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Performance Analysis of Optimization Methods for Machine Learning

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Performance Analysis of Optimization Methods for Machine Learning

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To the loving souls who have enriched my life with their presence

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Notation

Set notation

$\emptyset$	the empty set
$A \cup B$	the union of the sets A and B
$A \cap B$	the intersection of the sets A and B
$A \subseteq B$	A is a subset of B
$A \setminus B$	Set difference, i.e., $\{a \in A : a \notin B\}$
$\mathrm{cl}(A)$	the closure of A
$\mathrm{int}(A)$	the interior of A
$\mathrm{ri}(A)$	the relative interior of A
$\mathrm{dom}(f)$	domain of function f
$\mathrm{epi}\, f$	epigraph of function f
$\mathrm{co}(A)$	the convex hull of $A \subseteq \mathbb{R}^n$

Special sets

$[n]$	the set of integers from 1 to n
$\mathbb{N}$	the natural numbers
$\mathbb{N}^n$	the set of n-tuples of the natural numbers
$\mathbb{Z}$	the integer numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}_+$	the non-negative reals
$\mathbb{R}^n$	the n-dimensional vectors of reals
$\mathbb{R}^n_+$	the n-dimensional vectors of non-negative reals
$\mathbb{S}^n$	the set of $n \times n$ real symmetric matrices
$\mathbb{S}^n_+$	the set of $n \times n$ positive semidefinite matrices

Functions

f^*	conjugate of function f

Inner Products, dualities and Norms

$\langle \cdot, \cdot \rangle_X$	bilinear function for dual system (X, X^*)
$\langle \cdot, \cdot \rangle_x$	local inner product at x
$\| \cdot \|_x$	norm induced by $\langle \cdot, \cdot \rangle_x$

Linear Algebra

I	identity matrix (of fitting size)
I_n	$n \times n$ identity matrix
J	all-ones matrix (of fitting size)
J_n	$n \times n$ all-ones matrix
e	all-ones vector
e_i	i-th standard unit vector
Tr	the trace operator
$A \circ B$	Hadamard product of matrices A and B
span(A)	real span of elements of A
$v^\top$	transpose of vector v (or matrix)
$A \succ 0$	the matrix A is positive definite
$A \succeq 0$	the matrix A is positive semidefinite
$\lambda_{\min}(A)$	minimum eigenvalue of matrix A
$\lambda_{\max}(A)$	maximum eigenvalue of matrix A
A^{-1}	inverse of matrix A
A_{ij}	i-th and j-th element of matrix A

What we hope ever to do with ease, we must learn first to do with diligence.

Samuel Johnson

1 Introduction

Preamble

Optimization plays a crucial role in various fields, ranging from engineering and economics to machine learning and data analysis. Most of the optimization problems are hard to be solved analytically (or in some cases it is not efficient to do so), therefore they are solved using iterative methods to approximate the solution. As a result, the development of efficient algorithms for solving optimization problems is crucial. To understand if an algorithm is efficient for a specific class of problems it is important to understand the behaviour of the algorithm. Henceforth, analysis of convergence of algorithms is a fundamental research area in optimization theory. In this context, different mathematical frameworks and tools have emerged to study and estimate the performance of optimization algorithms.

In this section we will overview general optimization problems as well as some concepts that we will use in this thesis. We will end the chapter by providing the societal and scientific relevance of the topic. Then, we will present the outcomes of this research that appeared in peer-reviewed journals or are under review.

1.1 Nonlinear programming

The general form of a *nonlinear programming* problem is as follows

$$\begin{aligned} \inf_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad \text{for } i = 1,2,\ldots,m \\ & h_j(x) = 0, \quad \text{for } j = 1,2,\ldots,p \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)$ represents the decision variables, $f(x)$ is the objective function to be minimized, $g_i(x)$ are the inequality constraints, and $h_j(x)$ are the equality constraints [LY+84]. Any point that satisfies the constraints is called a feasible point, and the set of feasible points is called the feasible set. Alternatively, the feasible set can be represented by $\mathbb{X}$, defined as

$$\mathbb{X} = \left\{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ h_j(x) = 0,\ i \in \{1,\ldots,m\},\ j \in \{1,\ldots,p\}\right\}.$$

Therefore the optimization problem can be written as

$$\inf_{x\in\mathbb{X}} f(x).$$

Analyzing such a general optimization problem is difficult, therefore we review different classes of functions facilitating studying them by considering their properties.

1.2 Classes of functions

Suppose that we want to solve the following optimization problem

$$\inf_{x\in\mathbb{R}^n} f(x).$$

It is almost impossible to talk about the behaviour of an algorithm for a general optimization problem as above [Nes18, Chapter 2], even for convergence of the algorithm to a local minimum. Therefore, it is important to introduce some assumptions on the function in order to facilitate the study of the function class. In other words, we assume that the function under the study belongs to some function classes $\mathcal{F}$ and try to make these assumptions as little restrictive as possible.

In the literature convex or L-smooth functions are among the most important function classes. A function f is called *L-smooth* in $\mathbb{R}^n$ if for some $L > 0$ it satisfies the following condition,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\,\|x - y\| \ \text{ for all } x, y \in \mathbb{R}^n.$$

There are also other classes such as *μ-strongly convex functions*, where a function f is called *μ-strongly convex* if, for some $\mu \geq 0$, the function $f(x) - \frac{\mu}{2}\|x\|^2$ is convex. Note that a 0-strongly convex function is just a convex function.

We will also consider the class of *μ-strongly convex, L-smooth functions*, denoted by $\mathcal{F}_{\mu,L}(\mathbb{R}^n)$. A function f belongs to this class if it is both L-smooth and μ-strongly convex. Note that the class $\mathcal{F}_{\mu,L}(\mathbb{R}^n)$ is a convex set in a suitable function space.

As we will discuss these function classes and their properties in Chapter 2 in detail, for further discussion we refer the interested reader to look at Chapter 2 and [Nes18, Bec17].

1.3 Iterative first-order methods

Iterative methods are among the most widely used algorithms to solve an optimization problem. The class of iterative algorithms ranges from zero-order to higher order algorithms. These algorithms start from an initial point x^0 and iteratively generate subsequent points (iterates). *First-order methods* are among the iterative methods. First-order methods use only the information of the gradient and function value at a given point. These methods can be categorized by three strategies that they use to generate the next point, namely line search, trust region, and fixed step lengths [NW06]. In this thesis we mainly focus on methods that use fixed step lengths.

Line-search algorithms, in general, move in a search direction d^k from the previous point x^k to generate the new point x^{k+1}. In other words, they solve the following problem

$$x^{k+1} = \operatorname{argmin}_t f(x^k + td^k).$$

One choice for the search direction is the steepest descent direction $-\nabla f(x^k)$ [NW06, Section 2.2].

Sometimes, it is too expensive to solve the above problem. One way to deal with this problem is to choose the step length in advance, which sometimes is called fixed step length. The gradient descent method is a widely used fixed step first-order iterative method. It updates the decision variables by taking steps proportional to the negative gradient of the objective function, i.e.

$$x^{k+1} = x^k - t_k \nabla f(x^k).$$

To have an overview on first-order methods we refer the reader to some reference books, e.g., [NW06, Nes18, Bec17].

Nocedal and Wright [NW06] name three properties that an algorithm should possess to be called a good algorithm, namely robustness, efficiency and accuracy. Studying worst-case convergence allows us to understand the weaknesses and strengths of an algorithm.

1.4 Convergence rate

Convergence rate in the worst case is a crucial aspect of analyzing optimization algorithms' performance. In this section we present some fundamental concepts and definitions about convergence rates.

Suppose that a sequence $\{x^k\}$ has a limit point $x^\star$. We are interested in how fast $x^k - x^\star$ goes to zero. We consider two types of *convergence*, *Q-convergence* and *R-convergence* [BGLS06].

Convergence of the quotient series $q^k := \frac{\|x^{k+1}-x^\star\|}{\|x^k-x^\star\|}$ as $k \to \infty$ is called Q-convergence of the sequence $\{x^k\}$. We say the sequence $\{x^k\}$ converges to $x^\star$ *Q-linearly* if $\limsup_{k\to\infty} q^k < 1$. If $\limsup_{k\to\infty} q^k = 0$ we call $\{x^k\}$ *Q-superlinearly convergent* [BGLS06, Section 1.5]. We usually omit the Q when we talk about convergence rates, if it does not cause any confusion.

On the other hand, if $\|x^k - x^\star\| \leq \nu^k$ and ν^k is Q-convergent, then the sequence $\{x^k\}$ is called R-convergent [NW06, Section A.2]. We will refer back to R-convergence in Section 9.5. It is obvious that Q-convergence implies R-convergence.

Moreover, if $\frac{\|x^k-x^\star\|}{f(x^0)-f(x^\star)} \leq \frac{c}{k^\alpha}$ for some constants c and $\alpha > 0$, the convergence rate of $\{x^k\}$ is called *sub-linear*, and it is denoted by $\mathcal{O}\left(\frac{1}{k^\alpha}\right)$ [Bec17].

For a comprehensive exploration of convergence rates, we recommend the interested reader to delve into the references [NW06] and [BGLS06]. These books provide valuable insights and in-depth analyses on the topic of convergence rates in the context of optimization.

1.5 Performance estimation problems (PEPs)

The Performance Estimation Problem (PEP) is a mathematical framework that aims to estimate the worst-case convergence rates and performance of optimization algorithms, originally for calculating convergence rates of first-order algorithms. This strong tool was first introduced by Drori and Teboulle [DT14]. This method was improved by the work of Taylor et. al [THG17c] by introducing necessary and sufficient interpolation conditions for μ-strongly convex L-smooth functions. Since then, many scholars used this method to find the convergence rate of different algorithms; see Section 3.5 for the recent works on PEP.

Briefly speaking, PEP tries to model the convergence rate of the algorithm by modeling the convergence rate problem as a semidefinite programming problem. The idea is to find the worst objective function from a given class for a given algorithm. In other words, PEP aims to identify the worst-case input for a given iterative optimization algorithm. This is in itself an optimization problem in some function space, and the surprising thing is that it may sometimes be reformulated as a semidefinite program. We describe PEP in detail in Chapter 3.

1.6 Semidefinite programming

As mentioned above, PEP transforms convergence rate analysis of an iterative algorithm into a semidefinite programming problem. In this section, we briefly introduce semidefinite programming (SDP); more details are presented in Section 2.2.

Semidefinite programming is a generalization of linear programming to handle optimization problems involving positive semidefinite matrix variables. More precisely, SDP deals with optimizing a linear objective function over the set of positive semidefinite matrices subject to some linear constraints [VB96].

Mathematically, SDP can be formulated as

$$\begin{aligned} p^\star = \sup_X \quad & \langle C, X\rangle \\ \text{s.t.} \quad & \langle A_i, X\rangle = b_i, \quad i = 1, \ldots, m, \\ & X \succeq 0, \end{aligned}$$

where matrix X is the variable, $C, A_1, \ldots, A_m$ are given symmetric matrices, and $b \in \mathbb{R}^m$ is a given vector [LV12].

SDP has been extensively studied in the field of optimization. The works by De Klerk [DK06], and Vandenberghe and Boyd [VB96] provide a comprehensive overview of SDP and its applications.

1.7 Machine learning and optimization

Machine learning typically involves a training phase, where the model learns from data, followed by a testing phase to evaluate its performance on unseen data, ensuring it can generalize well. Neural networks, which are central to deep learning, consist of interconnected layers of neurons that learn complex patterns [GBC16]. Machine learning algorithms can be broadly divided into four main categories as supervised, semi-supervised, unsupervised, and reinforcement learning [SCZZ19]. Typically, these algorithms are used for solving optimization problems during the training phase. This process aims to determine the optimal parameters of a model by minimizing a loss function.

A loss function, often denoted as $\mathcal{L}(\theta)$, depends on the model parameters θ. The optimization problem is then to find the values of the model parameters that minimize this loss function. Mathematically, it can be expressed as

$$\min_{\theta} \mathcal{L}(\theta).$$

For a comprehensive review on loss functions we refer the interested reader to [WMZT20]. In the context of training deep neural networks, the objective is to minimize a cost function, which can be formulated as

$$\min_{\theta} J(\theta) := \mathbb{E}_{(x,y)\sim\hat{p}_{\text{data}}}\left[\mathcal{L}(f(x;\theta), y)\right].$$

Here, $\mathcal{L}(\cdot)$ represents the loss function, $f(x;\theta)$ denotes the predicted value for the input x, θ stands for the model parameters, y represents the actual output, and $\hat{p}_{\text{data}}$ corresponds to the empirical distribution [GBC16].

First-order methods in deep learning, such as Gradient Descent, SGD (Stochastic Gradient Descent), Momentum, and Adam, are crucial for optimizing neural networks. They work by using the gradient of the loss function to iteratively adjust the network's parameters, minimizing loss [BB24, GBC16].

The importance of the use of first-order methods in machine learning forms an additional motivation for gaining a better understanding of their convergence analysis.

1.8 Thesis overview

We now give a brief overview of the contents of this thesis, chapter by chapter. Moreover, a summary of the main results on the convergence rate in the thesis will be given later in Table 10.1.

Chapter 2

This chapter is dedicated to some preliminaries that will be used in the other chapters. In particular, we present some fundamental theorems on functions, semidefinite programming and most importantly, we present interpolation theorems that are foundations for the performance estimation method. These interpolation theorems are in the spirit of Whitney-type extension theorems that characterize when a function may be extended to a larger domain while keeping certain properties (like continuity, convexity, or L-smoothness).

Chapter 3

In this chapter, we introduce the performance estimation method. We show how to get a convergence rate of a first-order algorithm using this method. We define the Gram matrix semidefinite programming formulation and continue with dual presentation of the performance estimation problem. We present a simple example to show how the performance estimation method works. We conclude the chapter by reviewing some recent works on the performance estimation method.

Chapter 4

In this chapter, we study the convergence rate of the gradient (or steepest descent) method with fixed step lengths for finding a stationary point of an L-smooth function. We establish a new worst-case convergence rate, and show that the bound may be exact in some cases, in particular when all step lengths lie in the interval $(0, 1/L]$. In addition, we derive an optimal step length with respect to the new bound. In addition, we present an extension of L-smooth functions from a open convex set.

Chapter 5

In this chapter, we derive a new linear convergence rate for the gradient method with fixed step lengths for non-convex smooth optimization problems satisfying the Polyak-Łojasiewicz (PŁ) inequality. We establish that the PŁ inequality is a necessary and sufficient condition for linear convergence to the optimal value for this class of problems. We list some related classes of functions for which the gradient method may enjoy a linear convergence rate. Moreover, we investigate their relationship with the PŁ inequality.

Chapter 6

In this chapter, we study randomized and cyclic coordinate descent for convex unconstrained optimization problems. We improve the known convergence rates in some cases by using the numerical semidefinite programming performance estimation method. As a spin-off we provide a method to analyse the worst-case performance of the Gauss–Seidel iterative method for linear systems where the coefficient matrix is positive semidefinite with a positive diagonal. Moreover, we study the weighted Jacobi method for solving quadratic programming problems and revisit some well-known results in the literature.

