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Inheritance of properties in NTU communication situations

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Abstract

In this paper we consider communication situations in which utility is nontransferable. We compare this model with the more familiar model of transferable utility communication situations and point out an odd feature of the latter. We mainly focus on the inheritance of properties of the underlying game to the graph-restricted game.

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1 Introduction

In cooperative game theory the central question is how to divide the value of the grand coalition in a fair way, given the values of all subcoalitions. The value of a coalition is interpreted as the maximum (monetary) amount the members of that coalition can obtain if they cooperate. Often, however, this hypothetical maximum is based on some simplifying assumptions on the underlying problem. Eg, in linear production situations (cf. Owen (1975)), it is assumed that all the agents in a coalition are physically able to pool their resources. But one can imagine that as a result of transportation difficulties, cooperation between certain players is restricted.

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Myerson (1977) models such a problem as a communication situation, which consists of an underlying game (eg, linear production game) and an undirected graph representing the players' communication possibilities (eg, transport routes). A communication situation gives rise to a graph-restricted game, in which the value of a coalition of players reflects their underlying possibilities as well as their ability to realise them. A recent overview of communication situations and related models is provided by Slikker and Van den Nouweland (2001).

The literature on communication situations mainly focuses on the case in which the underlying game is a transferable utility (TU) game, which then gives rise to a TU graph-restricted game. In section 3, we explain why this approach seems an inadequate (perhaps even self-contradictory) way to model the communication restrictions. To address this, we consider nontransferable utility (NTU) communication situations.

Myerson (1977) proposes the Shapley value of the graph-restricted game (later called the Myerson value) as solution concept for TU communication situations. We use the MC value, which is an NTU generalisation of the Shapley value (cf. Otten et al. (1998)), to extend the Myerson value to the class of NTU communication situations.

Van den Nouweland and Borm (1991) and Slikker (2000) study the inheritance of properties in TU communication situations, ie, given a certain property of TU games, they provide necessary and sufficient conditions that a graph must satisfy such that for every game satisfying that property, the graph-restricted game satisfies the same property. We extend their analysis to NTU communication situations and relate the two models.

The paper is organised as follows. In section 2, we introduce some notation and basic definitions. In section 3, we define graph-restricted games, discuss the models of TU and NTU communication situations and extend the Myerson value. Finally, in section 4, inheritance of properties is analysed.

2 Notation and basic definitions

The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative reals by $\mathbb{R}_+$. By $\mathbb{R}^N$ we denote the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For $S \subset N, S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x_i)_{i \in S}$. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$ for all $i \in N$ and $y > x$

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The comprehensive convex hull of a set $A \subset \mathbb{R}^S, S \subset N, S \neq \emptyset$ is defined by

$$cc(A) = \{x \in \mathbb{R}^S \mid \exists_{x^1,\dots,x^t \in A} \exists_{(\lambda_1,\dots,\lambda_t) \in \Delta^t} : x \leq \sum_{i=1}^{t} \lambda_i x^i\},$$

where $\Delta^t = \{\lambda \in \mathbb{R}^t_+ \mid \sum_{i=1}^t \lambda_t = 1\}$. For $S \subset N, S \neq \emptyset$ and $x \in \mathbb{R}$ we define

$$Z^{S,x} = \{y \in \mathbb{R}^S \mid \sum_{i \in S} y_i \leq x\}$$

and

$$\bar{Z}^{S,x} = \{y \in \mathbb{R}^S \mid \sum_{i \in S} y_i \leq x, \forall_{i \in S} : y_i \leq x\}.$$

A *communication network* is an undirected graph (N, E), where the vertices $N = \{1, \dots, n\}$ represent the players and the edges $E \subset \{\{i, j\} \mid i, j \in N, i \neq j\}$ represent the (bilateral) communication links between the players. We denote the set of communication networks with player set N by CN^N.

Let $(N, E) \in CN^N$. For all $S \subset N$ we define

$$E(S) = \{\{i, j\} \in E \mid i, j \in S\},$$

the set of links between members of S.

A *path* in (N, E) is a sequence of players $(x_1, \dots, x_t)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{1, \dots, t-1\}$. A *cycle* is a path $(x_1, \dots, x_t)$ with $t \geq 4$, $x_t = x_1$ and $x_1, \dots, x_{t-1}$ all distinct points. Two players $i, j \in N, i \neq j$ are *connected* if there exists a path $(i, x_1, \dots, x_t, j)$.

(N, E) is called

- *empty* if $E = \emptyset$;
- *complete* if $E = \{\{i, j\} \mid i, j \in N, i \neq j\}$;
- *connected* if each pair $i, j \in N, i \neq j$ is connected;
- *cycle-free* if it does not contain a cycle;
- *cycle-complete* if for every cycle $(x_1, \dots, x_t)$, $\{x_i, x_j\} \in E$ for all $i, j \in \{1, \dots t\}, i \neq j$;
- a *star* if there exists an $i \in N$ such that $E = \{\{i, j\} \mid j \in N \backslash \{i\}\}$.

For $S \subset N$ we denote the components of S with respect to (N, E) by S/E, ie, $S/E = \{S_1, \ldots, S_m\}$ with

- $(S_i, E(S_i))$ is connected for all $i \in \{1, \ldots, m\}$;
- $S_i \cap S_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}, i \neq j$;
- $(S_i \cup S_j, E(S_i \cup S_j))$ is not connected for all $i, j \in \{1, \ldots, m\}, i \neq j$;
- $S_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$.

