ij}^h = p_{ji}^h$ for all h, i, j , and it is called *symmetric* if all of its relations are symmetric, i.e. if $R_i^\top = R_i$ for all i . It follows that a symmetric association scheme is

always commutative. Symmetric association schemes were introduced by Bose and Shimamoto [6]. Commutative association schemes were first considered by Delsarte [21].

Let A_i be the adjacency matrix of relation (graph) R_i , that is, the $\{0, 1\}$ -matrix indexed by V , with entry 1 in position (x, y) if and only if $(x, y) \in R_i$. Then the properties of the association scheme are translated into the following:

- 1) $A_0 = I$;
- 2) $A_0 + A_1 + \dots + A_d = J$, the all-ones matrix;
- 3) for all i there is an i' such that $A_i^\top = A_{i'}$;
- 4) $A_i A_j = \sum_h p_{ij}^h A_h$.

Thus the vector space $\langle A_0, A_1, \dots, A_d \rangle$ forms an algebra. This algebra is called the *Bose-Mesner algebra*. For a commutative association scheme the Bose-Mesner algebra is commutative.

Association schemes can be seen as combinatorial generalizations of the set of 2-orbits (or orbitals) of transitive permutation groups. The centralizer algebra of such a group, that is, the algebra of matrices commuting with all permutation matrices corresponding to the permutations in the group, is precisely the Bose-Mesner algebra of the association scheme of its 2-orbits.

One of the most important concepts of this paper is the concept of merging of classes, or *fusion*. We say that $\{S_0, S_1, \dots, S_e\}$ is a fusion of the association scheme $\{R_0, R_1, \dots, R_d\}$ if $S_0 = R_0$ and S_i is the union of some of the relations R_j , for all i . It is interesting to see which fusions of an association scheme are also association schemes.

Even more general than association schemes are the so-called coherent configurations. These satisfy all properties of an association scheme, except that the identity relation may now be a union of some of the relations. The corresponding algebra of a coherent configuration is called a coherent algebra. Coherent configurations are the combinatorial generalization of the set of 2-orbits of a permutation group (not necessarily transitive). Coherent configurations were introduced by Higman (cf. [31]), and independently by Weisfeiler and Leman (cf. [23]), who called them cellular algebras.

The coherent closure of a matrix A is the smallest coherent algebra containing A . Since we will mainly be interested in association schemes, it is a relevant question to ask when the coherent closure is the Bose-Mesner algebra of an association scheme.

Now consider a commutative association scheme. It is well known that since the Bose-Mesner algebra is commutative, it has a basis of common eigenvectors, and that \mathbb{C}^v can be decomposed into maximal common eigenspaces V_i , $i = 0, 1, \dots, d$. The idempotent (and mutually orthogonal) matrices E_j corresponding to the orthogonal projection on V_j form another basis for the Bose-Mesner algebra. Moreover, $A_i = \sum_j P_{ji} E_j$ for all i , where P_{ji} is the eigenvalue of A_i on the eigenspace V_j . The matrix P is called the *eigenmatrix* of the association scheme. Dually we can write $E_i = \frac{1}{v} \sum_j Q_{ji} A_j$ for all i (with v being the number of vertices), and we call Q the *dual eigenmatrix*. It follows that $PQ = QP = vI$. Another important property of P and Q is that $k_i Q_{ij} = m_j \overline{P_{ji}}$, where $m_j = \dim V_j$ and $k_i = p_{ii}^0$ is the valency of graph R_i . This property follows from taking traces in the equation $A_i^\top E_j = \overline{P_{ji}} E_j$, and expressing E_j on the left hand side in terms of the adjacency matrices. Finally, we call an association scheme *formally self-dual* if $P = Q$, possibly after some reordering of the eigenspaces V_j . For more details of the above we refer the interested reader to [3] or [7].

A *strongly regular graph* with parameters (v, k, λ, μ) is a graph on v vertices which is regular with

valency k and such that any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. It is well known that a strongly regular graph and its complement form a symmetric 2-class association scheme. A connected strongly regular graph has three distinct eigenvalues k , r , and s . A disconnected strongly regular graph is the disjoint union of cliques of the same size n . In this case it has eigenvalues $k = r = n - 1$ and $s = -1$. To unify the two cases, it will be convenient to distinguish between the eigenvalue k and the so-called *restricted eigenvalues* r and s , i.e., those eigenvalues that have an eigenvector orthogonal to the all-ones vector. The *restricted multiplicity* of a restricted eigenvalue is the dimension of the corresponding eigenspace intersected with the orthogonal complement of the all-ones vector. The restricted eigenvalues r and s of a strongly regular graph follow from the parameters by the equations $r + s = \lambda - \mu$ and $rs = \mu - k$. The corresponding restricted multiplicities f of r and g of s can be written as $f = -\frac{k+(v-1)s}{r-s}$ and $g = \frac{k+(v-1)r}{r-s}$.

An important role in this paper will be played by the strongly regular graphs of Latin square type, and the ones of negative Latin square type. A strongly regular graph is of Latin square type if $v = n^2$, $k = t(n - 1)$, $r = n - t$, and $s = -t$ for some positive integers n and t . For such a graph the other parameters are equal to $\lambda = n - 2 + (t - 1)(t - 2)$ and $\mu = t(t - 1)$. A strongly regular graph is of negative Latin square type if $v = n^2$, $k = t(n + 1)$, $r = t$, and $s = t - n$ for some positive integers n and t . For such a graph the other parameters are equal to $\lambda = -n - 2 + (t + 1)(t + 2)$ and $\mu = t(t + 1)$. Note that the parameters of a graph of negative Latin square type are obtained from those of one of Latin square type by replacing n and t by their opposites. Alternatively, one can introduce a parameter $\epsilon = \pm 1$ to denote the (positive or negative) type, and then the parameters become $v = n^2$, $k = t(n - \epsilon)$, $r = \epsilon(n - t)$, $s = -\epsilon t$, $\lambda = \epsilon n - 2 + (t - \epsilon)(t - 2\epsilon)$ and $\mu = t(t - \epsilon)$.

An example of a strongly regular graph of Latin square type with the above parameters is obtained from a set of $t - 2$ mutually orthogonal Latin squares of side n . The vertices are the n^2 cells of the Latin squares, and two distinct cells are adjacent if they are in the same row, the same column, or if they have the same symbol in one of the $t - 2$ Latin squares.

There exist strongly regular graphs of Latin square type which do not come from mutually orthogonal Latin squares as described above. However, Bruck [12] showed that if a strongly regular graph of Latin square type has the above parameters (such a graph is sometimes called a pseudolatin graph), with $n > \frac{1}{2}(t - 1)(t^3 - t^2 + t + 2)$, then the graph does come from a set of $t - 2$ mutually orthogonal Latin squares of side n .

For some recent examples of negative Latin square type graphs we refer to [46], which contains a prolific construction of strongly regular graphs of negative Latin square type on $16r^2$ vertices of valency $(4r + 1)(2r - 1)$. The construction is based on Hadamard matrices of order $4r$.

3 Introduction to amorphic association schemes

An association scheme is called *amorphic* if all of its fusions are also association schemes. The 2-class association schemes are trivially amorphic, hence we will focus on amorphic d -class association schemes with $d \geq 3$. It is easily proven that such an association scheme is symmetric, and that all of its graphs R_i , $i \neq 0$ are strongly regular. In the next section we shall see that all the graphs in an amorphic d -class association scheme with $d \geq 3$ are of Latin square type, or they are all of negative Latin square type.

