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A PROPORTIONAL INFLUENCE MEASURE  
FOR SEQUENTIAL PROJECTS WITH  
IMPERFECT RELIABILITY**

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# Axiomatic Characterizations of a Proportional Influence Measure for Sequential Projects with Imperfect Reliability

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## Abstract

We define and axiomatically characterize a new proportional influence measure for sequential projects with imperfect reliability. We consider a model in which a finite set of players aims to complete a project, consisting of a finite number of tasks, which can only be carried out by certain specific players. Moreover, we assume the players to be imperfectly reliable, i.e., players are not guaranteed to carry out a task successfully. To determine which players are most important for the completion of a project, we use a *proportional influence measure*. This paper provides two characterizations of this influence measure. The most prominent property in the first characterization is task decomposability. This property describes the relationship between the influence measure of a project and the measures of influence one would obtain if one divides the tasks of the project over multiple independent smaller projects. Invariance under replacement is the most prominent property of the second characterization. If in a certain task group a specific player is replaced by a new player who was not in the original player set, this property states that this should have no effect on the allocated measure of influence of any other original player.

*Keywords:* projects, reliability, proportional influence measure, axiomatic characterization

*JEL classification:* C71

## 1 Introduction

Projects are omnipresent in society. Whether it concerns an improvised explosive device (IED) that needs to be developed, moved and placed (see Lindelauf (2011) for a more detailed breakdown of the tasks in a typical IED-project) or a project involving architecture, engineering and

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construction (AEC). Generally speaking, projects consist of tasks, and each task can only be carried out by certain players, leading to a so-called task structure.

A significant part of quantitative research on projects is concerned with project planning. For example, Brown et al. (2009) develop interdiction actions that maximally delay completion of a (nuclear) weapons project. Estévez-Fernández et al. (2007) use so-called project games to analyze projects in which certain tasks can be delayed or expedited, leading to costs or rewards, respectively, that need to be allocated. The distribution of shared costs in delayed projects is also analyzed by Bergantiños and Sánchez (2002) and Brânzei et al. (2002). In this paper, we do not focus on project planning, but on measuring the relative control of each player in the completion of a project.

In the context of organized crime, an intelligence agency may be interested in determining who is the most important player for the completion of a project, in order to determine who should be eliminated or apprehended. Regarding AEC projects, completing the project may lead to some (monetary) reward allocation that should fairly reflect the contribution of the various players. As a common feature, the question at hand is which players are most influential for the completion of a project. To this aim, we define and axiomatically characterize a new *proportional influence measure* based on the task structure of a project.

Before elaborating on the new influence measure and its axiomatic characterizations, it is important to discuss what type of projects we are analyzing. The general definition of a task structure concerns projects for which a finite player set attempts to carry out a finite number of tasks. We assume that for the completion of a project, each task must be carried out successfully. Each task can only be carried out by a certain set of players, called the task group. The tasks of a project can be carried out *sequentially*. For example, this implies that if a player is a member of all task groups, this player could attempt to complete the project alone. We emphasize ‘attempt to’ here; when a player attempts to carry out a task, we generally do not assume this player is guaranteed to carry out the task successfully. Concretely, each player-task combination has a certain success probability, also referred to as the *reliability* of the player for that specific task. Depending on the context, reliability can reflect e.g., operational risk, trustworthiness, or quality of a player in a broader sense. While the reliability of players is often not incorporated in the literature, it is a key aspect of our model. Thus, we focus mainly on *sequential projects with imperfect (player) reliability*.

In sequential projects with imperfect reliability, the success probabilities of the individual players in each task group also lead to success probabilities of the tasks, which in turn yields a

success probability of the project as a whole. For the latter, we assume that the successes of different tasks are independent. In our model, all players in a task group can attempt to carry out the corresponding task. The probability that a task is not carried out successfully equals the probability that all players fail to carry out this task. This can be interpreted in two ways. First, it is possible that all players in a task group attempt to carry out the task in parallel, where the task is considered successful if at least one player manages to carry out the task. Alternatively, a project might be such that some (random) player in the task group attempts to carry out the task, after which another (random) player in the task group attempts the task only if the first player has failed. This process continues until the task is either carried out successfully, or all players (in the task group) have failed. We do not consider, e.g., additional set-up or delay costs associated with attempting to carry out the task multiple times.

We define a proportional influence measure which allocates the final success probability of a project over the players, to determine to which extent each player contributes to the completion of a project. For each task players can carry out, the increase in value of their influence measure corresponds to their relative success probability within the task group. Hence, the allocated value per task increases with the success probability of the player (for this task), but decreases with the ‘total’ success probability of the other players in the corresponding task group. This essentially reflects a balance between a player’s quality and a player’s replaceability.

We axiomatically characterize our proportional influence measure by proving it is the only allocation mechanism for sequential projects with imperfect reliability that satisfies two sets of logically independent properties. The first characterization is on the domain of projects with a fixed player set and its main property concerns the *task decomposability* of a project. It describes the relationship between the influence measure of a project and the measures of influence one would obtain if one divides the tasks of the project over multiple independent smaller projects. For the second characterization, we consider the domain of projects with a varying set of players. The characterization is mainly based on the property *invariance under replacement*. This property relates projects with different player sets. In particular, it prescribes the relation when in a certain task group a specific player is replaced by a new player who was not in the original player set. We also observe that both characterizations still work on the smaller domain of sequential projects with perfect reliability.

A potential application lies within the framework of construction projects. Matthews and Howell (2005) propose an Integrated Project Delivery (IPD) method that emphasizes coopera-

tion between various parties who share risk and reward. Despite the fact that IPD is regarded as an effective method, Teng et al. (2019) point out that the number of construction projects using IPD is limited, in part due to the lack of a fair mechanism to allocate profit. For IPD projects that fit the assumptions of our framework, the proportional influence measure could be used as such a mechanism.