A *transferable utility* or *TU game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the value function assigning to every coalition $S \subset N$ a value $v(S)$. By convention, $v(\emptyset) = 0$. For the sake of convenience, we also assume $v(\{i\}) = 0$ for all $i \in N$. We denote the class of TU games with player set N by TU^N.

A *cooperative game with nontransferable utility*, or *NTU game*, is described by a pair (N, V), where $N = \{1, \ldots, n\}$ is the set of players and V is the payoff map assigning to each coalition $S \subset N, S \neq \emptyset$ a subset $V(S)$ of $\mathbb{R}^S$ such that, for all $i \in N$,

$$V(\{i\}) = (-\infty, 0]$$

and for all $S \subset N, S \neq \emptyset$ we have

$V(S)$ is nonempty and closed,

$V(S)$ is comprehensive, ie, $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$,

$V(S) \cap \mathbb{R}^S_+$ is bounded.

Moreover, we assume that (N, V) is *monotonic*: for all $S \subset T \subset N, S \neq \emptyset$ and for all $x \in V(S)$ there exists a $y \in V(T)$ such that $y_S \geq x$. Note that we do not define $V(\emptyset)$. For ease of notation, we sometimes use V rather than (N, V) to denote an NTU game. The set of all NTU games with player set N is denoted by NTU^N.

Player $i \in N$ is called a *dummy player* in $(N, V) \in NTU^N$ if

$$V(S \cup \{i\}) = V(S) \times V(\{i\})$$

for all $S \subset N \backslash \{i\}, S \neq \emptyset$.

Let $(N, V) \in NTU^N$. The set of *weak Pareto efficient allocations* for coalition $S \subset N, S \neq \emptyset$, denoted by $WPar(V, S)$, is defined by

$$WPar(V, S) = \{x \in V(S) \,|\, \nexists_{y \in V(S)} : y > x\}$$

and the set of *individually rational allocations* for coalition $S \subset N, S \neq \emptyset$ is defined by

$$IR(V, S) = V(S) \cap \mathbb{R}^S_+.$$

The set of *reasonable* allocations for $S \subset N, S \neq \emptyset$ consists of all weak Pareto efficient and individually rational allocations:

$$R(V, S) = WPar(V, S) \cap IR(V, S).$$

The subgame of (N, V) with respect to coalition $S \subset N, S \neq \emptyset$ is defined as the NTU game (S, V_S) with $V_S(T) = V(T)$ for all $T \subset S, T \neq \emptyset$.

The *core* of (N, V) consists of those elements of $V(N)$ for which it holds that no coalition $S \subset N, S \neq \emptyset$ can improve its payoff on its own:

$$C(V) = \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\}.$$

The game (N, V) is called *balanced* if its core is nonempty and *totally balanced* if the cores of all its subgames are nonempty. This definition of the core generalises the definition of the core of a TU game (N, v):

$$C(v) = \{x \in \mathbb{R}^N \,|\, \sum_{i \in N} x_i = v(N), \forall_{S \subset N} : \sum_{i \in S} x_i \geq v(S)\}.$$

(N, V) is called *superadditive* if for all $S, T \subset N, S \cap T = \emptyset, S, T \neq \emptyset$ we have

$$V(S) \times V(T) \subset V(S \cup T)$$

and *individually superadditive* if for all $i \in N$ and for all $S \subset N \backslash \{i\}, S \neq \emptyset$ we have

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}).$$

Note that individual superadditivity is stronger than monotonicity.

An NTU game (N, V) is called *individual merge convex* (cf. Hendrickx et al. (2000)) if it is individually superadditive and it satisfies the individual merge property, ie, for all $k \in N$ and all $S \subsetneq T \subset N \backslash \{k\}$ such that $S \neq \emptyset$, the following statement is true: for all $p \in R(V, S)$, all $q \in V(T)$ and all $r \in V(S \cup \{k\})$ such that $r_S \geq p$ there exists an $s \in V(T \cup \{k\})$ such that

$$\begin{cases} \forall_{i \in T} : s_i \geq q_i, \\ s_k \geq r_k. \end{cases}$$

The idea behind the individual merge property is as follows: whatever allocations the coalitions S and T agree upon separately, given an allocation for coalition $S \cup \{k\}$ such that the members of S are willing to let player k join, player k can obtain a (weakly) better payoff by joining the larger coalition T.

A *population monotonic allocation scheme* or *pmas* for (N, V) is a collection of vectors $y^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$ such that

$$y^S \in WPar(V, S)$$

for all $S \subset N, S \neq \emptyset$ and

$$y_i^S \leq y_i^T$$

if $S, T \subset N$ and $i \in N$ are such that $S \subset T$ and $i \in S$. Note that this definition generalises the definition of pmas for TU games as given in Sprumont (1990) and correspondingly, a pmas induces a core element in every subgame.

An *ordering* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(k)$ denotes which player in N is at position k. The set of all $n!$ orderings of N is denoted by $\Pi(N)$. We define the *marginal vector* $M^\sigma(V)$ corresponding to the ordering $\sigma \in \Pi(N)$ by

$$\begin{aligned} M^\sigma_{\sigma(k)}(V) = \quad \max\{x_{\sigma(k)} \mid \quad & x \in V(\{\sigma(1), \ldots, \sigma(k)\}), \\ & \forall_{i \in \{1, \ldots, k-1\}} : x_{\sigma(i)} = M^\sigma_{\sigma(i)}(V)\} \end{aligned}$$

for all $k \in \{1, \ldots, n\}$. Note that the maximums are well defined as a result of the assumptions of monotonicity, closedness and boundedness. By construction, $M^\sigma(V) \in WPar(V, N)$. If a game is individually superadditive, then all marginal vectors belong to $IR(V, N)$. The marginal vector $m^\sigma(v)$ of a TU game is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}),$$

which is a special case of the NTU definition.