Chapter 7

In this chapter, we study the gradient descent-ascent method for convex-concave saddle-point problems. We derive a new non-asymptotic global convergence rate in terms of distance to the solution set by using the semidefinite programming performance estimation method. The given convergence rate incorporates most parameters of the problem and it is exact for a large class of strongly convex-strongly concave saddle-point problems for one iteration. We also investigate the algorithm without strong convexity and we provide some necessary and sufficient conditions under which the gradient descent-ascent enjoys linear convergence.

Chapter 8

In this chapter, we study the non-asymptotic convergence rate of the DCA (difference-of-convex algorithm), also known as the convex–concave procedure, with two different termination criteria that are suitable for smooth and non-smooth decompositions, respectively. The DCA is a popular algorithm for difference-of-convex (DC) problems and known to converge to a stationary point of the objective under some assumptions. We derive a $\mathcal{O}(1/\sqrt{N})$ worst-case convergence rate of the objective gradient norm after N iterations for certain classes of DC problems, without assuming strong convexity in the DC decomposition, and give an example which shows the convergence rate is exact. We also provide a new $\mathcal{O}(1/N)$ convergence rate for the DCA with another termination criterion to deal with the non-smooth case. Furthermore, we study the convergence rate for the proximal gradient method and the gradient descent method. Additionally, we study the impact of using regularization on DCA. Moreover, we derive a new linear convergence rate result for the DCA under the assumption of the Polyak–Łojasiewicz inequality. The novel aspect of our analysis is that it employs semidefinite programming performance estimation.

Chapter 9

In this chapter, we derive new non-ergodic convergence rates for the alternating direction method of multipliers (ADMM) by using performance estimation. We give some examples which show the exactness of the given bounds. We also study the linear and R-linear convergence of ADMM. We establish that ADMM enjoys a global linear convergence rate if and only if the dual objective satisfies the Polyak-Łojasiewicz (PŁ) inequality in the presence of strong convexity. In addition, we give an explicit formula for the linear convergence rate factor. Moreover, we study the R-linear convergence of ADMM under two new scenarios related to function classes and the rank of the matrix.

Chapter 10

We end the thesis by providing some concluding remarks and possible research questions for future work. Moreover, we provide a table summarizing the main results of the thesis.

Societal and scientific relevance of the thesis topics

In this section, we examine the influence of our endeavors on both society and the scientific community. As it is mentioned, optimization techniques are essential tools in a spectrum of fields, including mathematics, engineering, economics, computer science, and more.

From the society's perspective, the rising demand for optimization in various fields from healthcare, environmental sustainability, to financial market and machine learning, increases the demand for more efficient algorithms to solve these problems in an efficient way. With the increasing complexity of the systems and the proliferation of big data, there is an ever growing demand for more efficient and faster algorithms. The first-order algorithms demonstrate a good performance both in practice and in theory. They usually are easy to implement and have low costs. On the other hand, different problems have different structures that may fit better with one algorithm than the other. Studying first-order methods provides practitioners with a good perspective on selecting the most suitable algorithm for their problem with a theoretical guarantee. Moreover, first-order methods are the back-bone for optimizing the models used in machine learning and it goes without saying how important machine learning is in the modern world.

1.9 Contributions to the Literature

This thesis is based on the following articles:

[AdKZ22] Hadi Abbaszadehpeivasti, Etienne de Klerk, and Moslem Zamani. The exact worst-case convergence rate of the gradient method with fixed step lengths for L-smooth functions. *Optimization Letters*, 16(6):1649–1661, 2022.

[AdKZ23a] Hadi Abbaszadehpeivasti, Etienne de Klerk, and Moslem Zamani. Conditions for linear convergence of the gradient method for non-convex optimization. *Optimization Letters*, 17(5):1105–1125, 2023.

[AdKZ23b] Hadi Abbaszadehpeivasti, Etienne de Klerk, and Moslem Zamani. Convergence rate analysis of randomized and cyclic coordinate descent for convex optimization through semidefinite programming. *Applied Set-Valued Analysis and Optimization*, 5(2):141–153, 2023.

[ZAdK24] Moslem Zamani, Hadi Abbaszadehpeivasti, and Etienne de Klerk. Convergence rate analysis of the gradient descent-ascent method for convex-concave saddle-point problems. *Optimization Methods and Software*, 2024.

[AdKZ23c] Hadi Abbaszadehpeivasti, Etienne de Klerk, and Moslem Zamani. On the rate of convergence of the difference-of-convex algorithm (DCA). *Journal of Optimization Theory and Applications*, pages 1–22, 2023.

[ZAdK23] Moslem Zamani, Hadi Abbaszadehpeivasti, and Etienne de Klerk. The exact worst-case convergence rate of the alternating direction method of multipliers. *Mathematical Programming*, 2023.

These articles are used in the chapters of this thesis as follows:

Chapter 4	Based on [AdKZ22]
Chapter 5	Based on [AdKZ23a]

Chapter 6	Based on [AdKZ23b]
Chapter 7	Based on [ZAdK24]
Chapter 8	Based on [AdKZ23c]
Chapter 9	Based on [ZAdK23]

Now is no time to think of what you do not have. Think of what you can do with what there is.

The Old Man and the Sea, Ernest Hemingway

2

Preliminaries and interpolation theorems

Preamble

In this chapter, we delve into the study of interpolation theorems, a crucial ingredient extensively utilized throughout this thesis to analyze the complexity of first-order methods. In particular, these theorems sometimes enable us to reformulate the membership problem for some function classes as finite interpolation conditions.

In recent years, interpolation theorems have gained significant attention from researchers who aim to extend or leverage them for their analyses, particularly in the context of performance estimation. Before delving into these theorems, we will cover fundamental concepts in convex and smooth analysis. By establishing a solid foundation in these areas, we can better grasp the subsequent theorems and their implications.

2.1 Preliminaries

In this section, we introduce fundamental mathematical concepts and definitions that will serve as the foundation for our thesis. We provide a concise overview of these key notions to establish the groundwork for our study. For a more in-depth exploration of these mathematical concepts and the proofs, interested readers are encouraged to refer to supplementary reference books (e.g., [Roc97, Bec17, Nes18, HUL13]).

Let us begin with the definition of convex sets.

Definition 2.1. A set $\mathbb{S} \subseteq \mathbb{R}^n$ is said to be *convex* if, for all $x, y \in \mathbb{S}$ and $\lambda \in [0,1]$, we have:

$$\lambda x + (1-\lambda)y \in \mathbb{S}.$$

In simpler terms, this definition implies that any line segment connecting two points within the set also lies entirely within the set.

The notation $\mathbb{R}^n$ represents the Euclidean space of n dimensions. Now, let us define the inner product and *induced norm* with respect to a given matrix $A \in \mathbb{R}^{n\times m}$. Given vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, their inner product with respect to matrix A, denoted as $\langle x, y\rangle_A$, is defined as:

$$\langle x, y\rangle_A = \langle x, Ay\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes a reference inner product. Note that in the Euclidean space the reference inner product is defined by $\langle x, y\rangle = x^T y$ for any given vectors $x, y \in \mathbb{R}^n$. The *seminorm* with respect to matrix A and the *inner product* $\langle \cdot, \cdot \rangle$, denoted as $\|x\|_A$, is defined as:

$$\|x\|_A = \|Ax\| = \sqrt{\langle Ax, Ax\rangle}.$$

It is worth noting that if A has independent columns this is an induced norm; see [HJ12, Section 5.2] for more discussion on seminorms.

Now, let us briefly discuss some properties of a function. Consider an extended real-valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The effective domain of the function is defined as:

$$\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

To define the notion of relative interior we need to first define the affine hull of a set $\mathbb{S}$.

Definition 2.2. The *affine hull* of a set $\mathbb{S} \subseteq \mathbb{R}^n$ is given by

$$\operatorname{aff} \mathbb{S} = \{\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n \mid x_1, \ldots, x_n \in \mathbb{S},\ \theta_1 + \cdots + \theta_n = 1,\ \theta_1, \ldots, \theta_n \in \mathbb{R}\}.$$

Now we define the relative interior of the set $\mathbb{S}$.

Definition 2.3. The *relative interior* of a set $\mathbb{S}$ is defined by

$$\operatorname{ri} \mathbb{S} = \{x \in \operatorname{aff} \mathbb{S} \mid B(x,\epsilon) \cap \operatorname{aff} \mathbb{S} \subseteq \mathbb{S} \text{ for some } \epsilon > 0\},$$

where $B(x,\epsilon) = \{y \mid \|y - x\| \leq \epsilon\}$ is the ball around x with radius ϵ.

In case that the set $\mathbb{S}$ is convex, alternatively, $x \in \operatorname{ri}(\mathbb{S})$ if and only if, for every $y \in \mathbb{S} \setminus \{x\}$, there exists a $z \in \mathbb{S}$ such that x is on the line segment connecting y and z.

We end this subsection with the definition of proper functions.

Definition 2.4. A function f is called *proper* if there exists $x \in \mathbb{R}^n$ such that $f(x) < +\infty$.

Definition 2.5. For a set $X \subseteq \mathbb{R}^n$, we denote the *distance function* to the set X by $d_X(x) := \inf_{y \in X} \|y - x\|$ for any $x \in \mathbb{R}^n$. The set-valued mapping $\Pi_X(x)$ stands for the *projection* of x on X, that is, $\Pi_X(x) = \{y : \|y - x\| = d_X(x)\}$.

Note that, if X is a non-empty closed set, then $\Pi_X(x)$ is non-empty and well-defined.

2.1.1 Convex functions

Convex functions play a pivotal role in optimization and mathematical modeling. Intuitively, a univariate function is considered convex if, for any two points in its domain, the value of the function at any point on the line segment connecting these two points lies below or on the line connecting the function values at those points.

There are some alternative ways of defining a convex function mathematically which are considered interchangeable. One definition which is very intuitive is as follows.

Definition 2.6. Let $\mathbb{D} \subseteq \mathbb{R}^n$ be a convex set. A function $f : \mathbb{D} \to \mathbb{R}$ is *convex* if, for all x, y in its domain and for any λ with $0 \leq \lambda \leq 1$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

where $\mathbb{D}$ is the domain of the function f.

A function is called *strictly convex* if the above inequality strictly holds for $x \neq y$ and $0 < \lambda < 1$. To introduce the other definition we need to define the epigraph of a function.

Definition 2.7. For a function $f : \mathbb{D} \to \mathbb{R}$, the *epigraph* of f is defined by

$$\text{epi } f = \{(x,t) \in \mathbb{D} \times \mathbb{R} \mid f(x) \leq t\}.$$

We now define convex and closed functions using the epigraph of the function.

Definition 2.8. A function f is *convex* if its epigraph is a convex set.

Definition 2.9. A function f is *closed* if its epigraph is a closed set.

We can characterize lower semi-continuous functions through the definition of closed functions. In other words, a function $f : \mathbb{R}^n \to (-\infty, \infty]$ is *lower semi-continuous* function if and only if it is a closed function; see [Roc97, Theorem 7.1].

2.1.2 Subgradient

Another essential concept utilized in this thesis is the notion of subgradient. To formally define the subgradient, we first need to introduce the concept of dual spaces. Let us consider a vector space $\mathbb{E}$, and the set of all linear functionals on $\mathbb{E}$, termed the *dual space*, and denoted as $\mathbb{E}^*$. The norm in the dual space is called the *dual norm*, and it is defined as follows:

$$\|y\|_* = \sup_{x \in \mathbb{E}} \{y(x) : \|x\| \leq 1\}, \quad y \in \mathbb{E}^*.$$

In this thesis, we focus on utilizing the Euclidean norm. An essential characteristic of the Euclidean norm is its self-duality, expressed as $\|\cdot\| = \|\cdot\|_*$. Additionally, our study centers on the vector space $\mathbb{E} = \mathbb{R}^n$, which possesses a dual space denoted as $\mathbb{E}^* = \mathbb{R}^n$. In this case, any y in $\mathbb{E}^*$ may be identified with $x \mapsto \langle x, y \rangle$ for some y in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^n$. In other words, $\mathbb{E}^*$ is isomorphic to $\mathbb{R}^n$.

This understanding provides us with the necessary tools to present a formal definition of the subgradient.

Definition 2.10. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper function. If $x \in \text{dom}(f)$, a vector $g \in \mathbb{R}^n$ is called *subgradient* of f at x if

$$f(y) \geq f(x) + \langle g, y - x \rangle \ \ \forall y \in \mathbb{R}^n.$$

In a more intuitive sense, this inequality implies that the subgradient at a given point corresponds to an affine (linear) function, which underestimates the original function.

Now we can provide the formal definition of subdifferential.

Definition 2.11. The *subdifferential* of a function f at point x in its domain is the set of all its subgradients, denoted by $\partial f(x)$. In other words,

$$\partial f(x) = \{g \in \mathbb{R}^n : f(y) \geq f(x) + \langle g, y - x \rangle \ \forall y \in \mathbb{R}^n\}.$$

This definition allows us to handle cases where the function may have more than one subgradient associated with the given point. However, when a proper convex function is differentiable and $x \in \operatorname{int}(\operatorname{dom}(f))$, the set reduces to a singleton, equal to $\{\nabla f(x)\}$; e.g. see [Bec17, Theorem 3.33]. The subdifferential possesses several important properties. For a proper function $f : \mathbb{R}^n \to (-\infty, +\infty]$, its subdifferential at a specific point $x \in \mathbb{R}^n$ is a closed and convex set. Additionally, a proper function is said to be subdifferentiable at a point x if $\partial f(x) \neq \emptyset$, indicating that there exists at least one subgradient at that point. Another significant theorem related to the subdifferential of a function is known as Fermat's optimality condition, which can be stated as follows.

Theorem 2.12. *[E.g. Bec17, Theorem 3.63] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper convex function. Then, a point $x^\star$ is a global minimizer of the function f, i.e., $x^\star \in \arg\min_x \{f(x) : x \in \mathbb{R}^n\}$, if and only if $0 \in \partial f(x^\star)$.*

In other words, Fermat's optimality condition asserts that a point $x^\star$ is a global minimizer of the function f if and only if the subdifferential of f at that point contains the zero vector. This condition serves as a fundamental guideline to identify optimal solutions in convex optimization problems.

An important rule which is used in this thesis is the inclusion known as the weak sum rule of subdifferential calculus:

$$\sum_{i=1}^{m} \partial f_i(x) \subseteq \partial \left(\sum_{i=1}^{m} f_i \right)(x),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in \{1, \ldots, m\}$ are proper convex functions. Moreover, equality holds if the intersection of the relative interiors of the domains of the functions is nonempty, i.e., $\cap_{i=1}^m \operatorname{ri}(\operatorname{dom}(f_i)) \neq \emptyset$.

