GAMES AND ECONOMIC BEHAVIOR **15**, 165–176 (1996)
ARTICLE NO. 0064

The Split Core for Sequencing Games

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Received November 2, 1994

The split core is a refinement of the core for sequencing games. The split core arises from a generalization of the equal gain splitting (EGS) rule that is introduced by *Curiel et al.* (1989). It is pointed out that the split core is the convex hull of permutation-based gain splitting allocations and the EGS allocation is in the barycenter of the split core. Finally, an axiomatic characterization of the split core is provided. *Journal of Economic Literature* Classification Number: 026. © 1996 Academic Press, Inc.

1. INTRODUCTION

In one-machine sequencing situations each agent has one job that has to be processed on a single machine. Each job is specified by its processing time, the time the machine takes to handle the job. We assume that the cost of a player depends linearly on the completion time of his job. Finally, there is an initial order on the jobs of the agents before the processing of the machine starts.

Each group of agents (coalition) is allowed to obtain cost savings by rearranging their jobs in a way that is admissible with respect to the initial order. An optimal order of a coalition is an admissible rearrangement that maximizes the cost savings of this coalition. By defining the worth of a coalition as the (maximum) cost savings a coalition can make by an optimal admissible rearrangement, we obtain a cooperative sequencing game, related to the one-machine sequencing situation. This game theoretic approach has been taken in Curiel *et al.* (1989). They introduced the equal gain splitting (EGS) rule on the class of sequencing situations. The EGS rule is based on the fact that the optimal order of the grand coalition can be obtained from the initial order by consecutive switches of neighbors. According to the EGS rule each agent obtains half of the gains of all neighbor switches he is actually involved in to reach an optimal order.

* Financially supported by the Dutch Organization for Scientific Research (NWO).

0899-8256/96 $18.00

Note that the EGS rule is independent of the chosen optimal order and that the gain of a neighbor switch is independent of the position of the neighbors in the queue. It was shown that each EGS allocation is in the core of the corresponding sequencing game. Further, an axiomatic characterization of the EGS rule was provided. Curiel *et al.* (1993b) showed that the EGS allocation is the average of two marginal vectors of the corresponding sequencing game. Curiel *et al.* (1993a) presented an alternative characterization of the EGS rule. Moreover, they introduced the head–tail core for sequencing games and showed that the corresponding EGS allocation is in the barycenter of this core. They also showed that the EGS rule can be regarded as a general nucleolus (Maschler *et al.* 1992).

This paper considers a generalization of the EGS rule. For the EGS rule, the equal splitting of the gain of each neighbor switch can be interpreted as a result of a bargaining process. However, such a bargaining process can lead to other nonnegative divisions of the gain of each neighbor switch. Therefore, we study division rules for sequencing situations where each player obtains an arbitrary nonnegative part of the gains of all neighbor switches he is actually involved in to reach the optimal order. The union of all corresponding allocations is called the split core. Obviously, the EGS allocation is an element of the split core. It is shown that the split core of a sequencing situation is a subset of the core of the corresponding sequencing game. Further, it is shown that the split core is the convex hull of so-called permutation-based gain splitting allocations and that the corresponding EGS allocation is the average of these vectors. Finally, it is shown that the split core is the largest set-valued solution concept satisfying efficiency, the dummy property, and a monotonicity condition.

2. SEQUENCING GAMES

In a one-machine sequencing situation there is a queue of agents, each having one job that has to be processed by one machine. The finite set of agents is denoted by $N = \{1, \ldots, n\}$. The processing order of the agents is described by a bijection $\sigma\colon N \to \{1, \ldots, n\}$. Specifically, $\sigma(i) = j$ means that the job of agent i is the jth that will be processed. We assume that there is an initial processing order $\sigma_0\colon N \to \{1, \ldots, n\}$ on the jobs of the agents before the processing of the machine starts. The processing time p_i of the job of agent i is the time the machine takes to handle this job. Further, it is assumed that for each agent the cost for spending time in the system can be expressed in an infinitely divisible transferable commodity, i.e., money, and that these costs are described by an affine cost function $c_i\colon [0, \infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t + \beta_i$ with $\alpha_i > 0$, $\beta_i \in \mathbb{R}$. So $c_i(t)$ is the cost for agent i if he has spent t units of time in the system.

A sequencing situation as described above is denoted by (N, p, α, σ_0) where $N = \{1, \ldots, n\}$, $p = (p_i)_{i \in N} \in \mathbb{R}^n_+$, $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}^n_+$, and $\sigma_0\colon N \to \{1, \ldots, n\}$. The vector $\beta = (\beta_i)_{i \in N} \in \mathbb{R}^n$ is omitted in the description of the sequencing situation since the fixed costs it represents are independent of the processing order of the jobs.

