An Iterative Procedure for Evaluating Digraph Competitions

Peter Borm[¤] René van den Brink[y] Marco Slikker[z]

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Abstract

A competition which is based on the results of (partial) pairwise comparisons can be modelled by means of a directed graph. Given initial weights on the nodes in such digraph competitions, we view the measurement of the importance (i.e., the cardinal ranking) of the nodes as an allocation problem where we redistribute the initial weights on the basis of insights from cooperative game theory. After describing the resulting procedure of redistributing the initial weights, we describe an iterative process is described which repeats this procedure: at each step the allocation obtained in the previous step determines the new input weights. Existence and uniqueness of the limit is established for arbitrary digraphs. Applications to the evaluation of e.g. sport competitions and paired comparison experiments are discussed.

Keywords: cooperative games, digraph competitions, limit measure, relational power measure, Shapley value, stochastic processes.

[¤]CentER and Department of Econometrics, Tilburg University, PO Box 90153, 5000 LE Tilburg, The Netherlands. Corresponding author. e-mail address: p.e.m.borm@kub.nl.

[y]CentER and Department of Econometrics, Tilburg University, PO Box 90153, 5000 LE Tilburg, The Netherlands. This author is …nancially supported by the Netherlands Organization for Scienti…c Research (NWO), ESR-grant 510-01-0504.

[z]Department of Business Economics and Marketing, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands.

1 Introduction

A directed graph or digraph is a pair (N, D) where N is a finite set of nodes and $D \subseteq N \times N$ is a binary relation on N representing the set of (directed) arcs. Such a digraph can represent various hierarchical structures which are based on (partial) pairwise comparisons. To focus ideas we will mainly consider the example of a sports competition in which there are teams that play matches against each other. In this case the nodes represent the teams that participate in the competition, while $(i, j) \in D$ means that team i has won the match it played against team j. The model and its implications however seem applicable in a wide variety of instances. A digraph can also represent the results of paired comparison experiments (cf. David (1963)) for example within a group of alternative medicines for a specific (aspect of a) disease. Moreover, within the context of social choice theory, a digraph could summarize the aggregated pairwise preferences of a group of individuals (based for example on majority voting) over a certain set of alternatives. Again in a totally different setting a digraph could represent the hierarchical structure of certain economic organizations.

A digraph competition can be evaluated using a relational power measure being a function that assigns to each node a value representing its importance or 'relational power' in the competition. Such approaches are numerous in the literature and can be found in, e.g., Rubinstein (1980) who uses the (Copeland) score measure in ranking the nodes in tournaments (being complete, asymmetric digraphs), or Laffond, Laslier and LeBreton (1993) who use non-cooperative games in finding the 'winners' of a tournament.

The perspective of the present paper is to look at an arbitrary digraph competition (so not necessarily just tournaments) as a special type of allocation problem where we assume initially that each node is assigned equal weight (say equal to one). Measuring relational power then can be seen as 'fairly' redistributing these weights taking account the hierarchical structure that is represented by the digraph. An adequate analysis of such an allocation problem should not only be based on arguments of individual performance but also on arguments of more relative group performance. Hence we turn to associated cooperative games in coalitional form since this theory aims for fair redistributions based on coalitional considerations. Because of this explicit link to game theory, the nodes in a digraph from now on will be called players. This approach was followed by van den Brink and Borm (1994) who allocated the initial weights of the players in a digraph competition according to the Shapley value of the conservative score game corresponding to the digraph competition. The resulting relational power measure was shown to coincide with the BG-measure as de…ned in van den Brink and Gilles (1994).

As mentioned by van den Brink and Borm (1994) most of their results carry through for general weights, i.e., the initial weights need not be equal to one. Instead of taking initial weights equal to one, it seems natural to take weights that already, somehow, re‡ect the relational power of the players in a digraph competition. For example, one could take the Shapley value of the conservative score game as weights. We exploit this by investigating the limit behavior in the iterative process which repeats the procedure described above by considering the weights as prescribed by the Shapley value at each step as new input weights. Using standard techniques from the theory of stochastic processes, it is shown that this limit is uniquely determined. In particular the limit measure also is a stationary measure: a measure having the property that taking it as input weights yields the same weights as 'outputs'. In general however there can be more than one stationary power measure. Other stationary point or "limit of procedures" approaches can be found for example in Daniels (1969) and Keener (1993).

The same procedure as discussed above can be applied to weighted digraph competitions in which weights are assigned to the arcs to re‡ect in some sense the importance of the arcs. For many applications this seems useful. As an example we mention ranking the teams in a sports competition in which all teams play against each other twice (home and away), or a sports competition in which not all teams play against the other teams the same number of times. Then the weight assigned to an arc $(i;j)$ can be the number of times team i defeated team j. This application could be used when making a ranking list in order to determine the 'best' team in a number of competitions where the set of teams in every competition may di¤er. This is especially useful when making a ranking list based on matches played during a number of years. Another example can be found in paired comparison experiments, e.g., with respect to medicines. Letting the players correspond to the medicines, the weight associated to the arc $(i;j)$ could correspond to the relative performance or e¤ectivity rate of medicine i compared to medicine j. Within social choice theory, the set of players being a set of alternatives from which a speci…c group of agents has to choose, the weight assigned to the arc $(i;j)$ could equal the number of agents that prefer alternative i to alternative j.

We would like to emphasize that this paper is just meant to o¤er a new limit of measure procedure for digraph competitions based on cooperative game theoretical tools, and to discuss possible applications. A comparison of this new cardinal ranking procedure with other scoring or ranking methods on the basis of desirable properties within the general framework of social choice correspondences can be found in Borm, van den Brink, Levinsky and Slikker (2000).