From an affine plane of order n we can construct an amorphic association scheme as follows. The vertices of the association scheme are the points of the affine plane. For each of the $n + 1$ parallel classes of lines of the affine plane we define a graph (relation), in which two distinct vertices are

adjacent if and only if the line through them is in that particular parallel class. In this amorphic $(n+1)$ -class association scheme all of the graphs are disconnected with valency $n-1$, i.e. the graphs are of Latin square type (with $t=1$). On the other hand, any such association scheme comes from an affine plane by the above construction. We shall refer to it as a *complete affine association scheme*.

Analogously, we would like to define a complete negative affine association scheme on n^2 vertices as one with $n-1$ graphs having valency $n+1$, i.e. the graphs are of negative Latin square type (with $t=1$). However, such strongly regular graphs have $\lambda=4-n$, hence a complete negative affine association scheme can only exist for $n \leq 4$ (and they do for $n=2,3,4$).

Although the fusion of any amorphic association scheme is also an amorphic association scheme, it is not true that each amorphic association scheme is a fusion scheme of a complete (negative) affine association scheme.

4 A.V. Ivanov's classification

Our next goal is to show that all the graphs in an amorphic association scheme with at least three classes are of Latin square type, or they are all of negative Latin square type. We call this A.V. Ivanov's classification. The proof given is a combination of Ivanov's proof [29] and Higman's proof [32] (cf. [17]).

Theorem 1. *All graphs in an amorphic association scheme with at least three classes are strongly regular of Latin square type, or they are all of negative Latin square type.*

Proof. It suffices to prove the statement for amorphic association schemes with three classes; if the number of classes is larger, then we can fix any two classes, and fuse the remaining classes.

So let $G_i, i=1,2,3$ be the three strongly regular graphs in the amorphic association scheme, and let k_i be the valency and r_i, s_i be the restricted eigenvalues of G_i , for $i=1,2,3$. Here we do not assume that $r_i > s_i$. Since the eigenmatrix of the association scheme cannot have repeated rows (it is nonsingular) and the row sums of its bottom three rows are zero, we may assume without loss of generality that the association scheme has eigenmatrix

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 \\ 1 & r_1 & s_2 & s_3 \\ 1 & s_1 & r_2 & s_3 \\ 1 & s_1 & s_2 & r_3 \end{pmatrix}. \quad (1)$$

It then follows that $r_1 - s_1 = r_2 - s_2 = r_3 - s_3$. By considering the appropriate off-diagonal elements in the equation $PQ = vI$, and the property that $Q_{ij} = \frac{m_{ij}}{k_i} P_{ji}$, we obtain the equations

$$1 + \frac{r_1 s_1}{k_1} + \frac{r_2 s_2}{k_2} + \frac{s_3^2}{k_3} = 0 \text{ and} \quad (2)$$

$$1 + \frac{r_1 s_1}{k_1} + \frac{s_2^2}{k_2} + \frac{r_3 s_3}{k_3} = 0.$$

From this it follows that $\frac{s_2}{k_2} = \frac{s_3}{k_3}$. Similarly we obtain that $\frac{s_1}{k_1} = \frac{s_2}{k_2}$. From Equation (2) and the equation $1 + r_1 + s_2 + s_3 = 0$, we now obtain that $\frac{k_1}{s_1} = 1 - u$, where $u = r_2 - s_2$. Hence we have that $k_i = s_i(1-u)$ for $i=1,2,3$, and this in turn implies that $v = 1 + k_1 + k_2 + k_3 = 1 + (s_1 + s_2 + s_3)(1-u) = u^2$. If $u > 0$, then it follows that the three graphs are of Latin square type; if $u < 0$, then the three

graphs are of negative Latin square type. \square

Due to this classification of amorphic association schemes, these form a quite natural generalization of nets and affine planes.

Using Ivanov's classification, it was shown by Ito, Munemasa, and Yamada [35] that the eigenmatrix of an amorphic association scheme has some specific form, and that such a scheme is formally self-dual.

Proposition 1. *Let $\{R_0, R_1, \dots, R_d\}$ be an amorphic association scheme on n^2 vertices, with $d \geq 3$, for some integer n , and let $\epsilon = \pm 1$. If R_i has valency $t_i(n - \epsilon)$, for some integer t_i , for $i = 1, \dots, d$, then the eigenmatrix P can be written such that $P_{ji} = -\epsilon t_i$ if $i \neq j$ and $P_{ii} = \epsilon(n - t_i)$ for $i, j \neq 0$. Moreover, the association scheme is formally self-dual.*

Proof. The strongly regular graph R_i has valency $k_i = t_i(n - \epsilon)$; the other eigenvalues are $-\epsilon t_i$ and $\epsilon(n - t_i)$, for $i = 1, \dots, d$. We remark that depending on ϵ the graphs R_i are of Latin square type, or of negative Latin square type.

Since $n^2 = 1 + k_1 + \dots + k_d$, we derive that $t_1 + \dots + t_d = n + \epsilon$. Since the row sum $1 + P_{j1} + P_{j2} + \dots + P_{jd}$ is zero for $j \neq 0$, and since P_{ji} equals $-\epsilon t_i$ or $\epsilon(n - t_i)$, it follows that for each row of the eigenmatrix (except the first one) there is exactly one i such that the row contains eigenvalue $\epsilon(n - t_i)$. Since the i -th column of P ($i \neq 0$) contains at least one $\epsilon(n - t_i)$, the first part of the proposition follows.

Since R_i has eigenvalue $\epsilon(n - t_i)$ with multiplicity k_i (since R_i is of (negative) Latin square type), it follows that the multiplicities of the association scheme are also given by $m_i = k_i$, for $i = 0, 1, \dots, d$. From the equations $Q_{ij} = \frac{m_j}{k_j} P_{ji} = \frac{k_j}{k_i} P_{ji} = \frac{t_j}{t_i} P_{ji}$ it now follows that $P = Q$, that is, the association scheme is formally self-dual. \square

Corollary 1. *Let $\{R_0, R_1, \dots, R_d\}$ be an amorphic association scheme on n^2 vertices, with $d \geq 3$, for some integer n , and let $\epsilon = \pm 1$. If R_i has valency $t_i(n - \epsilon)$, for some integer t_i , for $i = 1, \dots, d$, then the intersection numbers are equal to $p_{ii}^i = \epsilon n - 2 + (t_i - \epsilon)(t_i - 2\epsilon)$, $p_{ii}^h = t_i(t_i - \epsilon)$, $p_{ih}^h = t_i(t_h - \epsilon)$, and $p_{ij}^h = t_i t_j$ for all distinct $h, i, j \neq 0$.*

Proof. This follows from the eigenmatrix in the previous proposition and the equations $p_{ij}^h = \frac{1}{vk_h} \sum_t m_t P_{th} P_{ti} P_{tj}$ (which follow from $\text{tr}(A_i A_j A_h) = \sum_{s,t} (A_i A_j)_{st} (A_h)_{ts} = vk_h p_{ij}^h$). \square

On the other hand, we have the following sufficient condition for the eigenmatrix of an association scheme to be amorphic.

