Link Formation in Cooperative Situations

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Abstract : In this paper we study the endogenous formation of cooperation structures or communication graphs between players in a superadditive TU game. For each cooperation structure that is formed, the payoffs to the players are determined by an exogenously given solution. We model the process of cooperation structure formation as a game in strategic form. It is shown that several equilibrium refinements predict the formation of the complete cooperation structure or some structure which is payoff-equivalent to the complete structure. These results are obtained for a large class of solutions for cooperative games with cooperation structures. A by-product of our analysis is a characterization of the class of weighted Myerson values.

1 Introduction

Usually, cooperative game theory is concerned with predicting how rational players will distribute the gains that are obtained through cooperation. The standard approach in the literature is to represent the underlying situation as a game in characteristic function form. A *solution concept* specifies the distribution of payoffs for each game. This formulation (usually) either implicitly assumes that the grand coalition will form, or specifies an *exogenous* coalition structure. However, the distribution of payoffs will depend on the structure of coalitions which form, since this will typically determine the total amount that is available for distribution. Moreover, the eventual coalitional structure itself will usually be influenced by what players expect to get in different coalitions. Hence, the ideal approach is one in which the coalition structure as well as the distribution of payoffs are determined *simultaneously*.

It is natural in this context to adopt the so-called *Nash program*, and try and support the prediction of any endogenous theory of coalition formation and associated solution concept as a noncooperative equilibrium outcome of a 'larger' game in which the negotiation process is embedded. Since the seminal work of Rubinstein (1982), there have been a number of papers in this tradition. Of particular relevance for present purposes are Binmore (1985), Chatterjee et al. (1993), Gul (1989), Perry and Reny (1994) and Selten (1981), where the negotiation process associated with characteristic function games is explicitly modelled. While Gul (1989) and Perry and Reny (1994) derived negotiation processes leading up to specific solution concepts (the Shapley value and the core respectively), Chatterjee et al. (1993) formulated a generalization of the Rubinstein alternating offers model to represent the negotiation process. Amongst other results, they also showed that the grand coalition need not always form even in strictly superadditive games. Of course, all these papers provide an endogenous theory of coalition formation as well as a prediction about the distribution of payoffs.[1]

There are two points of departure from this literature in the current paper. First, we focus attention on Myerson's (1977) *cooperation structures*[2], rather than coalition structures. A cooperation structure is a graph whose vertices are identified with the players. A link between two players means that these players can carry on meaningful direct negotiations with each other. Notice that a coalition structure is a special kind of cooperation structure where two members i and j are linked if and only if they are in the same coalition.[3] Second, following Aumann and Myerson (1988), we model situations in which the eventual distribution of payoffs is determined in *two* distinct stages or periods. The first period is devoted to link formation only. During this period, the players cannot enter into binding agreements of any kind, either on the nature of the link formation, or on the subsequent division of payoffs. In the second period, no new links can be formed, but players negotiate over the division of the payoff, *given* the cooperation structure which has formed in the first stage.

The goal of this paper is to analyse the endogenous formation of cooperation structures in this setting. In order to do this, we assume that in the first stage of the above process, agents' decisions on whether or not to form a link with other agents can be represented as a game in strategic form.[4] In the link

[1]See also Sengupta and Sengupta (1994), who determine the coalition structure and distribution of payoffs simultaneously, although not in the tradition of the Nash program.

[2]See van den Nouweland (1993) for a survey of recent research on games with coooperation structures.

[3]Aumann and Myerson (1988) give examples of negotiation situations which can be modelled by cooperation structures, but not by coalition structures.

[4]This game was originally introduced by Myerson (1991) (p. 448). See also Hart and Kurz (1983), who discuss a similar strategic-form game in the context of the endogenous formation of coalition structures. In contrast, Aumann and Myerson (1988) model the process of link

formation game, each player announces a set of players with whom he or she wants to form a link. A link is formed between i and j if both players want the link. Given the announcements of the n players, this specification gives the cooperation structure. Suppose there is a rule or solution which determines a distribution of payoffs for each cooperation structure. This, then, also gives the payoff function of the strategic form game. Since this is a well-defined strategic form game, we can use any noncooperative equilibrium concept to analyse the game.

Suppose now that the rule which determines payoffs for each cooperation structure has the property that no agent wants to unilaterally *break* a link with any player. Since no player wants to break a link, and it needs the consent of *two* players to form an additional link, any cooperation structure can be sustained as a Nash equilibrium. We, therefore, use refinements of the Nash equilibrium concept. In particular, we employ undominated Nash equilibrium, coalition-proof equilibrium, and the argmax set of *weighted potential* games.[5] Our principal conclusion is that for a wide class of solutions, these equilibrium refinements all lead to the formation of the full cooperation structure or cooperation structures which are *payoff-equivalent* to this structure. An important by-product of our analysis is a characterization of *weighted Myerson values* which are a generalization of weighted Shapley values to games with cooperation structures. We show that weighted Myerson values are the only solution concepts for games with cooperation structurs which satisfy an efficiency requirement and which generate linking games that are weighted potential games.

The plan of this paper is as follows. In section 2 we provide some basic definitions, including those of cooperation structures and solutions for games with cooperation structures. Some 'reasonable' properties on such solutions

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formation as a game of *perfect information.* We discuss this issue in more detail in section 3.

[5]The latter is defined in section 5.

are introduced, and some implications are derived. Section 3 contains a discussion of different ways of modelling the process of link formation. Endogenous cooperation structures corresponding to undominated Nash equilibrium and coalition-proof Nash equilibrium are determined in section 4. Section 5 contains the characterization of weighted Myerson values, and also shows that the argmax set of the weighted potential corresponds to the full cooperation structure and payoff-equivalent structures. We conclude in section 6.

2 Cooperation Structures and Solutions

Let (N, v) be a TU coalitional game, where $N = \{1, 2, \ldots, n\}$ denotes the finite player set and v is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. *Throughout this paper, we will assume that v is superadditive*[6].

A *cooperation structure* is a graph $g = (N, L)$ where N is the set of vertices, and L is the *edge set*. An edge will also be called a *link*, and denoted by l, l' etc. For any $S \subseteq N$, we say that players $i, j \in S$ are *connected* in S if there exists a path from i to j that uses only vertices in S. The relation 'connected in N' is an equivalence relation on N. The equivalence classes of this relation are the *connected components* of the graph g.

We follow Aumann and Myerson (1988) in interpreting a link between two players as meaning that these players can carry on meaningful direct negotiations with each other. The negotiation to form links takes place in a preliminary period when "for one reason or another, one cannot enter into binding agreements of any kind (such as those relating to subsequent divisions of the payoff..)".[7]

A *solution* is a mapping γ which assigns an element in $\mathbb{R}^n$ to each TU

[6] v is superadditive if for all $S, T \in 2^N$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$.

[7] Aumann and Myerson (1988), page 187. See also Myerson (1977).

game (N, v) and cooperation structure $g = (N, L)$. Since there will be no ambiguity about the underlying game (N, v), we will simply write $\gamma(L)$, $\gamma(L')$, etc., instead of writing $\gamma(N, v, L)$, $\gamma(N, v, L')$, etc.