Given a processing order $\sigma\colon N \to \{1, \ldots, n\}$, the completion time of the job of agent i equals $C(\sigma, i) = \sum_{j \in P(\sigma,i)} p_j + p_i$, where $P(\sigma, i) = \{j \mid \sigma(j) < \sigma(i)\}$ is the set of predecessors of agent's i job with respect to the processing order σ.

Because we assumed that the costs of the agents are expressed in money, the costs of an agent increase either when the time he spends in the system increases or when he has to pay an amount of money. Similarly, the costs of an agent decrease either when the time he spends in the systems decreases or when he receives an amount of money. Using these facts, agents can obtain cost savings in the following way. First we introduce an outside agent who acts like a bank. Next, consider a new processing order which differs from the initial processing order. Now, if an agent has moved forward in this new processing order, the time he spends in the system has decreased and hence, his costs have decreased. Such an agent has to pay an amount of money to the bank such that his new costs plus the amount of money paid equals the costs he had in the initial processing order. Similarly, an agent who has moved backward in the new processing order receives an amount of money from the bank such that his new costs minus the amount of money received equals the costs he had in the initial processing order. So each agent ends up with the same costs as in the initial processing order. Moreover, there is a surplus (or deficit) of money in the bank.

In this paper we focus on two issues, the processing order which maximizes the bank's surplus and how this surplus can be allocated over the agents.

For the first issue, consider an arbitrary processing order σ. The amount agent i has to pay to the bank equals $C(\sigma_0, i) - C(\sigma, i)$. Hence, the surplus of the bank equals $\sum_{i \in N}(C(\sigma_0, i) - C(\sigma, i))$. For the processing order which maximizes this expression we can rely on a result due to Smith (1956), which says that the cost savings are maximal if the jobs are processed in decreasing order with respect to α_i/p_i.

For the second issue we will make use of cooperative game theory. Therefore we start with some basic definitions from this area.

A cooperative game is a pair (N, v), where N is a finite set of players and v is a mapping $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ and 2^N the collection of all subsets of N. The function v is called the characteristic function of the game and $v(S)$ denotes the worth of the coalition $S \in 2^N$.

A game (N, v) is called convex if for all coalitions $S, T \in 2^N$ and all $i \in N$ with $S \subset T \subset N\backslash\{i\}$ it holds that

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Cooperative game theory focuses on division rules for the worth $v(N)$ of the grand coalition. A core element $x = (x_i)_{i \in N} \in \mathbb{R}^N$ is such that no coalition has an incentive to split off, i.e.,

$$\sum_{i \in N} x_i = v(N) \text{ and } x(S) \geq v(S) \qquad \text{for all } S \in 2^N,$$

where $x(S) = \sum_{i \in S} x_i$. The core $C(v)$ consists of all core elements.

Let (N, v) be a game and let Π_N be the set of all permutations of N. Then the kth coordinate of the marginal vector $m^\pi(v)$, $\pi \in \Pi_N$, is defined by

$$m_k^\pi(v) = v(\{j \mid \pi(j) \leq \pi(k)\}) - v(\{j \mid \pi(j) < \pi(k)\}).$$

Shapley (1971) and Ichiishi (1981) showed that the marginal vectors are the extreme points of the core if and only if the game is convex. Since the core is a convex set we have that the core of a convex game is the convex hull of its marginals.

To find "appropriate" allocations of the bank's surplus we will construct a cooperative game, with agent set N and characteristic function v, denoting the maximal cost savings each coalition can obtain by rearranging their positions in the processing order. For the grand coalition N we know that

$$v(N) = \max_{\sigma \in \Sigma_N} \sum_{i \in N} (C(\sigma_0, i) - C(\sigma, i)),$$

where Σ_N is the set of all processing orders $\sigma\colon N \to \{1, 2, \ldots, n\}$.

For determining $v(S)$ for coalitions $S \neq N$ we need to know which changes in the processing order are allowed for a coalition S and which are not. For instance, in a three-agent sequencing situation with initial processing order $(1, 2, 3)$, agents 1 and 3 cannot change places without the cooperation of agent 2. So the only processing order which is allowed for coalition $S = \{1, 3\}$ is the order $(1, 2, 3)$. In general, changes in processing orders where members of coalition S pass agents outside coalition S are not allowed. More formally, a processing order σ is admissible for coalition S if $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \backslash S$. The set of admissible processing orders for coalition S is denoted with Σ_S. The maximal cost savings a coalition $S \in 2^N$ can obtain is then defined by

$$v(S) = \max_{\sigma \in \Sigma_S} \sum_{i \in S} (C(\sigma_0, i) - C(\sigma, i)). \tag{1}$$

The cooperative game defined above is called a cooperative sequencing game. These games were first introduced by Curiel *et al.* (1989).