Now, let us consider the following constrained optimization problem:

$$\inf\{f(x) : x \in C\}, \tag{2.1}$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ is an extended real-valued proper convex function, and the set C is a convex set. The necessary and sufficient condition for optimality in this problem can be expressed through the following theorem:

Theorem 2.13. *[E.g. Bec17, Theorem 3.68] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended real-valued proper convex function, and let C be a convex set where $ri(dom(f)) \cap ri(C) \neq \emptyset$. Then, $x^\star \in C$ is an optimal solution of problem* (2.1) *if and only if there exists $g \in \partial f(x^\star)$ such that $\langle g, x - x^\star \rangle \geq 0$ for any $x \in C$.*

In other words, $x^\star$ is an optimal solution of the constrained optimization problem if and only if there exists a subgradient g of f at $x^\star$ such that the inner product of g with any direction $(x - x^\star)$ in the feasible set C is nonnegative. This condition is a fundamental result that helps identify the optimal solutions in convex constrained optimization problems.

For a comprehensive exploration of further properties related to the subdifferential and subgradient and the formal proofs of the theorems given in this section, we recommend interested readers to consult relevant reference books such as [Bec17].

2.1.3 Conjugate function

Another fundamental concept heavily employed in this thesis is the notion of the conjugate function.

Definition 2.14. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended real-valued function. The function $f^* : \mathbb{R}^n \to (-\infty, +\infty]$, defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle y, x \rangle - f(x)\}, \quad y \in \mathbb{R}^n,$$

is called the *conjugate* of f.

Hence, the conjugate function of f transforms a given point in the dual space $\mathbb{R}^n$ into a real value. The conjugate function plays a pivotal role in various mathematical and optimization analyses. The most important property of the conjugate function is its closeness and convexity, which can be formally stated as follows.

Theorem 2.15. *[E.g. Bec17, Theorem 4.3] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended real-valued function. Then the conjugate function f^* is closed and convex.*

Moreover, if f is a proper convex function, its conjugate is also proper. By definition of the conjugate, the following theorem can be derived, which is also known as *Fenchel's inequality*.

Theorem 2.16. *[E.g. Bec17, Theorem 4.6] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended real-valued proper function. For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have*

$$f(x) + f^*(y) \geq \langle y, x \rangle .$$

One might wonder what happens if we take the conjugate of the conjugate function, which is known as the *biconjugate*, denoted by f^{**} and defined by

$$f^{**}(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - f^*(y)\}, \quad x \in \mathbb{R}^n.$$

The biconjugate also possesses some interesting properties. For any extended real-valued function f and any x, we have $f(x) \geq f^{**}(x)$. Moreover, if f is a proper, closed, and convex function, then equality holds [Bec17, Theorem 4.8].

We can now establish a connection between the subdifferential and the conjugate function through the conjugate subgradient theorem.

Theorem 2.17. *[E.g. Roc97, Theorem 23.5] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper and convex function. The following two claims are equivalent for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$:*

(1) $\langle x, y \rangle = f(x) + f^*(y)$.

(2) $y \in \partial f(x)$.

If, in addition, f is closed, then (1) and (2) are equivalent to:

(3) $x \in \partial f^*(y)$.

In simpler terms, the conjugate subgradient theorem establishes the equivalence between three statements for a proper and closed convex function f, its conjugate function f^*, and their respective subdifferentials. This theorem provides essential insights into the relationship between the primal and dual spaces in optimization.

Now, let us discuss some calculation rules applicable to conjugate functions. We begin with the summation of separable functions:

Theorem 2.18. *[E.g. Bec17, Theorem 4.12] Let $g(x_1, x_2, \ldots, x_p) = \sum_{i=1}^p f_i(x_i)$ be a seperable function, where $f_i : \mathbb{R}^{n_i} \to (-\infty, +\infty]$ for $i \in 1, 2, \ldots, p$ are proper functions. Then, we have:*

$$g^*\left(y_1, y_2, \ldots, y_p\right) = \sum_{i=1}^p f_i^*(y_i) \ \text{ for any } y_i \in \mathbb{R}^{n_i}, \ \ i \in \{1, 2, \ldots, p\}.$$

This theorem allows us to express the conjugate of separable functions as the sum of the conjugates of each function. Next, we present another useful property of conjugate functions.

Theorem 2.19. *[E.g. Bec17, Theorem 4.14] Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be an extended real-valued function, and $\alpha \in \mathbb{R}_{++}$.*

- *If $g(x) = \alpha f(x)$, then $g^*(y) = \alpha f^*\left(\frac{y}{\alpha}\right)$, for $y \in \mathbb{R}^n$.*
- *If $h(x) = \alpha f\left(\frac{x}{\alpha}\right)$, then $h^*(y) = \alpha f^*(y)$, for $y \in \mathbb{R}^n$.*

These properties allow us to manipulate the conjugate function when the original function is scaled.

We conclude this section with *Fenchel's duality theorem*.

Theorem 2.20. *[E.g. Roc97, Theorem 31.1] Let f and g be extended real-valued proper convex functions. If the intersection of the relative interiors of the domains of f and g is nonempty, i.e., $ri(dom(f)) \cap ri(dom(g)) \neq \emptyset$, then the following relationship holds:*

$$\inf_{x \in \mathbb{R}^n} \{f(x) + g(x)\} = \sup_{y \in \mathbb{R}^n} \{-f^*(y) - g^*(-y)\},$$

and the supremum in the right-hand problem is attained whenever it is finite.

There are several other properties for conjugate functions, which will be discussed in the relevant chapters.

2.1.4 L-smooth functions

In this section, we will review the definition and properties of L-smooth functions, which constitute one of the essential classes of functions studied in this thesis. Let us begin by formally defining L-smooth functions.

Definition 2.21. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ and open set $\mathbb{D} \subseteq \mathbb{R}^n$ where f is differentiable over $\mathbb{D}$. f is called *L-smooth* over $\mathbb{D}$ for some $L > 0$ if it satisfies the following condition:

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \text{ for all } x, y \in \mathbb{D}.$$

The constant L in the definition is referred to as the smoothness modulus or parameter of the function. The definition of smoothness allows us to observe that if a function f is L_1-smooth, it is also L_2-smooth for every $L_2 \geq L_1$. Consequently, when we refer to a function as L-smooth, we specifically mean the smallest possible value of L that satisfies the smoothness condition.

The *descent lemma* is one of the most important properties of L-smooth functions.

Lemma 2.22. *[Nes03, Lemma 1.2.3] Suppose that $f : \mathbb{R}^n \to (-\infty, \infty]$ is an L-smooth function over an open convex set $\mathbb{D}$ for some $L > 0$. Then, for every $x, y \in \mathbb{D}$, we have:*

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|x - y\|^2 .$$

In particular, if $y = x - 1/L \nabla f(x)$, then

$$\frac{1}{2L} \|\nabla f(x)\|^2 \leq f(x) - f\left(x - \frac{1}{L} \nabla f(x)\right).$$

Additionally, we list some key properties of L-smooth functions.

Theorem 2.23. *[Nes18, Theorem 2.1.5] Let f be a differentiable convex function over $\mathbb{R}^n$, and let $L > 0$. Then all of the following statements are equivalent:*

- *f is L-smooth.*
- *$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|x - y\|^2$ for all $x, y \in \mathbb{R}^n$.*
- *$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|^2$ for all $x, y \in \mathbb{R}^n$.*
- *$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2$ for all $x, y \in \mathbb{R}^n$.*
- *$f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y) - \frac{L}{2} \lambda (1-\lambda) \|x - y\|^2$ for all $x, y \in \mathbb{R}^n$.*

Furthermore, if the Hessian of a function f exists, then the module of smoothness for the function can be defined by spectral radius of the Hessian. In other words, the spectral radius of the Hessian lies within the interval $[-L, L]$. It's important to note that the spectrum of a Hessian depends on the choice of the underlying inner product. Consequently, the value of L is dependent on the specific inner product used.

The following proposition states a well-known characterization of L-smooth functions that follows, e.g., from [Nes03, Lemma 1.2.3], [Nes03, Theorem 2.1.5] and [THG17a, Lemma 3.9].

Proposition 2.24. *Let $L > 0$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth and the gradient exists, if and only if it has an L-Lipschitz gradient.*

We denote the class of L-smooth functions by $\mathcal{F}_{-L,L}(\mathbb{D})$ where $\mathbb{D} \subseteq \mathbb{R}^n$.

2.1.5 Strong convexity

We begin this section by providing the formal definition of strongly convex functions.

Definition 2.25. [Nes18, Definition 2.1.3] Let $\mu \geq 0$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function over its domain, and for every $x, y \in \mathrm{dom}(f)$ the inequality

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^2$$

holds. In this case, f is called a *μ-strongly convex function*.

This definition implies that a function f is μ-strongly convex if and only if the function $f(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex. Similar to the L-smooth case, if f is strongly convex with modulus μ_1, it is also μ_2-strongly convex for every $0 < \mu_2 < \mu_1$. Hence, it is essential to find the largest strongly convex modulus. Furthermore, it is evident that a convex function is a 0-strongly convex function.

It is straightforward to observe that if $f : \mathbb{R}^n \to \mathbb{R}$ is a μ-strongly convex function for some $\mu > 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then the sum $f + g$ is also a μ-strongly convex function.

The following theorem presents some essential characteristics of μ-strongly convex functions.

Theorem 2.26. *[E.g. Bec17, Theorem 5.24] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper, closed, and convex function, and let $\mu > 0$. The following statements are equivalent:*

- *f is a μ-strongly convex function.*
- *For any $x \in \mathrm{dom}(\partial f)$, $y \in \mathrm{dom}(f)$, and $g \in \partial f(x)$, the inequality*

  $$f(y) \geq f(x) + \langle g, y - x \rangle + \frac{\mu}{2}\|y - x\|^2$$

  holds.
- *For any $x, y \in \mathrm{dom}(\partial f)$ and $g_x \in \partial f(x)$, $g_y \in \partial f(y)$, the inequality*

  $$\left\langle g_x - g_y, x - y \right\rangle \geq \mu\|x - y\|^2$$

  holds.

One of the most significant characteristics of strongly convex functions is their possession of a unique minimizer. The following theorem formalizes this property.

Theorem 2.27. *[E.g. Bec17, Theorem 5.25] If $f : \mathbb{R}^n \to \mathbb{R}$ is a proper, closed, and μ-strongly convex function for some $\mu > 0$, then f has a unique minimizer. Moreover, the following inequality holds for any $x \in dom(f)$ and $x^\star$, which is the unique minimizer of f:*

$$f(x) - f(x^\star) \geq \frac{\mu}{2} \|x - x^\star\|^2.$$

Suppose that the function f is twice continuously differentiable and the Hessian of the function is positive definite. Similar to L-smooth functions, the modulus of strong convexity of f is equal to the minimum eigenvalue of the Hessian. Therefore, one has

$$\mu I \preceq \nabla^2 f(x) \preceq LI.$$

The connection between strong convexity and L-smoothness of a function can be established using conjugate functions. The following theorem presents this connection formally.

Theorem 2.28. *[E.g. Bec17, Theorem 5.26] Suppose that $\mu > 0$. Then:*

- *If $f : \mathbb{R}^n \to \mathbb{R}$ is a $\frac{1}{\mu}$-smooth convex function, then its conjugate function f^* is a μ-strongly convex function.*
- *If $f : \mathbb{R}^n \to \mathbb{R}$ is a proper closed μ-strongly convex function, then its conjugate function f^* is a $\frac{1}{\mu}$-smooth function.*

The set of functions that are both μ-strongly convex and L-smooth is denoted by $\mathcal{F}_{\mu,L}(\mathbb{R}^n)$. Notably, it is worth mentioning that for μ-strongly convex and L-smooth functions, $\mu \leq L$ always holds. Furthermore, the class of μ-strongly convex functions which are not smooth is denoted by $\mathcal{F}_{\mu,\infty}(\mathbb{R}^n)$.

The definitions of L-smoothness and μ-strong convexity can be influenced by the chosen norm, raising the question of whether these definitions can be generalized. To address this concern, Lu et al. introduced the concepts of *relative smoothness* and *relative strong convexity* with respect to a reference function [LFN18]. Inspired by their work, we define c-strongly convex functions as follows:

Definition 2.29. Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a closed proper function, and let $A \in \mathbb{R}^{n\times m}$. We say f is c-strongly convex relative to $\|\cdot\|_A$ if the function $f - \frac{c}{2}\|\cdot\|_A^2$ is convex.

It is worth noting that in Definition 2.29 the value of c depends on the seminorm $\|\cdot\|_A$ that we use. Moreover, setting $A = I$ in the definition allows us to deduce the definition of μ-strong convexity. It is noteworthy that any function that is μ-strongly convex is also $\frac{\mu}{\lambda_{\max}(A^T A)}$-strongly convex with respect to the seminorm $\|\cdot\|_A$. However, the converse is not necessarily true unless A has full column rank. For further details on the strong convexity in relation to a given function, we refer the reader to [LFN18, BBT17].

The set of c-strongly convex functions relative to the seminorm $\|\cdot\|_A$ on $\mathbb{R}^n$ is denoted as $\mathcal{F}_c^A(\mathbb{R}^n)$.

2.1.6 Non-convex non-smooth functions

If a function f is non-convex and non-smooth, we will also need a more general notion of subgradients than in the convex case.

Definition 2.30. Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous.

- The vector $g \in \mathbb{R}^n$ is called a *regular subgradient* of f at $\bar{x}$, written $g \in \hat{\partial}_L f(\bar{x})$, if for all x in some neighborhood of $\bar{x}$

$$f(x) \geq f(\bar{x}) + \langle g, x - \bar{x} \rangle + o(\|x - \bar{x}\|).$$

- The vector $g \in \mathbb{R}^n$ is called a *general subgradient* of f at $\bar{x}$, written $g \in \partial_L f(\bar{x})$, if there exist sequences $\{x^i\}$ and $\{g^i\}$ with $g^i \in \hat{\partial}_L f(x^i)$ such that

$$x^i \to \bar{x},\ f\left(x^i\right) \to f(\bar{x}),\ g^i \to g \text{ as } i \to \infty.$$

It is worth mentioning that $\hat{\partial}_L f(\bar{x})$ is a closed convex set. In addition, $\partial_L f(\bar{x})$ is also closed but not necessarily convex. Note that when f is closed proper convex, then $\partial f(x) = \hat{\partial}_L f(x) = \partial_L f(x)$ for $x \in \mathrm{dom}(f)$. We refer the interested reader to [RW09, Chapter 8] for more discussions on regular and general subdifferentials.