A solution can for example be generated for any graph g by applying the usual or familiar cooperative solution concepts to the 'graph-restricted game' (N, v^g). This game is defined as follows. Let $S\backslash g$ denote the partition of S into subsets of players that are connected in S by g. That is,

$$S\backslash g = \{\{i \mid j \text{ and } i \text{ are connected in } S \text{ by } g\ \} \mid j \in S\} \tag{1}$$

Now, define $v^g : 2^N \to I\!R$ by

$$v^g(S) = \sum_{T \in S\backslash g} v(T) \tag{2}$$

For instance, for any $g = (N, L)$, the Shapley value of the associated game (N, v^g) is a solution for (N, v, L), and has come to be called the *Myerson value*.[8] Similarly, weighted Myerson values of (N, v, L) are the weighted Shapley values of (N, v^g).

A class of solutions which will play a prominent role in this paper is the class satisfying the following 'reasonable' properties on a solution γ below.

Component efficiency (CE) : For all cooperation structures (N, L) and all $S \in 2^N$, if S is a connected component of (N, L), then $\sum_{i \in S} \gamma_i(L) = v(S)$.

Weak link symmetry (WLS) : For all $i, j \in N$, and all cooperation structures (N, L), if $\gamma_i(L \cup \{i, j\}) > \gamma_i(L)$, then $\gamma_j(L \cup \{i, j\}) > \gamma_j(L)$.

Improvement property (IP) : For all $i, j \in N$ and all cooperation structures (N, L), if for some $k \in N\backslash\{i, j\}$, $\gamma_k(L \cup \{i, j\}) > \gamma_k(L)$, then $\gamma_i(L \cup \{i, j\}) > \gamma_i(L)$ or $\gamma_j(L \cup \{i, j\}) > \gamma_j(L)$.

[8] Myerson (1977) contains a characterization of the Myerson value. See also Jackson and Wolinsky (1994).

These properties all have very simple interpretations. Component efficiency, which was originally used by Myerson (1977), states that the players in a connected component S split the value $v(S)$ amongst themselves. The second property is a very weak form of symmetry. It says that if a new link between players i and j makes i *strictly* better off, then it must also strictly improve the payoff of player j. Finally, the improvement property states that if a new link between players i and j strictly improves the payoff of any other player k, then the payoff of *either* i *or* j must also strictly improve.

The class of *weighted Myerson values* satisfies all the properties listed above. There are also others. For instance, if (N, v) is a *convex* game, then the *egalitarian solution* of Dutta and Ray (1989) corresponding to the associated game (N, v^g) also satisfies these properties.

The three properties together imply an interesting fourth property. This is the content of the next lemma.

Lemma 1 *Let γ be any solution satisfying CE, WLS and IP. Then, for all $i, j \in N$, and all cooperation structures (N, L),*

$$\gamma_i(L \cup \{i, j\}) \geq \gamma_i(L). \tag{3}$$

Proof : Suppose for some $i, j \in N$ and (N, L), $\gamma_i(L) > \gamma_i(L \cup \{i, j\})$. Then, by WLS, we must also have $\gamma_j(L) \geq \gamma_j(L \cup \{i, j\})$. But then, since v is superadditive, and γ satisfies CE, there must exist $k \notin \{i, j\}$ such that $\gamma_k(L) < \gamma_k(L \cup \{i, j\})$. This shows that γ violates IP since $\gamma_i(L) > \gamma_i(L \cup \{i, j\})$ and $\gamma_j(L) \geq \gamma_j(L \cup \{i, j\})$. ■

Remark 1: We will denote the property incorporated in equation (3) by *Link Monotonicity*. Note that Link Monotonicity is an appealing property in its own right. It says that a player i should not be worse-off as a result of forming a new link with some player j.[9]

Remark 2: It is easy to construct examples to show that the Component Efficiency, Weak Link Symmetry, and Improvement properties are independent.

Another consequence of these three properties is derived in the next lemma. We show that if the formation of a link $\{i,j\}$ affects the payoff of some other player k, then it must also affect the payoffs of both players that formed the new link. This property will be used later on in the paper.

Lemma 2 *Let γ satisfy CE, WLS and IP. Then, for all $i,j \in N$, and all cooperation structures (N,L), if for some $k \in N\backslash\{i,j\}$, $\gamma_k(L \cup \{i,j\}) \neq \gamma_k(L)$, then $\gamma_i(L \cup \{i,j\}) \neq \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) \neq \gamma_j(L)$.*

Proof: Suppose for some $i,j \in N$ and $k \in N\backslash\{i,j\}$, $\gamma_k(L \cup \{i,j\}) \neq \gamma_k(L)$. If $\gamma_k(L \cup \{i,j\}) > \gamma_k(L)$, then from WLS and IP we must have $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$.

Suppose $\gamma_k(L \cup \{i,j\}) < \gamma_k(L)$. From WLS, either $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$, or $\gamma_i(L \cup \{i,j\}) \leq \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) \leq \gamma_j(L)$. But, in the latter case, CE and superadditivity imply that there exists a $l \notin \{i,j\}$ such that $\gamma_l(L \cup \{i,j\}) > \gamma_l(L)$. This would violate IP, so it must hold that $\gamma_i(L \cup \{i,j\}) > \gamma_i(L)$ and $\gamma_j(L \cup \{i,j\}) > \gamma_j(L)$. This establishes the lemma. ■

While the three properties are all appealing and are satisfied by a large class of solutions, there are other solutions outside this class that seem to be appealing. One such solution is defined below.

For any i and L, let $L_i = \{\{i,j\} \mid j \in N, \{i,j\} \in L\}$, the set of links that are adjacent to i, and $l_i = |L_i|$. Let $S_i(L)$ denote the connected component of

[9] Note that the game is superadditive.

L containing i. Then, the *Proportional Links Solution*, denoted γ^P, is given by

$$\gamma_i^P(L) = \begin{cases} \frac{l_i}{\sum_{j \in S_i(L)} l_j} v(S_i(L)) & \text{if } |S_i(L)| \geq 2 \\ v(\{i\}) & \text{if } S_i(L) = \{i\} \end{cases} \tag{4}$$

for all L and all $i \in N$. The solution γ^P captures the notion that the more links a player has with other players, the better are his *relative* prospects in the subsequent negotiations over the division of the payoff. Notice that this makes sense only when the players are equally 'powerful' in the game (N, v). Otherwise, a *big* player may get more than *small* players even if he has fewer links. We leave it to the reader to check that γ^P satisfies CE and IP, but not WLS.

3 Modelling Negotiation Processes

In this section, we use the 3-person majority game to illustrate some of the issues involved in the endogenous formation of cooperation structures. In particular, we discuss the Aumann-Myerson extensive form approach in some detail. We point out that this approach, which involves a *sequential* formation of links may be appropriate in situations where the negotiation process is 'public', and where for one reason or another, bilateral negotiations take place in some predetermined order. When these prerequisites are not met, it may be more appropriate to model the 'negotiation game' as a game in strategic form. This is the approach adopted in this paper, and it is defined more formally later on in this section.

Aumann and Myerson (1988) use the following extensive form. First, an exogenous *rule* determines the sequential order in which *pairs* of players negotiate to form a link. A link is formed if and only if both potential partners agree, and once formed, a link cannot be broken. Moreover, after the last pair in the order has decided on whether or not to form a link, all the remaining pairs are given another opportunity to form a link. The process stops if all pairs that did not form a link yet have had a last opportunity to do so. At any point of time, the entire history of links formed or rejected is known to the players, so that it is a game of perfect information. Some cooperation graph g will form at the end of the process. The payoff to player i is then $\phi_i(v^g)$, that is, the Shapley value of player i in the game v^g.