We conclude this section with a recapitulation of some results achieved by Curiel *et al.* (1989) which will be useful in the remainder of the paper. A coalition S is connected if for all $i, j \in S$ and all $k \in N$ with $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ it holds that $k \in S$. For connected coalitions S expression (1) can be written as

$$v(S) = \sum_{i,j \in S\colon \sigma_0(i) < \sigma_0(j)} g_{ij},$$

where $g_{ij} := \max\{p_i\alpha_j - p_j\alpha_i, 0\}$ represents the cost savings attainable by agents i and j when agent i is exactly in front agent j. In other words, g_{ij} equals the maximal amount of money that can be transferred from agents i and j to the bank. For a coalition T which is not connected it follows from the definition of Σ_T that

$$v(T) = \sum_{S \in T \backslash \sigma_0} v(S)$$

with $T\backslash\sigma_0$ the set of maximally connected components of T.

The EGS rule of a sequencing situation (N, p, α, σ_0) is defined by

$$\mathrm{EGS}_i(N, p, \alpha, \sigma_0) = \frac{1}{2} \sum_{j:\, \sigma_0(j) > \sigma_0(i)} g_{ij} + \frac{1}{2} \sum_{k:\, \sigma_0(k) < \sigma_0(i)} g_{ki}$$

for all $i \in N$. Note that the optimal processing order can be obtained from the initial processing order by consecutive switches of neighbors i, j with $g_{ij} \geq 0$ (*cf.* Smith, 1956). The EGS rule then divides the cost savings obtained in a neighbor switch equally among the two agents involved in the neighbor switch. Finally we recall that sequencing games are convex and that the EGS rule assigns to each sequencing situation an allocation that is in the core of the corresponding sequencing game.

EXAMPLE 2.1. Let $N = \{1, 2, 3\}$, $p = (2, 2, 1)$, $\alpha = (4, 6, 5)$, and $\sigma_0(i) = i$ for all $i \in N$. It follows that $g_{12} = g_{23} = 4$ and $g_{13} = 6$. Then $\mathrm{EGS}_1(N, p, \alpha, \sigma_0) = \frac{1}{2}(4+6) = 5$, $\mathrm{EGS}_2(N, p, \alpha, \sigma_0) = \frac{1}{2}(4+4) = 4$, and $\mathrm{EGS}_3(N, p, \alpha, \sigma_0) = \frac{1}{2}(6+4) = 5$.

3. THE SPLIT CORE

In this section we introduce the split core of a sequencing situation. It is shown that the split core of a sequencing situation is a subset of the core of the corresponding sequencing game. Further, we introduce permutation-based gain splitting allocations to describe the extreme points of the split core with. It is then shown that the EGS allocation is in the barycenter of the corresponding split core. Finally, the split core is axiomatically characterized by efficiency, the dummy property, and a form of monotonicity.

Generalizing the EGS rule we consider gain splitting (GS) rules in which each player obtains a nonnegative part of the cost savings of all neighbor switches he is actually involved in to reach the optimal order. Formally, we define for all $i \in N$ and all $\lambda \in \Lambda$

$$\mathrm{GS}_i^{\lambda}(N, p, \alpha, \sigma_0) = \sum_{j:\, \sigma_0(i) < \sigma_0(j)} \lambda_{ij} g_{ij} + \sum_{k:\, \sigma_0(k) < \sigma_0(i)} (1 - \lambda_{ki}) g_{ki},$$

where $\Lambda = \{\{\lambda_{ij}\}_{i,j\in N, i\neq j} \mid 0 \leq \lambda_{ij} \leq 1\}$. Note that for each $\lambda \in \Lambda$ we possibly obtain another allocation. Moreover, $\text{GS}^\lambda(N, p, \alpha, \sigma_0) = \text{EGS}(N, p, \alpha, \sigma_0)$ in case $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

EXAMPLE 3.1. Consider the situation described in Example 2.1. If we take $\lambda_{12} = \frac{3}{4}$, $\lambda_{13} = \frac{1}{3}$, and $\lambda_{23} = 1$, then $\text{GS}_1^\lambda(N, p, \alpha, \sigma_0) = 5$, $\text{GS}_2^\lambda(N, p, \alpha, \sigma_0) = 5$, and $\text{GS}_3^\lambda(N, p, \alpha, \sigma_0) = 4$.

The split core of a sequencing situation (N, p, α, σ_0) is defined by

$$\text{SPC}(N, p, \alpha, \sigma_0) = \{\text{GS}^\lambda(N, p, \alpha, \sigma_0) \mid \lambda \in \Lambda\}.$$

First it is shown that the split core is a subset of the core.

THEOREM 3.2. *Let (N, p, α, σ_0) be a sequencing situation and let (N, v) be the corresponding sequencing game. Then* $\text{SPC}(N, p, \alpha, \sigma_0) \subset C(v)$.