2.2 Semidefinite programming

In this section we discuss some concepts on *semidefinite programming* (SDP); for more details see [LV12, VB96]. The standard form of semidefinite programming as used in [LV12] (more common is 'inf' instead of 'sup') is given by

$$\begin{aligned} p^\star = \sup_X \quad & \langle C, X\rangle \\ \text{s.t.} \quad & \langle A_i, X\rangle = b_i, \quad i = 1,\dots,m, \\ & X \succeq 0, \end{aligned} \tag{2.2}$$

where matrix X is the variable, $C, A_1, \dots, A_m$ are given symmetric matrices, and $b \in \mathbb{R}^m$ is a given vector. The inner product $\langle\cdot,\cdot\rangle$ denotes the *Frobenius inner product*, that is $\langle A, B\rangle = \text{tr}(AB)$. If there is a positive definite matrix X that satisfies all the constraints and $X \succ 0$, then we call the problem strictly feasible; sometimes this is called Slater's condition [BV04, Chapter 5].

Similar to the linear programming setting we define the dual form of the primal problem 2.2, which is given by

$$\begin{aligned} d^\star = \inf_y \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i - C \succeq 0, \quad i = 1,\dots,m, \\ & y \in \mathbb{R}^m, \end{aligned} \tag{2.3}$$

where y is the variable and the constraint $\sum_{i=1}^m y_i A_i - C \succeq 0$ is also known as a linear matrix inequality (LMI). Now we provide some results on SDP which we will use in the following chapters.

Lemma 2.31. *[E.g. LV12, Lemma 2.1.1] Assume that X is a feasible solution to Problem 2.2 and y is a feasible solution to Problem* (2.3). *Then we have*

- *$p^\star \le d^\star$, which is known as weak duality.*
- *If $\langle C, X\rangle = b^T y$ then $p^\star = d^\star = \langle C, X\rangle = b^T y$.*

The difference between the supremum in the primal problem and the infimum in the dual problem is called the *duality gap*, defined as $d^\star - p^\star$. If there is no duality gap, then we say that *strong duality* holds. The next theorem states this in formal way.

Theorem 2.32. *[E.g. DK06, Theorem 2.2] Assume that X is a feasible solution to Problem 2.2 and y is a feasible solution to Problem* (2.3). *Then we have*

- *If the problem* (2.3) *is bounded from below (i.e., $d^\star > -\infty$) and is strictly feasible, then the primal problem* (2.2) *attains its supremum and there is no duality gap.*
- *If the problem* (2.2) *is bounded from above (i.e., $p^\star < \infty$) and is strictly feasible, then the primal problem* (2.3) *attains its infimum and there is no duality gap.*

Another important concept is complementary slackness.

Theorem 2.33. *Assume that X is a feasible solution of the primal problem* (2.2) *and y is a feasible solution of dual problem*(2.3)*. If (X, y) satisfy*

$$\left(\sum_{i=1}^{m} y_i A_i - C\right) X = 0,$$

which is known as the complementary slackness condition, then X is an optimal solution of the primal problem and y is an optimal solution of the dual problem.

Consider vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ with $k \geq 1$. Define the matrix $X = \text{Gram}(u_1, u_2, \ldots, u_k)$ such that $X_{ij} = u_i^T u_j$ for $i, j \in \{1, \ldots, k\}$. This matrix is referred to as the Gram matrix of $u_1, u_2, \ldots, u_k$. It is noteworthy that a matrix $X \in \mathbb{S}^k$ is positive semidefinite if and only if it can be expressed as the Gram matrix of some vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ and some $n \geq k$.

See [DK06] for a comprehensive overview of SDP and [MHA20] for new developments on SDP and its applications. Additionally, it's important to note that the MOSEK toolbox in MATLAB [ApS19] is used to solve the semidefinite programming (SDP) problems that appear in this thesis.

2.3 Interpolation theorems

In this section, our main focus is on interpolation theorems, which play a crucial role in formulating the performance estimation problem, as discussed in Chapter 3. The primary aim is to express a specific class of functions using a finite number of triplets, each containing a point, the function value, and the (sub-)gradient of the function at that point. Let us consider $\mathcal{I}$ be a finite index set, this enables us to determine, based on a finite number of such triplets $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}}$, whether there is a function with $f\left(x^i\right) = f^i$ and $g^i \in \partial f\left(x^i\right)$ that belongs to a specific class of functions. The formal definition of interpolability is given by the following definition.

Definition 2.34. A set of triplets $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}} \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ is called $\mathcal{F}_{\mu,L}(\mathbb{R}^n)$-interpolable if there is a function $f \in \mathcal{F}_{\mu,L}(\mathbb{R}^n)$ such that for $i \in \mathcal{I}$, $f\left(x^i\right) = f^i$ and $g^i \in \partial f\left(x^i\right)$.

To illustrate interpolation, let us consider the following example.

Example 2.35. *Consider* $\left\{\left(x^1, x^2, x^3\right) = (-1, 1.5, 4)\right\} \subseteq \mathbb{R}^3$, *the corresponding function values* $\left\{\left(f^1, f^2, f^3\right) = (3.07, 0.52, 1.91)\right\}$ *and the corresponding gradients* $\left\{\left(g^1, g^2, g^3\right) = (-1.6, -0.65, 2.8)\right\} \subseteq \mathbb{R}^3$ *as illustrated in Figure 2.1. Our goal is to determine if there exists a* μ*-strongly convex and* L*-smooth interpolating function with* $\mu = 0.21$ *and* $L = 3.74$*. If such a function exists, we say that the triplet* $\left\{\left(x^i; f^i; g^i\right)\right\}_{i\in\mathcal{I}}$*, where* $\mathcal{I} = \{1, 2, 3\}$*, is* $\mathcal{F}_{\mu,L}(\mathbb{R})$*-interpolable.*

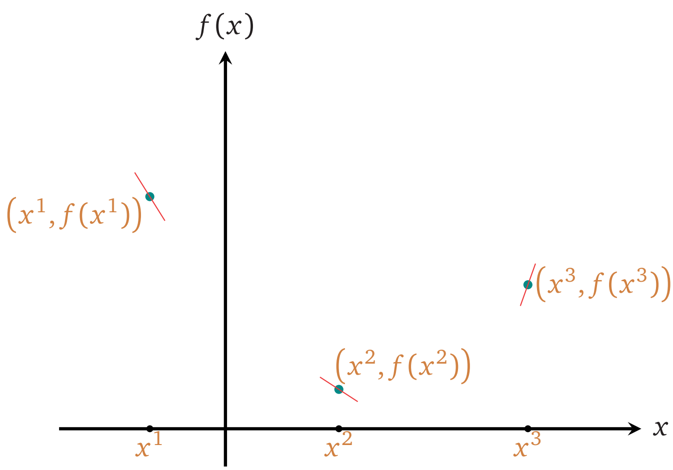


Figure 2.1: Three points x^1, x^2, x^3 along with their corresponding function values and gradients

In this example, we find that there does indeed exist a function that satisfies the conditions. It is presented in Figure 2.2 and it is given by

$$f(x) = \begin{cases} 0.7167x^2 + 2.3887 & x \in (-\infty, -1.5) \\ 0.02x^4 - 0.0533x^3 + .16x^2 - 1.04x + 1.8 & x \in [-1.5, 4.5] \\ 0.4944x^2 - 6.3112 & x \in (4.5, \infty). \end{cases}$$

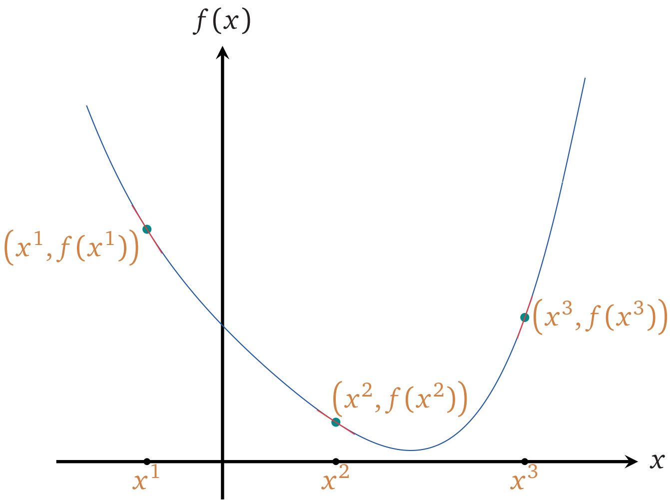


Figure 2.2: A μ-strongly convex and L-smooth function, with $\mu = 0.21$ and $L = 3.74$ modules, that passes through the points with the corresponding function values and gradients shown in Figure 2.1.

Now, we present the interpolation theorems utilized in this thesis. For a comprehensive understanding and detailed proofs, we direct the interested reader to [Tay17], unless specified otherwise.

The following theorem provides *convex interpolation*, which serves as the basis for the other theorems.

Theorem 2.36. *[THG17c, Theorem 1] The following statements are equivalent:*

1. $\left\{\left(x^i; f^i; g^i\right)\right\}_{i\in\mathcal{I}}$ *is* $\mathcal{F}_{0,\infty}(\mathbb{R}^n)$*-interpolable.*

2. *The following inequality holds for all* $i, j \in \mathcal{I}$*:*

$$f^i \geq f^j + \left\langle g^j, x^i - x^j \right\rangle.$$

The proof of the theorem comes from the definition of subgradient. We move on to a more general theorem, which provides necessary and sufficient conditions for the class of μ-strongly convex L-smooth functions. To prove these theorems one may use the properties of conjugate functions.

Theorem 2.37. *[THG17c, Theorem 4] Let* $0 \leq \mu < L \leq \infty$*. The following statements are equivalent:*

1. $\{(x^i; f^i; g^i)\}_{i\in I}$ *is* $\mathcal{F}_{\mu,L}(\mathbb{R}^n)$*-interpolable.*

2. *The following inequality holds for all* $i, j \in \mathcal{I}$*:*

$$\frac{1}{2\left(1-\frac{\mu}{L}\right)}\left(\frac{1}{L}\left\|g^i-g^j\right\|^2+\mu\left\|x^i-x^j\right\|^2-\frac{2\mu}{L}\left\langle g^j-g^i, x^j-x^i\right\rangle\right)\leq$$
$$f^i-f^j-\left\langle g^j, x^i-x^j\right\rangle. \tag{2.4}$$

The proof of this theorem relies on the properties of conjugate functions. Using this theorem, the following corollaries can be deduced.

Corollary 2.38. *The following statements are equivalent:*

1. $\{(x^i; f^i; g^i)\}_{i\in I}$ *is* $\mathcal{F}_{0,L}(\mathbb{R}^n)$*-interpolable.*

2. *The following inequality holds for all* $i, j \in \mathcal{I}$*:*

$$f^i \geq f^j + \left\langle g^j, x^i - x^j\right\rangle + \frac{1}{2L}\left\|g^i-g^j\right\|^2.$$

Corollary 2.39. *The following statements are equivalent:*

1. $\{(x^i; f^i; g^i)\}_{i\in I}$ *is* $\mathcal{F}_{\mu,\infty}(\mathbb{R}^n)$*-interpolable.*

2. *The following inequality holds for all* $i, j \in \mathcal{I}$*:*

$$f^i \geq f^j + \left\langle g^j, x^i - x^j\right\rangle + \frac{\mu}{2}\left\|x^i-x^j\right\|^2.$$

The next theorem gives necessary and sufficient conditions for $\mathcal{F}^A_{c,\infty}$-interpolablity. Analogous to that of [THG17c, Theorem 4] (Theorem 2.37) we have the following interpolation theorem.

Theorem 2.40. *Let* $c \in [0,\infty)$*. The set* $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}} \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *is* $\mathcal{F}^A_{c,\infty}$*-interpolable if and only if for any* $i, j \in \mathcal{I}$*, we have*

$$\tfrac{c}{2}\left\|x^i-x^j\right\|_A^2 \leq f^i - f^j - \left\langle g^j, x^i-x^j\right\rangle. \tag{2.5}$$

Proof. The triple $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}}$ is $\mathcal{F}^A_{c,\infty}$-interpolable if and only if the triple $\left\{\left(x^i; f^i - \tfrac{c}{2}\left\|x^i\right\|_A^2; g^i - cA^TAx^i\right)\right\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{0,\infty}$-interpolable. By Theorem 2.36, $\left\{\left(x^i; f^i - \tfrac{c}{2}\left\|x^i\right\|_A^2; g^i - cA^TAx^i\right)\right\}_{i\in\mathcal{I}}$ is $\mathcal{F}_{0,\infty}$-interpolable if and only if

$$f^i - \tfrac{c}{2}\left\|x^i\right\|_A^2 \geq f^j - \tfrac{c}{2}\left\|x^j\right\|_A^2 - \left\langle g^j - cA^TAx^j, x^i - x^j\right\rangle$$

which implies the desired inequality. □

The other interpolation theorem of interest pertains to the class of non-convex smooth functions.

Theorem 2.41. *[THG17a, Theorem 3.10][DS20, Theorem 7 in Appendix] Let* $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}} \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *with a given finite index set* $\mathcal{I}$ *and* $L > 0$*. There exists an* L*-smooth function* f *with*

$$f(x^i) = f^i, \nabla f(x^i) = g^i \quad \forall i \in \mathcal{I}, \tag{2.6}$$

if and only if

$$\tfrac{1}{2L}\left\|g^i - g^j\right\|^2 - \tfrac{L}{4}\left\|x^i - x^j - \tfrac{1}{L}(g^i - g^j)\right\|^2 \leq f^i - f^j - \left\langle g^j, x^i - x^j\right\rangle \quad \forall i, j \in \mathcal{I}. \tag{2.7}$$

In addition, if $\{(x^i; f^i; g^i)\}_{i\in\mathcal{I}}$ *satisfies* (2.7)*, then there exists a* L*-smooth function* f *for which* (2.6) *holds and* $\min_{x\in\mathbb{R}^n} f(x) = \min_{i\in\mathcal{I}} f_i - \frac{1}{2L}\|g^i\|^2$*. Moreover, letting* $i^* \in \arg\min_{i\in\mathcal{I}} f_i - \frac{1}{2L}\|g^i\|^2$*, a global minimizer of this function is given by* $x^\star = x^{i^*} - \frac{1}{L}g^{i^*}$*.*

Another proof of the first part of the above-mentioned theorem may be found in [Wel73, Theorem 2. Page 148].

Now we provide a theorem that extends the result to a more general case. Analogous to strongly convex functions, we define the concept of 'minimum curvature' with modulus μ for the function f if the function $f(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex. Notably, in this case, the minimum curvature parameter is allowed to take negative values, unlike in the convex case where μ can only take non-negative values. The class of functions that exhibit minimum curvature μ and are L-smooth is termed 'hypo-convex functions' and is denoted by $\mathcal{H}_{\mu,L}(\mathbb{R}^n)$. It is worth mentioning that when $\mu = -L$, this class of functions reduces to non-convex L-smooth functions. The following theorem provides the necessary and sufficient conditions for the interpolability of this class of functions.

