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ON THE EQUIVALENCE BETWEEN SOME LOCAL AND GLOBAL CHINESE POSTMAN AND TRAVELING SALESMAN GRAPHS.

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Abstract

A connected graph $G = (V, E)$, a vertex in V and a non-negative weight function defined on E can be used to induce Chinese postman and traveling salesman (cooperative) games. A graph $G = (V, E)$ is said to be locally (respectively, globally) Chinese postman balanced (respectively, totally balanced, submodular) if for at least one vertex (respectively, for all vertices) in V and any non-negative weight function defined on E, the corresponding Chinese postman game is balanced (respectively, totally balanced, submodular). Local and global traveling salesman balanced (respectively, totally balanced, submodular) graphs are similarly defined.

In this paper we study the equivalence between local and global Chinese postman balanced (respectively, totally balanced, submodular) graphs, and between local and global traveling salesman submodular graphs.

KEYWORDS: Cooperative game, Chinese postman, traveling salesman, core, submodularity.

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1 Introduction

We study in this paper two classes of cooperative combinatorial games: Chinese postman games (cf. Hamers et al. (1999)) and traveling salesman games (cf. Tamir (1989) and Potters et al. (1992)). For both these games, balancedness, totally balancedness and submodularity were characterized in terms of some specific classes of graphs. To present these characterizations, we first need to recall some game theoretical notions.

A *cooperative game* is an ordered pair (N, c), where $N = \{1, 2, .., n\}$ is a finite set of players and $c : 2^N \to I\!R$ is a map that assigns to each coalition $S \subseteq N$ a real number $c(S)$, called the cost of S, such that $c(\emptyset) = 0$. In this paper we will think of (N, c) as a *cost* game, in which case $c(S)$ is the cost incurred to members of S when they create their own coaltion. One of the most prominent solution concepts in cooperative game theory is the core of a game. It consists of all vectors which distribute the cost incurred to N, $c(N)$, among the players in such way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. That is, a vector x is in the core of a game (N, c) if[1] $x(N) = c(N)$ and $x(S) \leq c(S)$ for all $S \subset N$. A cooperative game whose core is not empty is said to be *balanced*, and if the core of any subgame of it is nonempty, it is said to be *totally balanced*. A well-known class of (totally) balanced games is the class of submodular games. A game (N, c) is called *submodular* if $c(S \cup T) + c(S \cap T) \leq c(S) + c(T)$ for any $S, T \subseteq N$.

Submodular games are known to have nice properties, in the sense that some solutions concepts for these games coincide and others have intuitive description. For example, for submodular games the core is equal to the convex hull of all marginal vectors (cf. Shapley (1971) and Ichiishi (1981)), and , as a consequence, Shapley value is the barycentre of the core (Shapley (1971)). Further, the bargaining set and the core coincide and the kernel coincides with the nucleolus (Maschler et al. (1972)). Moreover, some of these solution concepts can be computed more efficiently for submodular games. For example, each marginal vector, which is an extreme point in the core, can be computed in linear time, the nucleolus can be computed in a strongly polynomial time (Kuipers (1996)) and the τ-value can be easily calculated (Tijs (1981)).

Next, we recall the definitions of the Chinese postman and the traveling salesman games. Let $G = (V(G), E(G))$ be a connected undirected (resp., strongly connected directed) graph

[1] For a vector $x \in I\!R^N$ and $T \subseteq N$ we let $x(T) = \sum_{j \in T} x_j$.

in which $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges (arcs). Further, let $v_0 \in V(G)$ be a fixed vertex, which will be referred to as the *warehouse*. Let $t : E \to [0, \infty)$ be a non-negative weight function defined on the edges (arcs) of E.