Proof. Let $\lambda \in \Lambda$ and let S be a connected set. Then

$$\begin{aligned}
\sum_{i\in S} GS_i^\lambda(N, p, \alpha, \sigma_0) &= \sum_{i\in S}\left[\sum_{j:\, \sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k:\, \sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki})\right] \\
&\geq \sum_{i\in S}\left[\sum_{j\in S:\, \sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k\in S:\, \sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki})\right] \\
&= \sum_{i,j\in S:\, \sigma_0(i)<\sigma_0(j)} g_{ij} = v(S).
\end{aligned}$$

In case $S = N$ the inequality above becomes an equality. Hence, $GS^\lambda(N, p, \alpha, \sigma_0) \in C(v)$. ■

For describing the extreme points of the split core we assign to each permutation $\tau \in \Pi_N$ a vector $\lambda(\tau) \in \Lambda$ in the following way. For all $i, j \in \{1, \ldots, n\}$, $i \neq j$

$$\lambda_{ij}(\tau) = \begin{cases} 0 & \text{if } \tau(i) < \tau(j) \\ 1 & \text{if } \tau(i) > \tau(j) \end{cases}. \tag{2}$$

Then for each sequencing situation (N, p, α, σ_0) the collection of permutation-based gain splitting allocations is defined by

$$\text{PBGS}(N, p, \alpha, \sigma_0) = \{GS^{\lambda(\tau)}(N, p, \alpha, \sigma_0) \mid \tau \in \Pi_N\}.$$

Let (N, p, α, σ_0) be a sequencing situation. Then the corresponding switching game (N, w) is defined by

$$w(S) = \sum_{i,j\in S:\, \sigma_0(i)<\sigma_0(j)} g_{ij} \qquad \text{for all } S \subset N.$$

With (N, v) the corresponding sequencing game we have that $w(S) \geq v(S)$ if S is a disconnected coalition and that $w(S) = v(S)$ if S is a connected coalition. Hence, $C(w) \subset C(v)$. Further, it is easy to verify that each switching game is a convex game.

The next lemma states that the marginal vectors of the switching game coincide with the permutation-based gain splitting allocations of the sequencing situation.

LEMMA 3.3. *Let (N, p, α, σ_0) be a sequencing situation and (N, w) be the corresponding switching game. Then $m^\tau(w) = \mathrm{GS}^{\lambda(\tau)}(N, p, \alpha, \sigma_0)$ for each $\tau \in \Pi_N$.*

Proof. Let $i \in N$ and $\tau \in \Pi_N$. Define the set of predecessors (followers) of agent i with respect to a permutation τ by $P(\tau, i) = \{j \mid \tau(j) < \tau(i)\}$ ($F(\tau, i) = \{j \mid \tau(j) > \tau(i)\}$). Then

$$\begin{aligned}
GS_i^{\lambda(\tau)}(N, p, \alpha, \sigma_0) &= \sum_{j:\, \sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij}(\tau) + \sum_{k:\, \sigma_0(k)<\sigma_0(i)} g_{ki}(1 - \lambda_{ki}(\tau)) \\
&= \sum_{j \in P(\tau,i)\cap F(\sigma_0,i)} g_{ij}\lambda_{ij}(\tau) + \sum_{j \in F(\tau,i)\cap F(\sigma_0,i)} g_{ij}\lambda_{ij}(\tau) \\
&\quad + \sum_{k \in P(\tau,i)\cap P(\sigma_0,i)} g_{ki}(1 - \lambda_{ki}(\tau)) \\
&\quad + \sum_{k \in F(\tau,i)\cap P(\sigma_0,i)} g_{ki}(1 - \lambda_{ki}(\tau)) \\
&= \sum_{j \in P(\tau,i)\cap F(\sigma_0,i)} g_{ij} + \sum_{k \in P(\tau,i)\cap P(\sigma_0,i)} g_{ki} \\
&= \sum_{k,l \in P(\tau,i)\cup\{i\}:\, \sigma_0(k)<\sigma_0(l)} g_{kl} - \sum_{k,l \in P(\tau,i):\, \sigma_0(k)<\sigma_0(l)} g_{kl} \\
&= v(P(\tau, i) \cup \{i\}) - v(P(\tau, i)) = m_i^\tau(w),
\end{aligned}$$

where the third equality follows from (2). ■

The following theorem shows that the split core is the convex hull of all corresponding permutation-based gain allocations.