Theorem 2.42. *[RGP22, Theorem 3.1] Let* $\left\{\left(x^i; f^i; g^i\right)\right\}_{i\in\mathcal{I}} \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *with a given index set* $\mathcal{I}$ *and let* $L \in (0, \infty]$ *and* $\mu \in (-\infty, L)$*. There exists a function* $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ *with*

$$f\left(x^i\right) = f^i, \nabla f\left(x^i\right) = g^i \quad i \in \mathcal{I},$$

if and only if for every $i, j \in \mathcal{I}$

$$\frac{1}{2\left(1-\frac{\mu}{L}\right)}\left(\frac{1}{L}\left\|g^i - g^j\right\|^2 + \mu\left\|x^i - x^j\right\|^2 - \frac{2\mu}{L}\left\langle g^j - g^i, x^j - x^i\right\rangle\right) \leq$$
$$f^i - f^j - \left\langle g^j, x^i - x^j\right\rangle. \tag{2.8}$$

Homogeneous quadratic functions have a specific structure that allows us to simplify the interpolation theorem for them, making it worth mentioning. Consider the quadratic function $f(x) := \frac{1}{2}x^TQx$ where $\mu I \preceq Q \preceq LI$ for some $0 \leq \mu \leq L$. The gradient at a given point x can be calculated as $\nabla f(x) = Qx$. Moreover, the function can be expressed as $f(x) = \frac{1}{2}x^T\nabla f(x)$. Utilizing this representation, we can substitute the function value in the interpolation Theorem 2.37; see [BHG22] for more details. Consequently, the interpolation theorem takes the form:

$$\frac{1}{2\left(1-\frac{\mu}{L}\right)}\left(\frac{1}{L}\left\|g^i-g^j\right\|^2+\mu\left\|x^i-x^j\right\|^2-\frac{2\mu}{L}\left\langle g^j-g^i,x^j-x^i\right\rangle\right)\leq$$
$$\frac{1}{2}\left\langle g^i,x^i\right\rangle-\frac{1}{2}\left\langle g^j,x^j\right\rangle-\left\langle g^j,x^i-x^j\right\rangle.$$

We conclude this section by providing the following theorem for μ–strongly convex, L-smooth functions that are twice continuously differentiable in an open convex set $\mathbb{D} \subseteq \mathbb{R}^n$, denoted by $\mathcal{F}_{\mu,L}(\mathbb{D})$.

Theorem 2.43. *[DKGT20] Let $f : \mathbb{D} \to \mathbb{R}$ be twice continuously differentiable, defined on an open convex set $\mathbb{D}$. The following statements are equivalent:*

1. $f \in \mathcal{F}_{\mu,L}(\mathbb{D})$.

2. *for all $x \in \mathbb{D}$, $\mu I \preceq \nabla^2 f(x) \preceq LI$.*

3. *On the set $\mathbb{D}$, the functions $f(\cdot)-\frac{\mu}{2}\|\cdot\|^2$ and $\frac{L}{2}\|\cdot\|^2-f(\cdot)$ are convex.*

4. *For all $x, y \in \mathbb{D}$*

$$\frac{1}{1-\frac{\mu}{L}}\left(\frac{1}{L}\|\nabla f(x)-\nabla f(y)\|^2+\mu\|x-y\|^2-\right.$$
$$\left.\frac{2\mu}{L}\langle\nabla f(y)-\nabla f(x),y-x\rangle\right)\leq\langle\nabla f(x)-\nabla f(y),x-y\rangle.$$

By the example given by [Dro20], it is known that in the case that $\mathbb{D} \subsetneq \mathbb{R}^n$ the inequality (2.4) does not necessarily hold. In other words, in case that $\mathbb{D} = \mathbb{R}^n$, the last inequality can be replaced by stronger inequality (2.4). Indeed, finding interpolation theorems over general open convex sets is an open problem; see the discussion after Lemma 3.1 in [DKGT20].

2.4 Conclusion

In this chapter, we have presented some fundamental theorems that will serve as building blocks for the subsequent chapters. Initially, we reviewed key theorems and definitions in convex analysis and smooth functions. Then, we briefly introduced semidefinite programming. Subsequently, we focused on interpolation theorems, that are essential for performance estimation problem—the fundamental tool for conducting our worst-case performance analysis in the upcoming chapters. We ended the chapter by providing some general definition of subgradients which we will use in Chapter 6.

A wise man proportions his belief to the evidence.

David Hume

3

Performance estimation problems (PEPs)

Preamble

It is important to know that, in general, optimization problems are considered unsolvable; see [Nes18, Introduction] and [NY83, Chapter 1]. Consequently, optimization algorithms are developed to find at least stationary points for certain classes of functions. Some optimization algorithms are iterative methods, generating feasible points from an initial guess until, hopefully, an optimal solution is reached. These methods differ based on the information used, like the objective function, constraint functions, and derivatives. Some use past data, while others rely on just the information gathered from the current iteration [NW06]. It is important to know how rapidly or with what level of accuracy an algorithm approaches the optimal solution after a certain number of iterations, if it does converge. Therefore, it is important to understand the behaviour of these algorithm for different class of functions.

In this chapter, we present the performance estimation problem (PEP), which serves as the foundation for our subsequent convergence rate analysis. Let us consider the following optimization problem,

$$\inf_{x\in\mathbb{R}^n} f(x). \tag{3.1}$$

Here, f denotes a lower semi-continuous function, potentially convex and smooth, which is lower-bounded by some $f^\star > -\infty$.

In this thesis, we focus on the worst case convergence rate of first-order methods from black-box perspective. This means that we assume the availability of the (sub-)gradient and function value at a given point. To analyze the worst-case convergence of a function, we utilize the performance estimation problem (PEP) — described below — which uses interpolation theorems discussed in Chapter 2.

Generally, expressing the worst-case convergence rate of an algorithm as an optimization problem results in an intractable task. However, PEP provides us with valuable tool to reformulate the problem as a semidefinite problem, offering a tractable problem that is independent of the dimension of the problem (3.1).

In the subsequent sections, we begin by expressing the worst-case convergence rate of an algorithm as a mathematical programming problem. We then transform this problem into a semidefinite programming formulation. Finally, we demonstrate how to derive a convergence rate by solving its Lagrangian dual. For simplicity, in the rest of the chapter we consider that the function f is L-smooth with finite L; for the cases that the function is not necessarily L-smooth we refer the reader to Chapters 8 and 9.

3.1 Worst case formulation of convergence rate of iterative first-order methods

Performance estimation problems were introduced in the seminal paper by Drori and Teboulle [DT14]. We focus on studying the convergence rate of first-order methods from an oracle viewpoint. Studying complexity from the oracle viewpoint was initially introduced by Nemirovski and Yudin in their book [NY83]. This viewpoint involves considering the function value and the (sub-)gradient of the function at a given point, denoted as f^i, g^i, where g^i represents the (sub-)gradient of the function at point x^i.

Let us consider an iterative first-order algorithm $\mathcal{M}$ that starts from the initial point x^0. In each iteration, first-order algorithms use the function value and the gradient of the points generated in the previous iterations, as well as the initial point x^0. This combination is known as the *first-order oracle*, denoted by $\mathcal{O}_f$. In other words, the first-order oracle of function f at point x is defined as $\mathcal{O}_f(x) = \{f(x), \nabla f(x)\}$. Using this notation, we can express the points generated by *black-*

box first-order methods after N iterations as follows.

$$\begin{cases} x^1 = \mathcal{M}\left(x^0, \mathcal{O}_f(x^0)\right) \\ x^2 = \mathcal{M}\left(x^0, \mathcal{O}_f(x^0), \mathcal{O}_f(x^1)\right) \\ \vdots \\ x^N = \mathcal{M}\left(x^0, \mathcal{O}_f(x^0), \mathcal{O}_f(x^1), \ldots, \mathcal{O}_f(x^{N-1})\right). \end{cases}$$

To analyze an algorithm we need to determine a performance measure or criterion. Natural and common performance measures which are used in the literature, depending on the problem class, are distance of the point generated by the algorithm in last iterate from the optimal solution, $\|x^N - x^\star\|$, the distance of the function value from optimal value, $f(x^N) - f^\star$, and the norm of the gradient, $\|\nabla f(x^N)\|$, as well as other measures.

Let us consider a performance measure denoted by $\mathcal{P}(\mathcal{O}_f, x^0, x^1, \ldots, x^N, x^\star)$. In order to analyze an algorithm, we must establish an initial condition for the starting point x^0 to measure the algorithm's performance. A commonly used initial condition is to restrict the distance of the initial point from the optimal solution, given by $\|x^0 - x^\star\| \leq \Delta$, where Δ represents a predetermined constant. It is readily seen that this is problem-specific and it can be different for different algorithms and different function classes; for more initial conditions see the subsequent chapters.

For the remainder of this chapter, we will focus exclusively on the case where the function class is restricted to smooth convex or smooth strongly convex functions, for the sake of simplicity and illustration. Since the original problem (3.1) is a minimization problem, when evaluating the worst-case convergence rate, we aim to maximize the performance measure to evaluate the worst case scenario. Thus, we can formulate the worst-case convergence of the given algorithm $\mathcal{M}$ after N iterations within a specific class of functions $\mathcal{F}$ as follows.

$$\begin{aligned} w(\mathcal{F}, \Delta, \mathcal{M}, N, \mathcal{P}) = \sup_{f, x^1, x^2, \ldots, x^N, x^\star} \quad & \mathcal{P}(\mathcal{O}_f, x^0, x^1, \ldots, x^N, x^\star) \\ \text{s.t.} \quad & f \in \mathcal{F} \\ & x^\star \text{ is an optimal solution of } f \\ & x^1, x^2 \ldots, x^N \text{ are generated by the algorithm } \mathcal{M} \\ & \text{given } x^0 \text{ and } \mathcal{O}_f \\ & \|x^0 - x^\star\| \leq \Delta. \end{aligned} \tag{3.2}$$

This problem is intractable in general because we aim to optimize over the, typically infinite-dimensional, function class. To solve this issue, we use the interpolation theorems which are discussed in Chapter 2. Using these results we can rewrite the problem 3.2 as a finite-dimensional problem.

$$
\begin{aligned}
w_f(\Delta, \mathcal{M}, N, \mathcal{P}) = \sup_{\{x^i; f^i; g^i\}_{i\in\mathcal{I}} \in (\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)^{N+2}} \quad & \mathcal{P}\left(\left\{x^i; f^i; g^i\right\}_{i\in\mathcal{I}}\right) \\
\text{s.t.} \quad & \left\{x^i; f^i; g^i\right\}_{i\in\mathcal{I}} \text{ are } \mathcal{F}\text{-interpolable} \\
& \text{i.e. } f^i = f(x^i) \text{ and } g^i = \nabla f(x^i) \text{ for } i \in \mathcal{I} \\
& x^\star \text{ is an optimal solution of } f \\
& x^1, x^2 \ldots, x^N \text{ are generated by the algorithm } \mathcal{M} \\
& \|x^0 - x^\star\| \leq \Delta,
\end{aligned}
\tag{3.3}
$$

where $\mathcal{I} = \{0, 1, \ldots, N, \star\}$. Using interpolation theorems that give necessary and sufficient conditions for different class of functions, it can be easily seen that problems (3.2) and (3.3) are equivalent as the only difference is the first constraint. For a comprehensive understanding of this topic, we recommend interested readers to refer to Taylor's thesis on performance estimation problems [Tay17, Chapter 4] and Drori's thesis [Dro14, Chapter 2].

Problem (3.3) is finite-dimensional but still it is not a convex problem due to the non-convex quadratic constraints imposed by interpolation conditions. In the next section, we will provide a Gram matrix reformulation of the problem to make it a semidefinite programming problem.

3.2 Gram matrix reformulation

In this section, our objective is to formulate worst-case convergence rates of iterative first-order methods with a fixed step length as a mathematical programming problem. We start by introducing fixed step first-order methods. In general, *fixed step iterative methods* are characterized by the fact that the current point generated by the algorithm depends only on the starting point and the gradients of the points generated by the algorithm, using a pre-determined step length. Mathematically, the iterative process is expressed as follows.

$$
x^k = x^0 - \sum_{i=1}^{k} t_{ki} g^{i-1}, \tag{3.4}
$$

where:

- x^0 represents the starting point,
- g^i denotes the gradient of the point x^i, and
- t_{ki} are fixed step lengths for the iterations.

One of the most well-known first-order methods is the *gradient descent method with fixed step-size* described in Algorithm 3.1.

Algorithm 3.1 Gradient method with fixed step lengths

Set N and $\{t_k\}_{k=1}^N$ (step lengths) and pick $x^0 \in \mathbb{R}^n$.
For $k = 1, 2, \ldots, N$ perform the following step:
1. $x^k = x^{k-1} - t_k \nabla f(x^{k-1})$

For more details about this algorithm see Chapter 4 and Chapter 5. Note that Algorithm 3.1 fits in the general framework (3.4) by setting $t_{kk} = t_k$ and $t_{ki} = 0$ if $i \neq k$.

The function classes $\mathcal{F}_{\mu,L}$ along with the iterative first-order methods, are invariant concerning translations of the domain and shifts in function values, i.e.

$$f \in \mathcal{F}_{\mu,L}(\mathbb{R}^n) \implies x \mapsto f(x+c) + d \in \mathcal{F}_{\mu,L}(\mathbb{R}^n),$$

where $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ are fixed.

Remark 3.1. *Without loss of generality, we may therefore assume that $f^\star = 0$ and $x^\star$ is the zero vector. Additionally, when studying unconstrained optimization problems, so in problem 3.3 we can set $x^\star = 0$, $g^\star = 0$, $f^\star = 0$ without loss of generality because of the optimality conditions for the unconstrained problem 3.1.*

To proceed with performance estimation for this class of algorithms, we introduce a *Gram matrix*. Consider the following $n \times (N+2)$ matrix

$$P = [g^0 \ \ g^1 \ \ \ldots \ \ g^N \ \ x^0], \tag{3.5}$$

where g^i represents the gradient of the function at point x^i, and x^0 denotes the starting point. Now, let us define *Gram matrix* $G = \mathrm{Gram}(g^0, g^1, \ldots, g^N, x^0) = P^T P \in \mathbb{S}^{N+2}$, where $\mathbb{S}^{N+2}$ denotes the set of symmetric $(N+2)\times(N+2)$ matrices. The elements of G can be expressed as follows

$$G = \begin{pmatrix} \|g^0\|^2 & \ldots & \langle g^0, g^N \rangle & \langle g^0, x^0 \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle g^0, g^N \rangle & \ldots & \|g^N\|^2 & \langle g^N, x^0 \rangle \\ \langle g^0, x^0 \rangle & \ldots & \langle g^N, x^0 \rangle & \|x^0\|^2 \end{pmatrix}, \tag{3.6}$$

it is easily seen that this matrix is independent of the dimension of the x^0 and g^i.

We observe that by using (3.4), it becomes possible to express x^k as a linear combination of x^0 and g^i. This allows for the reformulation of the constraints within the context of problem (3.3) using both the Gram matrix G and the function values of f^i.