Chinese postman (CP) games[2] arise from situations $\Gamma_{CP} = (E(G), (G, v_0), t)$ in which the players are identified with the edges (arcs). A (directed) S-tour in G of a non-empty coalition $S \subseteq E(G)$ is a closed walk that starts at the warehouse v_0 and visits each edge (arc) that is included in S at least once. Then, the cost incurred to a non-empty coalition S in the CP game is defined as the cost of a (directed) minimum weight S-tour. Note that determining the cost of the grand coalition N is equivalent to solving the Chinese postman problem on G (cf. Edmonds and Johnson (1973)).

Traveling salesman (TS) games arise from situations $\Gamma_{TS} = (V^-(G), (G, v_0), t)$ in which the players are identified with the vertices, except v_0 (i.e. $V^-(G) = V(G) \backslash \{v_0\}$). A (directed) S-Steiner tour in G of a non-empty coalition $S \subseteq V^-(G)$ is a (directed) cycle, not necessarily simple, that includes all vertices of $S \cup \{v_0\}$. Then, the cost incurred to a non-empty coalition S in the TS game is defined as the cost of a minimum weight (directed) S-Steiner tour. Note, that determining the cost incurred to a coalition S is equivalent to solving a Steiner TSP problem (cf. Lawler et al. (1985)).

Next, we will briefly introduce some classes of graphs related to Chinese postman and traveling salesman games. An undirected (resp., directed) graph $G = (V(G), E(G))$ is said to be *globally Chinese postman (CP) balanced (resp., totally balanced, submodular)* if the induced CP game is balanced (resp., totally balanced, submodular) for any location of the warehouse (i.e. for any $v_0 \in V(G)$) and any non-negative weight function defined on the edges (arcs). *Globally traveling salesman (TS) submodular* undirected (resp., directed) graphs are similarly defined. Characterizations of globally CP balanced (resp., totally balanced, submodular) graphs and globally TS submodular graphs have already been established. Explicitly, Granot et al. (1999) have shown that an undirected graph is globally CP submodular if and only if it is globally CP totally balanced, which holds if and only if it is weakly[3] cyclic. They have further shown that an undirected graph G is globally CP balanced if and only if it is weakly[4] Eulerian. In contrast with the undirected case, Granot et al. (1999) have

[2] Hamers et al. (1999) introduced these games and refer to them as delivery games.
[3] A graph G is weakly cyclic if each biconnected component thereof is a circuit.
[4] A graph is weakly Eulerian if each biconnected component thereof is Eulerian.

shown that a directed graph is globally CP submodular if and only if it is weakly cyclic[5], and that any strongly connected directed graph is globally CP balanced.
Herer and Penn (1995) proved that an undirected graph G is globally TS submodular if and only if G can be obtained by 1-sums of copies of K_4 and outerplanar graphs. Finally, Granot et al. (2000) showed that a directed graph G is globally TS submodular if and only if G is a 1-sum of harmonic digraphs[6] each of which is outerplanar with a directed cycle on its outer boundary.

Requiring graphs to satisfy the various properties of balancedness, totally balancedness and submodularity globally, for all vertices in the graph G, may be unnecessarily restrictive. Indeed, to the extent that the location of the warehouse in G can be chosen, it would suffice to find a single vertex in G for which the induced CP and TS games have the desired properties. For this reason, we study in this paper the relationship between graphs that satisfy the various properties locally and globally. An undirected (resp., directed) graph $G = (V(G), E(G))$ is said to be *locally Chinese postman (CP) balanced (resp., totally balanced, submodular)* if the induced CP game is balanced (resp., totally balanced, submodular) for at least one location of the warehouse and any non-negative weight function defined on the edges (arcs). Undirected (resp., directed) *locally traveling salesman (TS) submodular* graphs are similarly defined.

In this paper we show that the local and global requirements are equivalent for undirected CP balanced, totally balanced and submodular graphs, for directed CP balanced graphs and for undirected TS submodular graphs. For the directed CP and TS cases, it is shown that the class of locally CP (resp., TS) submodular graphs properly contains the class of globally CP (resp., TS) submodular graphs.