THEOREM 3.4. *Let (N, p, α, σ_0) be a sequencing situation. Then*

$$\mathrm{SPC}(N, p, \alpha, \sigma_0) = \mathrm{conv}\{GS^{\lambda(\tau)}(N, p, \alpha, \sigma_0) \mid \tau \in \Pi_N\}.$$

Proof. Let (N, w) be the switching game corresponding to (N, p, α, σ_0). Since (N, w) is a convex game we have that $C(w) = \mathrm{conv}\{m^\tau(w) \mid \tau \in \Pi_N\}$. Lemma 3.3 implies that $\mathrm{conv}\{m^\tau(w) \mid \tau \in \Pi_N\} = \mathrm{conv}\{GS^{\lambda(\tau)}(N, p, \alpha, \sigma_0) \mid \tau \in \Pi_N\}$. Since $\mathrm{SPC}(N, p, \alpha, \sigma_0)$ is a convex set we have $C(w) \subset \mathrm{SPC}(N, p, \alpha, \sigma_0)$.

On the other hand, let $\lambda \in \Lambda$ and let $S \subset N$. Then

$$\begin{aligned}\sum_{i\in S}\mathrm{GS}_i^{\lambda}(N,p,\alpha,\sigma_0) &\geq \sum_{i\in S}\left[\sum_{j\in S:\ \sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k\in S:\ \sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki})\right]\\ &= \sum_{i,j\in S:\ \sigma_0(i)<\sigma_0(j)} g_{ij} = w(S).\end{aligned}$$

In case $S = N$ the inequality in the above calculation becomes an equality. Hence, $\mathrm{SPC}(N, p, \alpha, \sigma_0) \subset C(w)$. ■

From Theorem 3.4 it follows immediately that for each sequencing situation (N, p, α, σ_0) the set $\mathrm{PBGS}(N, p, \alpha, \sigma_0)$ is the set of all extreme points of $\mathrm{SPC}(N, p, \alpha, \sigma_0)$.

The next theorem shows that the EGS allocation of a sequencing situation is the average of all corresponding permutation-based gain splitting allocations.

THEOREM 3.5. *Let (N, p, α, σ_0) be a sequencing situation. Then*

$$\mathrm{EGS}(N,p,\alpha,\sigma_0) = \frac{1}{n!}\sum_{\tau\in\Pi_N}\mathrm{GS}^{\lambda(\tau)}(N,p,\alpha,\sigma_0).$$

Proof. For each $\tau \in \Pi_N$ there exists a unique $\tau^c \in \Pi_N$ such that $\lambda(\tau) + \lambda(\tau^c) = \lambda(e)$, where $\lambda(e) \in \Lambda$ with $\lambda(e)_{ij} = 1$ for all $i, j \in N$, $i \neq j$. Note that the definition of $\lambda(\tau)$ implies that $\{\tau \mid \tau \in \Pi_N\} = \{\tau^c \mid \tau \in \Pi_N\}$. Since $\mathrm{GS}^{\lambda(\tau)}(N, p, \alpha, \sigma_0) + \mathrm{GS}^{\lambda(\tau^c)}(N, p, \alpha, \sigma_0) = \mathrm{GS}^{\lambda(e)}(N, p, \alpha, \sigma_0) = 2\,\mathrm{EGS}(N, \sigma_0, p, \alpha)$ we have that

$$\begin{aligned}&\frac{1}{n!}\sum_{\tau\in\Pi_N}\mathrm{GS}^{\lambda(\tau)}(N,p,\alpha,\sigma_0)\\ &\quad= \frac{1}{n!}\sum_{\tau\in\Pi_N}\left(\frac{1}{2}\mathrm{GS}^{\lambda(\tau)}(N,p,\alpha,\sigma_0)+\frac{1}{2}\mathrm{GS}^{\lambda(\tau^c)}(N,p,\alpha,\sigma_0)\right)\\ &\quad= \frac{1}{n!}\sum_{\tau\in\Pi_N}\mathrm{EGS}(N,p,\alpha,\sigma_0) = \mathrm{EGS}(N,p,\alpha,\sigma_0).\end{aligned}$$

■

EXAMPLE 3.6. Let $N = \{1, 2, 3\}$, $p = (2, 2, 1)$, $\alpha = (4, 6, 5)$, and $\sigma_0(i) = i$ for all $i \in N$. Then $g_{12} = g_{23} = 4$ and $g_{13} = 6$, and the corresponding sequencing game is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{2, 3\}) = 4$, $v(\{1, 3\}) = 0$, and $v(\{1, 2, 3\}) = 14$. The extreme points of $\mathrm{SPC}(N, p, \alpha, \sigma_0)$