An alternative way to analyze sequential projects with imperfect reliability is to define an appropriate cooperative game and use existing game-theoretic solution concepts. We sketch a path for future research in this direction in the final section of the paper.

Section 2 formally introduces projects with imperfect reliability and the proportional influence measure, as well as their counterparts in case of perfect reliability. Section 3 discusses the first characterization, based on decomposing the tasks of a project. The second characterization, based on replacing players, is covered in Section 4. Section 5 concludes.

## 2 Projects and the Proportional Influence Measure

Let  $N$  denote the finite set of players and  $T$  the finite set of tasks to be carried out by these players. For each  $k \in T$ , let  $p^k \in [0, 1]^N$  denote a probability vector such that  $p_i^k$ ,  $i \in N$ , represents the success probability of player  $i$  to carry out task  $k$ . A *sequential project with imperfect reliability*  $P$  can thus be summarized by the tuple

$$P = (N, T, \{p^k\}_{k \in T}).$$

Additionally, for each  $k \in T$ ,  $N^k = \{i \in N \mid p_i^k > 0\}$  is called the *task group* of task  $k$ . We assume that  $N^k \neq \emptyset$  for all  $k \in T$ . Otherwise, the project cannot be completed.

The class of all projects is denoted by  $\mathcal{P}$ . If we restrict to the class of projects with some fixed player set  $N$ , we emphasize this in the notation using  $\mathcal{P}^N$ . This distinction of domains becomes relevant in the characterizations of the proportional influence measure later on.

All players in a task group can attempt to carry out the corresponding task, and all task must be carried out successfully in order to complete the project. The probability that a task is not carried out successfully is equal to the probability that all players fail to carry out this task. Assuming independence between the success of tasks, the success probability of a project can then be found by simply multiplying the success probabilities of each individual task. We denote the probability that a project  $P \in \mathcal{P}$  is completed successfully as  $q(P)$ . This can be

seen as the ‘quality’ of the player set (with respect to carrying out the tasks in the project), or simply as the success probability of the project. Formally, we define the success probability of a project as a function  $q : \mathcal{P} \rightarrow [0, 1]$  such that

$$q(P) = \prod_{k \in T} \left( 1 - \prod_{i \in N} (1 - p_i^k) \right)$$

for any  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ .

We now define a solution concept for sequential projects with imperfect reliability. We call  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  a solution concept on  $\mathcal{P}^N$ . For  $i \in N$ ,  $f_i(P)$  represents a measure of influence of player  $i$  on the completion of the project. Similarly,  $f$  is a solution concept on  $\mathcal{P}$  if it assigns a vector in  $\mathbb{R}^N$  to any  $P \in \mathcal{P}^N$  for any finite player set  $N$ . We propose a *proportional influence measure* for sequential projects with imperfect reliability, denoted by  $\rho$ , in which the total probability of success  $q(P)$  is allocated among the players in the following way. First, by the nature of a project,  $q(P)$  is shared equally among the tasks. Second, for each task  $k \in T$ ,  $q(P)/|T|$  is allocated to players proportional to their individual task-specific success probabilities provided by  $p^k$ .

### Definition 2.1

The *proportional influence measure*  $\rho : \mathcal{P}^N \rightarrow [0, 1]^N$  is defined by setting

$$\rho_i(P) = \frac{q(P)}{|T|} \sum_{k \in T} \frac{p_i^k}{\sum_{j \in N} p_j^k}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  and any  $i \in N$ .

Note that we define  $\rho$  as a solution concept on  $\mathcal{P}^N$  here. For the second characterization, we define properties of a solution concept on  $\mathcal{P}$ . In this case, we also interpret  $\rho$  as a solution concept on  $\mathcal{P}$ , simply by using the definition above for any finite  $N$ .

### Example 2.1

Consider a project  $P = (N, T, \{p^k\}_{k \in T})$  that consists of two tasks, to be carried out by a set of three players, with  $N = \{1, 2, 3\}$ ,  $T = \{a, b\}$ , and  $p^a = (0.8, 0.9, 0)$  and  $p^b = (0.8, 0, 1)$ . Clearly,

$$q(P) = (1 - (1 - p_1^a)(1 - p_2^a)(1 - p_3^a))(1 - (1 - p_1^b)(1 - p_2^b)(1 - p_3^b)) = 0.98.$$

Consequently,

$$\rho(P) = \frac{0.98}{2} \left( \frac{0.8}{1.7} + \frac{0.8}{1.8}, \frac{0.9}{1.7}, \frac{1}{1.8} \right) \approx (0.45, 0.26, 0.27). \quad \triangle$$

A project  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  is called a project with perfect reliability if  $p_i^k \in \{0, 1\}$  for all  $k \in T$  and  $i \in N$ . The subclass of  $\mathcal{P}^N$  of all projects with perfect reliability is denoted

by  $\mathcal{S}^N$ . Similarly,  $\mathcal{S}$  denotes the corresponding subclass of  $\mathcal{P}$ . Since  $N^k \neq \emptyset$  for all  $k \in T$ , note that we have  $q(P) = 1$  for all  $P \in \mathcal{S}$ . Clearly, for a project  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{S}^N$  with perfect reliability, we have

$$\rho_i(P) = \frac{1}{|T|} \sum_{k \in T: i \in N^k} \frac{1}{|N^k|}$$

for all  $i \in N$ .

### 3 Characterization Using the Decomposition of Tasks

In this section, we present our first axiomatic characterization of the proportional influence measure. The most prominent property of a solution concept in this characterization considers the effect of dividing the tasks of a project over multiple smaller projects which we ‘solve’ independently. In particular, it relates the solution of this project to the solutions of the smaller projects.

We use the following four properties of a solution concept  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  to axiomatically characterize  $\rho$  on  $\mathcal{P}^N$ .

We say that  $f$  satisfies efficiency if  $f$  allocates the total probability that a project is completed.