2 The undirected CP case

In this section we show that an undirected connected graph is locally CP balanced if and only if it is a weakly Eulerian graph. Moreover, it is shown that an undirected connected

[5] A directed weakly cyclic graph is a 1-sum of directed circuits, where a 1-sum of a graph G and H is defined as the graph derived from G and H by coalescing one vertex in G with another vertex in H.

[6] A digraph is said to be harmonic if each pair of directed circuits therein visit their common vertices in the same order.

graph is locally CP submodular if and only if it is locally CP totally balanced, which holds only if it is weakly cyclic.

Theorem 2.1 *The class of undirected locally CP balanced graphs coincides with the class of undirected globally CP balanced graphs.*

PROOF: From Granot et al. (1999), an undirected globally CP balanced graph is weakly Eulerian. Clearly, a globally CP balanced graph is locally CP balanced. So, let $G = (V(G), E(G))$ be a locally CP balanced graph and assume, on the contrary, that G is not weakly Eulerian. Then G contains a biconnected component, say C, that is not Eulerian. Thus, there exists a vertex w in C whose degree, k, in C is odd and larger than 3. Let $E(w)$ be the edge set that is incident to w in C. The removal of $E(w)$ divides G to two connected components, say $G_1 = (V(E_1), E_1)$ and $G_2 = (V(E_2), E_2)$, where $V(E_j), j = 1, 2$ are the vertices spanned by the edge set $E_j, j = 1, 2$, respectively. Without loss of generality, we assume that $(E(C)\backslash E(w)) \subset E_2$ (see Figure 2.1). Observe that $V(E_1)$ could possibly consist of the single vertex w.

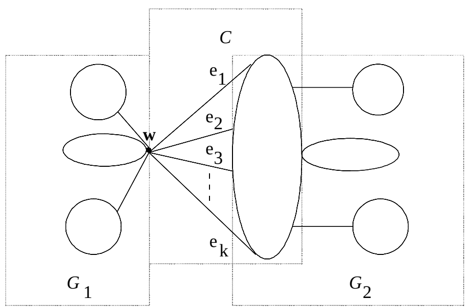

Figure 2.1: A non-Eulerian component C of G.

Consider the edge weight function t in which the costs of all edges in $E(w)$ are 1 and the costs of all other edges are zero. Let $v_0 \in V(E_i), i \in \{1, 2\}$. Then, we claim that the Chinese postman game $(E(G), c)$ corresponding to $(E(G), (G, v_0), t)$ has an empty core. Indeed, if the core is not empty, then there exists a vector x, $x \in I\!R^N$, such that

$$\begin{array}{lcll}
x(E(G)) & = & c(E(G)) = k+1 & \\
x(\{e_1, e_2, E_j\}) & \leq & c(\{e_1, e_2, E_j\} = 2, & j \in \{1,2\}, \quad j \neq i \\
x(\{e_2, e_3, E_j\}) & \leq & c(\{e_2, e_3, E_j\} = 2, & j \in \{1,2\}, \quad j \neq i \\
x(e_j, e_{j+1}) & \leq & c(\{e_j, e_{j+1})\}) = 2, & \text{for all } \; j = 3, 4, ..., k-1 \\
x(e_1, e_k) & \leq & c(\{e_1, e_k\}) = 2 & \\
x(E_i) & \leq & c(E_i) = 0 & .
\end{array} \qquad (1)$$

Summing the inequalities in (1) we obtain that

$$2x(E(G)) \leq 2k < 2(k+1) = 2c(E(G)).$$

We have obtained a contradiction, since it was assumed that $x(E(G)) = c(E(G))$, and we conclude that $(E(G), c)$ has an empty core. Since this result is independent of the choice of v_0, we can conclude that G is not locally CP balanced. □

Theorem 2.2 *Let G be a connected undirected graph. Then the following statements are equivalent:*
(i) G is globally CP submodular.
(ii) G is locally CP submodular.
(iii) G is globally CP totally balanced.
(iv) G is locally CP totally balanced.

