THE PEGS-RULE FOR PROBABILISTIC SEQUENCING SITUATIONS

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Abstract

This paper considers one-machine sequencing situations in which agents are in a fixed initial order before the processing of the machine starts. The agents are allowed to rearrange positions to save costs w.r.t. the costs given by the initial order. We will assume that the agents are not certain about the position they will take in the initial order. For these sequencing situations we introduce and characterize in two different ways the Probabilistic Equal Gain Splitting (PEGS) rule that assigns the expected cost savings to the agents.

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1 Introduction

In one-machine sequencing situations each player has one job which has to be processed on a single machine. The processing time is the time the machine needs to handle the job. For each player we assume that his cost depends linearly on the completion time of his job. Further, there is an initial order on the jobs of the agents before the processing of the machine starts.

For deterministic one-machine sequencing situations the agents can save costs by rearranging their position from the initial order to an optimal order before the machine starts processing. For this class of sequencing situations *Curiel, Pederzoli* and *Tijs* (1989) introduced the Equal Gain Splitting (EGS) rule. This rule is based on the fact that the optimal order of a sequencing situation can be obtained from the initial order by consecutive switches of neighbours with a non-negative gain. An agent obtains half of the gain of all neighbourhood switches in which he is actually involved. Note that the EGS-rule is independent of the chosen optimal order and that the gain of a neighbourhood switch is independent of the position of the neighbours in the queue. They characterized the EGS-rule using properties that describe the effect of neighbour switches. An alternative characterization of the EGS-rule, using a consistency property, is provided in *Suijs, Hamers* and *Tijs* (1995).

This paper assumes that there will be an initial processing order before the processing of the machine starts, but it is not exactly known which order this will be. This uncertainty will be described by a probability measure on all possible processing orders. We will refer to these one-machine sequencing situations as probabilistic sequencing situations. Note that a deterministic sequencing situation is a probabilistic sequencing situation in which the initial order is given with probability one.

We introduce the Probabilistic Equal Gain Splitting (PEGS) rule, which is a generalization of the EGS-rule, for probabilistic sequencing situations. The PEGS-rule is defined as the expectation of all EGS-rules of the corresponding deterministic situations. Both characterizations which will be provided of the PEGS-rule use reductions of probabilistic sequencing situations to a player.

Section 2 will describe formally probabilistic sequencing situations. The definition and the characterizations of the PEGS-rule are given in section 3.

2 Probabilistic sequencing situations

In a one-machine probabilistic sequencing situation there is a queue of agents, each with one job, before a machine. Each agent has to have his job processed on this machine. The finite set of agents is denoted by $N = \{1, \ldots, n\}$. The processing time p_i of the job of player i is the time the machine takes to handle the job. It is assumed that every agent has a linear cost function $c_i : [0, \infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. This means that the cost of agent i is $c_i(t)$ if he is t units of time in the system. By a bijection $\sigma : N \to \{1, \ldots, n\}$ we can describe the position of the agents in the queue. Specifically $\sigma(i) = j$ means that player i is in position j. We assume that there is a fixed processing order before the processing of the machine starts, but it is not exactly known which processing order this will be. This uncertainty is described by a probability measure μ on all possible processing orders Π_N, which is known to all players.

A probabilistic sequencing situation as described above is denoted by (N, μ, p, α), where $N = \{1, \ldots, n\}$, μ the probability measure on Π_N, $p = (p_i)_{i \in N} \in (0, \infty)^N$,$\alpha = (\alpha_i)_{i \in N} \in (0, \infty)^N$. The completion time $C(\sigma, i)$ of a player i w.r.t. the order σ is defined by $C(\sigma, i) = \sum_{j : \sigma(j) \leq \sigma(i)} p_j$.

A deterministic sequencing situation in which the fixed processing order σ_0 is given with probability one, denoted by (N, σ_0, p, α), is a special case of a probabilistic sequencing situation. These deterministic sequencing situations are extensively studied in a game theoretical context by *Curiel, Pederzoli* and *Tijs* (1989), *Curiel, Potters, Rajendra Prassad, Tijs* and *Veltman* (1993) and *Hamers, Borm* and *Tijs* (1995).