Proposition 2. *An association scheme with eigenmatrix P is amorphic if $P_{ii} - P_{ji}$ is the same for all pairs of distinct $i, j = 1, \dots, d$, possibly after reordering of the eigenspaces.*

Proof. The condition implies that each relation is strongly regular, with one of the restricted eigenvalues occurring in exactly one of the eigenspaces. Let's call this eigenspace the diagonal eigenspace (because the corresponding eigenvalue is on the diagonal of P). If one fuses two relations, then it follows that the union has the same eigenvalue on the two corresponding diagonal eigenspaces. Thus it follows that these eigenspaces fuse too, and hence we again have an association scheme. Moreover, this association scheme still satisfies the condition in the proposition. So by induction any fusion is an association scheme, and hence the association scheme is amorphic. \square

5 Constructions of amorphic association schemes

5.1 Mutually orthogonal Latin squares

Amorphic association schemes of Latin square type can be constructed from mutually orthogonal Latin squares, or equivalently, from orthogonal arrays. An orthogonal array $OA(n, m)$ is an $m \times n^2$ matrix M such that for any two of its rows i and i' the pairs $(M_{ij}, M_{i'j}), j = 1, \dots, n^2$ comprise all pairs from the symbols $1, \dots, n$. It is well-known that $m \leq n + 1$, with equality if and only if the rows represent the parallel classes of lines of an affine plane (cf. [1]). For constructions of orthogonal arrays we refer to [1]. Note that an $OA(n, 3)$ corresponds to a single Latin square: row 1 of M gives the row index of the Latin square, row 2 its column index, and row 3 the corresponding entry.

A d -class amorphic association scheme is constructed from the matrix M as follows. First we partition (the rows) $1, \dots, m$ into nonempty sets I_1, I_2, \dots, I_{d-1} . The vertices are (the columns) $1, \dots, n^2$. A pair of distinct vertices j, j' is in relation R_h if $M_{ij} = M_{i'j'}$ for some $i \in I_h$, for $h = 1, \dots, d - 1$, and in relation R_d if it is in none of the relations $R_h, h = 1, \dots, d - 1$ just defined. We say that this association scheme is of type $L_{|I_1|, \dots, |I_{d-1}|}(n)$. We remark that if $m = n + 1$, then R_d is empty, hence the number of classes is $d - 1$, instead of d .

We note that the earlier mentioned construction of the complete affine association scheme is a special case of the above.

5.2 Hadamard matrices

Goldbach and Claasen [28] showed that a 3-class association scheme with the same parameters as $L_{1,s}(2s)$ is equivalent to a symmetric Bush-type Hadamard matrix (also called checkered Hadamard matrix) of order $4s^2$. Janko, Kharaghani, and Tonchev [37] constructed a Bush-type Hadamard matrix of order 324, which was the first known symmetric one for odd $s > 1$. This Bush-type Hadamard matrix gives an amorphic association scheme with the same parameters as one of type $L_{1,9}(18)$. In [47], symmetric Bush-type Hadamard matrices of order $4s^2$ were constructed for all odd s .

Chuvaeva and Ivanov [15] constructed an amorphic association scheme with the same parameters as $L_{1,1,2m-1}(4m)$ from an Hadamard matrix H of order $4m$ as follows. Vertices are the $4m \times 4m$ cells of the matrix. Being in the same row is one relation, being in the same column a second. Two vertices (i, j) and (i', j') with $i \neq i', j \neq j'$ are in the third or fourth relation depending on whether $H_{ij}H_{i'j'}H_{i'j}H_{ij'}$ equals 1 or -1 .

5.3 Disjoint (m, n) -sets in projective planes

An (m, n) -set in a projective plane is a set of points such that any line intersects the set in either m or n points. An (m, n) -set of size h in the projective plane $PG(2, q)$ provides us with a strongly regular graph as follows. Embed the plane in $PG(3, q)$, and consider the affine space $AG(3, q) = PG(3, q) \setminus PG(2, q)$ as vertex set. Two vertices are adjacent if the line through them intersects the (m, n) -set. It follows that the eigenvalues of the graph are $k = qh - h$, $r = qn - h$, and $s = qm - h$ (cf. [14] or [27, Lemma 9.3]). Clearly, an (m, n) -set and an (m', n') -set which are disjoint give edge-disjoint strongly regular graphs on the same vertex set, hence it may be possible to construct amorphic association schemes (with at least three classes) in this way. A necessary condition for this is that the (disjoint) union of the (m, n) -set and the (m', n') -set is an (m'', n'') -set. It follows that this implies that the corresponding graphs are of Latin square or negative Latin square type.

It follows quite easily that the graph obtained from an (m, n) -set M is of Latin square type if and only if M is an $(m, m+p)$ -set of size $m(p^2 + p + 1)$ in $PG(2, p^2)$, where p is a prime power. Examples are given by Baer subplanes $PG(2, p)$ ($m = 1$). The graph is of negative Latin square type if and only if M is an $(m, m+p)$ -set of size $(m+p)(p^2 - p + 1)$, again in $PG(2, p^2)$, p a prime power. Examples are given by maximal arcs ($m = 0$) and unitals ($m = 1$).

Hence, a necessary condition for the existence of mutually disjoint (non-complementary) (m_i, n_i) -sets M_i , $i = 1, \dots, d-1$, $d \geq 3$ in $PG(2, q)$ such that the union of any of the M_i is again an (m, n) -set, is that $q = p^2$, $n_i = m_i + p$, and either $|M_i| = m_i(p^2 + p + 1)$, $i = 1, \dots, d-1$ or $|M_i| = (m_i + p)(p^2 - p + 1)$, $i = 1, \dots, d-1$. From the following it follows that this condition is also sufficient. (Note that here the m_i are not the multiplicities of the association scheme.)

By standard counting arguments it follows that if an (m, n) -set and a disjoint (m', n') -set are of the above (and "same") form, then indeed their union is a (m'', n'') -set. Indeed, for an $(m, m+p)$ -set M of size $(m+p)(p^2 - p + 1)$, it follows that the number of lines that intersect M in m points is equal to $(m+p)(p^2 - p + 1)$. The number of lines that intersect M in m points that go through a given point outside M is equal to $m+p$. For an $(m, m+p)$ -set M of size $(m+p)(p^2 - p + 1)$ and a disjoint $(m', m'+p)$ -set M' of size $(m'+p)(p^2 - p + 1)$, it follows by counting (in two ways) the pairs (P, L) , P a point of both L and M' , and L a line that intersects M in m points, that there are no lines that intersect M in m points and M' in m' points. Hence it follows that $M \cup M'$ is an $(m+m'+p, m+m'+2p)$ -set of size $(m+m'+2p)(p^2 - p + 1)$. The other case (positive type) is similar.

We may conclude that disjoint (m, n) -sets of the form as described above give rise to amorphic association schemes. An interesting example which was already observed in [13] is a partition of $PG(2, 4)$ into two hyperovals ($(0, 2)$ -sets of size 6) and a unital $((1, 3)$ -set of size 9). This gives a 3-class amorphic association scheme of negative Latin square type on 64 vertices with valencies 18, 18, and 27. Hamilton, Stoichev, and Tonchev [30] found three disjoint maximal arcs ($(0, 4)$ -sets of size 52) in $PG(2, 16)$. It follows that the remaining points form a $(5, 9)$ -set of size 117. This gives rise to a 4-class amorphic association scheme of negative Latin square type on 4096 vertices with valencies 780, 780, 780, and 1755.