Since the game is finite and of perfect information, it has subgame perfect equilibria in pure strategies. The 'prediction' of the model is that only cooperation structures associated with subgame perfect equilibria will actually form as a result of negotiations between players.

Consider, for instance, the TU game v on the player set $\{1, 2, 3\}$ defined by

$$v(S) = \begin{cases} 1 \text{ if } \mid S \mid \geq 2 \\ 0 \text{ otherwise} \end{cases}$$

Suppose also that the *rule* specifies that the order of pairs is $\{1, 2\}, \{1, 3\}, \{2, 3\}$. Then, the Aumann-Myerson prediction is that only one pair will form a link. Suppose the link $\{1, 2\}$ is formed. Notice that either of 1 and 2 gain by forming an additional link with 3, *provided* the other player does not form a link with 3. Two further points need to be noted. First, if player i forms a link with 3, then it is in the interest of j ($j \neq i$) to also link up with 3. Second, if *all* links are formed, then players 1 and 2 are worse-off compared to the graph in which they alone form a link. Hence, the structure $\{\{1, 2\}\}$ is sustained as an 'equilibrium' by a pair of mutual threats of the kind :

"If you form a link with 3, then so will I."

Of course, this kind of threat makes sense only if i will come to *know* whether j has formed a link with 3. Moreover, i can acquire this information only if the negotiation process is *public*. If bilateral negotiations are conducted secretly, then it may be in the interest of some pair to conceal the fact that they have formed a link until the process of bilateral negotiations has come to an end. It is also clear that if different pairs can carry out negotiations *simultaneously* and if links once formed cannot be broken, then the mutual threats referred to earlier cannot be carried out.[10]

Thus, there are many contexts where considerations other than threats may have an important influence on the formation of links. For instance, suppose players 1 and 2 have already formed a link amongst themselves. Suppose also that neither player has as yet started negotiations with player 3. If 3 starts negotiations *simultaneously* with both 1 and 2, then 1 and 2 are in fact faced with a *Prisoners' Dilemma* situation. To see this, denote l and nl as the strategies of forming a link with 3 and not forming a link with 3 respectively. Then, the payoffs to 1 and 2 are described by the following matrix (the first entry in each box is 1's payoff, while the second entry is 2's payoff).

	Player 2		
		l	nl
Player 1	l	$(\frac{1}{3}, \frac{1}{3})$	$(\frac{2}{3}, \frac{1}{6})$
	nl	$(\frac{1}{6}, \frac{2}{3})$	$(\frac{1}{2}, \frac{1}{2})$

Note that l, that is forming a link with 3, is a *dominant* strategy for both players! Obviously, the complete graph may well form simply because players 1 and 2 cannot sign a binding agreement to abstain from forming a link with 3.

In this paper, we model the negotiation process as a game in strategic form. The specific strategic form game that we will construct was first defined by Myerson (1991), and has subsequently been used by Qin (1993). This model is described below.

[10] Aumann and Myerson (1988) also stress the importance of perfect information in deriving their results.

Let γ be a solution. Then, the *linking game* $\Gamma(\gamma)$ associated with γ is given by the $(n+2)$-tuple $(N; S_1, \ldots, S_n; f^\gamma)$ where for each $i \in N$, S_i is player i's strategy set with $S_i = 2^{N \setminus \{i\}}$, and the payoff function is the mapping $f^\gamma : \Pi_{i \in N} S_i \to I\!R^n$ given by

$$f_i^\gamma(s) = \gamma_i(L(s)) \tag{5}$$

for all $s \in \Pi_{i \in N} S_i$, with

$$L(s) = \{\{i,j\} \mid j \in s_i,\ i \in s_j\} \tag{6}$$

The interpretation of (5) and (6) is straightforward. A typical strategy of player i in $\Gamma(\gamma)$ consists of the set of players with whom i wants to form a link. Then (6) states that a link between i and j is formed if and only if they both want to form this link. Thus, each strategy vector s gives rise to a unique cooperation structure $L(s)$. Finally, the payoff to player i associated with s is simply $\gamma_i(L(s))$[11], the payoff that γ associates with the cooperation structure $L(s)$.

We will let $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ denote the strategy vector such that $\bar{s}_i = N \setminus \{i\}$ for all $i \in N$, while $\bar{L} = \{\{i,j\} \mid i \in N,\ j \in N\} = L(\bar{s})$ denotes the complete edge set on N. A cooperation structure L is *essentially complete for* γ if $\gamma(L) = \gamma(\bar{L})$. Hence, if L is essentially complete for γ, but $L \neq \bar{L}$, then the links which are not formed in L are inessential in the sense that their absence does not change *the payoff vector* from that corresponding to $\bar{L}$. Notice that the property of "essentially complete" is specific to the solution γ - a cooperation structure L may be essentially complete for γ, but not for γ'.

We now define some equilibrium concepts for any $\Gamma(\gamma)$. These will be used in section 4 below.

[11] We again remind the reader that we have suppressed the underlying TU game (N, v) in order to simplify the notation.

The first equilibrium concept that we consider is the undominated Nash equilibrium. For any $i \in N, s_i$ *dominates* s'_i iff for all $s_{-i} \in S_{-i}$, $f_i^\gamma(s_i, s_{-i}) \geq f_i^\gamma(s'_i, s_{-i})$ with the inequality being strict for *some* s_{-i}. Let $S_i^u(\gamma)$ be the set of undominated strategies for i in $\Gamma(\gamma)$, and $S^u(\gamma) = \Pi_{i \in N} S_i^u(\gamma)$. A strategy tuple s is an *undominated Nash equilibrium* of $\Gamma(\gamma)$ if s is a Nash equilibrium and, moreover, $s \in S^u(\gamma)$.

The second equilibrium concept that will be discussed is the Coalition-Proof Nash Equilibrium. In order to define the concept of Coalition-Proof Nash Equilibrium of $\Gamma(\gamma)$, we need some more notation. For any $T \subset N$ and $s_T^* \in S_T := \Pi_{i \in T} S_i$, let $\Gamma(\gamma, s_{N \setminus T}^*)$ denote the game induced on subgroup T by the actions $s_{N \setminus T}^*$. So,

$$\Gamma(\gamma, s_{N \setminus T}^*) = \langle T, \{S_i\}_{i \in T}, \tilde{f}^\gamma \rangle$$

where for all $j \in T$, $\tilde{f}_j^\gamma : \Pi_{i \in T} S_i \to I\!R$ is given by $\tilde{f}_j^\gamma((s_i)_{i \in T}) = f_j^\gamma((s_i)_{i \in T}, s_{N \setminus T}^*)$ for all $(s_i)_{i \in T} \in S_T$.