An NTU game (N, V) is called *marginal convex* if for all $\sigma \in \Pi(N)$ we have

$$M^\sigma(V) \in C(V).$$

In Hendrickx et al. (2000), it is shown that individual merge convexity is a stronger property than marginal convexity.

One important aspect of both individual merge convexity and marginal convexity is that within the class of NTU games that correspond to TU games, they are equivalent and coincide with TU convexity:

$$v(S \cap T) + v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \subset N$. Another property of these concepts is that if an NTU game (N, V) satisfies some form of convexity, then all its subgames do.

The Shapley value of (N, v) (Shapley (1953)) is defined by

$$\Phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

An NTU generalisation of the Shapley value is the *marginal based compromise value* or *MC value*, introduced in Otten et al. (1998). It is defined as

$$MC(V) = \alpha_V \sum_{\sigma \in \Pi(N)} M^\sigma(V),$$

where $\alpha_V = \max\{\alpha \in \mathbb{R}_+ \mid \alpha \sum_{\sigma \in \Pi(N)} M^\sigma(V) \in V(N)\}$.

To round off this section, a *TU communication situation* is a triple (N, v, E), where (N, v) is an underlying TU game and (N, E) is a communication network with the same player set. Similarly, an *NTU communcation situation* is a triple (N, V, E). The classes of TU communication situatons and NTU communication situations with player set N are denoted by TUC^N and $NTUC^N$, respectively.

3 Graph-restricted games

In this section, we define graph-restricted games, starting with TU games. We point out why TU graph-restricted games might not be a satisfactory way of modelling the role of the communication restrictions. To address this, we define NTU graph-restricted games and compare the two models.

Let $(N, v, E) \in TUC^N$. The game $(N, v) \in TU^N$ represents the underlying possibilities of the players. However, these possibilities cannot come all to fruition because of the communication restrictions represented by $(N, E) \in CN^N$. The *graph-restricted game* (N, v^E) (Myerson (1977)) takes these restrictions into account by considering the values of the components that can communicate:

$$v^E(S) = \sum_{C \in S/E} v(C)$$

for all $S \subset N$.

The resulting graph-restricted game is again a TU game and hence, side payments between the players through binding contracts are assumed to be possible, even between players that cannot communicate.

The most obvious way to address this contradictory element of the model is to consider NTU games. So, let $(N, V, E) \in NTU^N$. The graph-restricted game (N, V^E) (Slikker and Van den Nouweland (2001)) is defined by

$$V^E(S) = \prod_{C \in S/E} V(C)$$

for all $S \subset N, S \neq \emptyset$. Note that the graph-restricted game V^E is again an element of NTU^N, satisfying all the basic assumptions as stated in the previous section.

In particular, NTU graph-restricted games can be constructed for NTU games that arise from TU games. So, a TU communication situation $(N, v, E) \in TUC^N$ gives rise to two graph-restricted games: (N, v^E) and (N, V^E). But whereas side payments between players that cannot communicate are still possible in (N, v^E), they are ruled out in (N, V^E). In the remainder of this section, we study the relation between these two graph-restricted games.

The difference between the two games is illustrated in the following example.

Example 3.1 Consider the communication situation $(N, v, E) \in TUC^N$ with $N = \{1, 2, 3\}$, $v(S) = 1$ for all $S \subset N, |S| \geq 2$ and $E = \{\{1, 2\}\}$, so players 1 and 2 can communicate, while player 3 cannot communicate with either of them.

In the TU graph-restricted game (N, v^E), the value of the grand coalition equals

$$v^E(N) = v(\{1, 2\}) + v(\{3\}) = 1 + 0 = 1,$$

while in the NTU graph-restricted game (N, V^E),

$$V^E(N) = V(\{1, 2\}) \times V(\{3\}) = \{x \in \mathbb{R}^N \mid x_1 + x_2 \leq 1, x_3 \leq 0\}.$$

So, whereas in the TU graph-restricted game player 3 can get a positive payoff due to side payments, in the NTU graph-restricted game he gets at most zero. ◁

Although the two graph-restricted games need not be the same, the have the same core, as is shown in the following proposition.

Proposition 3.1 *Let $(N, v, E) \in TUC^N$. Then $C(v^E) = C(V^E)$.*

Proof: "$\subset$" Let $x \in C(v^E)$. Because $\sum_{i \in N} x_i = v^E(N)$ and $v^E(N) = \sum_{C \in N/E} v(C)$, we have $\sum_{C \in N/E} \sum_{i \in C} x_i = \sum_{C \in N/E} v(C)$. Furthermore, $\sum_{i \in C} x_i \geq v^E(C) = v(C)$ for all $C \in N/E$, so $\sum_{i \in C} x_i = v(C)$ for all $C \in N/E$. From this it follows that $x \in V^E(N)$. Next, let $S \subset N, S \neq \emptyset$. Then $V^E(S) = \prod_{C \in N/E} V(C) = \prod_{C \in S/E} Z^{C, v(C)} \subset Z^{S, \sum_{C \in S/E} v(C)}$. Since $\sum_{i \in S} x_i \geq v^E(S) = \sum_{C \in S/E} v(C)$, there can exist no $y \in V^E(S)$ such that $y > x_S$. Hence, $x \in C(V^E)$.
"$\supset$" Let $x \in C(V^E)$. For all $C \in N/E$, $\nexists_{y \in V^E(C)} : y > x_C$ implies $\sum_{i \in C} x_i = v^E(C) = v(C)$. Hence, $\sum_{i \in N} x_i = \sum_{C \in N/E} \sum_{i \in C} x_i = \sum_{C \in N/E} v(C) = v^E(N)$. Next, let $S \subset N, S \neq \emptyset$. Then for all $C \in S/E$, $\nexists_{y \in V^E(C)} : y > x_C$ implies $\sum_{i \in C} x_i \geq v(C)$, and hence, $\sum_{i \in S} x_i = \sum_{C \in S/E} \sum_{i \in C} x_i \geq \sum_{C \in S/E} v(C) = v^E(S)$. Hence, $x \in C(v^E)$. □