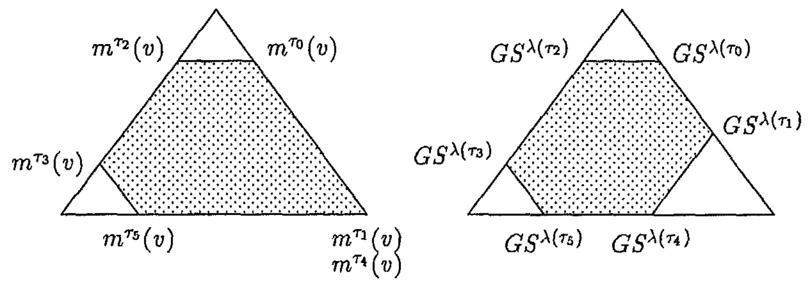

FIGURE 1

are

$$\begin{array}{ll}
\mathrm{GS}^{\lambda(\tau_0)}(N, p, \alpha, \sigma_0) = (0, 4, 10), & \text{where } \tau_0 = (1, 2, 3) \\
\mathrm{GS}^{\lambda(\tau_1)}(N, p, \alpha, \sigma_0) = (0, 8, 6), & \text{where } \tau_1 = (1, 3, 2) \\
\mathrm{GS}^{\lambda(\tau_2)}(N, p, \alpha, \sigma_0) = (4, 0, 10), & \text{where } \tau_2 = (2, 1, 3) \\
\mathrm{GS}^{\lambda(\tau_3)}(N, p, \alpha, \sigma_0) = (10, 0, 4), & \text{where } \tau_3 = (2, 3, 1) \\
\mathrm{GS}^{\lambda(\tau_4)}(N, p, \alpha, \sigma_0) = (6, 8, 0), & \text{where } \tau_4 = (3, 1, 2) \\
\mathrm{GS}^{\lambda(\tau_5)}(N, p, \alpha, \sigma_0) = (10, 4, 0), & \text{where } \tau_5 = (3, 2, 1)
\end{array}$$

and $\mathrm{EGS}(N, p, \alpha, \sigma_0) = \frac{1}{6}\sum_{\tau\in\Pi_N} \mathrm{GS}^{\lambda(\tau)}(N, p, \alpha, \sigma_0) = (5, 4, 5)$. Note that $m^{\tau_i}(v) = GS^{\lambda(\tau_i)}(N, p, \alpha, \sigma_0)$ for $i \in \{0, 2, 3, 5\}$ and that $m^{\tau_i}(v) \neq GS^{\lambda(\tau_i)}(N, p, \alpha, \sigma_0)$ for $i \in \{1, 4\}$. (See Fig. 1)

In Example 3.6 an extreme point of the core of a sequencing game coincides with an extreme point of the corresponding split core if the corresponding permutation is connected. Here, a permutation $\tau \in \Pi_N$ is called connected if for any $i \in N$ the set $P(\tau, i)$ is a connected set. The next proposition shows that this property holds for any sequencing situation.

PROPOSITION 3.7. *Let (N, p, α, σ_0) be a sequencing situation and (N, v) the corresponding sequencing game. Then $m^\tau(v) = \mathrm{GS}^{\lambda(\tau)}(N, p, \alpha, \sigma_0)$ if τ is connected.*

Proof. For any connected $\tau \in \Pi_N$ we have $m^\tau(v) = m^\tau(w)$. Lemma 3.3 then completes the proof. ∎

Example 3.6 shows that in case a permutation is not connected the corresponding permutation-based gain splitting allocation does not necessarily coincide with a marginal vector of the sequencing game. The following example illustrates that a $\lambda \in \Lambda$ with $\lambda_{ij} \in \{0, 1\}$ for all $i, j \in N$, $i \neq j$ that does not arise by a permutation $\tau \in \Pi_N$ as defined in (2) is not necessarily an extreme point of $\mathrm{SPC}(N, p, \alpha, \sigma_0)$.

EXAMPLE 3.8. Consider the game of Example 3.6. Consider λ defined by $\lambda_{12} = 1$, $\lambda_{13} = 0$, and $\lambda_{23} = 1$. Obviously λ cannot be constructed by means of a permutation τ as in (2). Note that $\text{GS}^{\lambda}(N, p, \alpha, \sigma_0) = (4, 6, 4)$ is not an extreme point of $\text{SPC}(N, p, \alpha, \sigma_0)$.

In the following we will give an axiomatic characterization of the split core. For this, let $\text{SEQ}(N)$ represent the class of all sequencing situations with agent set N. A set-valued solution concept γ assigns to each sequencing situation $\text{SEQ}(N)$ a nonempty subset of $\mathbb{R}^N$. We consider the following three properties of a solution concept γ.

(i) Efficiency: Let $(N, p, \alpha, \sigma_0) \in \text{SEQ}(N)$ and let $\hat{\sigma}$ be an optimal processing order for N. Then for any $x \in \gamma(N, p, \alpha, \sigma_0)$ we have that $\sum_{k \in N} x_k = C_N(\sigma_0) - C_N(\hat{\sigma})$.