**Efficiency (EFF)**  $f$  satisfies EFF on  $\mathcal{P}^N$  if  $\sum_{i \in N} f_i(P) = q(P)$  for any  $P \in \mathcal{P}^N$ .

For any  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  we define  $Z(P) = N \setminus \bigcup_{k \in T} N^k$  as the set of *null players with respect to  $P$* . These null players do not have a positive success probability for any task. We say that  $f$  satisfies the null player property if all null players with respect to  $P$  are allocated zero value.

**Null Player (NUL)**  $f$  satisfies NUL on  $\mathcal{P}^N$  if  $f_i(P) = 0$  for all  $P \in \mathcal{P}^N$  with  $i \in Z(P)$ .

Proportionality only applies to projects with a single task. For such projects, this property states that the values players get allocated are proportional to their success probabilities.

**Proportionality (PROP)**  $f$  satisfies PROP on  $\mathcal{P}^N$  if

$$\frac{f_i(P)}{p_i^l} = \frac{f_j(P)}{p_j^l}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $T = \{l\}$  and all  $i, j \in N$  such that  $p_i^l > 0$  and  $p_j^l > 0$ .

The final property of task decomposability only applies to projects  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  with  $|T| > 1$ . For such projects, this property describes the relationship between the solution of this project and the solutions of two projects  $P^{T^1}$  and  $P^{T^2}$  over which the tasks in  $T$  of the original project are divided into two disjoint sets  $T^1$  and  $T^2$ . In general, we denote smaller projects with  $P^S = (N, S, \{p^k\}_{k \in S}) \in \mathcal{P}^N$  for any  $S \subseteq T$ . Note that for all  $P^S, P^{S^1}, P^{S^2} \in \mathcal{P}^N$  with  $S \subseteq T$ ,  $S^1 \cup S^2 = S$  and  $S^1 \cap S^2 = \emptyset$ , we have  $q(P^S) = q(P^{S^1})q(P^{S^2})$ .

We say that  $f$  satisfies task decomposability if the value allocated by  $f$  for the original project can be written as a certain weighted average of the values allocated by  $f$  for the two corresponding smaller projects. These weights contain the relative number of tasks to be carried out and the success probabilities of the smaller projects. In particular, the higher the relative number of tasks, the higher the weight of that project. Further, the weight increases as the success probability of the *other* smaller project increases. The intuition behind this is that the original project is only completed if both smaller projects are completed, meaning completion of one smaller project should only count if the other smaller project is also successful.

**Task decomposability (DEC)**  $f$  satisfies DEC on  $\mathcal{P}^N$  if

$$f(P) = \frac{|T^1|}{|T|} q(P^{T^2}) f(P^{T^1}) + \frac{|T^2|}{|T|} q(P^{T^1}) f(P^{T^2})$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $|T| > 1$  and all  $P^{T^1}, P^{T^2} \in \mathcal{P}^N$  with  $|T^1| \geq 1$ ,  $|T^2| \geq 1$ ,  $T^1 \cup T^2 = T$  and  $T^1 \cap T^2 = \emptyset$ .

### Example 3.1

Reconsider the project  $P = (N, T, \{p^k\}_{k \in T})$  with  $N = \{1, 2, 3\}$ ,  $T = \{a, b\}$ ,  $p^a = (0.8, 0.9, 0)$  and  $p^b = (0.8, 0, 1)$ , as described in Example 2.1. We decompose this project into two smaller projects  $P^{T^1}$  and  $P^{T^2}$  with  $T^1 = \{a\}$  and  $T^2 = \{b\}$ , and note that  $q(P^{T^1}) = 0.98$  and  $q(P^{T^2}) = 1$ . Task decomposability is satisfied by  $\rho$  in this example, since

$$\begin{aligned} \frac{1}{2} \rho(P^{T^1}) + \frac{1}{2} 0.98 \rho(P^{T^2}) &= \frac{1}{2} \frac{0.98}{1} \left( \frac{0.8}{1.7}, \frac{0.9}{1.7}, 0 \right) + \frac{0.98}{2} \frac{1}{1} \left( \frac{0.8}{1.8}, 0, \frac{1}{1.8} \right) \\ &= \frac{0.98}{2} \left( \frac{0.8}{1.7} + \frac{0.8}{1.8}, \frac{0.9}{1.7}, \frac{1}{1.8} \right) = \rho(P). \end{aligned} \quad \triangle$$

Next, we show that  $\rho$  is the only solution concept for sequential projects with imperfect reliability that satisfies the four properties defined above. To do so, we first derive a consequence of the DEC property towards decomposing a project into single-task projects.



**Lemma 3.1**

Let  $f$  be a solution concept on  $\mathcal{P}^N$  that satisfies DEC. Then,

$$f(P) = \frac{1}{|T|} \sum_{k \in T} q(P^{T \setminus \{k\}}) f(P^{\{k\}}) \quad (1)$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $|T| > 1$ .

**Proof.** Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  with  $|T| > 1$ . We give a proof by induction on  $|T|$ . First, consider the base case  $T = \{k, l\}$  with  $k \neq l$ . By DEC, we have

$$f(P) = \frac{1}{2} q(P^{\{k\}}) f(P^{\{l\}}) + \frac{1}{2} q(P^{\{l\}}) f(P^{\{k\}})$$

as required.