Not only do the cores of the two graph-restricted games coincide, but also the Shapley/MC solutions. To show this, we first prove equality between the corresponding marginal vectors.

Lemma 3.2 *Let $(N, v, E) \in TUC^N$ and let $\sigma \in \Pi(N)$. Then $m^\sigma(v^E) = M^\sigma(V^E)$.*

Proof: Assume without loss of generality that $\sigma(i) = i$ for all $i \in N$. Define $S_i = \{1, \ldots, i\}$ for all $i \in N$. First, $M_1^\sigma(V_E) = 0 = m_1^\sigma(v^E)$. Next, let $k \in N \backslash \{n\}$ and assume that $M_j^\sigma(V^E) = m_j^\sigma(v^E)$ for all $j \in \{1, \ldots, k\}$. Let $T_k \subset S_k$ be such that $T_k \cup \{k+1\} \in S_{k+1}/E$ and define $\bar{T}_k = T_k \cup \{k+1\}$. Then,

$$\begin{aligned}
M_{k+1}^\sigma(V^E) &= \max\{x \,|\, (M_{S_k}^\sigma, x) \in V^E(S_{k+1})\} \\
&= \max\{x \,|\, (M_{T_k}^\sigma, x) \in V^E(\bar{T}_k)\} \\
&= \max\{x \,|\, \sum_{i \in T_k} M_i^\sigma(V^E) + x \leq v(\bar{T}_k)\} \\
&= v(\bar{T}_k) - \sum_{i \in T_k} M_i^\sigma(V^E) \\
&= v(\bar{T}_k) - \sum_{i \in T_k} m_i^\sigma(v^E) \\
&= m_{k+1}^\sigma(v^E)
\end{aligned}$$

□

The Shapley value of the TU graph-restricted game is called the *Myerson value* of the communication situation (Myerson (1977)). This value $\mu : TUC^N \to \mathbb{R}^N$ is defined as

$$\mu(N, v, E) = \Phi(v^E).$$

Theorem 3.3 *Let $(N, v, E) \in TUC^N$. Then $\mu(N, v, E) = MC(V^E)$.*

Proof: It follows from Lemma 3.2 that $\mu(N, v, E)$ is a compromise between 0 and $\sum_{\sigma\in\Pi(N)} M^\sigma(V^E)$. Since $V^E(N) \subset Z^{N,v^E(N)}$, it suffices to show that $\mu(N, v, E) \in V^E(N)$. By construction, $\sum_{i\in C} \mu_i(N, v, E) = v(C)$ for all $C \in N/E$. So, $\mu(N, v, E) \in V^E(N) = \prod_{C\in N/E} Z^{C,v(C)}$. □

4 Inheritance of properties

In this section we study the inheritance of properties in NTU communication situations. Ie, we provide necessary and sufficient conditions that a network $(N, E) \in CN^N$ must satisfy so that for every game $(N, V) \in NTU^N$ that satisfies a certain property, the graph-restricted game (N, V^E) satisfies the same property. Most of our results are based on their TU counterparts in Van den Nouweland and Borm (1991) and Slikker (2000) and in many proofs, counterexamples with NTU games arising from TU games are used. It turns out that the necessary and sufficient conditions on the graph are the same for TU and NTU games for nearly all properties, with the notable exception of (im) convexity.

First of all, we characterise when balancedness is inherited.

Theorem 4.1 *Let $(N, E) \in CN^N$. Then the following two statements are equivalent.*

(i) (N, E) is connected or empty.

(ii) For all balanced $(N, V) \in NTU^N$, (N, V^E) is balanced.

Proof: "(i)⇒(ii)" Assume that (i) holds. If (N, E) is empty, then for all $(N, V) \in NTU^N$, $V^E(S) = \bar{Z}^{S,0}$ for all $S \subset N, S \neq \emptyset$ and hence, (N, V^E) is balanced.

So, assume that (N,E) is connected and let $(N,V) \in NTU^N$ be balanced. Let $x \in C(V)$. Because (N,E) is connected, $x \in V^E(N) = V(N)$. Let $S \subset N, S \neq \emptyset$. Because $x \in C(V)$, there does not exist a $y \in V(C)$ such that $y > x_C$ for any $C \in S/E$. But since $V^E(S) = \prod_{C \in S/E} V(C)$, there does not exist a $y \in V^E(S)$ such that $y > x_S$. Hence, $x \in C(V^E)$ and (N, V^E) is balanced.
"(ii)⇒(i)" Assume that (ii) holds. If $|N| \leq 2$, the statement is trivial, so assume that $|N| \geq 3$ and suppose that (N,E) is not connected. Take $V(S) = Z^{S,-1}$ for all $S \subset N, 1 < |S| < |N|$ and $V(N) = Z^{N,0}$. Then (N,V) is balanced, since $(0,\ldots,0) \in C(V)$. By assumption, (N, V^E) is balanced as well, so let $y \in C(V^E)$. Because (N,E) is not connected, $V^E(N)$ is the cartesian product of components whose total equals 0 or -1. Because $y_i \geq V^E(\{i\}) = 0$ for all $i \in N$, N/E can only consist of singletons. Hence, (N,E) is empty. □

Contrary to balancedness, total balancedness is always inherited, as is shown in the following proposition.