The paper is organized as follows. In Section 2 we discuss the conservative score game and its Shapley value as a way to evaluate digraph competitions. Section 3 introduces the iterative procedure described above. It defines the limit measure and shows its uniqueness. In Section 4 we show that similar results also hold for weighted digraph competitions in which weights are assigned to the arcs, and apply the procedure to an example from the English soccer competition. Finally, in Section 5 we show that the limit measure remains the same if we 'disturb' the underlying Shapley value in an 'egalitarian' way.

2 Preliminaries: the Shapley value of a conservative score game

In this section we present the game and the (one step) measure by which we will evaluate digraph competitions $(N;D)$. We assume the binary relation D to be irreflexive, i.e., $(i;i) \notin D$ for all $i \in N$. The collection of all irreflexive digraphs[1] on N is denoted by $\mathcal{D}^N$.

Let $D \in \mathcal{D}^N$. For $i \in N$ the nodes in $S_D(i) := \{j \in N \mid (i;j) \in D\}$ are called the successors of i in D, and the nodes in $P_D(i) := \{j \in N \mid (j;i) \in D\}$ are called the predecessors of i in D. A relational power measure is a function $f : \mathcal{D}^N \to \mathbb{R}^N$ that assigns an $|N|$-dimensional real vector to every digraph on N. In this paper we consider the relational power measure that is derived from the Shapley value of a related cooperative game.

Van den Brink and Borm (1994) introduce to each digraph $D \in \mathcal{D}^N$ the conservative score game $(N;v_D^c)$ with $v_D^c : 2^N \to \mathbb{R}$ given by $v_D^c(E) = |\{j \in N \mid (P_D(j) \cup \{j\}) \subseteq E\}|$ for all $E \subseteq N$. This game can also be written as

$$v_D^c = \sum_{i \in N} u_{P_D(i) \cup \{i\}};$$

where $u_{P_D(i) \cup \{i\}}$ is the unanimity game[2] on $P_D(i) \cup \{i\}$. The game $(N;v_D^c)$, or shortly v_D^c;

[1] In the sequel we refer to irreflexive digraphs simply as digraphs.

[2] The unanimity game u_T; $T \subseteq N$, assigns the value 1 to all coalitions $E \supseteq T$, and the value 0 to all other coalitions.

assigns to every coalition $E \subset N$ the number of players in E that have no predecessors outside E. Thus, first we implicitly assign to every player an initial weight equal to one. Secondly, our interpretation is that all predecessors of a player (and the player himself) have a rightful direct claim on the weight of that specific player. Consequently, the value of a coalition of players in a conservative score game v_D^c can be seen as the maximal total weight for which there is no rightful direct claim from outside this coalition.

In particular, varying D, and applying a game theoretic solution concept to the game v_D^c yields a specific relational power measure. We will redistribute the initial weights in a digraph competition D by applying the Shapley value to the conservative score game v_D^c. The Shapley value (Shapley (1953)) of a game $(N; v)$ with $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$ is given by

$$Sh_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} \left(v(P(i;\pi) \cup \{i\}) - v(P(i;\pi))\right), \text{ for all } i \in N;$$

where $\Pi(N)$ denotes the collection of all permutations $\pi : N \to N$ on N and $P(i;\pi) = \{j \in N \mid \pi(j) < \pi(i)\}$ for every $\pi \in \Pi(N)$ and $i \in N$. Applying the Shapley value to conservative score games yields the BG-measure $\beta : \mathcal{D}^N \to \mathbb{R}^N$ (cf. van den Brink and Gilles (1994), also for axioms) given by

$$\beta_i(D) = Sh_i(v_D^c) = \sum_{j \in \{i\} \cup S_D(i)} \frac{1}{|P_D(j)| + 1} \text{ for all } i \in N. \qquad (1)$$

Thus, β distributes the initial weight of each node in a digraph equally over itself and all its predecessors[3].

Let us provide a brief motivation for choosing the Shapley value. Next to its rather appealing probabilistic interpretation in terms of the expected marginal contribution of a player when sequentially constructing the grand coalition N, the Shapley value of a conservative score game constitutes the barycentre of the core because of convexity of the game (cf. Shapley (1971)). Hence, the Shapley value can be seen as a nice compromise between all possible coalitionally stable allocations.

Example 2.1. Consider the digraph $D = \{(1,2), (1,3), (2,3), (2,4), (3,4), (4,1)\}$ on $N = \{1,2,3,4\}$. To have an interpretation, let N be a set of medicines and let D summarize the results of a (complete) paired comparison experiment.

According to the ranking by the (Copeland) score-measure medicines 1 and 2 then are ranked equally and are ranked higher than medicines 3 and 4 (which are both

[3] We remark that this is not the 'original' BG-measure considered in van den Brink and Gilles (1994), but the modified version considered in van den Brink and Borm (1994).

ranked equal). However, β provides a complete ranking with medicine 1 as the best one: $\beta(D) = \frac{1}{6}(8, 7, 4, 5)$. For example, $\beta_4(D) = \frac{5}{6}$ consist of two parts: $\frac{1}{2}$ (of the weight of player 1) and $\frac{1}{3}$ (of the weight of 4).