We remark that similar observations can be made in higher dimensional projective spaces.

5.4 Cyclotomic association schemes

Let $q = p^n$, where p is prime, and let d be a divisor of $q - 1$. The d -class cyclotomic association scheme on vertex set \mathbb{F}_q is defined as follows. Let α be a primitive element of \mathbb{F}_q . Then two vertices are in relation R_j if their difference equals α^{di+j} for some i , for $j = 1, \dots, d$. It was proven (in another context) by Baumert, Mills, and Ward [5, Thm. 4] that, for $d > 2$, this association scheme is amorphic (in their terminology: the cyclotomic numbers are uniform) if and only if -1 is a power of p modulo d , in other words, p is semiprimitive modulo d . The same result was later proven by Bannai and Munemasa [4]. Common factor of the proofs is the so-called Davenport-Hasse theorem. Also Brouwer, Wilson, and Xiang [9] use this theorem to essentially prove the sufficiency part of the above statement, i.e., Proposition 3. In this section we shall give a direct proof of this proposition. For the reverse, we do not know how to avoid the Davenport-Hasse theorem. The crucial part for this reverse is to show that if the d -class cyclotomic association scheme on \mathbb{F}_{q^2} is amorphic, and d divides $q - 1$, then the d -class cyclotomic association scheme on \mathbb{F}_q is also amorphic. Indeed, if we assume this to be true, and p^m is the smallest power of p such that the d -class cyclotomic association scheme on \mathbb{F}_{p^m} is amorphic, then by Theorem 1, p^m is a square, so m is even, and the association scheme is either of Latin square type or negative Latin square type. In the latter case d divides

$p^{m/2} - 1$, so by the assumption the association scheme on $\mathbb{F}_{p^{m/2}}$ is amorphic, which contradicts the minimality of p^m . In the first case d divides $p^{m/2} + 1$, so p is semiprimitive modulo d .

Before proving Proposition 3, let us first mention some generalities about cyclotomic association schemes, following the approach of Bannai and Munemasa [4]. Let $q = p^n$, and $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be the trace function, that is, $\text{Tr}(x) := x + x^p + \dots + x^{p^{n-1}}$. Each additive character χ_a ($a \in \mathbb{F}_q$) of \mathbb{F}_q is of the form $\chi_a(x) := \varepsilon_p^{\text{Tr}(ax)}$ where ε_p is a p -th primitive complex root of unity. For each subset $S \subset \mathbb{F}_q$ we set $\chi_a(S) := \sum_{s \in S} \chi_a(s)$. It follows from the definition that $\chi_a(S) = \chi_1(aS)$.

Consider now the d -class cyclotomic association scheme with valencies $k := (q - 1)/d$. The cyclotomic classes $S_i, i \in \mathbb{Z}_d$ are indexed by the elements of \mathbb{Z}_d in such a way that $S_i = \alpha^{-i}S_0$, where $S_0 = \langle \alpha^d \rangle = \{x \in \mathbb{F}_q^* \mid x^k = 1\}$ is a subgroup of \mathbb{F}_q^* of order k . Note that $\alpha S_i = S_{i-1}$ where the arithmetic is done modulo d . It follows that the eigenmatrix P of the association scheme is of the form

$$\begin{pmatrix} 1 & k & \dots & k \\ 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{pmatrix}$$

where P_0 is a $d \times d$ matrix, the entries of which are given by $(P_0)_{ij} = \chi_{\alpha^i}(S_j) = \chi_1(S_{j-i})$ where $i, j \in \mathbb{Z}_d$. In what follows we abbreviate $\eta_j := \chi_1(S_j) = \chi_1(S_0\alpha^{-j})$. Thus P_0 is a circulant matrix and may be written as $\sum_{j=0}^{d-1} \eta_j C^j$ where C is a circulant matrix for which $C_{ij} = 1$ if $j = i + 1 \pmod{d}$, and $C_{ij} = 0$ otherwise. It follows from the orthogonality relations that $P_0 J_d = -J_d$ and $P_0^* P_0 = qI_d - kJ_d$, where P_0^* is the conjugate transpose of P_0 .

We now claim that $\eta_{pi} = \eta_i$. To show this, we denote by Fr the Frobenius automorphism $x \mapsto x^p$. Because S_0 is a subgroup of \mathbb{F}_q^* , we have that $\text{Fr}(S_0) = S_0$, which implies that

$$\eta_{pi} = \chi_1(S_0\alpha^{-pi}) = \sum_{x \in S_0\alpha^{-pi}} \varepsilon_p^{\text{Tr}(x)} = \sum_{x \in \text{Fr}(S_0\alpha^{-i})} \varepsilon_p^{\text{Tr}(x)} = \sum_{x \in S_0\alpha^{-i}} \varepsilon_p^{\text{Tr}(x^p)} = \sum_{x \in S_0\alpha^{-i}} \varepsilon_p^{\text{Tr}(x)} = \eta_i.$$

Lemma 1. *Let p be semiprimitive modulo d , and $d > 2$. Then*

- (a) n is even;
- (b) $\eta_i \in \mathbb{Z}$ for each $i = 0, \dots, d - 1$;
- (c) $\eta_i = \eta_{-i}$ for each $i = 0, \dots, d - 1$.

Proof. Let ℓ be minimal with $p^\ell \equiv -1 \pmod{d}$.

(a) Since $d > 2$, the choice of ℓ implies that the order of p in \mathbb{Z}_d^* is 2ℓ . Together with $p^n \equiv 1 \pmod{d}$ we obtain $n = 2\ell m$ for some $m \in \mathbb{N}$.

(b) If $p = 2$, then we are done. If $p > 2$, then

$$k = \frac{q-1}{d} = \frac{p^{2\ell m} - 1}{d} = \frac{p^{2\ell m} - 1}{p^\ell + 1} \cdot \frac{p^\ell + 1}{d} = (p^\ell - 1)(p^{2\ell(m-1)} + \dots + p^{2\ell} + 1) \cdot \frac{p^\ell + 1}{d}$$

hence $(p-1) \mid k$. Therefore S_0 is a union of \mathbb{F}_p^* -cosets which implies that any additive character of \mathbb{F}_q is integral on S_0 . This implies that $\chi_1(S_0\alpha^i) \in \mathbb{Z}$ for each i .

(c) This part follows from repeatedly using that $\eta_{pi} = \eta_i$ and the condition $p^\ell \equiv -1 \pmod{d}$. \square

We can now prove the following.

Proposition 3. *Let p be semiprimitive modulo d , and $d > 2$. Then the d -class cyclotomic association scheme on \mathbb{F}_{p^n} is amorphic.*

Proof. We let m, ℓ have the same meaning as in Lemma 1. Since P_0 is an integral matrix, we obtain together with part (c) of Lemma 1 that

$$P_0^* = P_0^\top = \sum_{i=0}^{d-1} \eta_i (C^i)^\top = \sum_{i=0}^{d-1} \eta_i C^{-i} = \sum_{i=0}^{d-1} \eta_{-i} C^{-i} = P_0.$$

Thus we have the equation $P_0^2 = qI_d - kJ_d$. We also note that the row sums of P_0 equal -1 .