Next, assume the induction hypothesis that equation (1) holds for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $|T| = t$  for a given natural number  $t \geq 2$ . Then, let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  with  $|T| = t + 1$  and let  $l \in T$ . We get

$$\begin{aligned} f(P) &= \frac{t}{t+1} q(P^{\{l\}}) f(P^{T \setminus \{l\}}) + \frac{1}{t+1} q(P^{T \setminus \{l\}}) f(P^{\{l\}}) \\ &= \frac{t}{t+1} q(P^{\{l\}}) \frac{1}{t} \sum_{k \in T \setminus \{l\}} q(P^{T \setminus \{k, l\}}) f(P^{\{k\}}) + \frac{1}{t+1} q(P^{T \setminus \{l\}}) f(P^{\{l\}}) \\ &= \frac{1}{t+1} \sum_{k \in T \setminus \{l\}} q(P^{T \setminus \{k\}}) f(P^{\{k\}}) + \frac{1}{t+1} q(P^{T \setminus \{l\}}) f(P^{\{l\}}) \\ &= \frac{1}{t+1} \sum_{k \in T} q(P^{T \setminus \{k\}}) f(P^{\{k\}}), \end{aligned}$$

where we use the fact that  $f$  satisfies DEC in the first equality, the induction hypothesis in the second equality, and that  $q(P^{T \setminus \{k\}}) = q(P^{T \setminus \{k, l\}}) q(P^{\{l\}})$  for any  $k \in T \setminus \{l\}$  in the third equality.  $\square$

**Theorem 3.2**

Let  $f$  be a solution concept on  $\mathcal{P}^N$ . Then,  $f = \rho$  if and only if  $f$  satisfies EFF, NUL, PROP, and DEC.

**Proof.** We first show that  $\rho$  satisfies the four properties. EFF and NUL are obvious from the definition of  $\rho$ .

**PROP:** Consider  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $T = \{l\}$  and let  $i, j \in N$  be such that  $p_i^l > 0$  and  $p_j^l > 0$ . Then,

$$\frac{\rho_i(P)}{p_i^l} = \frac{q(P) \frac{p_i^l}{\sum_{r \in N} p_r^l}}{p_i^l} = \frac{q(P)}{\sum_{r \in N} p_r^l} = \frac{q(P) \frac{p_j^l}{\sum_{r \in N} p_r^l}}{p_j^l} = \frac{\rho_j(P)}{p_j^l}.$$

**DEC:** Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  be such that  $|T| > 1$  and let  $i \in N$ . Let  $T^1$  and  $T^2$  be such that  $|T^1| \geq 1$ ,  $|T^2| \geq 1$ ,  $T^1 \cup T^2 = T$  and  $T^1 \cap T^2 = \emptyset$  and consider  $P^{T^1}$  and  $P^{T^2}$ . Then,

$$\begin{aligned} \rho_i(P) &= \frac{q(P)}{|T|} \left( \sum_{k \in T^1} \frac{p_i^k}{\sum_{j \in N} p_j^k} + \sum_{k \in T^2} \frac{p_i^k}{\sum_{j \in N} p_j^k} \right) \\ &= \frac{|T^1|}{|T|} q(P^{T^2}) \frac{q(P^{T^1})}{|T^1|} \sum_{k \in T^1} \frac{p_i^k}{\sum_{j \in N} p_j^k} + \frac{|T^2|}{|T|} q(P^{T^1}) \frac{q(P^{T^2})}{|T^2|} \sum_{k \in T^2} \frac{p_i^k}{\sum_{j \in N} p_j^k} \\ &= \frac{|T^1|}{|T|} q(P^{T^2}) \rho_i(P^{T^1}) + \frac{|T^2|}{|T|} q(P^{T^1}) \rho_i(P^{T^2}). \end{aligned}$$

Next, let  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  satisfy the four properties. We show that  $f(P) = \rho(P)$  for all  $P \in \mathcal{P}^N$ . We first focus on projects with one task. Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  with  $T = \{k\}$ . Let  $i \in N$  such that  $p_i^k = 0$ . By NUL, we have  $f_i(P) = 0 = \rho_i(P)$ . Next, let  $i \in N$  such that  $p_i^k > 0$ . Since  $f$  satisfies PROP, we know that for any  $j \in N$  with  $p_j^k > 0$ , we have

$$f_j(P) = \frac{p_j^k}{p_i^k} f_i(P).$$

Using EFF, we get

$$q(P) = \sum_{j \in N} f_j(P) = \sum_{j \in N: p_j^k > 0} f_j(P) = \sum_{j \in N: p_j^k > 0} \frac{p_j^k}{p_i^k} f_i(P) = f_i(P) \frac{\sum_{j \in N} p_j^k}{p_i^k}.$$

From this, we may conclude

$$f_i(P) = q(P) \frac{p_i^k}{\sum_{j \in N} p_j^k} = \rho_i(P).$$

Next, let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  such that  $|T| > 1$ . By Lemma 3.1,

$$\begin{aligned} f_i(P) &= \frac{1}{|T|} \sum_{k \in T} q(P^{T \setminus \{k\}}) f_i(P^{\{k\}}) \\ &= \frac{1}{|T|} \sum_{k \in T} q(P^{T \setminus \{k\}}) q(P^{\{k\}}) \frac{p_i^k}{\sum_{j \in N} p_j^k} \\ &= \frac{q(P)}{|T|} \sum_{k \in T} \frac{p_i^k}{\sum_{j \in N} p_j^k} \\ &= \rho_i(P) \end{aligned}$$

for any  $i \in N$ , which concludes the proof.  $\square$

Finally, we observe that the four characterizing properties mentioned in the theorem are logically independent. For any subset of three properties, it suffices to find an alternative solution concept  $f$  with  $f \neq \rho$  that satisfies<sup>1</sup> these properties.

<sup>1</sup>Detailed arguments are available upon request.

**No EFF:** Consider  $f(P) = 2\rho(P)$  for all  $P \in \mathcal{P}^N$ .

**No NUL:** Consider  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  defined by

$$f(P) = \begin{cases} q(P)e_1 & \text{if } 1 \in Z(P) \\ \rho(P) & \text{if } 1 \notin Z(P) \end{cases}$$

for all  $P \in \mathcal{P}^N$ .<sup>2</sup>

**No PROP:** Let  $2^N$  denote the collection of subsets of  $N$ . Fix a representation function  $g : 2^N \setminus \{\emptyset\} \rightarrow N$  such that  $g(S) \in S$  for all  $S \in 2^N \setminus \{\emptyset\}$ . Consider  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  defined by

$$f(P) = \frac{q(P)}{|T|} \sum_{k \in T} e_{g(\{j \in N \mid p_j^k > 0\})}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$ .