3 An iterative procedure

In the measure β discussed in the previous section it is implicitly assumed that e-very node in a digraph competition has an initial weight equal to one, and measuring relational power is seen as fairly redistributing these weights based on the structure represented by the digraph. Instead of taking initial weights equal to one, it seems natural to take weights that already reflect the relational power of the players. If the measure $\beta^1 := \beta$ determines the weights in the redistibution method discussed in the previous section, then one obtains the second ordermeasure β^2. Of course, this second order measure can be used as new input weights, and so on, yielding higher order measures. We will show that the iterative process which repeats the procedure described above, by considering the $(t-1)^{\text{th}}$-order measure β^{t-1} as new input weights at the t^{th} step, has a unique limit.
This limit measure is a specific stationary power measure in the sense that when taking the weights equal to this measure yields these same weights as 'output' measure. In general however there can be more than one stationary power measure.

Formally, it is done as follows. We start with the game v_D^1 being the conservative score game and consider its Shapley value $\beta(D) = \beta^1(D)$. For $t \in \{2, 3, \ldots\}$ we recursively define the game $v_D^t = \sum_{i \in N} Sh_i(v_D^{t-1}) u_{P_D(i) \cup \{i\}}$. It is easy to verify that for the Shapley value $Sh(v_D^t) = \beta^t(D)$ of the game v_D^t it holds that

$$\beta^1(D) = \beta(D);$$

and

$$\beta_i^t(D) = \sum_{j \in S_D(i) \cup \{i\}} \frac{\beta_j^{t-1}(D)}{|P_D(j)| + 1} \text{ for all } i \in N \text{ and } t \in \{2, 3, \ldots\}.$$

Example 3.1. For the digraph competition D of Example 2.1 it holds that $\beta^2(D) = \frac{1}{36}(53, 39, 18, 34)$; yielding the same ranking as β^1.

In order to prove the existence of the limit measure we turn to the theory of stochastic processes, and present an alternative way to look at a digraph competition. For every

$D \in \mathcal{D}^N$ we define the transition matrix $P(D)$ as the matrix which components are given by $p_{ij} = \frac{1}{|P_D(j)|+1}$ if $(i,j) \in D$ or $i = j$, and $p_{ij} = 0$ otherwise. The system $(N, P(D))$ can be seen as a stochastic process in which N is a set of states an item possibly can be in at time periods $t = 0, 1, 2, \ldots$, and p_{ij} is the probability that the item is in state i at time $t > 0$ given that it was in state j at time $t - 1$. (Note that $P(D)$ is a stochastic matrix (nonnegative and the columns add up to one), with a positive diagonal.) Now, suppose that at $t = 0$ there is exactly one item in every state in N. Then, the measure $\beta^t(D)$ which by definition eqals $(P(T))^t \mathbf{1}$, yields[4] the expected number of items in the states at time $t \geq 1$. The limit measure yields the expected number of items in each state after repeating this procedure infinitely many times. The existence of this limit measure follows from standard results on stochastic matrices as documented in, e.g., Berger (1993), which for the sake of completeness are be presented below.

For the stochastic process $(N, P(D))$ and $i, j \in N$ we denote by ρ_{ij} the probability to ever arrive at i starting from j. Then $i \in N$ is called recurrent if $\rho_{ii} = 1$, and it is called transient if $\rho_{ii} < 1$, i.e., state i is recurrent if with probability one an item starting in state i ever returns to state i, and it is transient if there is a positive probability that the item will not return to state i. By N_R we denote the set of all recurrent states, and by N_T we denote the set of all transient states in $(N, P(D))$. A set $T \subset N$ is a closed set in $P(D)$ if $\rho_{ij} = 0$ for all $i \in T$, $j \notin T$. A set $T \subset N$ is a closed irreducibleset in $P(D)$ if it is closed and $\rho_{ij} > 0$ for all $i, j \in T$.

Closed irreducible sets in the stochastic process $(N, P(D))$ correspond to top cycles in D. The subset $T \subset N$ is a top cycle in $D \in \mathcal{D}^N$ if (i) for every $i, j \in T$ it holds that $(i, j) \in tr(D)$ where $tr(D)$ denotes the transitive closure[5] of D, and (ii) for every $i \in T$ and $h \in N \setminus T$ it holds that $(h, i) \notin D$. The set N_R of recurrent states then coincides with the set of nodes that belong to a top cycle, and the set N_T of transient states consists of the nodes that do not belong to any top cycle.

Proposition 3.2. For every stochastic process $(N, P(D))$ it holds that N_R is a finite union of closed irreducible sets.

Now, for every closed irreducible set T and $j \in N$, let $\rho_T(j)$ be the probability to ever arrive at a state in T starting at state j. Then (i) starting in a state in T we never leave state T, (ii) starting in a recurrent state outside T we never arrive at a state in T, and

[4] By $\mathbf{1}$ we denote the vector with all coordinates equal to one.

[5] The transitive closure $tr(D)$ of $D \in \mathcal{D}^N$ is given by $(i, j) \in tr(D)$ if and only if there exists a sequence of nodes $i_1, \ldots, i_m$ such that $i_1 = i$, $i_m = j$ and $(i_{k-1}, i_k) \in D$ for all $2 \leq k \leq m$.

(iii) starting in a transient state the probability to ever arrive at a state in T is equal to the probability we arrive at a state in T after one step plus the sum over all transient states of the probability that we arrive at that transient state after one step multiplied by the probability that we ever arrive at a state in T starting from this transient state. Thus,

(i) $\rho_T(j) = 1$ if $j \in T$,

(ii) $\rho_T(j) = 0$ if $j \in N_R \setminus T$, and

(iii) $\rho_T(j) = \sum_{i \in T} p_{ij} + \sum_{i \in N_T} p_{ij} \cdot \rho_T(i)$ if $j \in N_T$.

Note that this induces a system of $|N|$ inpependent equations in the $|N|$ unknown variables $\rho_T(j)$; $j \in N$. This yields the following proposition.