The Coalition-Proof Nash Equilibrium is defined inductively as follows: In a single player game, $s^* \in S$ is a *Coalition-Proof Nash Equilibrium* (CPNE) of $\Gamma(\gamma)$ iff s_i^* maximizes $f_i^\gamma(s)$ over S. Now, let $\Gamma(\gamma)$ be a game with n players, where $n > 1$, and assume that Coalition-Proof Nash Equilibria have been defined for games with less than n players. Then, a strategy tuple $s^* \in S_N := \Pi_{i \in N} S_i$ is called *self-enforcing* if for all $T \subsetneq N$, s_T^* is a CPNE in the game $\Gamma(\gamma, s_{N \setminus T}^*)$. A strategy tuple $s^* \in S_N$ is a CPNE of $\Gamma(\gamma)$ if it is self-enforcing and, moreover, there does not exist another self-enforcing strategy vector $s \in S_N$ such that $f_i^\gamma(s) > f_i^\gamma(s^*)$ for all $i \in N$.

Let CPNE (γ) denote the set of CPNE of $\Gamma(\gamma)$.[12] Notice that the notion of CPNE incorporates a kind of 'farsighted' thought process on the part of players

[12]See Bernheim, Peleg and Whinston (1987) for discussion of Coalition-Proof Nash Equilibrium.

since a coalition when contemplating a deviation takes into consideration the possibility of further deviations by subcoalitions.[13]

4 Equilibrium Cooperation Structures

In this section, we characterize the sets of equilibrium cooperation structures under the equilibrium concepts defined in the previous section.

Our principal objective is to show that the equilibrium concepts defined in section 3 all lead to essentially complete cooperation structures for solutions satisfying the properties that are listed in section 2.

Theorem 1 Let γ be a solution that satisfies CE, WLS and IP. Then, $\bar{s}$ is an undominated Nash equilibrium of $\Gamma(\gamma)$. Moreover, if s is an undominated Nash equilibrium of $\Gamma(\gamma)$, then $L(s)$ is essentially complete for γ.

Proof : First, we show that $\bar{s}_i$ is undominated for all $i \in N$ (in fact, we even show that it is weakly dominant).

So, choose $i \in N, s_i \in S_i$ and $s_{-i} \in S_{-i}$ arbitrarily. Let $L = L(\bar{s}_i, s_{-i})$ and $L' = L(s_i, s_{-i})$. Note that, since $s_i \subseteq \bar{s}_i$, $L' \subseteq L$. Also, if $l \in L \backslash L'$, then $i \in l$. So, from repeated application of link monotonicity (see lemma 1),

$$f_i^\gamma(\bar{s}_i, s_{-i}) = \gamma_i(L) \geq \gamma_i(L') = f_i^\gamma(s_i, s_{-i}) \tag{7}$$

Since s_i and s_{-i} were chosen arbitrarily, this shows that $\bar{s}_i \in S_i^u(\gamma)$. Further, putting $s_{-i} = \bar{s}_{-i}$ in (7), we also get that $\bar{s}$ is a Nash equilibrium of $\Gamma(\gamma)$. So, we may conclude that $\bar{s} \in S^u(\gamma)$.

[13]We mention this because Aumann and Myerson (1988) state that they do not use the 'usual, myopic, here-and-now kind of equilibrium condition', but a 'look ahead' one. Of course, farsightedness can be modelled in many different ways.

Now, we show that $L(s)$ is essentially complete for an undominated Nash equilibrium s. Choose $s \neq \bar{s}$ arbitrarily. Without loss of generality, let $\{i \in N \mid s_i \neq \bar{s}_i\} = \{1, 2, ..., K\}$. Construct a sequence $\{s^0, s^1, \ldots, s^K\}$ of strategy tuples as follows.

(i) $s^0 = s$

(ii) $s^k_k = \bar{s}_k$ for all $k = 1, 2, ..., K$.

(iii) $s^k_j = s^{k-1}_j$ for all $k = 1, 2, ..., K$, and all $j \neq k$.

Clearly, $s^K = \bar{s}$. Consider any s^{k-1} and s^k. By construction, $s^{k-1}_j = s^k_j$ for all $j \neq k$, while $s^k_k = \bar{s}_k$ and $s^{k-1}_k = s_k$. So, using link monotonicity, we have

$$f^\gamma_k(s^k) = \gamma_k(L(s^k)) \geq \gamma_k(L(s^{k-1})) = f^\gamma_k(s^{k-1}) \tag{8}$$

Suppose (8) holds with strict inequality. Then, we have demonstrated the existence of strategies s_{-k} such that

$$f^\gamma_k(\bar{s}_k, s_{-k}) > f^\gamma_k(s_k, s_{-k}) \tag{9}$$

But, (7) and (9) together show that $\bar{s}_k$ *dominates* s_k. So, if $s \in S^u(\gamma)$, then (8) must hold with equality. Then it follows from lemma 2 that the payoffs to all players remain unchanged when going from s^{k-1} to s^k, so

$$\gamma(L(s^k)) = \gamma(L(s^{k-1})) \tag{10}$$

Since this argument can be repeated for $k = 1, 2, ..., K$, we get $\gamma(L(s^0)) = \gamma(L(s^1)) = \cdots = \gamma(L(\bar{s}))$. Hence, if $s \in S^u(\gamma)$, then $L(s)$ is essentially complete. ■

Theorem 2 Let γ be a solution satisfying CE, WLS and IP. Then $\bar{s} \in$ CPNE (γ). Moreover, if $s \in$ CPNE (γ), then $L(s)$ is essentially complete for γ.

Proof : In fact, we will prove a slightly generalized version of the theorem and show that for each coalition $T \subseteq N$ and all $s_{N \setminus T} \in S_{N \setminus T}$ it holds that $\bar{s}_T \in$ CPNE $(\gamma, s_{N \setminus T})$ and that for all $s_T^* \in$ CPNE $(\gamma, s_{N \setminus T})$ it holds that $f^\gamma(s_T^*, s_{N \setminus T}) = f^\gamma(\bar{s}_T, s_{N \setminus T})$. We will follow the definition of Coalition-Proof Nash Equilibrium and proceed by induction on the number of elements of T. Throughout the following, we will assume $s_{N \setminus T} \in S_{N \setminus T}$ to be arbitrary.

Let $T = \{i\}$. Then by repeated application of Link Monotonicity we know that $f_i^\gamma(\bar{s}_i, s_{N \setminus \{i\}}) \geq f_i^\gamma(s_i, s_{N \setminus \{i\}})$ for all $s_i \in S_i$. From this it readily follows that $\bar{s}_i \in$ CPNE $(\gamma, s_{N \setminus \{i\}})$. Now, suppose $s_i^* \in$ CPNE $(\gamma, s_{N \setminus \{i\}})$. Then, since $f_i^\gamma(s_i^*, s_{N \setminus \{i\}}) \leq f_i^\gamma(\bar{s}_i, s_{N \setminus \{i\}})$, it follows that $f_i^\gamma(s_i^*, s_{N \setminus \{i\}}) = f_i^\gamma(\bar{s}_i, s_{N \setminus \{i\}})$ must hold. Now we use lemma 2 and see that $f^\gamma(s_i^*, s_{N \setminus \{i\}}) = f^\gamma(\bar{s}_i, s_{N \setminus \{i\}})$.