**No DEC:** Consider  $f : \mathcal{P}^N \rightarrow \mathbb{R}^N$  defined by

$$f_i(P) = \begin{cases} \rho(P) & \text{if } |T| = 1 \\ \frac{q(P)}{|N \setminus Z(P)|} & \text{if } i \in N \setminus Z(P) \text{ and } |T| > 1 \\ 0 & \text{if } i \in Z(P) \text{ and } |T| > 1 \end{cases}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}^N$  and any  $i \in N$ .

To conclude this section, we analyze the results for the special case of sequential projects with perfect reliability. Despite the fact that the class  $\mathcal{S}^N$  of projects becomes significantly smaller than  $\mathcal{P}^N$ , the proportional influence measure  $\rho$  is still the only solution concept on this subdomain that satisfies the four properties of Theorem 3.2.

For the sake of completeness, we provide the explicit reformulation of the four properties on  $\mathcal{S}^N$ . Let  $f : \mathcal{S}^N \rightarrow \mathbb{R}^N$  be a solution concept on  $\mathcal{S}^N$ .

**Efficiency (EFF)**  $f$  satisfies EFF on  $\mathcal{S}^N$  if  $\sum_{i \in N} f_i(P) = 1$  for all  $P \in \mathcal{S}^N$ .

**Null Player (NUL)**  $f$  satisfies NUL on  $\mathcal{S}^N$  if  $f_i(P) = 0$  for all  $P \in \mathcal{S}^N$  with  $i \in Z(P)$ .

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<sup>2</sup>Here,  $e_1$  denotes the unit vector of length  $|N|$  of which the first element equals one, or, more generally,  $e_i$  denotes such a unit vector of which the  $i$ -th element equals one.

**Proportionality (PROP)**  $f$  satisfies PROP on  $\mathcal{S}^N$  if  $f_i(P) = f_j(P)$  for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{S}^N$  such that  $T = \{l\}$  and all  $i, j \in N$  such that  $p_i^l = 1$  and  $p_j^l = 1$ .

**Task decomposability (DEC)**  $f$  satisfies DEC on  $\mathcal{S}^N$  if

$$f(P) = \frac{|T^1|}{|T|} f(P^{T^1}) + \frac{|T^2|}{|T|} f(P^{T^2})$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{S}^N$  such that  $|T| > 1$  and all  $P^{T^1}, P^{T^2} \in \mathcal{S}^N$  with  $|T^1| \geq 1$ ,  $|T^2| \geq 1$ ,  $T^1 \cup T^2 = T$  and  $T^1 \cap T^2 = \emptyset$ .

### Theorem 3.3

Let  $f$  be a solution concept on  $\mathcal{S}^N$ . Then,  $f = \rho$  if and only if  $f$  satisfies EFF, NUL, PROP, and DEC.

## 4 Characterization Using the Replacement of Players

Our second characterization is most prominently concerned with the behavior of the proportional influence measure when in a certain task group a specific player is replaced by a player who was not in the original player set. Clearly, this property requires the player set to change and the domain of solution concepts under consideration will become  $\mathcal{P}$  instead of  $\mathcal{P}^N$ .

The axiomatic characterization of  $\rho$  on  $\mathcal{P}$  is based on four properties of a solution concept  $f$  on  $\mathcal{P}$ .

The first two properties in this characterization are efficiency and the null player property.

**Efficiency (EFF)**  $f$  satisfies EFF on  $\mathcal{P}$  if  $f$  satisfies EFF on  $\mathcal{P}^N$  for any finite  $N$ .

**Null Player (NUL)**  $f$  satisfies NUL on  $\mathcal{P}$  if  $f$  satisfies NUL on  $\mathcal{P}^N$  for any finite  $N$ .

The next property only applies to projects  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  such that the collection  $\{N^k\}_{k \in T}$ , also called the *task structure* of a project, is a partition of  $N \setminus Z(P)$ . In this setting, partition proportionality states that the value allocated by  $f$  to each (non-null) player is proportional to the relative success probability of that player in the only task group to which the player belongs.

**Partition proportionality (PAP)**  $f$  satisfies PAP on  $\mathcal{P}$  if

$$\frac{\sum_{r \in N} p_r^l}{p_i^l} f_i(P) = \frac{\sum_{r \in N} p_r^m}{p_j^m} f_j(P)$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  such that  $\{N^k\}_{k \in T}$  is a partition of  $N \setminus Z(P)$  and all  $i, j \in N$  and  $l, m \in T$  such that  $p_i^l > 0$  and  $p_j^m > 0$ .

The final property of invariance under replacement states that when in a certain task group a specific player is replaced by exactly one player who was not in the original player set and who has the same success probability for that task, this does not affect any of the other non-null players. Before formally defining the property itself, we first introduce the general definition of a *replicate project*  $\bar{P}_i \in \mathcal{P}$  corresponding to a project  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ , in which a player  $i \in N$  is replaced by a ‘new’ player in one task group.<sup>3</sup>

**Definition 4.1**

Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  be a project and let  $i \in N$ ,  $l \in T$  such that  $p_i^l > 0$ . Then, a replicate project  $\bar{P}_i \in \mathcal{P}$  with replica  $\text{repl}(i)$  for player  $i$  is defined by  $\bar{P}_i = (\bar{N}, T, \{\bar{p}^k\}_{k \in T})$ , with  $\bar{N} = N \cup \{\text{repl}(i)\}$ ,  $\bar{p}_{\text{repl}(i)}^l = p_i^l$ ,  $\bar{p}_i^l = 0$ ,  $\bar{p}_j^l = p_j^l$  for all  $j \in N \setminus \{i\}$ , and, for all  $k \in T \setminus \{l\}$ ,  $\bar{p}_{\text{repl}(i)}^k = 0$  and  $\bar{p}_i^k = p_i^k$  for all  $i \in N$ .