Proposition 3.3. For every closed irreducible set T in a stochastic process $(N; P(D))$ the probabilities $\rho_T(j)$; $j \in N$; are uniquely determined.

So, for each state j and closed irreducible set T the probability for j to ever arrive at a state in T is uniquely determined. We are left to determine the probabilities that from state j one ever arrives at a particular state i in T. In order to do that we define a stationarydistribution π_T of the closed irreducible set T in the stochastic process $(N; P(D))$ as a distribution $\pi : T \to \mathbb{R}$ such that (i) $\sum_{i \in T} \pi(i) = 1$, and (ii) $P(D|_T)\pi = \pi$, where $D|_T \in D^T$ is given by $D|_T = \{(i; j) \in D \mid i; j \in T\}$.

Proposition 3.4 (see Berger (1993) Theorem III, p.105). Every closed irreducible set T in a stochastic process $(N; P(D))$ has a unique stationary distribution π_T.

From the propositions stated above it follows that the iterative procedure described in this section has a unique limit distribution.

Theorem 3.5 (see Berger (1993) Corollary X, p. 109). Let every closed irreducible set T in a stochastic process $(N; P(D))$ be a-periodic, i.e. $p_{ii} > 0$ for all $i \in T$. Then

$$\left(\lim_{t \to \infty} (P(D))^t\right)_{ij} = \begin{cases} 0 & \text{if } (i \in N_T) \text{ or } (i \in N_R \text{ and } j \in N_R \setminus T_i) \\ \pi_{T_i}(i) & \text{if } i \in N_R \text{ and } j \in T_i \\ \rho_{T_i}(j) \cdot \pi_{T_i}(i) & \text{if } i \in N_R,\ j \in N_T; \end{cases}$$

where T_i is the closed irreducible set in $(N; P(D))$ containing i.

This theorem immediately yields the following corollary.

Corollary 3.6. For every $D \in \mathcal{D}^N$ it holds that $\lim_{t\to\infty}(P(D))^t \mathbf{1}$. exists and is uniquely determined.

This makes it possible to define the limit measure $\lambda : \mathcal{D}^N \to \mathbb{R}^N$ given by $\lambda(D) = \lim_{t\to\infty}(P(D))^t \mathbf{1}$. The limit measure $\lambda(D)$ then is characterized by

$$\begin{aligned}
\lambda_i(D) &= \left(\left(\lim_{t\to\infty}(P(D))^t\right)\mathbf{1}\right)_i = \sum_{j\in N}\left(\lim_{t\to\infty} P(D)\right)_{ij} \\
&= \begin{cases} \sum_{j\in T_i} \pi_{T_i}(i) + \sum_{j\in N_T} \rho_{T_i}(j) & \text{if } i \in N_R \\ 0 & \text{if } i \in N_T \end{cases} \\
&= \begin{cases} \pi_{T_i}(i)\left(|T_i| + \sum_{j\in N_T} \rho_{T_i}(j)\right) & \text{if } i \in N_R \\ 0 & \text{if } i \in N_T. \end{cases}
\end{aligned}$$

where T_i is the closed irreducible set containing $i \in N_R$.

Example 3.7. Consider the digraph competition of Examples 2.1 and 3.1. Note that N is the only top cycle in this digraph, and thus the limit measure is obtained as the unique stationary power measure which adds up to 4: $\lambda(D) = \frac{4}{23}(8; 6; 3; 6)$. So, now medicines 2 and 4 are ranked equally as second-best.

A relational power measure $f : \mathcal{D}^N \to \mathbb{R}^N$ is a stationary power measure if $P(D)f(D) = f(D)$ for every $D \in \mathcal{D}^N$. The limit measure λ is a stationary power measure. Since the internal stationary distribution within a closed irreducible set only depends on the relations within that closed irreducible set it follows from Proposition 3.4 that the concept of a stationary power measure determines the proportional power distribution within sets. Since all transient states are assigned power value zero by all stationary power measures there is a unique stationary power measure (up to normalization) if there is one closed irreducible set. However, if there is more than one closed irreducible set then the concept of a stationary power measure does not determine the power distribution between closed irreducible sets. However, the limit measure λ does.

Using the limit measure in ranking the players in a digraph competition is a refinement of the top cycle procedure which ranks the players in two groups: the players that belong to a top cycle (or closed irreducible set) and the ones that do not. Not only does λ rank the players within top cycles, it also ranks players across top cycles which also is not done by the top cycle approach. The limit measure λ does not rank the players that do not belong to a top cycle. In a straightforward way this ranking can be refined by computing the limit measure $\lambda(D|_{N_T})$ of the digraph restricted to the transient states. Of course, the procedure can be refined further if necessary.

Example 3.8. Consider the digraph competition D on $N = \{1, 2, 3, 4\}$ given by $D = \{(1, 3), (1, 4), (2, 3), (3, 4)\}$. This digraph competition has two top cycles: $\{1\}$ and $\{2\}$. The limit measure of D is given by $\lambda(D) = \left(\frac{9}{4}, \frac{7}{4}, 0, 0\right)$. In order to rank the nodes 3 and 4 we can consider $\lambda(D|_{\{3,4\}}) = (1, 0)$. So we rank the nodes in the order $(1, 2, 3, 4)$.

We conclude this section by considering a digraph competition that is determined from a real sports competition.