We conclude this section by describing the maximum expected cost savings the agents can obtain. For a deterministic sequencing situation (N, σ, p, α) the costs $C_N(\sigma)$ of the collection N are equal to $\sum_{i \in N} \alpha_i C(\sigma, i)$. If $\hat{\sigma}$ is an optimal order of a deterministic sequencing situation (N, σ_0, p, α), then the maximal cost savings of N are equal to

$$C_N(\sigma_0) - C_N(\hat{\sigma}). \tag{1}$$

Curiel et al. (1989) showed that the cost savings where independent of the chosen optimal order. More precisely, they showed that for a deterministic sequencing situation (N, σ_0, p, α) expression (1) is equivalent to

$$\sum_{i,j \in N: \sigma_0(i) < \sigma_0(j)} g_{ij},$$

where $g_{ij} := \max\{\alpha_j p_i - \alpha_i p_j, 0\}$ represents the gain attainable for player i and j in case player i is directly in front of player j.

Now, it follows that for a probabilistic sequencing situation (N, μ, p, α) the expected cost savings are equal to

$$\sum_{\sigma \in \Pi_N} \mu(\sigma) \sum_{i,j \in N: \sigma(i) < \sigma(j)} g_{ij}. \tag{2}$$

3 Probabilistic equal gain splitting rule

In this section we recall the definition of the EGS-rule for deterministic sequencing situations. Then we will introduce the PEGS-rule for probabilistic sequencing situations that divides the expected cost savings of the set of agents N. Finally, we provide two different characterizations of the PEGS-rule.

The EGS-rule assigns to each agent half of the gains of all neighbour switches he is actually involved in reaching an optimal order from the fixed processing order. Formally, let (N, σ_0, p, α) be a deterministic sequencing situation, then the EGS-rule is defined for all $i \in N$ by

$$EGS_i(N, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j: \sigma_0(i) < \sigma_0(j)} g_{ij} + \frac{1}{2} \sum_{k: \sigma_0(k) < \sigma_0(i)} g_{ki}.$$

Note that the EGS-rule is independent of the chosen optimal order.

The Probabilistic Equal Gain Splitting (PEGS) rule for a probabilistic sequencing situations (N, μ, p, α) is defined as the expectation of all EGS-rules of the corresponding deterministic sequencing situations. Formally,

$$PEGS_i(N, \mu, p, \alpha) = \sum_{\sigma_0 \in \Pi_N} \mu(\sigma_0) EGS_i(N, \sigma_0, p, \alpha).$$

for all $i \in N$. The following example illustrates the PEGS-rule.

Example 1 Let $N = \{1, 2, 3\}, p = (1, 2, 2), \alpha = (3, 2, 4)$. Take $\sigma_1 = (1, 2, 3), \sigma_2 = (1, 3, 2), \sigma_3 = (2, 1, 3), \sigma_4 = (2, 3, 1), \sigma_5 = (3, 1, 2)$ and $\sigma_6 = (3, 2, 1)$. Let $\mu(\sigma_1) =$

$\frac{1}{6}, \mu(\sigma_2) = 0, \mu(\sigma_3) = \frac{1}{2}, \mu(\sigma_4) = \frac{1}{6}, \mu(\sigma_5) = \frac{1}{6}$ and $\mu(\sigma_6) = 0$. Then

$$\begin{aligned} EGS(N, \sigma_1, p, \alpha) &= (0, 2, 2) \\ EGS(N, \sigma_2, p, \alpha) &= (0, 0, 0) \\ EGS(N, \sigma_3, p, \alpha) &= (2, 4, 2) \\ EGS(N, \sigma_4, p, \alpha) &= (3, 4, 3) \\ EGS(N, \sigma_5, p, \alpha) &= (1, 0, 1) \\ EGS(N, \sigma_6, p, \alpha) &= (3, 2, 1) \end{aligned}$$

Hence, we may conclude that

$$PEGS(N, \mu, p, \alpha) = \sum_{i=1}^{6} \mu(\sigma_i) EGS(N, \sigma_i, p, \alpha) = (1\frac{2}{3}, 3, 2).$$