With this foundation in place, we can now proceed to introduce a semidefinite programming problem to address the performance estimation problem. In this regard, we consider the general case for the performance measure. We consider the performance measure as combination of function value as well as any quadratic function of the variable x^N and the gradient of the function at the last iterate. It can be easily seen that this can be generalized to other cases for example when we consider ergodic convergence rate, but for the simplicity in this part we just consider the most common criterion. In other words, we consider

$$\mathcal{P}\left(\left\{x^i; f^i; g^i\right\}_{i\in\mathcal{I}}\right) = b(f(x^N) - f^\star) + c_1\|g^N\|^2 + c_2\left\|x^0 - \sum_{i=1}^{N} t_i g^{i-1} - x^\star\right\|^2,$$

where b, c_1 and c_2 are some real numbers. By considering this criteria the semidefinite programming problem can be written as

$$\begin{aligned} w_f(\Delta, \mathcal{M}, N, \mathcal{P}) = \sup_{\{G\in\mathbb{S}^{N+2},\ f\in\mathbb{R}^{N+1}\}} \quad & bf^N + \mathrm{tr}(GC) \\ \text{s.t.}\quad & f^j - f^i + \mathrm{tr}(GA_{ij}) \leq 0 \quad \forall i, j \in \mathcal{I} \\ & \mathrm{tr}(GA_\Delta) \leq \Delta^2 \\ & G \succeq 0 \\ & \mathrm{rank}(G) \leq n. \end{aligned} \tag{3.7}$$

In this conceptual framework, the matrices denoted as A_{ij} arise from interpolation theorems, while the matrix A_Δ is related to the initial conditions. To show how to construct the matrices A_{ij} we use the same notation as used in [Tay17, Section 4.2.3]. Let us define $t_i \in \mathbb{R}^{N+2}$ for $i \in \{0, \ldots, N, \star\}$ by

$$t_i^T = (-t_{i,1}, -t_{i,2}, \ldots, -t_{i,i}, 0, \ldots, 0, 1),\ t_\star^T = (0, \ldots, 0).$$

By this definition and the definition of P as in (3.5) one can easily see that $x_i = Pt_i$. Also, define $u_i = e_{i+1} \in \mathbb{R}^{N+2}$ for $i = 0, 1, \ldots, N+1$ and $u_\star = 0^{N+2}$. For the purpose of easy reference, consider the interpolation constraint given by the interpolation

Theorem 2.4,

$$\frac{1}{2\left(1-\frac{\mu}{L}\right)}\left(\frac{1}{L}\left\|g^i-g^j\right\|^2+\mu\left\|x^i-x^j\right\|^2-\frac{2\mu}{L}\left\langle g^j-g^i,x^j-x^i\right\rangle\right)\leq$$
$$f^i-f^j-\left\langle g^j,x^i-x^j\right\rangle.$$

By the defined Gram matrix (3.6), we can write this as follows.

$$f_i \geq f_j+\frac{L}{L-\mu}\left(u_j^T G t_i-u_j^T G t_j\right)+\frac{1}{2(L-\mu)}\left(u_i-u_j\right)^T G\left(u_i-u_j\right)+$$
$$\frac{\mu}{L-\mu}\left(u_i^T G t_j-u_i^T G t_i\right)+\frac{L\mu}{2(L-\mu)}\left(t_i-t_j\right)^T G\left(t_i-t_j\right).$$

Using this inequality, A_{ij} is defined as

$$2A_{ij}=\frac{L}{L-\mu}\left(u_j\left(t_i-t_j\right)^T+\left(t_i-t_j\right)u_j^T\right)+\frac{1}{L-\mu}\left(u_i-u_j\right)\left(u_i-u_j\right)^T+$$
$$\frac{\mu}{L-\mu}\left(u_i\left(t_j-t_i\right)^T+\left(t_j-t_i\right)u_i^T\right)+\frac{L\mu}{L-\mu}\left(t_i-t_j\right)\left(t_i-t_j\right)^T.$$

The matrix A_Δ can be defined similarly as

$$A_\Delta=u_{N+1}u_{N+1}^T.$$

Also, the matrix C is constructed in the same way using values of c_1, c_2, and the recursion formula for x^N.

As it is discussed the initial conditions can be different based on the problem under study. In this context, we primarily focus on introducing the commonly employed initial condition for the sake of simplicity. Subsequent chapters will delve into specific problems, each accompanied by distinct initial conditions related to the particular problems.

If n is equal to or greater than $N+2$, it is possible to reformulate the problem as a standard semidefinite programming problem [THG17c, Theorem 5]. This reformulation eliminates the rank constraint, effectively transforming the problem into a convex programming problem. Consequently, it can be solved to an optimal solution, if such a solution exists.

$$\begin{aligned} w_f^{\mathrm{sdp}}(\Delta,\mathcal{M},N,\mathcal{P})=\sup_{\{G\in\mathbb{S}^{N+2},\ f\in\mathbb{R}^{N+1}\}} & \ b f^{N+1}+\operatorname{tr}(GC) \\ \text{s.t. } & f^j-f^i+\operatorname{tr}(GA_{ij})\leq 0, \quad \forall i,j\in\mathcal{I} \\ & \operatorname{tr}(GA_\Delta)\leq\Delta^2 \\ & G\succeq 0, \end{aligned} \tag{3.8}$$

where $\mathcal{I} = \{0, 1, \ldots, N+1, \star\}$.

In this thesis we just consider the case that $n \geq N+2$. As it is shown in Section 2.2, under this condition, the Problems (3.7) and (3.8) attain the same optimal values. Otherwise, when this condition is not met, Problem (3.8) is a relaxation of Problem (3.7) and we get an upper bound for the convergence rate by solving Problem (3.8). The assumption $n \geq N+2$ is satisfied for many real-world problems. Moreover, from the complexity perspective, one is usually interested in the asymptotic performance as n grows to infinity. For a more comprehensive understanding of how the matrices are constructed to establish problem (3.8), we provide a simple example in Section 3.4; also see subsequent chapters where each chapter is dedicated to a specific problem.

3.3 Dual multipliers

Solving the dual of (3.8) provides us with useful information about the convergence rate of a given algorithm by providing dual multipliers, which help to deduce an analytical proof for the convergence rate. According to the weak duality theorem (see Lemma 2.31), the dual formulation (problem (3.9) below) can yield an upper bound for problem (3.8). As our goal is to find an upper bound on the convergence rate of an algorithm, a solution to the dual problem offers precisely that; see Section 2.2. Within this section, we proceed to present the Lagrangian dual formulation of the problem (3.8) as

$$
\begin{aligned}
\inf_{\lambda_{ij},\tau}\ & \tau\Delta^2 \\
\text{s.t. } & \tau A_\Delta - C + \sum_{i,j\in\mathcal{I}} \lambda_{ij} A_{ij} \succeq 0 \\
& b - \sum_{i,j\in\mathcal{I}} \lambda_{ij}(u_j - u_i) = 0 \\
& \lambda_{ij} \geq 0, \quad \forall i,j \in \mathcal{I} \\
& \tau \geq 0,
\end{aligned}
\tag{3.9}
$$

where $\mathcal{I} = \{0, 1, \ldots, N+1, \star\}$, u_i is defined as in Section 3.2, the dual variable τ refers to the constraint related to the initial condition in the primal problem, and dual variable λ_{ij} refers to the constraints related to the interpolation constraints which corresponds to the pairs of the points i and j. The parameter t_{ij} is step length defined in relation (3.4). It is shown in the next theorem that under the assumption $t_{i,i} \neq 0$ for $i = 1 \ldots, N$, there is no duality gap between problem (3.8) and (3.9), since the Slater condition holds in this case (see Section 2.2 and [BV04] for more details); for more details see the proof of Theorem 4.7 in [Tay17, Section 4.2.4]. Recall that the assumption $t_{i,i} \neq 0$ means that each generated point depends on the previously generated point.

Theorem 3.2. *[THG17c, Theorem 6] If $t_{i,i} \neq 0$ for all $i \in \{1, \ldots, N\}$ and $0 \leq \mu < L < \infty$, the optimal values of problem* (3.8) *and* (3.9) *are the same and finite.*

To prove this theorem, Taylor et al. provide a quadratic function that satisfies the constraints of Problem (3.8) with $G \succ 0$. This means that the Slater condition holds and the duality gap between both problems is equal to zero [THG17c]. The question that naturally arises at this point is whether this method exclusively provides us with a numerical solution for a specific set of parameter values. In essence, how can one derive an analytical solution for the algorithm's convergence rate in a general case that can be mathematically verified?

To determine a convergence rate through this method, one must solve problem (3.9) for various parameters that are Δ, N, $t_{i,j}, L, \mu$. By using these diverse combinations, one can estimate the optimal values and dual multipliers for the primal problem. Subsequently, applying weak duality, it becomes possible to construct an analytical proof for the convergence rate. To illustrate this procedure we provide a simple example.

3.4 A simple example

In this section, we will demonstrate the proofs of the performance estimation method by applying it to a straightforward example introduced in [THG17c]. Our objective is to determine the worst-case performance of a single iteration of the gradient descent method (see Algorithm 3.1) with a step length of $t = \frac{3}{2L}$ when addressing the unconstrained optimization problem (3.1). We make the assumption that the function f is a smooth convex function with Lipschitz constant L. As performance measure, we will use the metric $f(x^1) - f^\star$, along with the initial condition $\|x^1 - x^\star\| = \Delta$.

The optimal solution to the corresponding dual SDP problem (3.9) associated with this problem is $\frac{L\Delta^2}{8}$, and it is achieved with the dual variables $\lambda_{01} = \lambda_{\star 0} = \lambda_{\star 1} = \frac{1}{2}$ and $\tau = \frac{L}{8}$. Additionally, the optimal positive semidefinite dual slack matrix is

$$S = \frac{1}{2}\begin{pmatrix} \frac{1}{L} & \frac{1}{L} & \frac{-1}{2} \\ \frac{1}{L} & \frac{1}{L} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{L}{4} \end{pmatrix} \succeq 0,$$

which is a rank one matrix. After retrieving the dual multipliers corresponding to the constraints, one can prove the inequality below (3.10) by constructing the following inequality, which can be obtained by multiplication of the corresponding dual variables to the constraints

$$\begin{aligned}
&f(x^1)-f(x^\star)-\frac{L}{8}\|x^0-x^\star\|^2- \\
&\frac{1}{2}\left(f(x^1)-f(x^0)+\langle\nabla f(x^1),x^0-x^1\rangle+\frac{1}{2L}\|\nabla f(x^0)-\nabla f(x^1)\|^2\right)- \\
&\frac{1}{2}\left(f(x^0)-f(x^\star)+\langle\nabla f(x^0),x^\star-x^0\rangle+\frac{1}{2L}\|\nabla f(x^0)-\nabla f(x^\star)\|^2\right)- \\
&\frac{1}{2}\left(f(x^1)-f(x^\star)+\langle\nabla f(x^1),x^\star-x^1\rangle+\frac{1}{2L}\|\nabla f(x^1)-\nabla f(x^\star)\|^2\right)= \\
&-\frac{L}{2}\left\|\frac{1}{2}(x^0-x^\star)-\frac{\nabla f(x^0)}{L}-\frac{\nabla f(x^1)}{L}\right\|^2\leq 0.
\end{aligned}$$

To verify the validity of the equality, one can set $\nabla f(x^\star)=0$ and through elementary calculus check it. Given that the terms related to the interpolation constraints are negative, we can deduce the following result

$$f(x^1)-f(x^\star)\leq\frac{L}{8}\|x^0-x^\star\|^2. \tag{3.10}$$

It is important to notice that this example is just for illustration purpose. If the criteria and initial condition are not the same, one cannot deduce a general convergence rate only after studying one iteration of the algorithm. To do so, it is necessary to provide the proof for an arbitrary number of iterations N. Here, for the purpose of illustration we just provided the proof for only one iteration which shows the behaviour of the function value with respect to the distance of the initial point from the optimal solution.

To determine the tightness of this bound, it is necessary to identify a function that achieves this bound following a single iteration of the algorithm. To find such a function, we may need to solve the primal problem (3.8). If the optimal value of the primal problem equals the optimal value of the corresponding dual problem, then by strong duality theorem, we can identify the worst-case function. In the case of this example, the following primal solution is optimal:

$$f^0 = \frac{L\Delta^2}{2},\ f^1 = \frac{L\Delta^2}{8},\ \text{and } G = L\Delta^2 \begin{pmatrix} L & \frac{-L}{2} & 1 \\ \frac{-L}{2} & \frac{L}{4} & \frac{-1}{2} \\ 2 & \frac{-1}{2} & L \end{pmatrix} \succeq 0,$$

where G is a rank one matrix. Now one can set the values $f(x^0) = \frac{L\Delta^2}{2}$, $f(x^1) = \frac{L\Delta^2}{8}$, $\nabla f(x^0) = L\Delta$, $\nabla f(x^1) = \frac{-L\Delta}{2}$, and $x^0 = \Delta$. This corresponds to the function $f(x) = \frac{L}{2}x^2$. In other words, it can be checked that one iteration of the algorithm over the function $f(x) = \frac{L}{2}x^2$ attains the bound (3.10) if the step length is equal to $\frac{3}{2L}$, i.e. $x^1 = x^0 - \frac{3}{2L}\nabla f(x^0)$ with the starting point $x^0 = \Delta$.

One might wonder about the process of generating a function that achieves the bound and belongs to the function class. Generally, this process is based on an educated guess. Initially, it involves identifying suitable function values, gradients, and points that meet the optimal solution of Problem (3.8). Subsequently, using the properties of subgradients and conjugate functions, it is possible to conjecture a function that belongs to the specified function class and obtains the worst-case convergence rate.

As it is discussed in the previous section, PEP is considered as a computer-assisted method for finding worst-case convergence rates. To this end, we formulate the corresponding dual problem. Subsequently, we try to solve the problem using different settings of the parameters. This helps us to guess parametric optimal solutions to the dual problem as well as the optimal dual multipliers. Following this, one needs to verify the proof analytically, similar to the one in the example. Therefore, the computer is just used for finding the dual variables but the proofs are independent of the computer.

3.5 Some recent works on/with PEP

To conclude this chapter, in this section we provide a brief overview on some recent research done by other scholars using PEP without getting into all the details. As mentioned, the performance estimation method is a strong tool in worst-case convergence rate analysis of first-order methods, used by a growing number of scholars to evaluate the convergence rate of different algorithms in different settings, initially introduced in Drori and Teboulle's seminal paper [DT14].

The gradient descent method, originally introduced by Cauchy in 1847 [Cau47], has been a topic of interest for researchers aiming to understand its computational complexity for different classes of functions. Rotaru et al. studied this method [RGP22]. They extended the result provided in [AdKZ22] to the generalization of $\mathcal{F}_{\mu,L}$, where μ is allowed to take negative values, and proved that the given bound is tight for this class of functions with some step lengths [RGP22].