Example 3.9. The first round of the FIFA Soccer World Championship 1998 consisted of (eight) groups of four teams such that each team played each other team in its group exactly once. The scoring rule that was used gave a team three points for a win, one point for a draw, and zero points for a loss. One group consisted of the teams from Brazil (B), Norway (N), Morrocco (M) and Scotland (S). If we represent a draw between two teams by no relation between the two then the results of the matches are described by the digraph competition D on $N = \{B, N, M, S\}$ given by $D = \{(B, M), (B, S), (N, B), (M, S)\}$. The FIFA-scores assigned to the teams are 6 for Brazil, 5 for Norway, 4 for Morrocco and 1 for Scotland. So, Brazil is ranked highest in this group with Norway ranked second.
Looking at the digraph D we see that it has one top-cycle which consists only of Norway. So, our limit measure would rank Norway highest. Applying the limit measure to the competition restricted to the other three teams we see that Brazil is the only top cycle in the restricted competition, so Brazil is ranked second. So, comparing these two scoring rules we see that the FIFA score ranks Brazil highest, while the limit measure ranks Norway highest.

4 Weighted digraph competitions

In this section we generalize the results discussed before to weighted digraph competitions in which we put weights on the arcs. A weighted digraph on N is a function $\omega : N \times N \to \mathbb{R}_+$. We assume that $\omega(i, i) > 0$ for all $i \in N$. The collection of all weighted digraphs on N is denoted by W^N.

The direct generalization of β to weighted digraphs is given by

$$\beta_i(\omega) = \sum_{j \in N} \left(\frac{\omega(i,j)}{\sum_{h \in N} \omega(h,j)} \right) \quad \text{for all } i \in N \text{ and } \omega \in W^N \tag{2}$$

For every $\omega \in W^N$ we define the transition matrix $P(\omega)$ as the matrix with entries $p_{ij} = \frac{\omega(i,j)}{\sum_{h \in N} \omega(h,j)}$ ($i \in N$, $j \in N$). Multiplying $(P(\omega))^t$, $t \in N$, with the vector 1 with all elements equal to one, one obtains the t^{th}-order weighted measure $\beta^t(\omega)$. Since $P(\omega)$ is a stochastic matrix with a positive diagonal similar results as stated for (non-weighted) digraph competitions can be derived. A closed irredecible set in $P(\omega)$ now coincides with a set of nodes $T \subset N$ such that (i) $\omega(i,j) = 0$ for all $i \in N \setminus T$, $j \in T$, and (ii) for every $i, j \in T$ there exists a sequence of nodes $h_1, \ldots, h_m$ such that $h_1 = i$, $h_m = j$, and $\omega(h_t, h_{t+1}) > 0$ for all $t \in \{1, \ldots, m-1\}$. Consequently, $\lambda(\omega) = \lim_{t \to \infty} \beta^t(\omega)$ is well-defined.

Generalizing the results to weighted digraph competitions broadens the possibilities of applications of the (generalized) limit measure λ. As an example we mention ranking the teams in a sports competitions in which all teams play against each other twice (home and away), or a sports competitions in which not all teams play against other teams the same number of times. Then the weight $\omega(i,j)$ can be the number of times team i defeated team j. This application can be used when making a ranking list in order to determine the 'best' team in a number of competitions such that the set of teams in every competition differs (for example by promoting to and from higher, respectively, lower competitions).

We assumed the weights $\omega(i,i)$, $i \in N$ to be positive. The exact value of these weights depends on the specific application one has in mind. In the example of a sports competition in which each team plays against each other team twice and $\omega(i,j)$, $i \neq j$, being the number of times team i defeated team j, it seems natural to take $\omega(i,i) = 2$ for all $i \in N$. This can be seen as i playing two matches against itself (similarly as against all other teams).
We conclude this section by computing the limit measure λ for the digraph competition that is derived from the Carling Premier League 1997-1998.

Example 4.1. In this example we compare the ranking of the soccer teams that participated in the Carling Premier League 1997-1998 (the highest division in English soccer in that year) based on the limit measure to the ranking actually used. The second column of Table 1 provides the names of the teams in the order they were actually ranked. This

	Team	Wins	Draws	Losses	CPL-score	¸	¸-ranking
1	Arsenal	23	9	6	78	1.8742	1
2	Manchester United	23	8	7	77	1.6309	2
3	Liverpool	18	11	9	65	1.5217	3
4	Chelsea	20	3	15	63	1.1518	7
5	Leeds United	17	8	13	59	1.0995	8
6	Blackburn Rovers	16	10	12	58	1.1956	6
7	Aston Villa	17	6	15	57	1.2936	4
8	West Ham United	16	8	14	56	1.0161	11
9	Derby County	16	7	15	55	1.2133	5
10	Leicester City	13	14	11	53	1.0787	9
11	Coventry City	12	16	10	52	0.8438	13
12	Southampton	14	6	18	48	1.0391	10
13	Newcastle United	11	11	16	44	0.5937	18
14	Tottenham Hotspur	11	11	16	44	0.5956	17
15	Wimbledon	10	14	14	44	0.6528	15
16	She¢eld Wednesday	12	8	18	44	0.9645	12
17	Everton	9	13	16	40	0.6589	14
18	Bolton Wanderers	9	13	16	40	0.5676	19
19	Barnsley	10	5	23	35	0.6144	16
20	Crystal Palace	8	9	21	33	0.4212	20

Table 1. Scores and rankings Carling Premier League 1997-1998.

ranking is based on the CPL-score that is given the sixth column. This score is obtained by giving each team three points for a win, one point for a draw and zero points for a loss (so, it is the same scoring rule as used by the FIFA in Example 3.9). The number of matches won, draws and matches lost are given in columns 3, 4 and 5, respectivly. The score according to the limit measure ¸ is given in the eighth column, with corresponding ranking in the ninth column. In order to determine ¸ we need the actual digraph competition, i.e., we need to know for every match which team won or whether there was a draw. These data with the corresponding transition matrix are given in the appendix.