**Invariance under replacement (IUR)**  $f$  satisfies IUR on  $\mathcal{P}$  if

$$f_j(P) = f_j(\bar{P}_i)$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ , all  $i \in N \setminus Z(P)$  and  $j \in N \setminus (Z(P) \cup \{i\})$ , and all replicate projects  $\bar{P}_i$ .

**Example 4.1**

Reconsider the project  $P = (N, T, \{p^k\}_{k \in T})$  with  $N = \{1, 2, 3\}$ ,  $T = \{a, b\}$ ,  $p^a = (0.8, 0.9, 0)$  and  $p^b = (0.8, 0, 1)$ , as described in Example 2.1, and note that player 1 is in both task groups. We now consider the replicate project  $\bar{P}_1$  in which player 1 is replaced by a new player 4 in the second task group, so  $\bar{P}_1 = (\bar{N}, T, \{\bar{p}^k\}_{k \in T})$  with  $\bar{N} = \{1, 2, 3, 4\}$ ,  $p^a = (0.8, 0.9, 0, 0)$  and  $p^b = (0, 0, 1, 0.8)$ . Since  $q(\bar{P}_1) = 0.98$ , we have

$$\rho(\bar{P}_1) = \frac{0.98}{2} \left( \frac{0.8}{1.7}, \frac{0.9}{1.7}, \frac{1}{1.8}, \frac{0.8}{1.8} \right).$$

Indeed, we find that IUR is satisfied in this example, as  $\rho_2(\bar{P}_1) = \rho_2(P)$  and  $\rho_3(\bar{P}_1) = \rho_3(P)$ . Note that we also have  $\rho_1(\bar{P}_1) + \rho_4(\bar{P}_1) = \rho_1(P)$ .  $\triangle$

<sup>3</sup>For the definition of the IUR property, the specific task group in which the player is replaced is not important. Therefore, this task group is not explicitly reflected in the notation.

In the example, the value allocated to the player who is replaced (in one task group) in the project is equal to the sum of the values allocated to this player and the replica in the replicate project. In fact, this holds for any solution concept  $f$  that satisfies EFF, NUL, and IUR on  $\mathcal{P}$ .

**Lemma 4.2**

Let  $f$  be a solution concept on  $\mathcal{P}$  that satisfies EFF, NUL, and IUR. Then,

$$f_i(P) = f_i(\bar{P}_i) + f_{repl(i)}(\bar{P}_i)$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ , all  $i \in N \setminus Z(P)$ , and all replicate projects  $\bar{P}_i$  with replica  $repl(i)$  for player  $i$ .

**Proof.** Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ , let  $i \in N$  and  $l \in T$  be such that  $p_i^l > 0$ , and let  $\bar{P}_i = (\bar{N}, T, \{\bar{p}^k\}_{k \in T}) \in \mathcal{P}$  be the replicate project in which replica  $repl(i)$  replaces player  $i$  for task  $l$ . First, note that the success probabilities of  $P$  and  $\bar{P}_i$  are equal, since

$$\begin{aligned} q(P) &= \prod_{k \in T} \left( 1 - \prod_{j \in N} (1 - p_j^k) \right) \\ &= \left( 1 - (1 - p_i^l) \prod_{j \in N \setminus \{i\}} (1 - p_j^l) \right) \prod_{k \in T \setminus \{l\}} \left( 1 - \prod_{j \in N} (1 - p_j^k) \right) \\ &= \left( 1 - (1 - \bar{p}_{repl(i)}^l) \prod_{j \in \bar{N} \setminus \{i, repl(i)\}} (1 - \bar{p}_j^l) \right) \prod_{k \in T \setminus \{l\}} \left( 1 - \prod_{j \in \bar{N}} (1 - \bar{p}_j^k) \right) \\ &= \prod_{k \in T} \left( 1 - \prod_{j \in \bar{N}} (1 - \bar{p}_j^k) \right) \\ &= q(\bar{P}_i), \end{aligned}$$

where we use  $\bar{p}_{repl(i)}^k = 0$  for all  $k \in T \setminus \{l\}$  in the third equality and  $\bar{p}_i^l = 0$  in the fourth equality. It follows that

$$f_i(P) = q(P) - \sum_{j \in N \setminus (Z(P) \cup \{i\})} f_j(P) = q(\bar{P}_i) - \sum_{j \in N \setminus (Z(P) \cup \{i\})} f_j(\bar{P}_i) = f_i(\bar{P}_i) + f_{repl(i)}(\bar{P}_i),$$

where we use the fact that  $f$  satisfies EFF and NUL in the first and third equality, and we use  $q(P) = q(\bar{P}_i)$  and the fact that  $f$  satisfies IUR in the second equality.  $\square$

We now show that  $\rho$  is the only solution concept for sequential projects with imperfect reliability satisfying the four properties defined above.

**Theorem 4.3**

Let  $f$  be a solution concept on  $\mathcal{P}$ . Then,  $f = \rho$  if and only if  $f$  satisfies EFF, NUL, PAP, and IUR.

**Proof.** We first show that  $\rho$  satisfies the four properties. Clearly, EFF and NUL directly follow from the definition of  $\rho$ .

**PAP:** Consider  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  such that  $\{N^k\}_{k \in T}$  is a partition of  $N \setminus Z(P)$ . Let  $i, j \in N$  and  $l, m \in T$  be such that  $p_i^l > 0$  and  $p_j^m > 0$ . Then,

$$\frac{\sum_{r \in N} p_r^l}{p_i^l} \rho_i(P) = \frac{\sum_{r \in N} p_r^l q(P)}{p_i^l |T|} \sum_{k \in T} \frac{p_i^k}{\sum_{r \in N} p_r^k} = \frac{\sum_{r \in N} p_r^l q(P)}{p_i^l |T|} \frac{p_i^l}{\sum_{r \in N} p_r^l} = \frac{q(P)}{|T|},$$

independent of  $i$  and  $l$ . So, analogously,  $\rho_j(P) \sum_{r \in N} p_r^m / p_j^m = q(P) / |T|$ .