PROOF:
By Granot et al. (1999), (i) and (iii) are equivalent, and by definition we have that (i) implies (ii). From Shapley (1971) it follows that (ii) implies (iv). Thus, it remains to show that (iv) implies (iii). By Granot et al. (1999), a globally CP totally balanced graph is weakly cyclic. So, let G be a locally CP totally balanced graph and assume, on the contrary, that G is not weakly cyclic. Then, G contains a connected subgraph $G^* = (V(G^*), E(G^*))$ of the form shown in Figure 2.2.

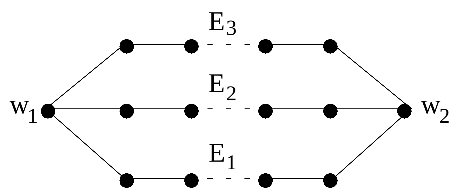

Figure 2.2: The graph G^* which is not weakly cyclic.

Let $E(G^*) = E_1 \cup E_2 \cup E_3$, where E_1, E_2 and E_3 are the edges as depicted in Figure 2.2, and let w_1, w_2, as indicated in Figure 2.2, be the vertices in G^* of degree 3. Let v_0 be somewhere located in G, and let $(E(G), c)$ be the CP game that arises from the situation $(E(G), (G, v_0), t)$. Since G is connected, there exists a path P_1 from v_0 to some vertex $v \in V(G^*)$ such that no other vertex of $V(G^*)$ is contained in P_1. Note, that P_1 consists only of v_0, if $v_0 \in V(G^*)$. If $v \neq w_1, w_2$, let P_2 be a path from v to w_1 that only consists of edges in E_j for some $j \in \{1, 2, 3\}$. Consider the weight function t, where $t(e) = 0$ for all edges $e \in E(P_1) \cup E(P_2)$, $t(E_j \backslash E(P_2)) = 1$ for all $j = 1, 2, 3$ and $t(e) = 100$ for all other edges, where $E(P_j)$ is the edge set of $P_j, j \in \{1, 2\}$. We claim that the core of the subgame $(E(G^*), c_{E(G^*)})$ is empty. Indeed, if the core is not empty, then there exists a vector x, $x \in I\!R^N$, such that

$$\begin{aligned}
x(E(G^*)) &= c(E(G^*)) = 4 \\
x(E_1 \cup E_2) &\leq c(E_1 \cup E_2) = 2 \\
x(E_1 \cup E_3) &\leq c(E_1 \cup E_3) = 2 \qquad (2) \\
x(E_2 \cup E_3) &\leq c(E_2 \cup E_3) = 2.
\end{aligned}$$

Summing the inequalities in (2) we obtain that $2x(E(G^*)) \leq 6 < 8 = 2(E(G^*))$, which is a contradiction, since it was assumed that $x(E(G^*)) = c(E(G^*))$, and we conclude that $(E(G^*), c)$ has an empty core. Hence, $(E(G), c)$ is not totally balanced. Since this result is independent of the choice of v_0, we have reached a contradiction since it was assumed that G is locally CP totally balanced, and the proof of Theorem 2.2 is complete. □

3 The directed CP case

In this section we show that any strongly connected directed graph is locally CP balanced, and that the class of directed globally CP submodular graphs is properly contained in the class of directed locally CP submodular graphs.

The first result of this section follows immediately from Granot et al. (1999), since they showed that any strongly connected directed graph is globally CP balanced.

Proposition 3.1 *A strongly connected directed graph is locally CP balanced.*

Granot et al. (1999) proved that a strongly connected directed graph is globally CP submodular if and only if it is directed weakly cyclic. The next proposition shows that this statement does not hold for directed locally CP submodular graphs.