Without giving a full comparison of the two rankings we just point to some striking di¤erences. At the top nothing changes with respect to the three highest ranked teams

(Arsenal, Manchester United and Liverpool). However, at the bottom we see that Barnsley is ranked nineteen according to the CPL-score, while according to λ it is ranked sixteen. This has the following important consequence. The three lowest ranked teams at the end of the competition are relegated and have to leave the Premier League. So, Barnsley is actually relegated by the CPL-score, but would not have been relegated if λ had been used. Instead, Newcastle United (ranked thirteen according to CPL-score) should have been relegated according to λ with rank eighteen.

5 Compound measures

In Section 3 we applied an iterative procedure to the measure β and showed the existence and uniqueness of the limit distribution λ. This limit remains the same if we 'disturb' β in an 'egalitarian' way. Formally this can be done by taking convex combinations of β and the egalitarian measure $\gamma : \mathcal{D}^N \to \mathbb{R}^N$ which assigns to every node in a digraph competition $D \in \mathcal{D}^N$ a value equal to one, i.e., $\gamma_i(D) = 1.$ Convex combinations $\beta^{\alpha}(D) = \alpha\beta(D) + (1-\alpha)\gamma(D)$ for $\alpha \in [0,1]$ and digraph[6] $D \in \mathcal{D}^N$ are refered to as compound measures.

We now apply our iterative procedure with respect to the compound measure β^{α} and initial distribution 1, i.e., each node having equal initial weight one. Clearly, for $\alpha = 0$ the limit distribution equals 1. Take $\alpha \in (0,1]$ and consider the stochastic process $(N, P^{\alpha}(D))$ with the stochastic matrix $P^{\alpha}(D) = \alpha P(D) + (1-\alpha)I$, where I denotes the identity matrix. The transition matrix $P^{\alpha}(D)$ has components given by $p^{\alpha}_{ij} = \alpha p_{ij}$ if $i \neq j$, and $p^{\alpha}_{i,j} = \alpha p_{i,j} + (1-\alpha)$ if $i = j$. Following the same reasoning as before, it readily follows that closed irreducible sets in $(N, P(D))$ and $(N, P^{\alpha}(D))$ are the same. The probabilities $\rho^{\alpha}_T(j)$ that we ever arrive at a state in closed irreducible set T given that we start at state j are given by

(i) $\rho^{\alpha}_T(j) = 1$ if $j \in T$;

(ii) $\rho^{\alpha}_T(j) = 0$ if $j \in N_R \backslash T$, and

(iii) for every $j \in N_T$ it holds that $\rho^{\alpha}_T(j) = \Sigma_{i\in T} p^{\alpha}_{ij} + \Sigma_{i\in N_T} p^{\alpha}_{ij} \cdot \rho^{\alpha}_T(i) =$

$= \Sigma_{i\in T}\alpha p_{ij} + \Sigma_{i\in N_T}\alpha p_{ij} \cdot \rho^{\alpha}_T(i) + (1-\alpha)\rho^{\alpha}_T(j);$

which yields that $\alpha\rho^{\alpha}_T(j) = \Sigma_{i\in T}\alpha p_{ij} + \Sigma_{i\in N_T}\alpha p_{ij} \cdot \rho^{\alpha}_T(i)$, and thus

[6] In the context of arbitrary cooperative games, convex combinations of the egalitarian rule and Shapley value are considered in Joosten (1996).

$$\rho_T^{\alpha}(j) = \Sigma_{i \in T}\, p_{ij} + \Sigma_{i \in N_T}\, p_{ij} \cdot \rho_T^{\alpha}(i).$$

Comparing this to the corresponding equations (i), (ii), (iii) in section 3 (between Propositions 3.2 and 3.3) yields that $\rho_T^{\alpha}(j) = \rho_T(j)$. Further, for closed irreducible set T it also holds that $P^{\alpha}(D|_T)\pi = \pi$ if and only if $P(D|_T)\pi = \pi$, and thus stationary distribution $\pi_T^{\alpha} = \pi_T$. Hence, Theorem 3.5 is also valid for $P^{\alpha}(D)$ such that all limit measures for $\alpha \in (0,1]$ yield the same limit measure λ.

Theorem 5.1. For every $D \in \mathcal{D}^N$ and $\alpha \in (0,1]$ it holds that $\lim_{t \to \infty} (P^{\alpha}(D))^t 1$ equals $\lambda(D)$.

Appendix

The results of the matches played in the Carling Premier League 1997-1998 of Example 4.1 are given in Table 2 (Arsenal (A), Aston Villa (AV), Barsnely (B), Blackburn Rovers (BR), Bolton Wanderers (BW), Crystal Palace (CP), Chelsea (C), Coventry City (CC), Derby County (DC), Everton (E), Leeds United (LU), Leicester City (LC), Liverpool (L), Manchester United (MU), Newcastle United (NU), Sheffield Wednesday (SW), Southampton (S), Tottenham Hotspur (TH), West Ham United (WHU), and Wimbledon (W)). In the top half (above the empty diagonal) in every cel we first put the number of goals scored by the row player and second the number of goals scored by the column player in the match corresponding to that cel. In the bottom half (under the diagonal) this is the other way around.

The corresponding transition matrix is obtained by letting $\omega(i,j)$ be the number of matches that team i won from team j (with $\omega(i,i) = 2$ for every $i \in N$) and given in Table 3.