**IUR:** Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ , let  $i \in N, l \in T$  be such that  $p_i^l > 0$ , let  $j \in N \setminus (Z(P) \cup \{i\})$ , and let  $\bar{P}_i = (\bar{N}, T, \{\bar{p}^k\}_{k \in T}) \in \mathcal{P}$  be a replicate project. Since  $q(P) = q(\bar{P}_i)$  and, for all  $k \in T$ ,  $\sum_{r \in N} p_r^k = \sum_{r \in \bar{N}} \bar{p}_r^k$  and  $p_j^k = \bar{p}_j^k$ , we have

$$\rho_j(P) = \frac{q(P)}{|T|} \sum_{k \in T} \frac{p_j^k}{\sum_{r \in N} p_r^k} = \frac{q(\bar{P}_i)}{|T|} \sum_{k \in T} \frac{\bar{p}_j^k}{\sum_{r \in \bar{N}} \bar{p}_r^k} = \rho_j(\bar{P}_i).$$

Next, let  $f$  be a solution concept on  $\mathcal{P}$  satisfying the four properties. We show that  $f(P) = \rho(P)$  for all  $P \in \mathcal{P}$ .

Let  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ . To be able to use the PAP property, which holds specifically for projects with a task structure that is a partition of all non-null players, we first construct a project  $\bar{P} = (\bar{N}, T, \{\bar{p}^k\}_{k \in T})$  without null players and in which all players have a strictly positive probability to successfully carry out exactly one task only. To define  $\bar{P}$ , first choose mutually disjoint sets  $\{R_i\}_{i \in N}$  such that

$$|R_i| = |\{k \in T \mid p_i^k > 0\}|$$

for all  $i \in N$ . Note that  $R_i = \emptyset$  if and only if  $i \in Z(P)$ . Set

$$\bar{N} = \bigcup_{i \in N} R_i.$$

Let  $k \in T$ . To define  $\bar{p}^k \in [0, 1]^{\bar{N}}$ , choose a bijection

$$g_i : R_i \rightarrow \{k \in T \mid p_i^k > 0\}$$

for all  $i \in N$  and set, for all  $r \in \bar{N}$  and  $k \in T$

$$\bar{p}_r^k = \begin{cases} p_i^k & \text{if } r \in R_i \text{ and } g_i(r) = k \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $Z(\bar{P}) = \emptyset$  and that the task groups  $\bar{N}^k$ ,  $k \in T$ , partition  $\bar{N}$ . Moreover, obviously,  $q(P) = q(\bar{P})$ .

For  $r \in R_i$ , let  $k(r) = g_i(r)$  denote the unique task group this player belongs to. Fix  $t \in \bar{N}$ . Then,

$$\begin{aligned}
q(\bar{P}) &= \sum_{r \in \bar{N}} f_r(\bar{P}) \\
&= \sum_{r \in \bar{N}} \frac{\sum_{s \in \bar{N}} \bar{p}_s^{k(r)}}{\bar{p}_r^{k(r)}} f_r(\bar{P}) \frac{\bar{p}_r^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_s^{k(r)}} \\
&= \frac{\sum_{s \in \bar{N}} \bar{p}_s^{k(t)}}{\bar{p}_t^{k(t)}} f_t(\bar{P}) \sum_{r \in \bar{N}} \frac{\bar{p}_r^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_s^{k(r)}} \\
&= \frac{\sum_{s \in \bar{N}} \bar{p}_s^{k(t)}}{\bar{p}_t^{k(t)}} f_t(\bar{P}) \sum_{k \in T} \frac{\sum_{s \in \bar{N}} \bar{p}_s^k}{\sum_{s \in \bar{N}} \bar{p}_s^k} \\
&= \frac{\sum_{s \in \bar{N}} \bar{p}_s^{k(t)}}{\bar{p}_t^{k(t)}} f_t(\bar{P}) |T|,
\end{aligned}$$

where we use the fact that  $f$  satisfies EFF in the first equality, that  $f_t(\bar{P}) \sum_{s \in \bar{N}} \bar{p}_s^{k(t)} / \bar{p}_t^{k(t)} = f_r(\bar{P}) \sum_{s \in \bar{N}} \bar{p}_s^{k(r)} / \bar{p}_r^{k(r)}$  for all  $r \in \bar{N}$  since  $f$  satisfies PAP in the third equality, and that  $\bar{p}_r^k = 0$  for all  $k \in T \setminus \{k(r)\}$ ,  $r \in \bar{N}$  in the fourth equality. Hence, since  $q(P) = q(\bar{P})$ ,

$$f_t(\bar{P}) = \frac{q(P)}{|T|} \frac{\bar{p}_t^{k(t)}}{\sum_{s \in \bar{N}} \bar{p}_s^{k(t)}}. \quad (2)$$

Now, let  $i \in N \setminus Z(P)$ . Then,

$$\begin{aligned}
f_i(P) &= \sum_{r \in R_i} f_r(\bar{P}) \\
&= \sum_{r \in R_i} \frac{q(P)}{|T|} \frac{\bar{p}_r^{k(r)}}{\sum_{s \in \bar{N}} \bar{p}_s^{k(r)}} \\
&= \frac{q(P)}{|T|} \sum_{r \in R_i} \frac{p_i^{k(r)}}{\sum_{s \in N} p_s^{k(r)}} \\
&= \frac{q(P)}{|T|} \sum_{k \in T: p_i^k > 0} \frac{p_i^k}{\sum_{s \in N} p_s^k} \\
&= \frac{q(P)}{|T|} \sum_{k \in T} \frac{p_i^k}{\sum_{s \in N} p_s^k} \\
&= \rho_i(P),
\end{aligned}$$

where we use the fact that  $f$  satisfies EFF, NUL, and IUR to apply IUR and in particular Lemma 4.2 repeatedly in the first equality, equation (2) in the second equality, the definition



of  $\bar{p}_r^k$  for every replica  $r \in R_i$  in the third equality, and the one-to-one correspondence between  $R_i$  and  $\{k \in T \mid p_i^k > 0\}$  in the fourth equality.