First let us show that there exists an integer x such that $(P_0 - xJ_d)^2 = qI_d$. Direct calculations give us

$$P_0^2 - 2xP_0J_d + x^2J_d^2 = qI_d - kJ_d + 2xJ_d + x^2dJ_d = qI_d + (-k + 2x + dx^2)J_d.$$

Now the equation $-k + 2x + dx^2 = 0$ has solutions

$$x = \frac{-2 \pm \sqrt{4 + 4kd}}{2d} = \frac{-1 \pm \sqrt{q}}{d} = \frac{-1 \pm p^{m\ell}}{d}.$$

Since $p^\ell \equiv -1 \pmod{d}$, one of these solutions is an integer. We choose this solution for x .

Next, let u denote the largest integer such that p^u divides all entries of $P_0 - xJ_d$. Then $M := \frac{1}{p^u}(P_0 - xJ_d)$ is an integer circulant matrix satisfying the equation $M^2 = p^{2(m\ell-u)}I_d$. Denote the first row of M by $(\mu_0, \dots, \mu_{d-1})$, i.e., $M = \sum_{i=0}^{d-1} \mu_i C^i$. Clearly $u \leq m\ell$, but let's assume that $u < m\ell$. Then $M^2 \equiv 0 \pmod{p}$. But then also

$$0 \equiv M^p \equiv \sum_{i=0}^{d-1} \mu_i^p C^{ip} \equiv \sum_{i=0}^{d-1} \mu_i C^{ip} \pmod{p}.$$

Because d and p are relatively prime, raising the matrices C^i to the power p just permutes them, so we may conclude that $\mu_i \equiv 0 \pmod{p}$ for all i , and so $M \equiv 0 \pmod{p}$. This however contradicts the maximality of u , so it follows that $u = m\ell$, and hence $M^2 = I_d$.

This implies that $\sum_{i=0}^{d-1} \mu_i^2 = 1$, hence it follows that $M = \pm C^i$ for some i , and so $P_0 = \pm p^u C^i + xJ_d$. It is clear now that (after rearrangement) the eigenmatrix P satisfies the conditions of Proposition 2, so that the association scheme is amorphic. □

Van Lint and Schrijver [42] were the first to use cyclotomic association schemes to construct exotic examples of strongly regular graphs, in their case by fusing relations in the 8-class cyclotomic association scheme on 81 vertices (which itself is not amorphic). It follows from their results that it is possible to fuse this 8-class association scheme into a 3-class amorphic association scheme with valencies 20, 30, and 30. Another amorphic fusion scheme of the 8-class scheme is the 4-class cyclotomic association scheme on \mathbb{F}_{81} .

Also De Lange [41] and Ikuta and Munemasa [34] constructed strongly regular graphs in cyclotomic association schemes. These will play an important role in counterexamples of Ivanov's conjecture in Section 7.5.

Fujisaki [24, 25] constructed amorphic association schemes by fusing relations in the direct product scheme of two pseudo-cyclic association schemes with the same parameters. In case the pseudo-cyclic association schemes are cyclotomic association schemes on \mathbb{F}_q , the constructed association scheme is isomorphic to a fusion scheme of the amorphic $(q+1)$ -class cyclotomic association scheme on \mathbb{F}_{q^2} .

5.5 Partial difference sets

Let H be a finite group of order v . A subset $S \subseteq H$ of cardinality k is called a (v, k, λ, μ) -partial difference set over H if for each $h \in H \setminus \{1\}$ the equation $h = xy^{-1}, x, y \in S$ has λ or μ solutions depending on whether h belongs to S or not. If the set S is symmetric, that is $s^{-1} \in S$ for each $s \in S$, then a partial difference set S gives rise to a strongly regular Cayley graph with parameters (v, k, λ, μ) . The vertices of the graph are the elements of H , and $x, y \in H$ are adjacent if and only if $xy^{-1} \in S$. Note that S is symmetric whenever $\lambda \neq \mu$. We say that a partial difference set is of (negative) Latin square type if the corresponding strongly regular graph is.

Davis and Xiang [20] constructed an infinite series of partial difference sets of negative Latin square type over abelian groups of type $\mathbb{Z}_4^k \otimes \mathbb{Z}_2^{4\ell-4k}, 0 \leq k \leq \ell$. They also constructed several infinite series of amorphic schemes of negative Latin square type over abelian 2-groups of exponent 4. Some of the series of schemes constructed in [20] have unbounded number of classes.

Jørgensen and Klin [38] constructed several partial difference sets (some of Latin square type, and some of negative Latin square type) over a non-abelian group of order 100. They also constructed amorphic 3-class association schemes of Latin square type with valencies 9, 36, 45. The number of non-isomorphic copies mentioned in [38, Corollary 13] was incorrect. The correct number is given in [39, Section 12.6]. In total, there are at least five non-isomorphic schemes of this type, instead of the seven mentioned in [38].

Polhill [48] constructed amorphic association schemes of negative Latin square type with 4 and 3 classes using partial difference sets over abelian 2- and 3-groups.

6 Classification of small amorphic association schemes

By combining the feasible parameter sets of strongly regular graphs of Latin square type and negative Latin square type, we find the feasible parameter sets for amorphic association schemes with at least three classes and at most 49 vertices, as given in Table 1. By $\#$ we denote the number of nonisomorphic such association schemes. As can be seen, we classified all amorphic association schemes on at most 36 vertices (and at least three classes). How we obtained these results will be explained next.

There are two parameter sets for amorphic association schemes of negative Latin square type. The first one is of a 3-class association scheme on 16 vertices, where all graphs are Clebsch graphs. According to [29] there are two such association schemes (one is the cyclotomic association scheme). The second one is a 3-class association scheme on 49 vertices, where all graphs have valency 16. Since there are no such strongly regular graphs, as was proved by Bussemaker, Haemers, Mathon, and Wilbrink [10], there are also no such 3-class association schemes.

Concerning the classification of the amorphic association schemes of Latin square type, we make the following observations. The association schemes of type $L_{1,1}(n)$ are clearly uniquely determined by the parameters. Any association scheme which has the same parameters as the amorphic association schemes of type $L_{|I_1|, \dots, |I_{d-1}|}(n)$, where $|I_i| \leq 2$ for $i = 1, \dots, d-1$ is indeed such an association scheme, i.e. it comes from mutually orthogonal Latin squares, unless maybe when $n = 4$. This is a consequence of the fact that the strongly regular lattice graph $L_2(n)$ is determined by its parameters unless $n = 4$, in which case there is also the so-called Shrikhande graph.