Now, let $|T| > 1$ and assume that we already proved that for all R with $|R| < |T|$ and all $s_{N \setminus R} \in S_{N \setminus R}$ it holds that $\bar{s}_R \in$ CPNE $(\gamma, s_{N \setminus R})$ and that for all $s_R^* \in$ CPNE $(\gamma, s_{N \setminus R})$ it holds that $f^\gamma(s_R^*, s_{N \setminus R}) = f^\gamma(\bar{s}_R, s_{N \setminus R})$. Then it readily follows from the first part of the induction hypothesis that $\bar{s}_R \in$ CPNE $(\gamma, \bar{s}_{T \setminus R}, s_{N \setminus T})$ for all $R \subsetneq T$. This shows that $\bar{s}_T$ is self-enforcing.
Suppose $s_T^* \in S_T$ is also self-enforcing, i.e. $s_R^* \in$ CPNE $(\gamma, s_{T \setminus R}^*, s_{N \setminus T})$ for all $R \subsetneq T$. We will start by showing that $f_i^\gamma(\bar{s}_T, s_{N \setminus T}) \geq f_i^\gamma(s_T^*, s_{N \setminus T})$ for all $i \in T$, which proves that $\bar{s}_T \in$ CPNE $(\gamma, s_{N \setminus T})$. So, let $i \in T$ be fixed for the moment. Then repeated application of Link Monotonicity implies that $f_i^\gamma(\bar{s}_T, s_{N \setminus T}) \geq f_i^\gamma(s_i^*, \bar{s}_{T \setminus i}, s_{N \setminus T})$. Further, since $s_{T \setminus \{i\}}^* \in$ CPNE $(\gamma, s_i^*, s_{N \setminus T})$, it follows from the second part of the induction hypothesis that $f^\gamma(s_i^*, \bar{s}_{T \setminus \{i\}}, s_{N \setminus T}) = f^\gamma(s_T^*, s_{N \setminus T})$. Combining the two last (in)equalities we find that $f_i^\gamma(\bar{s}_T, s_{N \setminus T}) \geq f_i^\gamma(s_T^*, s_{N \setminus T})$.

Note that we will have completed the proof of the theorem if we show that, in addition to $f_i^\gamma(\bar{s}_T, s_{N \setminus T}) \geq f_i^\gamma(s_T^*, s_{N \setminus T})$ for all $i \in T$, it

holds that either $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) > f_i^\gamma(s_T^*, s_{N\backslash T})$ for all $i \in T$ (and, consequently, $s_T^* \notin$ CPNE $(\gamma, s_{N\backslash T})$) or $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) = f_i^\gamma(s_T^*, s_{N\backslash T})$ for all $i \in T$ (and $s_T^* \in$ CPNE $(\gamma, s_{N\backslash T})$). So, suppose $i \in T$ is such that $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) > f_i^\gamma(s_T^*, s_{N\backslash T})$. Because s_T^* is self-enforcing, we know that $s_{T\backslash\{j\}}^* \in$ CPNE $(\gamma, s_j^*, s_{N\backslash T})$ for each $j \in T$, and it follows from the induction hypothesis that $f^\gamma(s_T^*, s_{N\backslash T}) = f^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$ for each $j \in T$. Let $j \in T\backslash\{i\}$ be fixed. Then we have just shown that $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) > f_i^\gamma(s_T^*, s_{N\backslash T}) = f_i^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$. We know by repeated application of Link Monotonicity that $f_j^\gamma(\bar{s}_T, s_{N\backslash T}) \geq f_j^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$. However, if this should hold with equality, $f_j^\gamma(\bar{s}_T, s_{N\backslash T}) = f_j^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$ then repeated application of lemma 2 would imply that $f^\gamma(\bar{s}_T, s_{N\backslash T}) = f^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$, which contradicts that $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) > f_i^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$. Hence, we may conclude that $f_j^\gamma(\bar{s}_T, s_{N\backslash T}) > f_j^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T})$. Since $f_j^\gamma(s_j^*, \bar{s}_{T\backslash j}, s_{N\backslash T}) = f_j^\gamma(s_T^*, s_{N\backslash T})$, we now know that $f_j^\gamma(\bar{s}_T, s_{N\backslash T}) > f_j^\gamma(s_T^*, s_{N\backslash T})$.
This shows that either $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) > f_i^\gamma(s_T^*, s_{N\backslash T})$ for all $i \in T$ or $f_i^\gamma(\bar{s}_T, s_{N\backslash T}) = f_i^\gamma(s_T^*, s_{N\backslash T})$ for all $i \in T$. ■

Remark 3: We have an example of a solution satisfying CE, WLS and IP, for which CPNE $(\gamma) \neq \{s \mid L(s)$ is essentially complete$\}$. In other words, there may be s which is not in CPNE (γ), though $L(s)$ is essentially complete.

We defined the Proportional Links Solution γ^P in section 2, and pointed out that it does not satisfy WLS. It also turns out that the conclusions of theorem 2 are no longer valid in the linking game $\Gamma(\gamma^P)$. While we do not have any general characterization results for $\Gamma(\gamma^P)$, we show below that complete structures will not necessarily be coalitionproof equilibria of $\Gamma(\gamma^P)$ in the special case of the 3-player *majority* game.[14]

[14] v is a majority game if a majority coalition has worth 1, and all other coalitions have zero worth.

Proposition 1 *Let N be a player set with $|N| = 3$, and let v be the majority game on $|N|$. Then, $s \in$ CPNE (γ^P) iff $L(s) = \{\{i,j\}\}$, i.e., only one pair of agents forms a link.*

Proof : Suppose only i and j form a link according to s. Then, $f_i^{\gamma^P}(s) = f_j^{\gamma^P}(s) = \frac{1}{2}$. Check that if i deviates and forms a link with k, then i's payoff remains at $\frac{1}{2}$. Also, clearly i and j together do not have any profitable deviation. Hence, s is coalitionproof.

Now, suppose that N is a connected set according to s. There are two possibilities.

Case (i) : $L(s) = \bar{L}$. In that case, $f_i^{\gamma^P} = \frac{1}{3}$ for all $i \in N$. Let i and j deviate and break links with k. Then, both i and j get a payoff of $\frac{1}{2}$. Suppose i makes a further deviation. The only deviation which needs to be considered is if i re-establishes a link with k. Check that i's payoff *remains* at $\frac{1}{2}$. So, in this case s cannot be a coalition proof equilibrium.

Case (ii) : $L(s) \neq \bar{L}$. Since N is a connected set in $L(s)$, the only possibility is that there exist i and j such that both are connected to k, but not to each other. Then, both i and j have a payoff of $\frac{1}{4}$. Let now i and j deviate, break links with k and form a link between each other. Then, their payoff increases to $\frac{1}{2}$. Check that neither player has any further profitable deviation. Again, this shows that s is not coalitionproof. ■

5 Weighted Potential Games

Monderer and Shapley (1993) prove various properties of the class of *potential games*.[15] Their results make potential games particularly interesting, and

[15] Rosenthal (1973) was the first to (implicitly) use potential functions for games in strategic form.

prompt us to study the class of linking games which are also potential games.

Let $\Gamma = (N; S_1, \ldots, S_n; \pi)$ be a game in strategic form, where for each $i \in N$, S_i is the strategy set of player i, and π is the payoff function. Let $w = (w_i)_{i \in N}$ be a vector of positive numbers, to be called *weights* for the players. A function $P^w : \Pi_{i \in N} S_i \rightarrow I\!\!R$ is a *w-potential* for Γ if for every $i \in N$ and for all $s_i \in S_i$ and $t_i \in S_i$

$$\pi_i(s_i, s_{-i}) - \pi_i(t_i, s_{-i}) = w_i(P^w(s_i, s_{-i}) - P^w(t_i, s_{-i})) \tag{11}$$

The game Γ is called a *w-potential game* if it admits a w-potential.