Proposition 4.2 *Let $(N,V,E) \in NTUC^N$. If (N,V) is totally balanced, then (N,V) is totally balanced.*

Proof: Assume that (N,V) is totally balanced. Let $T \subset N, T \neq \emptyset$. Then there exists an $x^C \in C(V_C) = C(V_C^E)$ for all $C \in T/E$. Define $x = (x^C)_{C \in T/E} \in \mathbb{R}^T$. It suffices to show that $x \in C(V_T^E)$.
Since $x^C \in V_C^E(C) = V_T^E(C)$ for all $C \in T/E$, we have $x \in V_T^E(T)$. Next, let $S \subset T, S \neq \emptyset$. Suppose there exists a $y \in V_T^E(S)$ such that $y > x_S$. Let $D \in S/E$ and let $C \in T/E$ be such that $D \subset C$. But then $y_D > x_D$ and $y_D \in V_T^E(D)$ contradict $x^C \in C(V_C^E)$. Hence, there exists no $y \in V_T^E(S)$ such that $y > x_S$ and so, (N, V^E) is totally balanced. □

Superadditivity is also inherited for every communication network.

Proposition 4.3 *Let $(N,V,E) \in NTUC^N$. If (N,V) is superadditive, then (N,V^E) is superadditive.*

Proof: Assume that (N,V) is superadditive. Let $S,T \subset N, S \cap T = \emptyset, S,T \neq \emptyset$. Note that $(S/E) \cup (T/E)$ is a finer partition of $S \cup T$ than $(S \cup T)/E$. Now,

$$V^E(S) \times V^E(T) = \prod_{C \in S/E} V(C) \times \prod_{C \in T/E} V(C) \subset \prod_{C \in (S \cup T)/E} V(C) = V^E(S \cup T),$$

where the inclusion follows from combining the components of $(S/E) \cup (T/E)$ and using superadditivity of (N, V). □

In a similar way, one can prove that individual superadditivity is always inherited.

Lemma 4.4 *Let $(N, V, E) \in NTUC^N$. If (N, V) is individually superadditive, then (N, V^E) is individually superadditive.*

Van den Nouweland and Borm (1991) shows that TU convexity is inherited for all cycle-complete graphs. The following lemma shows that cycle-completeness is also necessary for NTU games.

Lemma 4.5 *Let $(N, E) \in CN^N$ be not cycle-complete. Then there exists an individual merge convex game $(N, V) \in NTU^N$ such that (N, V^E) is not individual merge convex.*

Proof: Van den Nouweland and Borm (1991) show that there exists a convex game $(N, v) \in TU^N$ such that (N, v^E) is not convex. Let $(N, v) \in TU^N$ be such a game and let (N, V) be the corresponding NTU game. Then (N, V) is individual merge convex. Because a TU game is convex if and only if all its marginal vectors belong to the core, there exists a $\sigma \in \Pi(N)$ such that $m^\sigma(v^E) \notin C(v^E)$. But then, because of Proposition 3.1 and Lemma 3.2, $M^\sigma(V^E) \notin C(V^E)$. Hence, (N, V^E) is not marginal convex and therefore not individual merge convex. □

However, cycle-completeness is not sufficient for inheritance of individual merge convexity, as is shown in the following example.

Example 4.1 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3, 4\}$,

$$\begin{aligned}
V(S) &= \bar{Z}^{S,1} \text{ if } S \in \{\{1,3\}, \{3,4\}, \{2,3,4\}\}, \\
V(\{1,2,3\}) &= cc(\{(1,0,0), (0,2,0), (0,0,1)\}), \\
V(\{1,3,4\}) &= \bar{Z}^{\{1,3,4\},2}, \\
V(N) &= cc(\bar{Z}^{N,2} \cup \{(0,2,0,1)\}), \\
V(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset,
\end{aligned}$$

and E is such that $\{\{1,2\},\{2,3\},\{3,4\}\} \subset E$ and $\{1,3\} \notin E$.

This game is individual merge convex. The graph-restricted game (N, V^E) is given by

$$\begin{aligned} V(S) &= \bar{Z}^{S,1} \text{ if } S \in \{\{3,4\},\{2,3,4\}\}, \\ V(\{1,2,3\}) &= cc(\{(1,0,0),(0,2,0),(0,0,1)\}), \\ V(\{1,3,4\}) &= cc(\{(0,1,0),(0,0,1)\}), \\ V(N) &= cc(\bar{Z}^{N,2} \cup \{(0,2,0,1)\}), \\ V(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset. \end{aligned}$$

The graph-restricted game is not individual merge convex: take $k = 2$, $S = \{1,3\}$ and $T = \{1,3,4\}$, and take $p = (0,0) \in R(V^E, S)$, $q = (0,1,0) \in V^E(T)$ and $r = (0,2,0) \in V^E(S \cup \{k\})$. Then there does not exist an $s \in V^E(T \cup \{k\})$ such that $s_2 \geq 2$ and $s_3 \geq 1$. Hence, (N, V^E) is not individual merge convex. $\triangleleft$

It turns out that for NTU games, individual merge convexity is inherited for graphs whose components are either complete or a star.