The association schemes of type $L_{1,1,1}(n)$ and $L_{1,2}(n)$ are such association schemes, and they are equivalent to Latin squares (the latter for $n \neq 4$; the association schemes with parameters of $L_{1,2}(4)$ were classified in [29]). Latin squares can be classified according to several rules (cf. [16]). Two Latin squares are called isotopic if there is a bijection between the rows of one square and the rows

v	valencies	#	remarks
4	1, 1, 1	1	$L_{1,1}(2)$
9	2, 2, 4	1	$L_{1,1}(3)$
9	2, 2, 2, 2	1	$L_{1,1,1}(3)$
16	3, 3, 9	1	$L_{1,1}(4)$
16	3, 6, 6	4	[29]
16	5, 5, 5	2	[29]
16	3, 3, 3, 6	2	$L_{1,1,1}(4)$
16	3, 3, 3, 3, 3	1	$L_{1,1,1,1,1}(4)$
25	4, 4, 16	1	$L_{1,1}(5)$
25	4, 8, 12	2	$L_{1,2}(5)$
25	8, 8, 8	2	$L_{2,2,2}(5)$
25	4, 4, 4, 12	2	$L_{1,1,1}(5)$
25	4, 4, 8, 8	2	$L_{1,1,2,2}(5)$
25	4, 4, 4, 4, 8	1	$L_{1,1,1,1,2}(5)$
25	4, 4, 4, 4, 4, 4	1	$L_{1,1,1,1,1,1}(5)$
36	5, 5, 25	1	$L_{1,1}(6)$
36	5, 10, 20	17	$L_{1,2}(6)$
36	5, 15, 15	0	[11]
36	10, 10, 15	0	$L_{2,2}(6)$
36	5, 5, 5, 20	12	$L_{1,1,1}(6)$
36	5, 5, 10, 15	0	$L_{1,1,2}(6)$
36	5, 10, 10, 10	0	$L_{1,2,2,2}(6)$
36	5, 5, 5, 5, 15	0	$L_{1,1,1,1}(6)$
36	5, 5, 5, 10, 10	0	$L_{1,1,1,2,2}(6)$
36	5, 5, 5, 5, 5, 10	0	$L_{1,1,1,1,1,2}(6)$
36	5, 5, 5, 5, 5, 5	0	$L_{1,1,1,1,1,1,1}(6)$
49	6, 6, 36	1	$L_{1,1}(7)$
49	6, 12, 30	324	$L_{1,2}(7)$
49	6, 18, 24	≥ 2	
49	12, 12, 24	≥ 3	$L_{2,2}(7)$
49	12, 18, 18	≥ 3	
49	16, 16, 16	0	no SRG [10]
49	6, 6, 6, 30	147	$L_{1,1,1}(7)$
49	6, 6, 12, 24	≥ 3	$L_{1,1,2}(7)$
49	6, 6, 18, 18	≥ 3	
49	6, 12, 12, 18	4	$L_{1,2,2,3}(7)$
49	12, 12, 12, 12	4	$L_{2,2,2,2}(7)$
49	6, 6, 6, 6, 24	≥ 2	$L_{1,1,1,1}(7)$
49	6, 6, 6, 12, 18	3	$L_{1,1,1,2,3}(7)$
49	6, 6, 12, 12, 12	5	$L_{1,1,2,2,2}(7)$
49	6, 6, 6, 6, 6, 18	1	$L_{1,1,1,1,1,3}(7)$
49	6, 6, 6, 6, 12, 12	3	$L_{1,1,1,1,2,2}(7)$
49	6, 6, 6, 6, 6, 6, 12	1	$L_{1,1,1,1,1,1,2}(7)$
49	6, 6, 6, 6, 6, 6, 6	1	$L_{1,1,1,1,1,1,1,1}(7)$

Table 1: Amorphic association schemes on at most 49 vertices

of the other square, a bijection between the columns of one square and the columns of the other, and a bijection between the symbols of one square and the symbols of the other, such that the three bijections combined map one square to the other. Two Latin square are called conjugate if they come from the same orthogonal array (as described in Section 5.1), where possibly the rows of the array are permuted. Two Latin square graphs are called paratopic (also called main class isotopic, cf. [16]) if one square is isotopic to a conjugate of the other. It is easy to see that paratopy of Latin squares corresponds to isomorphism of association schemes of type $L_{1,1,1}(n)$. Thus the number of nonisomorphic association schemes of type $L_{1,1,1}(n)$ for $n \leq 10$ can be seen from the table of main classes (or main class isotopy classes) of Latin squares in [44].

In [17] it was incorrectly stated that isomorphism of association schemes of type $L_{1,2}(n)$ ($n \neq 4$) is the same as isotopy of the corresponding Latin squares. Consequently wrong numbers of such association schemes for $n = 6, 7, 8$ appeared in the tables of [17]. For the correct numbers we need the classification of Latin squares into so-called types. We say two Latin squares are of the same type if one square is isotopic to the other or its transpose. In [44], the number of types of Latin squares are computed for $n \leq 10$. Each type corresponds to an isomorphism class of the corresponding association schemes of type $L_{1,2}(n)$.

In analyzing the remaining parameter sets we distinguish the ones on 36 vertices from the others. First we look at the ones on 36 vertices. There is no amorphic association scheme on 36 vertices with valencies 5, 15, and 15. This follows from the result by Bussemaker, Haemers, and Spence [11] that there is no strongly regular $(36, 20, 10, 12)$ graph with a spread (a partition of the vertices into 6-cliques). The existence of one of the remaining association schemes on 36 vertices would imply an orthogonal array $OA(6, m)$ with $m \geq 4$. It is well-known that such an orthogonal array does not exist (cf. [1]), hence the remaining association schemes on 36 vertices do not exist.

For the remaining parameter sets we compute the number of association schemes that come from a Desarguesian affine plane $AG(2, q)$, (for $q = 4, 5, 7$). The remaining association schemes on 16 and 25 vertices clearly do come from such an affine plane. For the ones on 49 vertices we can say the following. If an association scheme has the same parameters as an association scheme of type $L_{|I_1|, \dots, |I_{d-1}|}(7)$, where $|I_i| \leq 2$ for $i = 1, \dots, d-1$ and $\sum_{i=1}^{d-1} |I_i| \geq 5$, then it is such an association scheme, and it comes from an $OA(7, m)$, with $m \geq 5$. From [22, Table 27.13] it follows that such an orthogonal array can be completed to an $OA(7, 8)$ (affine plane $AG(2, 7)$), hence the association scheme is of type $L_{|I_1|, \dots, |I_{d-1}|, |I_d|}(7)$ with $\sum_{i=1}^d |I_i| = 8$.

The number of association schemes of type $L_{n_1, \dots, n_d}(q)$, with $\sum_{i=1}^d n_i = q + 1$, coming from the affine plane $AG(2, q)$, is equal to the number of ways the set of parallel classes can be partitioned into parts of sizes n_1, \dots, n_d , i.e. it is equal to the number of orbits of $PGL(2, q)$ acting on partitions of the projective line $PG(1, q)$ into parts of sizes n_1, \dots, n_d (the projective line is a line at infinity of $AG(2, q)$ corresponding to the directions of its parallel classes). It follows that the association schemes of types $L_{1,1,1,1,1}(4)$, $L_{1,1,1,1,1,1}(5)$, $L_{1,1,1,1,2}(5)$, $L_{1,1,1,1,1,1,1}(7)$, $L_{1,1,1,1,1,2}(7)$, and $L_{1,1,1,1,1,3}(7)$ are uniquely determined by the parameters (since $PGL(2, q)$ is 3-transitive on $PG(1, q)$). It is moreover easily checked by hand now that there are 2 association schemes with parameters of type $L_{2,2,2}(5)$ and 2 association schemes with parameters of type $L_{1,1,2,2}(5)$. Using the computer program GAP [26], we computed the number of orbits of $PGL(2, 7)$ acting on partitions of the projective line $PG(1, 7)$. These numbers gave us the lower bounds or exact number of association schemes for the remaining parameter sets on 49 vertices.