Next, we provide a characterization of the PEGS-rule. For this characterization we define a reduction of a probabilistic sequencing situation (N, μ, p, α) to one certain player. Let PSEQ(N) denote the class of probabilistic sequencing situations with player set N. Let $\mid N \mid \geq 2$. Take $(N, \mu, p, \alpha) \in PSEQ(N)$ and $i \in N$. Then $(N\backslash\{i\}, \mu_{-i}, p_{-i}, \alpha_{-i})$ is called the reduced form of (N, μ, p, α) w.r.t i. Here, μ_{-i} is the probability measure on $\Pi(N\backslash\{i\})$ defined by

$$\mu_{-i}(\sigma_0) = \sum_{\tau \in \Pi_N(\sigma_0, i)} \mu(\tau)$$

for all $\sigma_0 \in \Pi_{N\backslash\{i\}}$ with $\Pi_N(\sigma_0, i)$ the set of bijections defined by

$\Pi_N(\sigma_0, i) = \{\tau \in \Pi_N \mid$ for all $k, l \in N\backslash\{i\} : \tau(k) < \tau(l)$ if and only if $\sigma_0(k) < \sigma_0(l)\}$.

Further, the vectors $p_{-i} \in \mathbb{R}^{N\backslash\{i\}}$ and $\alpha_{-i} \in \mathbb{R}^{N\backslash\{i\}}$ are obtained from p and α, respectively, by removing the coordinate corresponding to player i.

Consider the following properties for a rule $f : PSEQ(N) \to [0, \infty)^N$.

Efficiency: Let $(N, \mu, p, \alpha) \in PSEQ(N)$ and let $\hat{\sigma}$ be an optimal order for all corresponding deterministic sequencing situations. Then

$$\sum_{i \in N} f_i(N, \mu, p, \alpha) = \sum_{\sigma_0 \in \Pi_N} \mu(\sigma_0)(C_N(\sigma) - C_N(\hat{\sigma})).$$

Exchange Property: Let $(N, \mu, p, \alpha) \in PSEQ(N)$ and $i, j \in N, i \neq j$. Then

$$f_i(N, \mu, p, \alpha) - f_i(N\backslash\{j\}, \mu_{-j}, p_{-j}, \alpha_{-j})$$

$$= f_j(N, \mu, p, \alpha) - f_j(N\backslash\{i\}, \mu_{-i}, p_{-i}, \alpha_{-i}).$$

Efficiency states that the maximal expected cost savings the grand coalition can obtain is allocated to the players. For a sequencing situation with at least two players, the exchange property states that the change of the allocation of player i when player j is excluded equals the change of the allocation of player j when player i is excluded.

Lemma 1 *Let $(N,\mu,p,\alpha) \in PSEQ(N)$ and let $i,j \in N, i \neq j$. Then*

$$PEGS_j(N,\mu,p,\alpha) - PEGS_j(N\backslash\{i\},\mu_{-i},p_{-i},\alpha_{-i})$$
$$= \sum_{\pi\in\Pi_N:\pi(i)<\pi(j)} \mu(\pi)\frac{1}{2}g_{ij} + \sum_{\pi\in\Pi_N:\pi(i)>\pi(j)} \mu(\pi)\frac{1}{2}g_{ji}$$