Monderer and Shapley (1993) point out that the argmax set of a weighted potential does not depend on a particular choice of a weighted potential, and hence can be used as an equilibrium refinement. They also remark that this refinement concept is supported by some experimental results.[16] Moreover, the *Fictitious Play* process converges to the equilibrium set in a class of games that contains the finite weighted potential games.

In this section, we first show that the class of *weighted Myerson values* is precisely the class of solutions γ which satisfy component efficiency and which generate linking games that are weighted potential games. Second, we show that strategies in the argmax set of these potential games result in the formation of essentially complete cooperation structures.[17] The second result, in conjunction with the results of the previous section, strengthens the case for the formation of essentially complete structures if the negotiation process is simultaneous.

Some more definitions and lemmas precede the main results of this section.

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[16]Monderer and Shapley (1993) point out that this may be a mere coincidence. See also Van Huyck et al. (1990) and Crawford (1991).

[17]These results generalise analogous results of Qin (1993), who was concerned only with *Myerson values* and potential games.

A solution γ is *w-fair* if for all cooperation structures L and for all $i, j \in N$

$$\frac{1}{w_i}(\gamma_i(L) - \gamma_i(L\backslash\{\{i,j\}\})) = \frac{1}{w_j}(\gamma_j(L) - \gamma_j(L\backslash\{\{i,j\}\})) \tag{12}$$

A coalition S is a *partnership* in (N, v) if for each $T \subsetneq S$ and each $R \subseteq N\backslash S$, $v(R \cup T) = v(R)$. A solution[18] ϕ satisfies *partnership consistency* if for each partnership S in (N, v) and for the *unanimity* game u_S

$$\phi_i(v) = \phi_i\Big(\phi_S(v)u_S\Big) \text{ for each } i \in S.$$

Kalai and Samet (1988) use partnership consistency as one of the conditions in their characterization of the class of weighted Shapley values.

The next lemma illustrates an important property of the weighted Myerson values, and is of independent interest. We remind the reader that weighted Myerson values are weighted Shapley values of the graph-restricted game (N, v^g) (see page 5).

Lemma 3 *The weighted Myerson value μ^w is the unique rule that is component efficient and w-fair.*

Proof : We first prove that μ^w satisfies the two properties mentioned. To prove component efficiency, let $g = (N, L)$ be a cooperation structure and let S be a connected component of L. Then the associated game v^g can be split up in two games v^S and $v^{N\backslash S}$ as follows. Define

$$v^S(T) := v^g(T \cap S)$$
$$v^{N\backslash S}(T) := v^g(T\backslash S)$$

for all $T \subseteq N$. We then have $v^g = v^S + v^{N\backslash S}$ because S is a connected component of L. Then, because all $i \in S$ are dummy players in the game $v^{N\backslash S}$,

[18]Note that here, we are using the term 'solution' as the rule specifying payoffs to the players for classes of TU games.

we know by the dummy player property of the weighted Shapley value that $\phi_i^w(v^{N\backslash S}) = 0$ for all $i \in S$. Similarly, $\phi_i^w(v^S) = 0$ for all $i \in N\backslash S$. Using additivity of the weighted Shapley value we now obtain

$$\begin{aligned} \sum_{i\in S} \phi_i^w(v^g) &= \sum_{i\in S} \phi_i^w(v^S) + \sum_{i\in S} \phi_i^w(v^{N\backslash S}) \\ &= \sum_{i\in S} \phi_i^w(v^S) = v^S(N) = v^g(S) = v(S), \end{aligned}$$

where we use efficiency of the weighted Shapley value in the third equality.

To prove w-fairness, let $g = (N, L)$ be a cooperation structure and choose $i, j \in N$ such that $\{i, j\} \in L$. Define $\tilde{L} := L\backslash\{\{i, j\}\}$, $\tilde{g} = (N, \tilde{L})$, and define $\tilde{v} := v^g - v^{\tilde{g}}$. Then, for each $T \subseteq N$ with $\{i, j\} \nsubseteq T$

$$\tilde{v}(T) = \sum_{R\in T\backslash g} v(R) - \sum_{R\in T\backslash \tilde{g}} v(R) = 0,$$

where we use the fact that $T\backslash g = T\backslash\tilde{g}$. Hence, $\{i, j\}$ is a partnership in $\tilde{v}$. Also, since weighted Shapley values satisfy partnership consistency, we have

$$\phi_i^w(\tilde{v}) = \phi_i^w\left(\left(\phi_i^w(\tilde{v}) + \phi_j^w(\tilde{v})\right) u_{\{i,j\}}\right)$$

Now, note that

$$\phi_i^w\left(\left(\phi_i^w(\tilde{v}) + \phi_j^w(\tilde{v})\right) u_{\{i,j\}}\right) = \frac{w^i}{w^i + w^j}\left(\phi_i^w(\tilde{v}) + \phi_j^w(\tilde{v})\right)$$

A similar expression can be found for j. From this we see that

$$\frac{\phi_i^w(\tilde{v})}{w^i} = \frac{\phi_j^w(\tilde{v})}{w^j}$$

This gives us

$$\frac{\mu_i^w(L) - \mu_i^w(\tilde{L})}{w^i} = \frac{\phi_i^w(\tilde{v})}{w^i} = \frac{\phi_j^w(\tilde{v})}{w^j} = \frac{\mu_j^w(L) - \mu_j^w(\tilde{L})}{w^j}$$

We now show that there exists at most one rule that satisfies component efficiency and w-fairness. Suppose to the contrary that γ^1 and γ^2 are two rules satisfying the two properties, and let (N, L) be a cooperation structure with a minimum number of links such that $\gamma^1(L) \neq \gamma^2(L)$. Let $\{i, j\} \in L$. Then by w-fairness of γ^1,

$$\frac{1}{w_i}(\gamma_i^1(L) - \gamma_i^1(L \backslash \{\{i,j\}\})) = \frac{1}{w_j}(\gamma_j^1(L) - \gamma_j^1(L \backslash \{\{i,j\}\}))$$

Hence, using the minimality of L,

$$\begin{aligned} w_j\gamma_i^1(L) - w_i\gamma_j^1(L) &= w_j\gamma_i^1(L \backslash \{\{i,j\}\}) - w_i\gamma_j^1(L \backslash \{\{i,j\}\}) \\ &= w_j\gamma_i^2(L \backslash \{\{i,j\}\}) - w_i\gamma_j^2(L \backslash \{\{i,j\}\}) \\ &= w_j\gamma_i^2(L) - w_i\gamma_j^2(L)\,, \end{aligned}$$

where the last equality follows from w-fairness of γ^2. So, now we have

$$w_j(\gamma_i^1(L) - \gamma_i^2(L)) = w_i(\gamma_j^1(L) - \gamma_j^2(L)) \tag{13}$$

It can be easily seen that equality (13) holds for all i, j that are in the same connected component of (N, L). Therefore, we can find for each connected component S of (N, L) a number $d(S)$ such that for all $i \in S$

$$\frac{1}{w_i}(\gamma_i^1(L) - \gamma_i^2(L)) = d(S) \tag{14}$$

However, both γ^1 and γ^2 are component efficient solutions, so for each connected component S of (N, L), we have

$$\sum_{i \in S} \gamma_i^1(L) = \sum_{i \in S} \gamma_i^2(L) = v(S) \tag{15}$$

Now, combining (14) and (15), we find that $d(S) = 0$ for each connected component S, and, hence, that $\gamma^1(L) = \gamma^2(L)$. ∎

We now show that weighted Myerson values are the *only* solutions which are component efficient and which generate linking games that are weighted potential games. This follows from the previous lemma and the following lemma.