7 Strongly regular decompositions of the complete graph

The graphs in an amorphic association schemes decompose the edge set of the complete graph into strongly regular graphs. Such a decomposition is called a strongly regular decomposition of the complete graph, and these were studied in [19]. In this section, we shall mention several results on strongly regular decompositions of the complete graph in relation to amorphic association schemes. Among others, we shall give a new proof of the remarkable fact that such a decomposition with three graphs is always an amorphic association scheme. For decompositions into four (or more) strongly regular graphs this is not the case. For example, consider the (unique) example on 6 vertices, where one graph is the union of two triangles, and the other three are matchings. This example is not an association scheme. The results in this section are partly motivated by Ivanov's (false, cf. Section 7.5) conjecture that if a strongly regular decomposition is an association scheme, then it is amorphic.

7.1 Decompositions into three strongly regular graphs

In 1983, Schwenk [51] posed the question whether the edges of the complete graph on 10 vertices can be decomposed into three Petersen graphs. The answer is negative, and O.P. Lossers and Schwenk [43] gave the following beautiful proof.

Proposition 4. *The complete graph on 10 vertices cannot be decomposed into three Petersen graphs.*

Proof. Suppose there is such a decomposition. Let $A_i, i = 1, 2, 3$ be the adjacency matrices of the three Petersen graphs, then these all have eigenvalues 3, 1, and -2 , with multiplicities 1, 5, and 4, respectively. The all-ones vector \mathbf{j} is a common eigenvector for $A_i, i = 1, 2, 3$ with eigenvalue 3 (the valency of the Petersen graph). Each of the $A_i, i = 1, 2, 3$ has a 5-dimensional eigenspace for eigenvalue 1, and this eigenspace is orthogonal to the all-ones vector. Since the orthogonal complement of the all-ones vector is 9-dimensional, two 5-dimensional spaces contained in this orthogonal complement intersect non-trivially. This implies among others that A_1 and A_2 have a common eigenvector with eigenvalue 1. Since $I + A_1 + A_2 + A_3 = J$ (the all-ones matrix), it follows that A_3 also has this common eigenvector, but with eigenvalue -3 . Since the Petersen graph does not have -3 as an eigenvalue, this gives a contradiction. \square

The simple linear algebraic idea from this proof is one of the ideas which is used in [19] to prove that any decomposition of a complete graph into three strongly regular graphs forms an amorphic association scheme. Here we shall give an entirely different proof of this remarkable result, which was already claimed to be true in [29], but of which no proof was given. Proofs of special cases were given by Rowlinson [50] and independently Michael [45], who showed that any decomposition of a complete graph into three isomorphic strongly regular graphs forms an amorphic association scheme.

Theorem 2. *Let G_1, G_2, G_3 form a strongly regular decomposition of the complete graph. Then G_1, G_2, G_3 form an amorphic 3-class association scheme.*

Proof. Let G_i have adjacency matrix A_i and valency k_i , for $i = 1, 2, 3$, and let v be the number of vertices. Let $\mathbf{A} = \langle I, A_1, A_2, A_3 \rangle = \langle I, A_1, A_2, J \rangle$. To prove the theorem we want to show that \mathbf{A} is an algebra.

First of all, we note that since G_i is strongly regular, it follows that $A_i, A_i^2, A_i J = J A_i \in \mathbf{A}$ for all i . By working our $A_3^2 = (J - I - A_1 - A_2)^2$, it follows that $A_1 A_2 + A_2 A_1 \in \mathbf{A}$. Hence if A_1 and A_2 commute, then \mathbf{A} is an algebra.

So let's assume that A_1 and A_2 do not commute. Now let $\mathbf{B} = \langle I, A_1, A_2, J, A_1 A_2 \rangle$. We claim that \mathbf{B} is an algebra. It is easy to see that $A_i A_j \in \mathbf{B}$ for all i, j , and that $A_1 A_2 J = J A_1 A_2 \in \mathbf{B}$. Now

$A_1 \cdot A_1 A_2 \in \mathbf{A} A_2 \subseteq \mathbf{B}$, and similarly $A_1 A_2 \cdot A_2 \in \mathbf{B}$. Also, $A_2 \cdot A_1 A_2 = (A_1 A_2 + A_2 A_1) A_2 - A_1 A_2 A_2 \in \mathbf{A} A_2 + \mathbf{B} = \mathbf{B}$, and similarly $A_1 A_2 \cdot A_1 \in \mathbf{B}$. Hence \mathbf{B} is a non-commutative algebra, and its dimension is 5.

Now we need some facts from abstract algebra (for details, we refer the reader to [40, Ch. 10]). An algebra is semi-simple if its radical is zero. In order to prove that \mathbf{B} is semi-simple we shall use that the radical is a nilpotent ideal of the algebra. Let $X \in \text{rad}(\mathbf{B})$. Since $X^* \in \mathbf{B}$, where X^* is the complex conjugate transpose of X , and since $\text{rad}(\mathbf{B})$ is an ideal in \mathbf{B} , it follows that $XX^* \in \text{rad}(\mathbf{B})$. Since the radical is nilpotent, it follows that $(XX^*)^n = 0$ for some n . But this implies that $\text{tr}(XX^*) = 0$, and hence $X = 0$. So we conclude that \mathbf{B} is a semi-simple algebra.

Any semi-simple algebra over the complex numbers \mathbb{C} is isomorphic to a direct sum of complete matrix algebras over \mathbb{C} (cf. [40, Thm. 10.4.3]), hence so is \mathbf{B} . Since \mathbf{B} is non-commutative of dimension 5, it follows that \mathbf{B} is isomorphic to $\mathbb{C} \oplus M_2(\mathbb{C})$. Here $M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices over \mathbb{C} .

It follows that \mathbf{B} has exactly two irreducible representations ρ_0 and ρ_1 of degrees 1 and 2, respectively (a representation of degree n is a homomorphism from \mathbf{B} into $M_n(\mathbb{C})$). Since every matrix X in \mathbf{B} has constant row sums $r(X)$, the map sending X to $r(X)$ defines the representation ρ_0 of degree one. Let ρ be the standard representation of \mathbf{B} of degree v (i.e. any matrix X in \mathbf{B} is mapped to itself). Then ρ_0 has multiplicity 1 in ρ (because for example the all-ones matrix J has eigenvalue $r(J)$ with multiplicity one), and therefore $\rho \cong \rho_0 + m\rho_1$. From this it follows that $v = 1 + 2m$.

Now let χ, χ_0, χ_1 be the characters of the representations ρ, ρ_0, ρ_1 , respectively (i.e. $\chi(X) = \text{tr}(\rho(X))$, etc.). Then $0 = \chi(A_i) = \chi_0(A_i) + m\chi_1(A_i) = k_i + m\chi_1(A_i)$, and since $\chi_1(A_i)$ is an algebraic integer, it follows that k_i is a multiple of m , for $i = 1, 2, 3$. But then $v = 1 + k_1 + k_2 + k_3 \geq 1 + 3m$, which contradicts the fact that $v = 1 + 2m$. Hence the proof is finished. \square

Our next goal is to consider other strongly regular decompositions of the complete graph, and to investigate when these form an association scheme, and if they do, if they are amorphic.