PROOF: Let $(N,\mu,p,\alpha) \in PSEQ(N)$. Take $i,j \in N, i \neq j$, then

$$\begin{aligned}
& PEGS_j(N,\mu,p,\alpha) - PEGS_j(N\backslash\{i\},\mu_{-i},p_{-i},\alpha_{-i}) \\
= & \sum_{\pi\in\Pi_N} \mu(\pi)EGS_j(N,\pi,p,\alpha) \\
& - \sum_{\sigma\in\Pi_{N\backslash\{i\}}} \mu_{-i}(\sigma)EGS_j(N\backslash\{i\},\sigma,p_{-i},\alpha_{-i}) \\
= & \sum_{\pi\in\Pi_N} \mu(\pi)\frac{1}{2}[\sum_{k\in N:\pi(k)<\pi(j)} g_{kj} + \sum_{l\in N:\pi(j)<\pi(l)} g_{jl}] \\
& - \sum_{\sigma\in\Pi_{N\backslash\{i\}}} \mu_{-i}(\sigma)\frac{1}{2}[\sum_{k\in N\backslash\{i\}:\sigma(k)<\sigma(j)} g_{kj} + \sum_{l\in N\backslash\{i\}:\sigma(j)<\sigma(l)} g_{jl}] \\
= & \sum_{\pi\in\Pi_N} \mu(\pi)\frac{1}{2}[\sum_{k\in N:\pi(k)<\pi(j)} g_{kj} + \sum_{l\in N:\pi(j)<\pi(l)} g_{jl}] \\
& - \sum_{\sigma\in\Pi_{N\backslash\{i\}}} \sum_{\tau\in\Pi_N(\sigma,i)} \mu(\tau)\frac{1}{2}[\sum_{k\in N\backslash\{i\}:\tau(k)<\tau(j)} g_{kj} + \sum_{l\in N\backslash\{i\}:\tau(j)<\tau(l)} g_{jl}] \\
= & \sum_{\pi\in\Pi_N} \mu(\pi)\frac{1}{2}[\sum_{k\in N:\pi(k)<\pi(j)} g_{kj} + \sum_{l\in N:\pi(j)<\pi(l)} g_{jl}] \\
& - \sum_{\tau\in\Pi_N} \mu(\tau)\frac{1}{2}[\sum_{k\in N\backslash\{i\}:\tau(k)<\tau(j)} g_{kj} + \sum_{l\in N\backslash\{i\}:\tau(j)<\tau(l)} g_{jl}] \\
= & \sum_{\pi\in\Pi_N:\pi(i)<\pi(j)} \mu(\pi)\frac{1}{2}g_{ij} + \sum_{\pi\in\Pi_N:\pi(i)>\pi(j)} \mu(\pi)\frac{1}{2}g_{ji}.
\end{aligned}$$

□

Now we can characterize the PEGS-rule.

Theorem 1 *The PEGS-rule is the unique rule for probabilistic sequencing situations that satisfies efficiency and the exchange property.*

PROOF: Efficiency follows from the definition of the PEGS rule. The exchange property follows from lemma 1.

Let $f : PSEQ(N) \to [0,\infty)^N$ be a rule that satisfies the two properties. Let $(N,\mu,p,\alpha) \in PSEQ(N)$. The proof will be by induction to the number of players. Let $\mid N \mid = 2$. Then the exchange property yields that $f_1(N,\mu,p,\alpha) = f_2(N,\mu,p,\alpha)$. Combining this with efficiency we have that $f(N,\mu,p,\alpha) = PEGS(N,\mu,p,\alpha)$. Suppose that $f(N,\mu,p,\alpha) = PEGS(N,\mu,p,\alpha)$ for all $(N,\mu,p,\alpha) \in PSEQ(N)$ with $\mid N \mid = m$. Let $(N,\mu,p,\alpha) \in PSEQ(N)$ be such that $\mid N \mid = m+1$. In the following system of $m+1$ linear equations the first equation follows from efficiency and the second up to the last equation follow from the exchange property and induction hypothesis.

$$\begin{cases} \sum_{i\in N} f_i(N,\mu,p,\alpha) = \sum_{\sigma\in\Pi_N} \mu(\sigma)(C_N(\sigma) - C_N(\hat{\sigma})) \\ f_1(N,\mu,p,\alpha) - f_j(N,\mu,p,\alpha) \\ = PEGS_1(N\backslash\{j\},\mu_{-j},p_{-j},\alpha_{-j}) - PEGS_j(N\backslash\{1\},\mu_{-1},p_{-1},\alpha_{-1}) \quad \text{for all} j \in N\backslash\{1\}. \end{cases}$$

Since $PEGS(N,\mu,p,\alpha)$ satisfies efficiency and the exchange property it is a solution of this system of equations. Since this system of equations is determined by a square non-singular matrix we have that this system has a unique solution. Consequently, $f(N,\mu,p,\alpha) = PEGS(N,\mu,p,\alpha)$. □

Now, we provide an alternative characterization of the PEGS-rule. Consider the following property for a rule $f : PSEQ(N) \to [0,\infty)^N$.