Lemma 4 *Let γ be a component efficient solution generating a linking game $\Gamma(\gamma)$ that is a weighted potential game. Then there exist weights w such that γ is w-fair.*

Proof : Since the linking game $\Gamma(\gamma)$ is a weighted potential game, we can find positive weights w and a w-potential P^w for the game $\Gamma(\gamma)$. We will show that γ is w-fair. Let L be a cooperation structure and let $i, j \in N$. We define for all $k \in N$ $s_k := \{l \in N \mid \{k,l\} \in L\}$. Then, $L(s) = L$, and

$$P^w(s_i \backslash \{j\}, s_j, s_{-ij}) = P^w(s_i \backslash \{j\}, s_j \backslash \{i\}, s_{-ij}) = P^w(s_i, s_j \backslash \{i\}, s_{-ij}),$$

because all these strategy tuples result in the formation of the same cooperation structure, namely $L \backslash \{i, j\}$. Hence,

$$\begin{aligned} & \frac{1}{w_i}\Big(\gamma_i(L) - \gamma_i(L \backslash \{\{i,j\}\})\Big) \\ = & \frac{1}{w_i}\Big(f_i^\gamma(s) - f_i^\gamma(s_i \backslash \{j\}, s_{-i})\Big) \\ = & P^w(s) - P^w(s_i \backslash \{j\}, s_{-i}) \\ = & P^w(s) - P^w(s_j \backslash \{i\}, s_{-j}) \\ = & \frac{1}{w_j}\Big(f_j^\gamma(s) - f_j^\gamma(s_j \backslash \{i\}, s_{-j})\Big) \\ = & \frac{1}{w_j}\Big(\gamma_j(L) - \gamma_j(L \backslash \{\{i,j\}\})\Big) \end{aligned}$$

This proves that γ is w-fair. ■

The following lemma shows that each weighted Myerson value generates a linking game that is a weighted potential game.

Lemma 5 *The linking game $\Gamma(\mu^w)$ is a w-potential game.*

Proof : See the appendix.

Combining the results we have obtained so far, we obtain our characterization of the class of weighted Myerson values.

Theorem 3 Let γ be a component efficient solution. Then, the linking game $\Gamma(\gamma)$ is a weighted potential game iff γ is a weighted Myerson value.

Remark 4: Monderer and Shapley (1993) consider *participation* games, strategic form games in which each player has the option of either cooperating with *all* the other players or being on his own. Such a game is in fact a linking game in which the strategy set of a player i is limited to two strategies, $\emptyset$ and $N \backslash \{i\}$. Monderer and Shapley prove the theorem stated above using their version of the linking game. However, the severe restriction on the players' strategy sets implies that the participation games are of limited interest.

Note that in theorem 1, we proved that if γ satisfies CE, WLS and IP, then $\bar{s}_i$ is a *weakly dominating* strategy in $\Gamma(\gamma)$. Noting that weighted Myerson values satisfy these properties and any n-tuple of weakly dominating strategies must be in the argmax set of the corresponding weighted potential game, we obtain the first part of the following theorem.

Theorem 4 Let w be a set of positive weights and let P^w be a weighted potential for the linking game $\Gamma(\mu^w)$. Then $\bar{s} \in$ argmax P^w. Moreover, if $s \in$ argmax P^w, then $L(s)$ is essentially complete for μ^w.[19]

Remark 5: One can construct examples showing that the second statement of theorem 4 cannot be strengthened. That is, if $L(s)$ is essentially complete for μ^w, then s is not necessarily in argmax P^w.

[19]The proof of the latter half of the theorem is similar to the corresponding part of the proof of theorem 1, and is therefore omitted.

Conclusion

In this paper, we have studied the endogenous formation of *cooperation structures* in superadditive TU-games using a strategic game approach. In this strategic game, each player announces the set of players with whom he or she wants to form a link, and a link is formed if both players want to form the link. Given the resulting cooperation structure, the payoffs are determined by some exogenous solution for cooperative games with cooperation structures. We have concentrated on the class of solutions satisfying three appealing properties. We have shown that in this setting both the undominated Nash equilibrium and the Coalition-Proof Nash Equilibrium predict the formation of the full cooperation structure or some payoff equivalent structure.

We also considered linking games that are weighted potential games and their argmax sets. It turned out that, under an efficiency requirement, the class of solutions generating linking games that are weighted potential games is the class of weighted Myerson values. Further, the argmax set of the linking games that are weighted potential games predicts the formation of the full cooperation structure or some payoff equivalent structure.

The results obtained in this paper all point in the direction of the formation of the full cooperation structure in a superadditive environment. However, as indicated in the text, these results are sensitive to the assumptions on solutions for cooperative game with cooperation structures that we made in the paper. Further, the discussion in section 3 shows that in a context where links are formed sequentially rather than simultaneously other predictions may prevail.

Appendix

We provide here the proof of lemma 5. To simplify this proof, we recall some results of Kalai and Samet (1988) and we prove an intermediary claim.

Kalai and Samet (1988) gave the following probabilistic definition of the weighted Shapley value ϕ^w:

$$\phi_i^w(v) = \sum_{\sigma \in \Sigma(N)} p^w(\sigma)\Big(v(PR_i^\sigma \cup \{i\}) - v(PR_i^\sigma)\Big),$$

where $\Sigma(N)$ denotes the set of permutations of N, $PR_i^\sigma = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ denotes the set of predecessors of player i according to σ, and where for each $\sigma \in \Sigma(N)$ the probability $p^w(\sigma)$ is given by

$$p^w(\sigma) = \prod_{i=1}^{n} \left(\frac{w_{\sigma(i)}}{\sum_{j=1}^{i} w_{\sigma(j)}} \right)$$

With respect to the probabilities $p^w(\sigma)$ we derive the following result, which plays an important role in the proof of lemma 5.