Proposition 3.2 *The class of directed globally CP submodular graphs is properly contained in the class of directed locally CP submodular graphs.*

PROOF: It is sufficient to provide an example of a directed locally CP submodular graph that is not directed weakly cyclic. Granot et al. (1999) showed that a strongly connected graph that is not weakly cyclic contains vertices w_1, w_2 and three internally vertex-disjoint directed paths $P_1 : w_1 \to w_2$, $P_2 : w_1 \to w_2$ and $P_3 : w_2 \to w_1$. Let G consist of these three paths. Let $v_0 \neq w_1, w_2$ be a vertex in P_1, and let E_1, E_2, E_3 and E_4 be the sets of arcs contained in, respectively, the subpath of P_1 from w_1 to v_0, the subpath of P_1 from v_0 to w_2, P_2 and P_3. Let $(E(G), c)$ be the CP game corresponding to $(E(G), (G, v_0), t)$. Then, $c(S) = c(E(G))$ if $S \cap E_3 \neq \emptyset$ and $c(S) = c(E(G)) - t(E_3)$ if $S \cap E_3 = \emptyset$. Now, it is straightforward to check that $(E(G), c)$ is a submodular game. Hence, G is a directed locally CP submodular graph. □

4 The undirected TS case

In this section we show that an undirected locally TS submodular graph can be obtained by 1-sums of copies of K_4 and outerplanar graphs. Before we state the main result of this section, we recall the notion of a vertex-cutset. If v_s and v_t are two vertices in a graph G, then an $s - t$ vertex-cutset is a set of vertices whose removal from G, together with the edge (v_s, v_t), if such an edge exists, results with a disconnected graph where v_s and v_t are not contained in the same component. A *minimal* $s-t$ vertex-cutset does not contain a proper subset which is an $s-t$ vertex-cutset. The *cut condition*, as introduced by Here and Penn (1995), can be stated as follows: for all pairs of vertices v_s and v_t in G, every minimal $s-t$ vertex-cutset has a cardinality of at most two. Herer and Penn (1995) proved that a graph G is globally TS submodular if and only if G satisfies the cut condition.

Theorem 4.1 *An undirected graph is locally TS submodular if and only if it is globally TS submodular.*

PROOF: Clearly, a globally TS submodular graph is locally TS submodular. From Herer and Penn (1995) it follows that it is sufficient to show that a locally TS submodular graph satisfies the cut condition. So, let G be a locally TS submodular graph and assume, on the contrary, that it does not satisfy the cut condition. Hence, G has a minimal $s-t$ vertex-cutset K of cardinality greater or equal to three. Fix K, v_s and v_t, and let $(V^-(G), c)$ be the TS game corresponding to $(V^-(G), (G, v_0), t)$. We need to consider the following four exhaustive cases.

Case 1: $v_0 \in K$.

Let v_0, v_1, v_2 be three distinct vertices in K. Define the weight function t as follows: all edges not incident to any vertex in K have weight zero, all edges incident to precisely one vertex in K have weight one, and all edges incident to two vertices in K have weight 2. The edge (v_s, v_t), if such an edge exists, has weight two (see Figure 4.1).

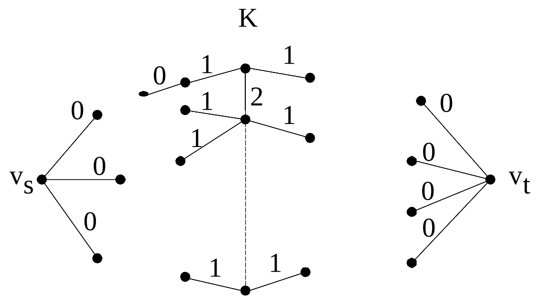

Figure 4.1: The graph G and the cutset K.

Let $S = \{v_s, v_t\}, L = \{v_1\}$ and $M = \{v_2\}$. Then, by definition of t, it is easy to verify that $c(S \cup L \cup M) \geq 6$. Further, the minimality of K guarantees that for every vertex in K there exists an $s - t$ path that traverses that vertex and not any other vertex in K, and that every non-trivial[7] $s - t$ path traverses some vertex in K. This implies that $c(S) = c(S \cup L) = c(S \cup M) = 4$. Thus, $c(S \cup L \cup M) - c(S \cup L) > c(S \cup M) - c(S)$. Hence, $(V^-(G), c)$ is not submodular.