Add Property: Let $(N,\mu,p,\alpha) \in PSEQ(N)$ and let $i \in N$. Then

$$f_i(N,\mu,p,\alpha) = \sum_{j\in N\backslash\{i\}} \{f_j(N,\mu,p,\alpha) - f_j(N\backslash\{i\},\mu_{-i},p_{-i},\alpha_{-i})\}.$$

The add property states that the profit a player i obtains in a situation (N,μ,p,α) is equal to the difference of the profit that the players $N\backslash\{i\}$ can make with i in the situation (N,μ,p,α) and the profit these players can make in the reduced situation w.r.t. i.

Now, we can give another characterization of the PEGS-rule.

Theorem 2 *The PEGS rule is the unique rule for probabilistic sequencing situations that satisfies efficiency and the add property.*

Proof: First we show that the PEGS rule satisfies the add property. Let $(N, \mu, p, \alpha) \in PSEQ(N)$. Take $i \in N$, then

$$\begin{aligned}
& PEGS_i(N, \mu, p, \alpha) \\
= & \sum_{\pi \in \pi_N} \mu(\pi) \frac{1}{2} [\sum_{k \in N : \pi(k) < \pi(i)} g_{ki} + \sum_{l \in N : \pi(i) < \pi(l)} g_{il}] \\
= & \sum_{j \in N \setminus \{i\}} \{ \sum_{\pi \in \Pi_N : \pi(i) < \pi(j)} \mu(\pi) \frac{1}{2} g_{ij} + \sum_{\pi \in \Pi_N : \pi(i) > \pi(j)} \mu(\pi) \frac{1}{2} g_{ji} \} \\
= & \sum_{j \in N \setminus \{i\}} [PEGS_j(N, \mu, p, \alpha) - PEGS_j(N \setminus \{i\}, \mu_{-i}, p_{-i}, \alpha_{-i})]
\end{aligned}$$

where the third equality holds by lemma 1.

Let $f : PSEQ(N) \to [0, \infty)^N$ be a rule that satisfies the two properties. Let $(N, \mu, p, \alpha) \in PSEQ(N)$. The proof will be by induction to the number of players. Let $\mid N \mid = 2$. Then the add property yields that $f_1(N, \mu, p, \alpha) = f_2(N, \mu, p, \alpha)$. Combining this with efficiency we have that $f(N, \mu, p, \alpha) = PEGS(N, \mu, p, \alpha)$. Suppose that $f(N, \mu, p, \alpha) = PEGS(N, \mu, p, \alpha)$ for all $(N, \mu, p, \alpha) \in PSEQ(N)$ with $\mid N \mid = m$. Let $(N, \mu, p, \alpha) \in PSEQ(N)$ be such that $\mid N \mid = m + 1$. Then for any $i \in N$ we have

$$\begin{aligned}
& 2 f_i(N, \mu, p, \alpha) \\
= & \sum_{j \in N} f_j(N, \mu, p, \alpha) - \sum_{j \in N \setminus \{i\}} f_j(N \setminus \{i\}, \mu_{-i}, p_{-i}, \alpha_{-i}) \\
= & \sum_{\pi \in \Pi_N} \mu(\pi) \sum_{k, l \in N : \pi(k) < \pi(l)} g_{kl} - \sum_{\sigma \in \Pi_{N \setminus \{i\}}} \mu_{-i}(\sigma) \sum_{k, l \in N \setminus \{i\}} g_{kl} \\
= & \sum_{\pi \in \Pi_N} \mu(\pi) [\sum_{k \in N : \pi(k) < \pi(i)} g_{ki} + \sum_{l \in N : \pi(i) < \pi(l)} g_{il}] \\
= & 2 PEGS_i(N, \mu, p, \alpha)
\end{aligned}$$

The first equality holds by the add property, the second equality holds by efficiency and (2) and the third follows from a similar calculation as in the proof of lemma 1. Now, we have $f_i(N, \mu, p, \alpha) = PEGS_i(N, \mu, p, \alpha)$. □

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