(ii) Dummy property: Let $(N, p, \alpha, \sigma_0) \in \text{SEQ}(N)$ and let $\hat{\sigma}$ be an optimal processing order for N. If $P(\sigma_0, k) = P(\hat{\sigma}, k)$ for some $k \in N$, then for all $x \in \gamma(N, p, \alpha, \sigma_0)$ it holds that $x_k = 0$.

(iii) Monotonicity: Let (N, p, α, σ_0), $(N, p, \alpha, \sigma_1) \in \text{SEQ}(N)$ and $i, j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1$, $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N \backslash \{i, j\}$. Then for all $x \in \gamma(N, p, \alpha, \sigma_0)$ there exists a $y \in \gamma(N, p, \alpha, \sigma_1)$ such that

(a) $x_k = y_k$ for all $k \in N \backslash \{i, j\}$ and $x_i \geq y_i$, $x_j \geq y_j$, or
(b) $x_k = y_k$ for all $k \in N \backslash \{i, j\}$ and $x_i \leq y_i$, $x_j \leq y_j$.

Efficiency states that the maximum cost savings of the grand coalition is divided among the players. The dummy property states that if a player does not contribute to the cost savings of the grand coalition, then he will obtain no share of these profits. For the explanation of monotonicity, note that if two neighbors i and j change places in a sequencing situation with initial order σ_0, the resulting situation can be interpreted as a sequencing situation with initial order σ_1. Then monotonicity states that for each allocation of the solution concept with respect to the game with initial order σ_0, there exists an allocation of the solution concept of the game with initial order σ_1, such that the remaining agents receive the same payoff and the agents i and j either both lose or both gain something. Note that the combination of monotonicity and efficiency yields that agents i and j will indeed change places if this is necessary to reach the optimal order.

The following theorem shows that the split core is the "maximal" solution concept that satisfies efficiency, the dummy property, and monotonicity. Here, maximality means that any solution concept that satisfies these three properties assigns to each sequencing situation a subset of the split core.

THEOREM 3.9. *The split core is a solution concept on* $\text{SEQ}(N)$ *that satisfies efficiency, the dummy property, and monotonicity. Let* γ *be a solution concept on*

SEQ(N) *that satisfies efficiency, the dummy property, and monotonicity. Then*

$$\gamma(N, p, \alpha, \sigma_0) \subset \mathrm{SPC}(N, p, \alpha, \sigma_0) \qquad \textit{for all } (N, p, \alpha, \sigma_0) \in \mathrm{SEQ}(N).$$

Proof. Obviously, the split core assigns to each sequencing situation in SEQ(N) a nonempty subset of $\mathbb{R}^N$. First we show that the split core satisfies the three properties. Let $(N, p, \alpha, \sigma_0) \in \mathrm{SEQ}(N)$. Efficiency follows immediately from the fact that SPC(N, p, α, σ_0) is a subset of the core of the corresponding sequencing game. If player k is a dummy player we have that $g_{ik} = 0$ for all $i \in N$ with $\sigma_0(i) < \sigma_0(k)$ and $g_{kj} = 0$ for all $j \in N$ with $\sigma_0(k) < \sigma_0(j)$. This implies that $\mathrm{GS}_k^\lambda(N, p, \alpha, \sigma_0) = 0$ for any $\lambda \in \Lambda$ and consequently the split core satisfies the dummy property. For monotonicity, let $\lambda \in \Lambda$ and let $i, j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1$ and take σ_1 such that $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N\backslash\{i, j\}$. From the definition of a gain splitting allocation it readily follows that

$$\mathrm{GS}_k^\lambda(N, p, \alpha, \sigma_0) = \mathrm{GS}_k^\lambda(N, p, \alpha, \sigma_1) \qquad \text{for all } k \in N\backslash\{i, j\}. \tag{3}$$

Further, we have that

$$\mathrm{GS}_i^\lambda(N, p, \alpha, \sigma_0) - \mathrm{GS}_i^\lambda(N, p, \alpha, \sigma_1) = g_{ij}\lambda_{ij} - g_{ji}(1 - \lambda_{ji}) \tag{4}$$

and

$$\mathrm{GS}_j^\lambda(N, p, \alpha, \sigma_0) - \mathrm{GS}_j^\lambda(N, p, \alpha, \sigma_1) = g_{ij}(1 - \lambda_{ij}) - g_{ji}\lambda_{ji}. \tag{5}$$

If $\alpha_j p_i - \alpha_i p_j \geq 0$ then $g_{ij} \geq 0$ and $g_{ji} = 0$, which implies that $\mathrm{GS}_m^\lambda(N, p, \alpha, \sigma_0) \geq \mathrm{GS}_m^\lambda(N, p, \alpha, \sigma_1)$ for $m \in \{i, j\}$. On the other hand, if $\alpha_j p_i - \alpha_i p_j < 0$ then $g_{ij} = 0$ and $g_{ji} > 0$, which implies $\mathrm{GS}_m^\lambda(N, p, \alpha, \sigma_0) \leq \mathrm{GS}_m^\lambda(N, p, \alpha, \sigma_1)$ for $m \in \{i, j\}$. Hence, the split core satisfies monotonicity.