	A	AV	B	BR	BW	CP	C	CC	DC	E	LU	LC	L	MU	NU	SW	S	TH	WHU	W
A		0-0	5-0	1-3	4-1	1-0	2-0	2-0	1-0	4-0	2-1	2-1	0-1	3-2	3-1	1-0	3-0	0-0	4-0	5-0
AV	1-0		0-1	0-4	1-3	3-1	0-2	3-0	2-1	2-1	1-0	1-1	2-1	0-2	0-1	2-2	1-1	4-1	2-0	1-2
B	0-2	0-3		1-1	2-1	1-0	0-6	2-0	1-0	2-2	2-3	0-2	2-3	0-2	2-2	2-1	4-3	1-1	1-2	2-1
BR	1-4	5-0	2-1		3-1	2-2	1-0	0-0	1-0	3-2	3-4	5-3	1-1	1-3	1-0	7-2	1-0	0-3	3-0	0-0
BW	0-1	0-1	1-1	2-1		5-2	1-0	1-5	3-3	0-0	2-3	2-0	1-1	0-0	1-0	3-2	0-0	1-1	1-1	1-0
CP	0-0	1-1	0-1	1-2	2-2		0-3	0-3	3-1	1-3	0-2	0-3	0-3	0-3	1-2	1-0	1-1	1-3	3-3	0-3
C	2-3	0-1	2-0	0-1	2-0	6-2		3-1	4-0	2-0	0-0	1-0	4-1	0-1	1-0	1-0	4-2	2-0	2-1	1-1
CC	2-2	1-2	1-0	2-0	2-2	1-1	3-2		1-0	0-0	0-0	0-2	1-1	3-2	2-2	1-0	1-0	4-0	1-1	0-0
DC	3-0	0-1	1-0	3-1	4-0	0-0	0-1	3-1		3-1	0-5	0-4	1-0	2-2	1-0	3-0	4-0	2-1	2-0	1-1
E	2-2	1-4	4-2	1-0	3-2	1-2	3-1	1-1	1-2		2-0	1-1	2-0	0-2	0-0	1-3	0-2	0-2	2-1	0-0
LU	1-1	1-1	2-1	4-0	2-0	0-2	3-1	3-3	4-3	0-0		0-1	0-2	1-0	4-1	1-2	0-1	1-0	3-1	1-1
LC	3-3	1-0	1-0	1-1	0-0	1-1	2-0	1-1	1-2	0-1	1-0		0-0	0-0	0-0	1-1	3-3	3-0	2-1	0-1
L	4-0	3-0	0-1	0-0	2-1	2-1	4-2	1-0	4-0	1-1	3-1	1-2		1-3	1-0	2-1	2-3	4-0	5-0	2-0
MU	0-1	1-0	7-0	4-0	1-1	2-0	2-2	3-0	2-0	2-0	3-0	0-1	1-1		1-1	6-1	1-0	2-0	2-1	2-0
NU	0-1	1-0	2-1	1-1	2-1	1-2	3-1	0-0	0-0	1-0	1-1	3-3	1-2	0-1		2-1	2-1	1-0	0-1	1-3
SW	2-0	1-3	2-1	0-0	5-0	1-3	1-4	0-0	2-5	3-1	1-3	1-0	3-3	2-0	2-1		1-0	1-0	1-1	1-1
S	1-3	1-2	4-1	3-0	0-1	1-0	1-0	1-2	0-2	2-1	0-2	2-1	1-1	1-0	2-1	2-3		3-2	3-0	0-1
TH	1-1	3-2	3-0	0-0	1-0	0-1	1-6	1-1	1-0	1-1	0-1	1-1	3-3	0-2	2-0	3-2	1-1		1-0	0-0
WHU	0-0	2-1	6-0	2-1	3-0	4-1	2-1	1-0	0-0	2-2	3-0	4-3	2-1	1-1	0-1	1-0	2-4	2-1		3-1
W	0-1	2-1	4-1	0-1	0-0	0-1	0-2	1-2	0-0	0-0	1-0	2-1	1-1	2-5	0-0	1-1	1-0	2-6	1-2	