Theorem 4.6 *Let $(N,E) \in CN^N$. Then the following two statements are equivalent.*

(i) For all $C \in N/E$, $(C, E(C))$ is complete or a star.

(ii) For all individual merge convex $(N,V) \in NTU^N$, (N, V^E) is individual merge convex.

Proof: "(i)$\Rightarrow$(ii)" Assume that (i) holds. Let $(N,V) \in NTU^N$ be an individual merge convex game. Then it follows from Lemma 4.4 that (N, V^E) is individually superadditive. For the individual merge property, let $k \in N$, $S \subsetneq T \subset N \backslash \{k\}, S \neq \emptyset$, and let $p \in R(V^E, S)$, $q \in V^E(T)$ and $r \in V^E(S \cup \{k\})$ such that $r_S \geq p$. Let $C^k \in N/E$ be such that $k \in C^k$ and denote $S^k = S \cap C^k$ and $T^k = T \cap C^k$. Note that it suffices to show that $(q_{T^k}, r_k) \in V^E(T^k \cup \{k\})$.
First, suppose that C^k is complete. Then $S^k \in S/E$, $T^k \in T/E$ and $S^k \cup \{k\} \in (S \cup \{k\})/E$, so $p_{S^k} \in R(V, S^k)$, $q_{T^k} \in V(T^k)$ and $r_{S^k \cup \{k\}} \in V(S^k \cup \{k\})$. Since (N,V) is individual merge convex, $(q_{T^k}, r_k) \in V(T^k \cup \{k\}) = V^E(T^k \cup \{k\})$.

Second, suppose that C^k is a star. If k is at the centre of this star, then $(S^k, E(S^k))$ and $(T^k, E(T^k))$ are empty and $(q_{T^k}, r_k) \in V^E(T \cup \{k\})$ by individual superadditivity. If k is not at the centre, but a member of S^k is, then the same argument as in the case where C^k is complete can be used. If neither k nor a member of S^k us at the centre, then $r_k = 0$ and $(q_{T^k}, r_k) \in V^E(T \cup \{k\})$ by individual superadditivity.
"(ii)⇒(i)" Assume that (ii) holds. It follows from Lemma 4.5 that (N, E) is cycle-complete. Suppose there exists a component $C \in N/E$ which is not complete or a star. Then it follows from Lemma 4.2 in Slikker (2000) that there exist four players in that component, without loss of generality players $M = \{1, \ldots, 4\} \subset C$, such that $\{\{1,2\},\{2,3\},\{3,4\}\} \subset E$ and $\{1,3\} \notin E$. Take $(N, V) \in NTU^N$ such that the subgame (M, V_M) equals the game in Example 4.1 and the players in $N \backslash M$ are dummy players. This game is individual merge convex, but the graph-restricted game (N, V^E) is not. This contradicts (ii), so every every component must be complete or a star. □

Although cycle-completeness is not sufficient to ensure inheritance of individual merge convexity for arbitrary NTU games, it is sufficient for NTU games arising from TU games.

Proposition 4.7 *Let $(N, E) \in CN^N$. Then the following two statements are equivalent.*

(i) (N, E) is cycle-complete.

(ii) For every convex game $(N, v) \in TU^N$, (N, V^E) is individual merge convex.

Proof: "(i)⇒(ii)" Assume that (i) holds. Let $(N, v) \in TU^N$ be a convex game. Then the corresponding NTU game (N, V) is individually superadditive and by Lemma 4.4, (N, V^E) is individually superadditive as well.
For the individual merge property, let $k \in N$, $S \subsetneqq T \subset N \backslash \{k\}, S \neq \emptyset$ and let $p \in R(V^E, S)$, $q \in V^E(T)$, $r \in V^E(S \cup \{k\})$ such that $r_S \geq p$. Define $\mathcal{C} = \{C \in S/E \,|\, \exists_{i \in C} : \{i,k\} \in E\}$, $\mathcal{C}' = (S/E) \backslash \mathcal{C}$, $\mathcal{D} = \{D \in T/E \,|\, \exists_{i \in D} : \{i,k\} \in E\}$ and $\mathcal{D}' = (T/E) \backslash \mathcal{D}$. Because $p \in WPar(V^E, S)$, we have that $\sum_{i \in C} p_i = v(C)$ for all $C \in S/E$. Similarly, $\sum_{i \in D} q_i \leq v(D)$ for all $D \in T/E$, $r_k + \sum_{i \in \bigcup_{C \in \mathcal{C}} C} r_i \leq v(\{k\} \cup \bigcup_{C \in \mathcal{C}} C)$ and $\sum_{i \in C} r_i \leq v(C)$ for all $C \in \mathcal{C}'$. From the proof of Theorem 1 in Van den Nouweland and Borm (1991), we know that

$$v(\{k\} \cup \bigcup_{C \in \mathcal{C}} C) - \sum_{C \in \mathcal{C}} v(C) \leq v(\{k\} \cup \bigcup_{D \in \mathcal{D}} D) - \sum_{D \in \mathcal{D}} v(D).$$

So, subsequently we have

$$\begin{aligned} v(\{k\} \cup \bigcup_{C \in \mathcal{C}} C) + \sum_{C \in \mathcal{C}'} v(C) - \sum_{C \in S/E} v(C) &\leq v(\{k\} \cup \bigcup_{D \in \mathcal{D}} D) - \sum_{D \in \mathcal{D}} v(D), \\ r_k + \sum_{i \in \bigcup_{C \in \mathcal{C}} C} r_i + \sum_{i \in \bigcup_{C \in \mathcal{C}'} C} r_i - \sum_{i \in S} p_i &\leq v(\{k\} \cup \bigcup_{D \in \mathcal{D}} D) - \sum_{i \in \bigcup_{D \in \mathcal{D}} D} q_i \\ v(\{k\} \cup \bigcup_{D \in \mathcal{D}} D) &\geq r_k + \sum_{i \in \bigcup_{D \in \mathcal{D}} D} q_i. \end{aligned}$$