Let γ be a set-valued solution concept on SEQ(N) that satisfies efficiency, the dummy property, and monotonicity, and let $(N, p, \alpha, \sigma_0) \in \mathrm{SEQ}(N)$. To show that $\gamma(N, p, \alpha, \sigma_0) \subset \mathrm{SPC}(N, p, \alpha, \sigma_0)$ we proceed by induction to the number of misplacements $M_\sigma = \{(i, j) \mid \sigma_0(i) < \sigma_0(j), g_{ij} > 0\}$. If $|M_{\sigma_0}| = 0$ then σ_0 is an optimal order and the dummy property implies that $\gamma(N, p, \alpha, \sigma_0) = \{(0, \ldots, 0)\} = \mathrm{SPC}(N, p, \alpha, \sigma_0)$. Assume that $\gamma(N, p, \alpha, \sigma) \subset \mathrm{SPC}(N, p, \alpha, \sigma)$ for all $\sigma \in \sum_N$ with $|M_\sigma| \leq m$. Let σ_0 be such that $|M_{\sigma_0}| = m + 1$. We will show that $\gamma(N, p, \alpha, \sigma_0) \subset \mathrm{SPC}(N, p, \alpha, \sigma_0)$. Take $x \in \gamma(N, p, \alpha, \sigma_0)$ and let $i, j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1$ and (i, j) is a misplacement of σ_0. Take σ_1 such that $\sigma_1(i) = \sigma_0(j)$, $\sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N\backslash\{i, j\}$. Note that $g_{ij} > 0$ since (i, j) is a misplacement. For any $z \in \gamma(N, p, \alpha, \sigma_1)$ we have by efficiency that

$$\sum_{k \in N} x_k - \sum_{k \in N} z_k = C_N(\sigma_0) - C_N(\sigma_1) = g_{ij}, \tag{6}$$

where the last equality follows from the definition of σ_1 and the fact that (i, j) is a misplacement. From (6) and monotonicity it follows that there exists a $y \in \gamma(N, p, \alpha, \sigma_1)$ such that

$$(x_i+x_j)-(y_i+y_j) = g_{ij} \text{ and } x_i \geq y_i, x_j \geq y_j, x_k = y_k \qquad \text{for all } k \in N\backslash\{i, j\}. \tag{7}$$

Since $|M_{\sigma_1}| = m$ the induction hypothesis yields that there exists a $\lambda \in \Lambda$ such that $y = \mathrm{GS}^{\lambda}(N, p, \alpha, \sigma_1)$. Substitution in (7) gives

$$\begin{array}{lcll} x_i + x_j & = & \mathrm{GS}_i^{\lambda}(N, p, \alpha, \sigma_1) + \mathrm{GS}_j^{\lambda}(N, p, \alpha, \sigma_1) + g_{ij} & \\ x_i & \geq & \mathrm{GS}_i^{\lambda}(N, p, \alpha, \sigma_1) & \\ x_j & \geq & \mathrm{GS}_j^{\lambda}(N, p, \alpha, \sigma_1) & \\ x_k & = & \mathrm{GS}_k^{\lambda}(N, p, \alpha, \sigma_1) \qquad \text{for all } k \in N\backslash\{i, j\}. & \end{array} \tag{8}$$

Since x is a solution of the system (8), there exists an $s^* \in [0, 1]$ such that

$$\begin{array}{lcl} x_i & = & \mathrm{GS}_i^{\lambda}(N, p, \alpha, \sigma_1) + s^* g_{ij} \\ x_j & = & \mathrm{GS}_j^{\lambda}(N, p, \alpha, \sigma_1) + (1 - s^*) g_{ij} \\ x_k & = & \mathrm{GS}_k^{\lambda}(N, p, \alpha, \sigma_1) \qquad \text{for all } k \in N\backslash\{i, j\}. \end{array} \tag{9}$$

Take $\lambda^* \in \Lambda$ such that $\lambda^*_{ij} = s^*$ and $\lambda^*_{kl} = \lambda_{kl}$ for all $k, l \in \{1, \ldots, n\}$, $k \neq l$ and $(k, l) \neq (i, j)$. Then from (3), (4), (5), (9), and the fact that $g_{ji} = 0$ we have $x = \mathrm{GS}^{\lambda^*}(N, p, \alpha, \sigma_0)$. Consequently, $x \in \mathrm{SPC}(N, p, \alpha, \sigma_0)$. ∎

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