In the setting of strongly convex functions, the gradient descent method with fixed step-length is studied in [Tay17] and when the line search is implemented in each iteration [dKGT17] proved a tight convergence rate by providing a convex quadratic function that attains the bound. Moreover, the worst case convergence rate for L-smooth convex functions with step-length less than $\frac{1}{L}$ is given by [DT14]. Recently, using PEP, Grimmer has shown that for L-smooth convex functions instead of considering fixed step-length, taking long steps periodically in the iterations of the gradient descent method improves the convergence rate [Gri23]. In the same spirit, Grimmer et al. [GSW23] introduced long steps for the gradient method and provided a new accelerated convergence rate of the algorithm for smooth convex functions. Also, more recently, Altschuler and Parrilo provided new results on the long step gradient descent method [AP23a, AP23b].

One variation on the gradient method is the coordinate descent method; see Chapter 6 for more details of the algorithm. For the class of smooth convex functions, [KHG23] proposed a method to study the convergence of this type of algorithm.

Using PEP Zamani and Glineur managed to find the tight convergence rate for the subgradient methods considering the last iterate of the algorithm [ZG23]. Fixing the number of iterations N, they provide the optimal constant step size based on the convergence rate. The optimal subgradient method introduced in the paper matches the best known lower bound.

Bousselmi et al. extended the PEP framework to analyze linear operators [BHG23]. Using this framework they were able to analyze composed objective function and also the Chambolle-Pock method.

Beyond first-order methods, de Klerk et al. using PEP managed to find worst-case convergence rate for one iteration of Newton method which is a second order method [DKGT20]. This line of work shows that in some cases PEP may be used to find convergence rate of second order methods as well.

Ryu et al. extended performance estimation to find the tight contraction factors for operator splitting methods [RTBG20]. They proposed some interpolation theorems and called their methodology *operator splitting performance estimation*.

Kim and Fessler [KF18a] presented a relaxed PEP to recover the results of fast iterative shrinkage/thresholding algorithm (FISTA), originally proposed in

[BT09], and introduced a generalized version of the algorithm. The same authors in another paper used PEP to generalize the optimized gradient method [KF18b]. In other papers, Kim and Fessler considered the step length of the first-order methods as a variable. In this case, PEP transforms to a bilinear optimization problem to find the best step length that minimizes the worst-case convergence of the algorithm with regard to the criterion. With this, they could manage to find new algorithms for convex smooth optimization problem [KF16, KF21]. To design new first-order algorithms, Gupta et al. [DGVPR23] developed a branch-and-bound performance estimation programming (BnB-PEP) method to efficiently deal with nonconvexity that arises in optimizing a first-order method. Moreover, using this tool, Jang et al. presented a new method called OptISTA which is developed based on FISTA method [JGR23].

Finding Lyapunov functions is of importance in studying first-order methods. Recently, Moucer et al. [MTB23] developed a systematic way using PEP to find and verify Lyapunov functions. PEP is also extended by Upadhyaya et al. to find quadratic Lyapunov inequalities to derive tight convergence rates for some classes of first-order methods [UBTG23].

Under negative comonotonicity assumptions Gorbunov et al. studied some algorithms for variational inequality and min-max optimization using PEP [GTHG23]. Moreover, Gorbunov et al. using this method could manage to derive the known convergence rate for the Past Extragradient (PEG) method with fewer assumptions [GTG22], in fact they removed the assumptions on the Lipschitz Jacobian which was used to prove the convergence rate. Moreover, for maximally monotone operators, Kim developed an accelerated proximal point using PEP [Kim21]. On the other hand, Gu and Yang could manage to find tight ergodic convergence rate of a proximal point algorithm for monotone variational inequalities [GY22].

Furthermore, performance estimation method and integral quadratic constraints (IQC) formulation is combined to find new algorithms by Lessard et al. [LRP16].

To make the use of this computer-assisted method easy, in two papers the authors developed a package (toolbox) to be used in Python (MATLAB) [GMG+22, THG17b], called PEPit (PESTO). Using these packages the user can easily numerically compute the complexity of some of the well-known first-order methods for different settings of the parameters.

There are some other works that are done using PEP; e.g. [PR23, DT20, DTdB21, TD23, THG18, TVSL18]. This list is not exhaustive, though.

Men pass away, but their deeds abide.

Augustin-Louis Cauchy

4

The exact worst-case convergence rate of the gradient method with fixed step lengths for L-smooth functions

Preamble

In this chapter, we study the convergence rate of the gradient (also known as steepest descent) method with fixed step lengths, which is introduced earlier in Algorithm 3.1, for finding a stationary point of an L-smooth function. We establish a new convergence rate, and show that the bound may be exact in some cases, in particular when all step lengths lie in the interval $(0, 1/L]$. In addition, we derive an optimal step length with respect to the new bound. This chapter is based on the paper [AdKZ22], except for Section 4.4 that deals with extensions of L-smooth functions.

4.1 Introduction

We consider the non-convex unconstrained optimization problem

$$\inf_{x\in\mathbb{R}^n} f(x), \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is bounded from below, and let a real number $f^\star$ denote a lower bound for problem (4.1). In addition, we assume throughout the chapter that f has an L-Lipschitz gradient, that is, f satisfies the conditions in Definition 2.21 for some (known) Lipschitz constant $L > 0$.

Problem (4.1) arises naturally in many applications including machine learning, signal and image processing, to name but a few [BCN18, JK+17]. One of the historic solution methods for problem (4.1) is the gradient method, proposed by Cauchy in 1847 [Cau47].

The gradient method with fixed step lengths may be described as follows.

Algorithm 4.1 Gradient method with fixed step lengths

Set N and $\{t_k\}_{k=1}^N$ (step lengths) and pick $x^0 \in \mathbb{R}^n$.
For $k = 1, 2, \dots, N$ perform the following step:
1. $x^k = x^{k-1} - t_k \nabla f(x^{k-1})$

Nesterov [Nes03, page 28] gives the following convergence rate (to a stationary point) for Algorithm 4.1 when $t_k \in (0, \frac{2}{L})$, $k \in \{1, \dots, N\}$:

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{f(x^0) - f^\star}{\left(\sum_{k=1}^N t_k(1-\frac{1}{2}Lt_k)\right) + \frac{1}{2L}} \right)^{1/2}.$$

In the special case $t_k = \frac{1}{L}$, $k \in \{1, \dots, N\}$, the last bound becomes

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{2L(f(x^0) - f^\star)}{N+1} \right)^{1/2}.$$

By employing the performance estimation method, Taylor [Tay17, page 190], without giving a proof, states the following convergence rate

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{4L(f(x^0) - f^\star)}{3N} \right)^{1/2}, \tag{4.2}$$

for $t_k = \frac{1}{L}$, $k \in \{1, \ldots, N\}$. Drori and Shamir [DS20, Corollary 1 in Appendix] considers the case that all step lengths are smaller than $\frac{1}{L}$, and proves the following convergence rate

$$\min_{0 \le k \le N} \left\| \nabla f(x^k) \right\| \le \left(\frac{4(f(x^0) - f^\star)}{\sum_{k=1}^{N} t_k(4 - L t_k)} \right)^{1/2}. \tag{4.3}$$

It can be observed that when the step lengths are the same for each iteration and tend to $\frac{1}{L}$, the bound (4.3) reduces to Taylor's convergence rate.

In this chapter, we investigate the convergence rate of Algorithm 4.1 further. By using the performance estimation method, we provide a converge rate, which is tighter than all aforementioned bounds. For example, as a part of our main result in Theorem 4.3, we improve on (4.3) by showing, for any choice of $t_k \in (0, \sqrt{3}/L)$ ($k \in \{1, \ldots, N\}$), that

$$\min_{0 \le k \le N} \left\| \nabla f(x^k) \right\| \le \left(\frac{4(f(x^0) - f^\star)}{\sum_{k=1}^{N} \min(-L^2 t_k^3 + 4t_k, -L t_k^2 + 4t_k) + \frac{2}{L}} \right)^{1/2}. \tag{4.4}$$

As a consequence, we also prove and improve on (4.2) by showing, in the special case where all $t_k = 1/L$ ($k \in \{1, \ldots, N\}$), that

$$\min_{0 \le k \le N} \left\| \nabla f(x^k) \right\| \le \left(\tfrac{4L(f(x^0) - f^\star)}{3N+2} \right)^{1/2}.$$

In addition, we construct an L-smooth function that attains the given bound in Theorem 4.3 for certain step lengths. We also propose an optimal step length that minimizes the right-hand-side of the bound (4.4), namely $t_k = \frac{\sqrt{4/3}}{L}$ for all $k \in \{1, \ldots, N\}$.

Outline

The chapter is organized as follows. We describe the performance estimation technique for this specific problem in Section 4.2. In Section 4.3, we study the convergence rate by using performance estimation. Section 4.4 is dedicated to study of extension of L-smooth functions. Finally, we conclude the chapter with a conjecture.

4.2 Performance estimation

As it is mentioned in Chapter 3, computation of the worst-case convergence rate for a given iterative method and a given class of functions is an infinite-dimensional optimization problem; for more details of the performance estimation see Chapter 3.

Similar to problem (P) in [DT14], the worst-case convergence rate of Algorithm 4.1 may be formulated as the following abstract optimization problem,

$$
\begin{aligned}
& \max \left(\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \right) \\
& \text{s. t. } f(x^0) - f^\star \le \Delta \\
& \quad x^N, x^{N-1}, \ldots, x^1 \text{ are generated by Algorithm 4.1 w.r.t. } f, x^0 \\
& \quad f(x) \ge f^\star \ \forall x \in \mathbb{R}^n \\
& \quad f \in \mathcal{F}_{-L,L}(\mathbb{R}^n) \\
& \quad x^0 \in \mathbb{R}^n,
\end{aligned}
\tag{4.5}
$$

where $\Delta \ge 0$ denote the difference between the given lower bound, $f^\star$, and the value of f at the starting point. In problem (4.5), f and x^0 are decision variables. This is an infinite-dimensional optimization problem with infinite number of constraints, and consequently intractable in general. In what follows, we provide a semidefinite programming relaxation for the problem.

The following well-known result is a fundamental property of gradient descent for L-smooth functions, if the step length $1/L$ is used.

Proposition 4.1. *[Nes03, page 26] If $f : \mathbb{R}^n \to \mathbb{R}$ is L-smooth, and $x \in \mathbb{R}^n$, then*

$$f\left(x - \frac{1}{L}\nabla f(x)\right) \le f(x) - \frac{1}{2L}\|\nabla f(x)\|^2.$$

Using this and Theorem 2.41, we will formulate problem (4.5) as a finite dimensional optimization problem.

$$
\begin{aligned}
& \max \left(\min_{0\le k\le N} \left\|g^k\right\| \right) \\
& \text{s. t. } \tfrac{1}{2L}\left\|g^i - g^j\right\|^2 - \tfrac{L}{4}\left\|x^i - x^j - \tfrac{1}{L}(g^i - g^j)\right\|^2 \le f^i - f^j - \\
& \qquad \left\langle g^j, x^i - x^j \right\rangle \quad i, j \in \{0, \ldots, N\} \\
& \quad x^k = x^{k-1} - t_k g^{k-1} \quad k \in \{1, \ldots, N\} \\
& \quad f^k \ge f^\star \quad k \in \{0, \ldots, N\} \\
& \quad f^0 - f^\star \le \Delta.
\end{aligned}
\tag{4.6}
$$

In the above formulation, $x^k,\ g^k,\ f^k$, $k \in \{0,\dots,N\}$, are decision variables. Note that in the above formulation, the constraints $f(x) \geq f^\star$ for each $x \in \mathbb{R}^n$ are replaced by the weaker condition $f^k \geq f^\star,\ \ k \in \{0,\dots,N\}$. Therefore, the optimal value of (4.5) and (4.6) may not be equal in general. However, if an optimal solution of problem (4.6) satisfies $f^\star = \min_{0\leq k\leq N} f^k - \frac{1}{2L}\|g^k\|^2$, then the formulation will be exact; see the second part of Theorem 2.41. By Proposition 4.1, we have $f(x) - \frac{1}{2L}\|\nabla f(x)\|^2 \geq f^\star$ for $x \in \mathbb{R}^n$. Hence, we replace the constraint $f^k \geq f^\star$ by $f^k - \frac{1}{2L}\|g^k\|^2 \geq f^\star$ and consider the following problem:

$$
\begin{aligned}
&\max \left(\min_{0\leq k\leq N} \left\|g^k\right\|\right)\\
&\text{s.t. } \tfrac{1}{2L}\left\|g^i - g^j\right\|^2 - \tfrac{L}{4}\left\|x^i - x^j - \tfrac{1}{L}(g^i - g^j)\right\|^2 \leq f^i - f^j - \\
&\qquad \left\langle g^j, x^i - x^j\right\rangle \ \ i,j \in \{0,\dots,N-1,N\}\\
&\quad x^k = x^{k-1} - t_k g^{k-1} \ \ k \in \{1,\dots,N\}\\
&\quad f^k - \tfrac{1}{2L}\|g^k\|^2 - f^\star \geq 0 \ \ k \in \{0,\dots,N\}\\
&\quad f^0 - f^\star \leq \Delta.
\end{aligned}
\tag{4.7}
$$

From the constraint $x^k = x^{k-1} - t_k g^{k-1}$, we get $x^i = x^0 + \sum_{k=0}^{i-1} t_{k+1} g^k$, $i \in \{1,\dots,N\}$. By using this relation to eliminate the x^i ($i \in \{1,\dots,N\}$), problem (4.7) may be written as follows:

$$
\begin{aligned}
&\max \ \ell\\
&\text{s.t. } f^i - f^j - \tfrac{1}{2L}\left\|g^i - g^j\right\|^2 + \tfrac{L}{4}\left\|-\sum_{k=j}^{i-1} t_{k+1} g^k + \tfrac{1}{L}(g^i - g^j)\right\|^2 + \left\langle g^j, \sum_{k=j}^{i-1} t_{k+1} g^k\right\rangle \geq 0 \ \ i > j\\
&\quad f^i - f^j - \tfrac{1}{2L}\left\|g^i - g^j\right\|^2 + \tfrac{L}{4}\left\|\sum_{k=i}^{j-1} t_{k+1} g^k - \tfrac{1}{L}(g^i - g^j)\right\|^2 - \left\langle g^j, \sum_{k=i}^{j-1} t_{k+1} g^k\right\rangle \geq 0 \ \ i < j\\
&\quad f^k - \tfrac{1}{2L}\|g^k\|^2 - f^\star \geq 0 \ \ k \in \{0,\dots,N\}\\
&\quad f^\star - f^0 + \Delta \geq 0\\
&\quad \left\|g^k\right\|^2 - \ell \geq 0 \ \ \ k \in \{0,\dots,N\},
\end{aligned}
\tag{4.8}
$$

where ℓ is an auxiliary variable to convert problem (4.7) into a quadratic program. Problem (4.8) is a non-convex quadratic program with quadratic constraints. In the following proposition, we show that the optimal values of problems (4.5) and (4.7) (or equivalently problem (4.8)) are the same for step lengths in the interval $(0, \frac{2}{L})$.