Claim Let $i, j \in N$ and let $S \subseteq N \backslash \{i, j\}$. Define $p^w(PR_i^\sigma = S \cup \{j\}) := \sum_{\sigma \in \Sigma(N):\, PR_i^\sigma = S \cup \{j\}} p^w(\sigma)$ and define $p^w(PR_j^\sigma = S \cup \{i\})$ correspondingly. Then

$$\frac{p^w(PR_i^\sigma = S \cup \{j\})}{p^w(PR_j^\sigma = S \cup \{i\})} = \frac{w_i}{w_j}$$

Proof : To simplify the proof we introduce some notation. For $\sigma \in \Sigma(N)$ and $k \in N$ we name $N \backslash (PR_k^\sigma \cup \{k\})$ the *tail of* k *in* σ. We simply say that T *is a tail in* σ if T it the tail of k in σ for some $k \in N$. With this notation we have

$$p^w(PR_i^\sigma = S \cup \{j\}) = p^w(N \backslash (S \cup \{i, j\}) \text{ is the tail of } i \text{ in } \sigma)$$

We denote the set $N \backslash (S \cup \{i, j\})$ by N^{-Sij}. Using this notation and conditional probabilities, we see that the last expression is equal to

$$p^w(N^{-Sij} \text{ is the tail of } i \mid N^{-Sij} \text{ is a tail }) \cdot p^w(N^{-Sij} \text{ is a tail })$$

The conditional probability mentioned here is, by definition of the probabilities p^w, equal to $\frac{w_i}{\sum_{k\in S\cup\{i,j\}} w^k}$. Hence, we find

$$p^w(PR_i^\sigma = S\cup\{j\}) = \frac{w_i}{\sum_{k\in S\cup\{i,j\}} w^k}\ p^w(N^{-Sij} \text{ is a tail }) \qquad \text{(A.1)}$$

In a similar way we can show that

$$p^w(PR_j^\sigma = S\cup\{i\}) = \frac{w_j}{\sum_{k\in S\cup\{i,j\}} w_k}\ p^w(N^{-Sij} \text{ is a tail }) \qquad \text{(A.2)}$$

Combining expressions (A.1) and (A.2) gives the desired result. ■

Lemma 5 *The linking game $\Gamma(\mu^w)$ is a w-potential game.*

Proof: We will prove that for the game $\Gamma(\mu^w)$, for $i,j\in N$, $s_{-ij}\in \Pi_{k\in N\setminus\{i,j\}}S_k$ and $s_i^1, s_i^2\in S_i$, $s_j^1, s_j^2 \in S_j$, we have

$$\begin{aligned}\Delta :=&\\ &\frac{\mu_i^w(L(b))-\mu_i^w(L(a))}{w_i}+\frac{\mu_j^w(L(c))-\mu_j^w(L(b))}{w_j}+\\ &\frac{\mu_i^w(L(d))-\mu_i^w(L(c))}{w_i}+\frac{\mu_j^w(L(a))-\mu_j^w(L(d))}{w_j}=0 \qquad \text{(A.3)}\end{aligned}$$

where $a := (s_i^1, s_j^1, s_{-ij})$, $b := (s_i^2, s_j^1, s_{-ij})$, $c := (s_i^2, s_j^2, s_{-ij})$, and $d := (s_i^1, s_j^2, s_{-ij})$. It can be shown analogously to theorem 2.8 of Monderer and Shapley (1993) that the property described in (A.3) is satisfied if and only if $\Gamma(\mu^w)$ is a w-potential game.

To prove that (A.3) is satisfied, we use the probabilistic definition of the weighted Shapley value ϕ^w of Kalai and Samet (1988). To simplify notations we will denote for each $\alpha\in\{a,b,c,d\}$ the game $v^{L(\alpha)}$ by v^α. We consider, for the moment, the term

$$\begin{aligned}&\mu_i^w(L(b))-\mu_i^w(L(a))\\ =&\sum_{\sigma\in\Sigma(N)} p^w(\sigma)\Big(v^b(PR_i^\sigma\cup\{i\})-v^b(PR_i^\sigma)-v^a(PR_i^\sigma\cup\{i\})+v^a(PR_i^\sigma)\Big)\\ =&\sum_{\sigma\in\Sigma(N)} p^w(\sigma)\Big(v^b(PR_i^\sigma\cup\{i\})-v^a(PR_i^\sigma\cup\{i\})\Big),\end{aligned}$$

where the last equality follows from the fact that $v^a(T) = v^b(T)$ for all $T \subseteq N$ that do not contain i. We can go through the same procedure for the other three terms appearing in Δ and this results in

$$\Delta = \sum_{\sigma \in \Sigma(N)} p^w(\sigma) \Big(\frac{v^b(PR_i^\sigma \cup \{i\}) - v^a(PR_i^\sigma \cup \{i\})}{w_i} + \frac{v^c(PR_j^\sigma \cup \{j\}) - v^b(PR_j^\sigma \cup \{j\})}{w_j} + \frac{v^d(PR_i^\sigma \cup \{i\}) - v^c(PR_i^\sigma \cup \{i\})}{w_i} + \frac{v^a(PR_j^\sigma \cup \{j\}) - v^d(PR_j^\sigma \cup \{j\})}{w_j} \Big)$$

Rearranging the terms, we obtain

$$\Delta = \sum_{\sigma \in \Sigma(N)} p^w(\sigma) \Big(\frac{v^b(PR_i^\sigma \cup \{i\}) - v^c(PR_i^\sigma \cup \{i\})}{w_i} + \frac{v^d(PR_i^\sigma \cup \{i\}) - v^a(PR_i^\sigma \cup \{i\})}{w_i} + \frac{v^c(PR_j^\sigma \cup \{j\}) - v^d(PR_j^\sigma \cup \{j\})}{w_j} + \frac{v^a(PR_j^\sigma \cup \{j\}) - v^b(PR_j^\sigma \cup \{j\})}{w_j} \Big)$$

Now, we can use the fact that $v^a(S) = v^b(S) = v^c(S) = v^d(S)$ for all $S \subseteq N \backslash \{i, j\}$ and we obtain

$$\begin{aligned} \Delta = \; & \textstyle\sum_{\sigma \in \Sigma(N):\, j \in PR_i^\sigma} \frac{p^w(\sigma)}{w_i} \Big(v^b(PR_i^\sigma \cup \{i\}) - v^c(PR_i^\sigma \cup \{i\}) + \\ & \qquad v^d(PR_i^\sigma \cup \{i\}) - v^a(PR_i^\sigma \cup \{i\}) \Big) + \\ & \textstyle\sum_{\sigma \in \Sigma(N):\, i \in PR_j^\sigma} \frac{p^w(\sigma)}{w_j} \Big(v^c(PR_j^\sigma \cup \{j\}) - v^d(PR_j^\sigma \cup \{j\}) + \\ & \qquad v^a(PR_j^\sigma \cup \{j\}) - v^b(PR_j^\sigma \cup \{j\}) \Big) \end{aligned}$$

This last expression can be seen to be equal to 0 if we consider the terms per strategy profile. Let $\alpha \in \{a, b, c, d\}$ and consider

$$\sum_{\sigma \in \Sigma(N):\, j \in PR_i^\sigma} \frac{p^w(\sigma)}{w_i} \Big(v^\alpha(PR_i^\sigma \cup \{i\}) \Big) - \sum_{\sigma \in \Sigma(N):\, i \in PR_j^\sigma} \frac{p^w(\sigma)}{w_j} \Big(v^\alpha(PR_j^\sigma \cup \{j\}) \Big)$$

This is equal to

$$\sum_{S \subseteq N \backslash \{i,j\}} v^\alpha(S \cup \{i, j\}) \Big(\sum_{\sigma \in \Sigma(N):\, PR_i^\sigma = S \cup \{j\}} \frac{p^w(\sigma)}{w_i} - \sum_{\sigma \in \Sigma(N):\, PR_j^\sigma = S \cup \{i\}} \frac{p^w(\sigma)}{w_j} \Big)$$

Now, we use the claim, and conclude that the last expression equals 0. ■

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