Because

$$V^E(T \cup \{k\}) = Z^{\bigcup_{D \in \mathcal{D}} D \cup \{k\}, v(\bigcup_{D \in \mathcal{D}} D\{k\})} \times \prod_{D \in \mathcal{D}'} Z^{D, v(D)},$$

we have that $(q, r_k) \in V^E(T \cup \{k\})$ and hence, (N, V^E) is individual merge convex.
"(ii)⇒(i)" Follows from the proof of Lemma 4.5. □

As is the case for TU games, existence of a pmas is always inherited for NTU games, as is shown in the following proposition.

Proposition 4.8 *Let $(N, V, E) \in NTUC^N$. If (N, V) has a pmas, then (N, V^E) has a pmas.*

Proof: Assume that (N, V) has a pmas $(y^S)_{S \subset N, S \neq \emptyset}$. For $i \in S$, denote by $C_i(S)$ the component in S/E to which i belongs. Define $x_i^S = y_i^{C_i(S)}$ for all $S \subset N, i \in S$. Because $y^C \in WPar(V, C)$ for all $C \in S/E$, we have that $x^S \in WPar(V^E, S)$. Furthermore, for all $i \in S \subset T \subset N$,

$$x_i^S = y_i^{C_i(S)} \leq y_i^{C_i(T)} = x_i^T,$$

because $C_i(S) \subset C_i(T)$ and $(y^S)_{S \subset N, S \neq \emptyset}$ is a pmas. Hence, $(x^S)_{S \subset N, S \neq \emptyset}$ is a pmas for the game (N, V^E). □

Now we turn our attention to the MC value. One interesting question is whether the MC value is an element of the core. The following proposition shows the relationship between an NTU game and its graph-restricted game in terms of this question.

Proposition 4.9 *Let $(N, E) \in CN^N$. Then the following two statements are equivalent.*

(i) (N, E) *is complete or empty.*

(ii) For every $(N, V) \in NTU^N$ *with* $MC(V) \in C(V)$, $MC(V^E) \in C(V^E)$.

Proof: "(i)⇒(ii)" Trivial.
"(ii)⇒(i)" Assume that (ii) holds. Suppose that (N, E) is not complete or empty. Then, along the lines of Theorem 4.1 in Slikker (2000), for the game (N, V) described in the proof of Theorem 4.1, we have that $MC(V) \in C(V)$, but $MC(V^E) \notin C(V^E)$. This contradicts (ii), so (i) must hold. □

Another interesting question is whether the MC allocation scheme $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for the NTU game (N, V). To characterise when this property is inherited, we need the following lemma.

Lemma 4.10 *Let* $(N, V) \in NTU^N$ *and let* $i \in N$ *be a dummy player. Then* $MC_i(V) = 0$ *and* $MC_{N \backslash \{i\}}(V) = MC(V_{N \backslash \{i\}})$.

Proof: By construction, $M_i^\sigma(V) = 0$ for all $\sigma \in \Pi(N)$ and hence, $MC_i(V) = \alpha_V \sum_{\sigma \in \Pi(N)} M_i^\sigma(V) = 0$. Furthermore, $M_j^\sigma(V) = M_j^{\sigma|N \backslash \{i\}}(V_{N \backslash \{i\}})$ for all $j \in N \backslash \{i\}, \sigma \in \Pi(N)$ and $V(N \backslash \{i\}) = V_{N \backslash \{i\}}(N \backslash \{i\})$, so $MC_{N \backslash \{i\}}(V) = MC(V_{N \backslash \{i\}})$ □

For TU games, the property that the Shapley allocation scheme is a pmas is inherited for graphs whose components are all complete. This is also necessary for NTU games, as is illustrated in the following example.

Example 4.2 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{2, 3\}\}$, $V(S) = Z^{S,2}$ for $S \subset N, |S| = 2$ and $V(N) = Z^{N,3}$. It is readily checked that the MC allocation scheme $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V). However, in the graph-restricted game (N, V^E),

$$MC_1(V^E_{\{1,2\}}) = 1 > \frac{2}{3} = MC_1(V^E).$$

◁

However, completeness of every component is not sufficient, as is shown in the following example.

Example 4.3 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, \ldots, 4\}$, $E = \{1,2\}, \{3,4\}\}$ and

$$\begin{aligned} V(\{1,2\}) &= Z^{\{1,2\},1}, \\ V(\{3,4\}) &= \{x \in \mathbb{R}^{\{3,4\}} \,|\, x_3 \leq 1, x_4 \leq 1\}, \\ V(S \cup \{i\}) &= V(S) \times V(\{i\}) \text{ for } S \in \{\{1,2\},\{3,4\}\}, i \in N \backslash S, \\ V(N) &= Z^{N,4}, \\ V(S) &= \bar{Z}^{S,0} \text{ for other } S \subset N, S \neq \emptyset. \end{aligned}$$

It is readily checked that $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V). However, in the graph-restricted game (N, V^E),

$$MC_3(V^E_{\{3,4\}}) = 1 > \frac{1}{2} = MC_3(V^E).$$

Furthermore, because the MC allocation scheme is a pmas, $MC(V_S) \in C(V_S)$ for all $S \subset N, S \neq \emptyset$. However,

$$MC(V^E) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin C(V^E).$$

◁

Proposition 4.11 *Let $(N, E) \in CN^N$. Then the following two statements are equivalent.*

(i) If $C \in N/E, |C| > 1$, then $(C, E(C))$ is complete and $|D| = 1$ for all $D \in N/E, D \neq C$.