Case 2: $v_0 \notin K$ and any path from v_0 to v_s or from v_0 to v_t contains a vertex of K. Let v_1, v_2, v_3 be three distinct vertices in K. Let P be a path from v_0 to v_3, which does not contain any other vertex of K. Define the weight function, t, as it is done in Case 1, except that $t(e(v_3)) = 0$, where $e(v_3)$ is that edge in P that is incident to v_3, and let the sets S, L and M be defined as in Case 1. As a consequence of our assumption on v_0 in this case, and since the cost of the path P is zero, one can easily verify that the values of the coalitions $S, S \cup L$, $S \cup M$ and $S \cup L \cup M$ are the same as in Case 1. Hence, $(V^-(G), c)$ is not submodular.

Case 3: $v_0 \notin K$ and there exists a path from v_0 to v_t that is vertex-disjoint from K. Let v_1, v_2, v_3 be three distinct vertices in K. Let the weight function be as defined in Case 1, and let $S = \{v_s, v_2\}$, $L = \{v_1\}$ and $M = \{v_3\}$. Similarly, as it was in Case 1 we find that $(V^-(G), c)$ is not submodular.

[7]A non-trivial path has two or more edges.

Case 4: $v_0 \notin K$ and there exists a path from v_0 to v_s that is vertex-disjoint from K. Let v_1, v_2, v_3 be three distinct vertices in K. Let the weight function be as defined in Case 1, and let $S = \{v_t, v_2\}$, $L = \{v_1\}$ and $M = \{v_3\}$. Similarly, as it was done in Case 1, we find that $(V^-(G), c)$ is not submodular.

Thus, for all possible locations of v_0, the associated game $(V^-(G), c)$ is not TS submodular, contradicting our assumption that G is locally TS submodular. Therefore, we can conclude that G satisfies the cut condition. □

Clearly, since a globally TS submodular graph can be constructed by 1-sums of copies of K_4 and outerplanar graphs (Herer and Penn (1995)), it follows from Theorem 4.1, that a locally TS submodular graph can be similarly constructed.

5 The directed TS case

In this section we show that the class of directed globally TS submodular graphs is properly contained in the class of directed locally TS submodular graphs.

First, we define two simple directed graphs F_1 and F_2, as shown in Figure 5.1 below.

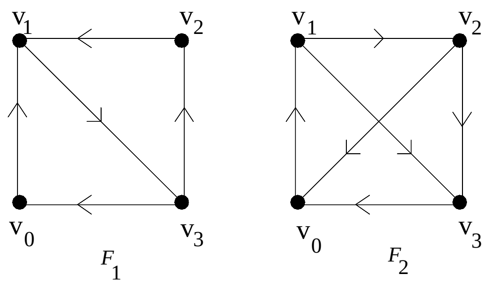

Figure 5.1: The graphs F_1 and F_2.

The following Theorem, due to Granot et al. (2000), characterizes directed globally TS submodular graphs.

Theorem 5.1 *Let G be a strongly connected directed graph. Then the following are equivalent:*
(i) G is globally TS submodular.
(ii) G does not contain a subdivision of F_1 and F_2.
(iii) G is 1-sum of harmonic digraphs, each of which is outerplanar with a directed cycle on its outer boundary.

The next proposition shows that this statement does not hold for directed locally TS submodular graphs.

Proposition 5.2 *The class of directed globally TS submodular graphs is properly contained in the class of directed locally TS submodular graphs.*

PROOF: From Theorem 5.1 it follows that it is sufficient to show that the directed graph F_1 is locally TS submodular. Let $(V^-(G), c)$ be the TS game that arises from the situation $(V^-(G), (G, v_0), t)$. Then, for any weight function we have $c(S) = c(V^-(G))$ if $v_2 \in S$ and $c(S) \leq c(V^-(G))$ if $v_2 \notin S$. It is straightforward to verify that $(V^-(G), c)$ is submodular. Hence F_1 is locally TS submodular. □

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