Finally, let  $i \in Z(P)$ . Since  $f$  satisfies NUL, we get  $f_i(P) = 0 = \rho_i(P)$ .

We conclude that  $f_i(P) = \rho_i(P)$  for any  $i \in N$ . □

Similar to the previous subsection, one can show that the aforementioned four properties are logically independent. For each subset of three properties, we define an alternative solution concept  $f$  on  $\mathcal{P}$  with  $f \neq \rho$  that satisfies<sup>4</sup> these properties.

**No EFF:** Consider  $f(P) = 2\rho(P)$  for all  $P \in \mathcal{P}$ .

**No NUL:** For any finite  $N$ , fix a representation function  $g_N : 2^N \setminus \{\emptyset\} \rightarrow N$  such that  $g_N(S) \in S$  for all  $S \in 2^N \setminus \{\emptyset\}$ . Consider the solution concept  $f$  on  $\mathcal{P}$  defined by

$$f(P) = \begin{cases} \rho(P) & \text{if } Z(P) = \emptyset \\ \frac{q(P)}{2} e_{g_N(Z(P))} + \frac{\rho(P)}{2} & \text{if } Z(P) \neq \emptyset \end{cases}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$ .

**No PAP:** Let  $2^T$  denote the collection of subsets of  $T$ . Fix a representation function  $g : 2^T \setminus \{\emptyset\} \rightarrow T$  such that  $g(S) \in S$  for all  $S \in 2^T \setminus \{\emptyset\}$ . Consider the solution concept  $f$  on  $\mathcal{P}$  defined by

$$f_i(P) = q(P) \frac{p_i^{g(T)}}{\sum_{j \in N} p_j^{g(T)}}$$

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  and any  $i \in N$ .

**No IUR:** Fix a representation function  $g : 2^T \setminus \{\emptyset\} \rightarrow T$  such that  $g(S) \in S$  for all  $S \in 2^T \setminus \{\emptyset\}$ . Let  $T_i = \{k \in T \mid p_i^k > 0\}$ . Consider the solution concept  $f$  on  $\mathcal{P}$  defined by

$$f_i(P) = \begin{cases} q(P) \frac{p_i^{g(T_i)}}{\sum_{r \in N} p_r^{g(T_i)}} & \text{if } i \notin Z(P) \\ \sum_{j \in N \setminus Z(P)} \left( \frac{p_j^{g(T_j)}}{\sum_{r \in N} p_r^{g(T_j)}} \right) & \\ 0 & \text{if } i \in Z(P) \end{cases}$$

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<sup>4</sup>Again, detailed arguments are available upon request.

for all  $P = (N, T, \{p^k\}_{k \in T}) \in \mathcal{P}$  and any  $i \in N$ .

To conclude this section, we note that similar to the first characterization, the proportional influence measure  $\rho$  is still the only solution concept on the subdomain  $\mathcal{S}$  of sequential projects with perfect reliability that satisfies the four properties. Here, we omit the (direct) reformulations of the properties on  $\mathcal{S}$ .

**Theorem 4.4**

*Let  $f$  be a solution concept on  $\mathcal{S}$ . Then,  $f = \rho$  if and only if  $f$  satisfies EFF, NUL, PAP, and IUR.*

## 5 Project Games

The proportional influence measure is a solution concept based directly on the task structure of a project. This solution concept takes into account the success probabilities of all players in  $N$ , but does not explicitly take into account the ability of subsets of  $N$  (coalitions) to complete the project. To analyze this interesting topic, we can model the situation as a *project game*<sup>5</sup>, in which appropriate values for coalitions are quantified. Based on such a project game, the influence of all players in  $N$  can then be measured using a game-theoretic solution concept, like the Shapley value (Shapley, 1953).

To give an impression of how to define an appropriate associated project game, let  $P = (N, T, \{p^k\}_{k \in T})$  be a sequential project with imperfect reliability. One possible corresponding game  $v^P$  could be defined by setting the value of a coalition equal to the success probability of that coalition by means of players in  $S$  only:

$$v^P(S) = \prod_{k \in T} \left( 1 - \prod_{i \in S} (1 - p_i^k) \right)$$

for any coalition  $S$ . Note that  $v^P(N) = q(P)$ .

**Example 5.1**

Reconsider the project  $P = (N, T, \{p^k\}_{k \in T})$  of Example 2.1, with  $N = \{1, 2, 3\}$ ,  $T = \{a, b\}$ ,  $p^a = (0.8, 0.9, 0)$  and  $p^b = (0.8, 0, 1)$ . The project game  $v^P$  is given in Table 1.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$N$
$v^P(S)$	0.64	0	0	0.784	0.8	0.9	0.98

Table 1: The project game  $v^P$  of Example 5.1

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<sup>5</sup>Not to be confused with the project games defined by Estévez-Fernández et al. (2007).

In the allocation of  $v^P(N)$ , it cannot be guaranteed that a coalition  $S$  is allocated (irrespective of coalition  $N \setminus S$ ) at least its value, as demonstrated by  $v^P(\{1\})$  and  $v^P(\{2, 3\})$ . Notwithstanding this modeling drawback, the project game is a consistent representation from which we can derive an allocation of influence. For example, the Shapley value of this game is given by  $(0.504, 0.234, 0.242)$ . Note that this allocation is largely similar to the proportional influence measure  $\rho(P) \approx (0.45, 0.26, 0.27)$ .  $\triangle$

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