(ii) For all $(N, V) \in NTU^N$ such that $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V), $(MC(V^E_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V^E).

Proof: "(i)⇒(ii)" Assume that (i) holds. If $|C| = 1$ for all $C \in N/E$, then (ii) holds trivially. So, let $C \in N/E, |C| > 1$, and let $(N, V) \in NTU^N$ such that the MC allocation scheme is a pmas. By Lemma 4.10, $MC_i(V^E_S) = 0$ for all $S \subset N, i \in S \backslash C$ and $MC_C(V^E) = MC(V^E_C)$. Because C is complete, $MC(V^E_S) = MC(V_S)$ for all $S \subset C, S \neq \emptyset$. So, since $(MC(V_S))_{S \subset C, S \neq \emptyset}$ is a pmas for (C, V_C), it is also a pmas for (C, V^E_C) and (ii) holds.

"(ii)⇒(i)" Assume that (ii) holds. Suppose that there is a component $C \in N/E$ such that $(C, E(C))$ is not complete. Then there exists, without loss of generality, $M = \{1, 2, 3\} \subset C$ such that $E(M) = \{\{1, 2\}, \{2, 3\}\}$. Take $(N, V) \in NTU^N$ such that (M, V_M) is the game in Example 4.2 and the players in $N \backslash M$ are dummy players. Then, as a result of Lemma 4.10, $(MC(V_S^E))_{S \subset N, S \neq \emptyset}$ is not a pmas for (N, V^E), although $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V). Contradiction, so there is no incomplete component.
Next, suppose there exist two complete components with more than one player. Then, without loss of generality, there exists $M = \{1, \ldots, 4\} \subset N$ such that $E(M) = \{\{1, 2\}, \{3, 4\}\}$. Take $(N, V) \in NTU^N$ such that (M, V_M) is the game in Example 4.3 and the players in $N \backslash M$ are dummy players. Then $(MC(V_S))_{S \subset N, S \neq \emptyset}$ is a pmas for (N, V), but $(MC(V_S^E))_{S \subset N, S \neq \emptyset}$ is not a pmas for (N, V^E). This contradicts (ii), so (i) must hold. □

Finally, we consider the MC values of all subgames. Slikker (2000) shows that for TU games, the property that the Shapley value of each subgame lies in its corresponding core is inherited for graphs with complete or star components. In the following example, we show that for NTU games, completeness is necessary.

Example 4.4 Consider $(N, V, E) \in NTUC^N$ with $N = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{2, 3\}\}$ and

$$\begin{aligned} V(\{1,2\}) &= \{x \in \mathbb{R}^{\{1,2\}} \,|\, x_1 \leq 1, x_2 \leq 2\}, \\ V(\{1,3\}) &= \{x \in \mathbb{R}^{\{1,3\}} \,|\, x_1 \leq 2, x_3 \leq 0\}, \\ V(\{2,3\}) &= Z^{\{2,3\},1}, \\ V(N) &= Z^{N,3}. \end{aligned}$$

It is readily checked that $MC(V_S) \in C(V_S)$ for all $S \subset N, S \neq \emptyset$. However, in the graph-restricted game (N, V^E),

$$MC(V^E) = (\frac{5}{6}, \frac{9}{6}, \frac{4}{6}),$$

which is not a core element because of coalition $\{1, 2\}$. ◁

Furthermore, we need the following lemma.

Lemma 4.12 *Let $(N, V, E) \in NTUC^N$ and let $i \in N$ be a dummy player. Then*

$$C(V) = \{(x, 0) \in \mathbb{R}^N \mid x \in C(V_{N\setminus\{i\}})\}.$$

Using this, we can state our final result.

Proposition 4.13 *Let $(N, E) \in CN^N$. Then the following two statements are equivalent.*

(i) If $C \in N/E, |C| > 1$, then $(C, E(C))$ is complete and $|D| = 1$ for all $D \in N/E, D \neq C$.

(ii) For all $(N, V) \in NTU^N$ such that $MC(V_S) \in C(V_S)$ for all $S \subset N, S \neq \emptyset$, $MC(V_S^E) \in C(V_S^E)$ for all $S \subset N, S \neq \emptyset$.

Proof: "(i)⇒(ii)" Follows immediately from Lemmas 4.10 and 4.12.
"(ii)⇒(i)" Assume that (ii) holds. Suppose that there exists a $C \in N/E$ which is not complete. Then the game in Example 4.4 can be used to contradict (ii). If there is more than one component with more than one player, Example 4.3 can be used. Hence, (i) must hold. □

To round of this section, we summarise our inheritance results in the following table, together with the corresponding results for TU games from Van den Nouweland and Borm (1991) and Slikker (2000).

Property	*TU inheritance condition*	*NTU inheritance condition*
Balancedness	connected or empty	connected or empty
Total balancedness	no condition	no condition
Superadditivity	no condition	no condition
Individual merge convexity	cycle-comple	every component complete or star
Existence of pmas	no condition	no condition
MC value in core	complete or empty	complete or empty
MC allocation scheme pmas	every component complete	one component complete, others singletons
MC value of every subgame in core	every component complete or star	one component complete, others singletons

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