Since amorphic association schemes are strongly regular decompositions of the complete graph into graphs of (negative) Latin square type, we consider these first.

7.2 Decompositions into (negative) Latin square type graphs

It is shown in [19] that besides the amorphic association schemes there are no other strongly regular decompositions of the complete graph into strongly regular graphs, that are either all of Latin square type, or all of negative Latin square type. Its proof basically consist of considering dimensions of common eigenspaces. It generalizes the result by Ito, Munemasa, and Yamada [35] who showed that there are no other such association schemes.

Theorem 3. *Let G_1, G_2, \dots, G_t form a strongly regular decomposition of the complete graph, with all graphs G_i of Latin square type or all of negative Latin square type. Then the decomposition forms an amorphic association scheme.*

In [19], an example of a strongly regular decomposition where some graphs are of Latin square type and the others of negative Latin square type is given. This example does not even form an association scheme.

7.3 Strongly regular decompositions and fusions

An amorphic association scheme is an association scheme for which each fusion is also an association scheme. As an analogue, we may consider strongly regular decompositions for which each fusion is also a strongly regular decomposition. It is perhaps not surprising that these are precisely the amorphic association schemes. However, we can prove the following much stronger result.

Theorem 4. *Let G_1, G_2, \dots, G_t form a strongly regular decomposition of the complete graph, such that the union of any two graphs G_i and G_j is also strongly regular. Then the decomposition forms an amorphic association scheme.*

Proof. Fix arbitrary i and j , and consider the graphs G_i and G_j . Because the union of these two graphs is strongly regular, also its complement is strongly regular. Thus $\{G_i, G_j, \cup_{h \neq i, j} G_h\}$ is a strongly regular decomposition of the complete graph. By Theorem 2 it forms an amorphic association scheme, and so by Theorem 1, G_i and G_j are both of Latin square type or both of negative Latin square type. Since i and j are arbitrary, it follows that all graphs are of Latin square type, or all graphs are of negative Latin square type. Theorem 3 finishes the proof. \square

7.4 Decompositions into four strongly regular graphs

In general it seems hard to classify the decompositions into four strongly regular graphs if we do not assume that the graphs (their adjacency matrices) commute. This may be surprising since we only know of one non-commutative example, the earlier mentioned one on 6 vertices.

In the case where at least three of the four graphs are disconnected, it is however possible to give a classification (cf. [19]). Besides the amorphic association schemes, and the example on 6 vertices, there is only one other possibility: one graph is a complete multipartite graph $K_{4,4,\dots,4}$, and the other three graphs are matchings. These four graphs form an association scheme, the wreath product of a complete graph and $L_{1,1,1}(2)$. We note that this association scheme is not amorphic.

Proposition 5. *Let G_1, G_2, G_3, G_4 form a strongly regular decomposition of the complete graph, for which at least three of the four graphs are disconnected. Then the decomposition is the example on 6 vertices in which three graphs are matchings, and the fourth graph consists of two disjoint triangles; or the wreath product (association scheme) of a complete graph and $L_{1,1,1}(2)$; or an association scheme of type $L_{1,1,1}(n)$, $n > 2$.*

In any other example of a decomposition into four strongly regular graphs at least two of the graphs must be primitive. The following parameter sets seem feasible for such a decomposition:

on $v = 40$ vertices: three graphs have parameters $(40, 12, 2, 4)$, the remaining graph is a disjoint union of 4-cliques;

on $v = 45$ vertices: three graphs have parameters $(45, 12, 3, 3)$, the remaining graph is a disjoint union of 9-cliques;

on $v = 50$ vertices: three graphs have parameters $(50, 7, 0, 1)$, i.e. they are Hoffman-Singleton graphs, the remaining graph has parameters $(50, 28, 15, 16)$.

We leave the construction of these decompositions as an open problem. In any case, the parameters are such that these decompositions will not be commutative. This follows from Theorem 5 in [19].

7.5 A.V. Ivanov's conjecture

A.V. Ivanov [36, Problem 1.3] conjectured that any association scheme consisting of strongly regular graphs only must be amorphic. The first counterexamples, given in [18], were constructed from

affine spaces $AG(n, q)$, $n > 2$, similarly as the complete affine association scheme was constructed in Section 3. The above mentioned wreath product of a complete graph and $L_{1,1,1}(n)$ provides other examples. Both families of counterexamples are imprimitive, but in [19] a first primitive counterexample was constructed. This counterexample is constructed as a fusion scheme of the cyclotomic 45-class association scheme on \mathbb{F}_{4096} . Let α be a primitive element in this field satisfying $\alpha^{12} = \alpha^6 + \alpha^4 + \alpha + 1$. Let two distinct vertices be adjacent in H_j if their difference is of the form α^{45i+j} for some i ($j = 1, \dots, 45$). De Lange [41] found that $G_2 = H_{45} \cup H_5 \cup H_{10}$ is strongly regular graph with eigenvalues $k = 273$, $r = 17$ and $s = -15$. The graphs $G_3 = H_{15} \cup H_{20} \cup H_{25}$ and $G_4 = H_{30} \cup H_{35} \cup H_{40}$ are isomorphic to G_2 , and the union of the three graphs G_2, G_3, G_4 is one of the graphs in the cyclotomic 5-class amorphic association scheme on \mathbb{F}_{4096} . Hence the complement G_1 of this union is strongly regular, with eigenvalues $k_1 = 3276$, $r_1 = 12$ and $s_1 = -52$. These four strongly regular graphs G_1, \dots, G_4 form a primitive 4-class association scheme which is not amorphic.

Ikuta and Munemasa [34] constructed a similar counterexample using the cyclotomic 75-class association scheme on \mathbb{F}_{220} .

8 Miscellaneous

A pseudo-automorphism of an association scheme is an automorphism of its Bose-Mesner algebra with respect to both ordinary and Hadamard multiplication. Ikuta, Ito, Munemasa [33] mention that a pseudo-cyclic d -class association scheme is amorphic if and only if its group of pseudo-automorphisms is the symmetric group S_d . A d -class pseudo-cyclic association scheme is called of G -type if the principal part of the eigenmatrix P (i.e., its lower right $d \times d$ submatrix) corresponds to the multiplication table of the group G . Ikuta, Ito, Munemasa [33] showed that if G is an elementary abelian 2-group, then any association scheme of G -type is amorphic. Bannai [2] gave a short proof of this result.

Ito, Munemasa, and Yamada [35] gave a more general definition of amorphic association schemes (and a more restrictive definition of fusions). In the symmetric case the two definitions are equivalent, but the more general definition also allows for examples in the non-symmetric case.

Barghi and Ponomarenko [49] introduced amorphic C-algebras by axiomatizing the property that each partition of a standard basis leads to a fusion algebra. They showed that each amorphic C-algebra is determined up to isomorphism by the multiset of its degrees and an additional integer $\epsilon = \pm 1$ (reflecting the positive or negative Latin square type). They also obtained the intersection numbers of amorphic association schemes as in Corollary 1.

Related to one of the open problems from Section 7.4, and the problem of decomposing K_{10} into three Petersen graphs is the following problem. Is it possible to decompose the complete graph on 50 vertices into 7 Hoffman-Singleton graphs? An attempt to solve this problem by Šiagiová and Meszka [52] resulted in a packing of 5 Hoffman-Singleton graphs.

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