Table 2. Results of all matches played in Carling Premier League 1997-1998

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	A	AV	B	BR	BW	CP	C	CC	DC	E	LU	LC	L	MU	NU	SW	S	TH	WHU	W
A	$\frac{2}{8}$	0	$\frac{2}{25}$	$\frac{1}{14}$	$\frac{2}{18}$	$\frac{1}{23}$	$\frac{2}{17}$	$\frac{1}{12}$	$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{15}$	$\frac{1}{13}$	0	$\frac{2}{9}$	$\frac{2}{18}$	$\frac{1}{20}$	$\frac{2}{20}$	0	$\frac{1}{16}$	$\frac{2}{16}$
AV	$\frac{1}{8}$	$\frac{2}{17}$	$\frac{1}{25}$	0	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	$\frac{2}{12}$	$\frac{2}{17}$	$\frac{2}{18}$	$\frac{1}{15}$	0	$\frac{1}{11}$	0	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{16}$	0
B	0	$\frac{1}{17}$	$\frac{2}{25}$	0	$\frac{1}{18}$	$\frac{2}{23}$	0	$\frac{1}{12}$	$\frac{1}{17}$	0	0	0	$\frac{1}{11}$	0	0	$\frac{1}{20}$	$\frac{1}{20}$	0	0	$\frac{1}{16}$
BR	$\frac{1}{8}$	$\frac{2}{17}$	$\frac{1}{25}$	$\frac{2}{14}$	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{2}{17}$	0	$\frac{1}{17}$	$\frac{1}{18}$	0	$\frac{1}{13}$	0	0	$\frac{1}{18}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{16}$	$\frac{1}{16}$
BW	0	$\frac{2}{17}$	0	$\frac{1}{14}$	$\frac{2}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	0	0	0	$\frac{1}{13}$	0	0	$\frac{1}{18}$	$\frac{1}{20}$	$\frac{1}{20}$	0	0	$\frac{1}{16}$	
CP	0	0	0	0	0	$\frac{2}{23}$	0	0	$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{15}$	0	0	0	$\frac{1}{18}$	$\frac{2}{20}$	0	$\frac{1}{18}$	0	$\frac{1}{16}$
C	0	$\frac{1}{17}$	$\frac{2}{25}$	0	$\frac{1}{18}$	$\frac{2}{23}$	$\frac{2}{17}$	$\frac{1}{12}$	$\frac{2}{17}$	$\frac{1}{18}$	0	$\frac{1}{13}$	$\frac{1}{11}$	0	$\frac{1}{18}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{2}{18}$	$\frac{1}{16}$	$\frac{1}{16}$
CC	0	0	$\frac{1}{25}$	$\frac{1}{14}$	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	$\frac{2}{12}$	$\frac{1}{17}$	0	0	0	0	$\frac{1}{9}$	0	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{18}$	0	$\frac{1}{16}$
DC	$\frac{1}{8}$	0	$\frac{1}{25}$	$\frac{1}{14}$	$\frac{1}{18}$	0	0	$\frac{1}{12}$	$\frac{2}{17}$	$\frac{2}{18}$	0	$\frac{1}{13}$	$\frac{1}{11}$	0	$\frac{1}{18}$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{1}{18}$	$\frac{1}{16}$	0
E	0	0	$\frac{1}{25}$	$\frac{1}{14}$	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	0	0	$\frac{2}{18}$	$\frac{1}{15}$	$\frac{1}{13}$	$\frac{1}{11}$	0	0	0	0	0	$\frac{1}{16}$	0
LU	0	0	$\frac{2}{25}$	$\frac{2}{14}$	$\frac{2}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	0	$\frac{2}{17}$	0	$\frac{2}{15}$	0	0	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{18}$	$\frac{1}{16}$	0
LC	0	$\frac{1}{17}$	$\frac{2}{25}$	0	0	$\frac{1}{23}$	$\frac{1}{17}$	$\frac{1}{12}$	$\frac{1}{17}$	0	$\frac{2}{15}$	$\frac{2}{13}$	$\frac{1}{11}$	$\frac{1}{9}$	0	0	0	$\frac{1}{18}$	$\frac{1}{16}$	0
L	$\frac{2}{8}$	$\frac{1}{17}$	$\frac{1}{25}$	0	$\frac{1}{18}$	$\frac{2}{23}$	$\frac{1}{17}$	$\frac{1}{12}$	$\frac{1}{17}$	0	$\frac{2}{15}$	0	$\frac{2}{11}$	0	$\frac{2}{18}$	$\frac{1}{20}$	0	$\frac{1}{18}$	$\frac{1}{16}$	$\frac{1}{16}$
MU	0	$\frac{2}{17}$	$\frac{2}{25}$	$\frac{2}{14}$	0	$\frac{2}{23}$	$\frac{1}{17}$	$\frac{1}{12}$	$\frac{1}{17}$	$\frac{2}{18}$	$\frac{1}{15}$	0	$\frac{1}{11}$	$\frac{2}{9}$	$\frac{1}{18}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{18}$	$\frac{1}{16}$	$\frac{2}{16}$
NU	0	$\frac{2}{17}$	$\frac{1}{25}$	0	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	0	0	$\frac{1}{18}$	0	0	0	0	$\frac{2}{18}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{16}$	0
SW	$\frac{1}{8}$	0	$\frac{1}{25}$	0	$\frac{1}{18}$	0	0	0	0	$\frac{2}{18}$	$\frac{1}{15}$	$\frac{1}{13}$	0	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{1}{18}$	0	0
S	0	0	$\frac{1}{25}$	$\frac{1}{14}$	0	$\frac{1}{23}$	$\frac{1}{17}$	0	0	$\frac{2}{18}$	$\frac{1}{15}$	$\frac{1}{13}$	$\frac{1}{11}$	$\frac{1}{9}$	$\frac{1}{18}$	0	$\frac{2}{20}$	$\frac{1}{18}$	$\frac{2}{16}$	0
TH	0	$\frac{1}{17}$	$\frac{1}{25}$	$\frac{1}{14}$	$\frac{1}{18}$	$\frac{1}{23}$	0	0	$\frac{1}{17}$	$\frac{1}{18}$	0	0	0	0	$\frac{1}{18}$	$\frac{1}{20}$	0	$\frac{2}{18}$	$\frac{1}{16}$	$\frac{1}{16}$
WHU	0	$\frac{1}{17}$	$\frac{2}{25}$	$\frac{1}{14}$	$\frac{1}{18}$	$\frac{1}{23}$	$\frac{1}{17}$	$\frac{1}{12}$	0	0	$\frac{1}{15}$	$\frac{1}{13}$	$\frac{1}{11}$	0	$\frac{1}{18}$	$\frac{1}{20}$	0	$\frac{1}{18}$	$\frac{2}{16}$	$\frac{2}{16}$
W	0	$\frac{2}{17}$	$\frac{1}{25}$	0	0	$\frac{1}{23}$	0	0	0	0	$\frac{1}{15}$	$\frac{2}{13}$	0	0	$\frac{1}{18}$	0	$\frac{2}{20}$	0	0	$\frac{2}{16}$

Table 3. Transition matrix of Carling Premier League 1997-1998