Proposition 4.2. *If $t_k \in (0, \frac{2}{L})$, $k \in \{1,\dots,N\}$, then problems* (4.5) *and* (4.7) *(or equivalently problem* (4.8)*) share the same optimal value.*

Proof. Clearly, problem (4.7) is a relaxation of problem (4.5). Therefore, we only need to show that, for any feasible solution of (4.7), say $\{(\bar{x}^i; \bar{g}^i; \bar{f}^i)\}_0^N$, there exists an L-smooth function f with

$$f(\bar{x}^i) = \bar{f}^i, \ \nabla f(\bar{x}^i) = \bar{g}^i, \quad 0 \leq i \leq N,$$

and $\min_{x\in\mathbb{R}^n} f(x) \geq f^\star$. The existence such of a function follows from Theorem 2.41, as all assumptions of Theorem 2.41 are satisfied. □

To obtain a tractable form of problem (4.8), we relax it to a semidefinite program, similar to that of Chapter 3. To this end, we define the $(N+1)\times(N+1)$ positive semidefinite matrix G as,

$$G = \begin{pmatrix} (g^0)^T \\ \vdots \\ (g^N)^T \end{pmatrix} \begin{pmatrix} g^0 & \dots & g^N \end{pmatrix} = \begin{pmatrix} \|g^0\|^2 & \dots & \langle g^0, g^N \rangle \\ \vdots & \ddots & \vdots \\ \langle g^0, g^N \rangle & \dots & \|g^N\|^2 \end{pmatrix}.$$

We may now formulate the following semidefinite program,

$$\begin{aligned} \max\ & \ell \\ \text{s.t. } & f^i - f^j + \mathrm{tr}(A^{ij}G) \geq 0 \quad i \neq j \in \{0,\dots,N\} \\ & f^k - \tfrac{1}{2L} G_{kk} - f^\star \geq 0 \quad k \in \{0,\dots,N\} \\ & f^\star - f^1 + \Delta \geq 0 \\ & G_{kk} - \ell \geq 0 \quad k \in \{0,\dots,N\} \\ & G \succeq 0, \end{aligned} \tag{4.9}$$

where the $(N+1)\times(N+1)$ matrices A^{ij}, $i \neq j \in \{0,\dots,N\}$, are formed according to the constraints (4.8), and G, ℓ, f^i, $i \in \{0,\dots,N\}$, are decision variables. Problem (4.9) is a relaxation of (4.8), but if $n \geq N+1$ the relaxation is exact, that is the optimal values of (4.8) and (4.9) are the same. Indeed, if $n \geq N+1$, and G is a feasible matrix in (4.9), then G is the Gram matrix of $N+1$ vectors in $\mathbb{R}^n$, and these vectors may be identified with $g^0,\dots,g^N$; see Chapter 3 for more details.

4.3 Worst-case convergence rate

In this section, we investigate the convergence rate of gradient method with fixed step lengths. The next theorem gives the worst-case convergence rate of Algorithm 4.1 to a stationary point of an L-smooth function. The technique of the proof, as is usual for SDP performance estimation, is to use weak duality. In particular, we will in fact construct a feasible solution to the dual SDP problem of (4.9), and thus derive an upper bound for problem (4.8).

In practice, this dual feasible solution is constructed in a computer-assisted manner, by solving the primal and dual SDP problems for different fixed values of the parameters, and subsequently guessing the values of the dual multipliers. In the proof of Theorem 4.3, we simply verify that these 'guesses' are correct.

Theorem 4.3. *Let $t_k \in (0, \frac{\sqrt{3}}{L})$ for $k \in \{1,\dots,N\}$. Consider N iterations of Algorithm 4.1 with step lengths t_k ($k \in \{1,\dots,N\}$), applied to some L-smooth function f with minimum value $f^\star$, with the starting point x^0 satisfying $f(x^0)-f^\star \le \Delta$, for some given $\Delta > 0$.*

Then, if $x^1,\dots,x^N$ denote the iterates of Algorithm 4.1, one has

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{4\Delta}{\sum_{k=1}^N \min(-L^2t_k^3+4t_k, -Lt_k^2+4t_k)+\frac{2}{L}}\right)^{1/2}. \qquad (4.10)$$

In particular, if $t_k = \frac{\sqrt{4/3}}{L}$ for $k \in \{1,\dots,N\}$, we get

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{6\sqrt{3}L(f(x^0)-f^\star)}{8N+3\sqrt{3}}\right)^{1/2}. \qquad (4.11)$$

Similarly, if $t_k = \frac{1}{L}$ for $k \in \{1,\dots,N\}$, one has

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| \le \left(\frac{4L(f(x^0)-f^\star)}{3N+2}\right)^{1/2}. \qquad (4.12)$$

Proof. Let U denote the square of the right-side of inequality (4.10) and let $B = \frac{U}{\Delta}$. To establish this bound, we show that U is an upper bound for problem (4.8). Consider the feasible point $\left(\{g^k; f^k\}_0^N; \ell\right)$ for problem (4.8). Suppose that

$$\alpha_k = \tfrac{B}{2}\max\{2, t_kL+1\} \quad k \in \{1,\dots,N\}.$$

In addition, we define σ_1 and σ_k, respectively, as follows:

$$\begin{aligned}\sigma_1 &= \tfrac{B}{4}\min\left\{-Lt_1^2+3t_1, -L^2t_1^3+3t_1\right\},\\ \sigma_k &= \tfrac{B}{4}\min\left\{-Lt_k^2+3t_k+t_{k-1}, -L^2t_k^3+3t_k+t_{k-1}\right\} \quad k \in \{2,\dots,N\},\end{aligned}$$

and $\sigma_{N+1} = 1-\sum_{k=1}^N \sigma_k = \frac{B}{4L}(2+Lt_N)$. As $t_k \in (0, \frac{\sqrt{3}}{L})$ for $k \in \{1,\dots,N\}$, the σ_k's will be non-negative. It is seen that

$$\sigma_k + (2\alpha_k - B)\frac{Lt_k^2}{4} - \frac{Bt_k}{2} = \tfrac{B}{4}(t_k+t_{k-1}) \quad k \in \{2,\dots,N\}.$$

By using the last equality, one may verify directly through elementary algebra that

$$\ell - U + \sum_{k=1}^{N+1} \sigma_k \left(\left\| g^{k-1} \right\|^2 - \ell \right) + B\left(f^\star - f^0 + \Delta\right) + B\left(f^N - \tfrac{1}{2L}\|g^N\|^2 - f^\star\right)$$
$$+ \sum_{k=1}^{N} \alpha_k \Big(f^{k-1} - f^k - \tfrac{1}{2L} \left\| g^{k-1} - g^k \right\|^2 + \tfrac{L}{4} \left\| t_k g^{k-1} - \tfrac{1}{L}\left(g^{k-1} - g^k\right) \right\|^2$$
$$- \left\langle g^k, t_k g^{k-1} \right\rangle \Big) + \sum_{k=1}^{N} (\alpha_k - B) \Big(f^k - f^{k-1} - \tfrac{1}{2L} \left\| g^k - g^{k-1} \right\|^2$$
$$+ \tfrac{L}{4} \left\| -t_k g^{k-1} - \tfrac{1}{L}\left(g^k - g^{k-1}\right) \right\|^2 - \left\langle g^{k-1}, -t_k g^{k-1} \right\rangle \Big) = \tfrac{-(2\alpha_1 - B)}{4L} \left\| g^0 - g^1 \right\|^2$$
$$+ \tfrac{Bt_1}{4} \left\| g^0 \right\|^2 - \tfrac{Bt_1}{2} \left\langle g^0, g^1 \right\rangle + \tfrac{Bt_N}{4} \left\| g^N \right\|^2$$
$$+ \sum_{k=2}^{N} \Big(\tfrac{-(2\alpha_k - B)}{4L} \left\| g^{k-1} - g^k \right\|^2 + \tfrac{B(t_k + t_{k-1})}{4} \left\| g^{k-1} \right\|^2 - \tfrac{Bt_k}{2} \left\langle g^{k-1}, g^k \right\rangle \Big) =$$
$$- \sum_{k=1}^{N} Q_k,$$

where

$$Q_k = \begin{cases} \frac{B}{4}\left(\frac{1}{L} - t_k\right) \left\| g^{k-1} - g^k \right\|^2 & t_k < \frac{1}{L} \\ 0 & t_k \geq \frac{1}{L}. \end{cases}$$

Since $\sum_{k=1}^N Q_k$ is a non-negative quadratic function and the given dual multipliers are non-negative, we have $\ell \leq U$ for any feasible solution of (4.8). □

The special step length $t_k = \frac{\sqrt{4/3}}{L}$ for $k \in \{1, \ldots, N\}$ used to obtain (4.11) will be motivated later in Theorem 4.5. Note that (4.12) gives a formal proof (with a small improvement) of the bound claimed by Taylor [Tay17, page 190]; see (4.2).

An important question concerning the bound (4.10) is its difference with the optimal value of (4.5). It is known that the lower bound for Algorithm 4.1 is of the order $\Omega\left(\frac{1}{\sqrt{N}}\right)$ [CGT10, CDHS20]. In what follows, we establish that the bound (4.10) is exact in some cases.

Proposition 4.4. *The value*

$$\left(\frac{4\Delta}{\sum_{k=1}^N \min\left(-L^2 t_k^3 + 4t_k, -L t_k^2 + 4t_k\right) + \frac{2}{L}} \right)^{1/2}$$

is the optimal value of (4.5) *when all step lengths satisfy* $t_k \in (0, \frac{1}{L}]$, $k \in \{1, \ldots, N\}$.

Proof. It suffices for a given N to demonstrate an L-smooth function f and a point x^1 such that

$$\min_{0\le k\le N} \left\|\nabla f(x^k)\right\| = \left(\frac{4\Delta}{\sum_{k=1}^N \min\left(-L^2t_k^3+4t_k, -Lt_k^2+4t_k\right)+\frac{2}{L}}\right)^{1/2}. \tag{4.13}$$

Suppose now that $t_k \in (0, \frac{1}{L}]$, $k \in \{1,\dots,N\}$, and U denotes the right-hand-side of equality (4.13). We set $t_{N+1} = \frac{1}{L}$. Let

$$l_i = U\left(\sum_{k=i}^N t_{k+1}\right), \quad f^i = \Delta - \frac{U^2}{4}\left(\sum_{k=1}^i -Lt_k^2 + 4t_k\right) \qquad i \in \{0,\dots,N\},$$

and $l_{N+2} = 0$. By elementary calculus, one can check that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{L}{2}(x-l_0)^2 + U(x-l_0) + f^0 & x \in \left[\frac{1}{2}(l_0+l_1), \infty\right) \\ \frac{-L}{2}(x-l_i)^2 + U(x-l_i) + f^i & x \in \left[l_i, \frac{1}{2}(l_{i-1}+l_i)\right] \\ \frac{L}{2}(x-l_i)^2 + U(x-l_i) + f^i & x \in \left[\frac{1}{2}(l_i+l_{i+1}), l_i\right] \\ \frac{L}{2}x^2 & x \in \left(-\infty, \frac{1}{2}l_N\right] \end{cases} \tag{4.14}$$

for $i \in \{1,\dots,N\}$, is L-smooth with the optimal value $f^\star = 0$ and the optimal solution $x^\star = 0$. In addition, we have equality (4.13) for $x^1 = l_1$. Indeed,

$$\begin{aligned} x^i &= l_i && i \in \{0,\dots,N\} \\ \nabla f(x^i) &= U && i \in \{0,\dots,N\} \\ f(x^i) &= f^i && i \in \{0,\dots,N\}. \end{aligned}$$

□

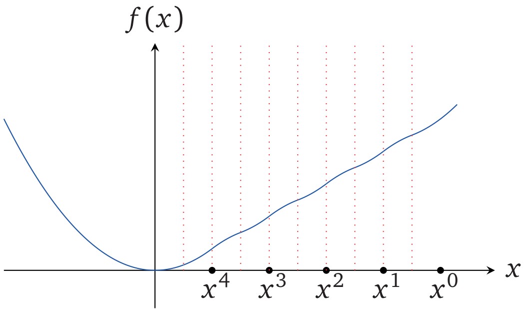
Figure 4.1 represents the plot of function f as constructed in the proof of Proposition 4.4 for different parameters and the fixed step length $t_k = \frac{1}{L}$ for all k.

Note that, though we have only shown the exactness of the bound (4.10) for step lengths in the interval $(0, \frac{1}{L}]$, we also conjecture that the bound (4.10) is in fact exact for all step lengths in the interval $(0, \frac{\sqrt{3}}{L})$.

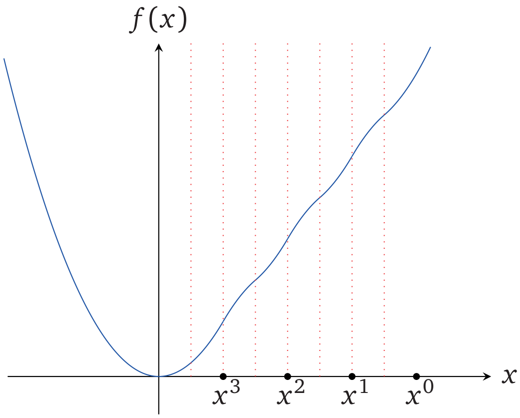
By minimizing the right-hand-side of (4.10), the next theorem gives the 'optimal' step lengths with respect to the bound.

Theorem 4.5. *Let f be an L-smooth function. Then the optimal step size for gradient method with respect to bound* (4.10) *is given by*

$$t_k = \frac{\sqrt{\frac{4}{3}}}{L} \quad \forall k \in \{1,\dots,N\},$$



(a) $N = 4,\ \Delta = 2,\ L = 1$



(b) $N = 3,\ \Delta = 4,\ L = 2$

Figure 4.1: Plot of the function f in (4.14) for different parameters and $t_k = \frac{1}{L}$. (Dotted lines denote the endpoints of intervals.)

provided that $t_k \in (0, \frac{\sqrt{3}}{L})$ *for all* $k \in \{1, \ldots, N\}$.

Proof. We minimize the right-hand-side of (4.10), that is

$$\min_{t_k \in (0, \frac{\sqrt{3}}{L})} \left(\frac{4\Delta}{\sum_{k=1}^{N} \min\left(-L^2 t_k^3 + 4t_k, -L t_k^2 + 4t_k\right) + \frac{2}{L}} \right)^{1/2},$$

which is equivalent to maximizing

$$\max_{t \in \left(0, \frac{\sqrt{3}}{L}\right)^N} H(t) := \sum_{k=1}^{N} \min\left(-L^2 t_k^3 + 4t_k, -L t_k^2 + 4t_k\right).$$

Since H is a strictly concave function on $\left(0, \frac{\sqrt{3}}{L}\right)^N$ and at $\bar{t}$ given by

$$\bar{t}_k = \frac{\sqrt{\frac{4}{3}}}{L} \quad \forall k \in \{1, \ldots, N\},$$

we have $\nabla H(\bar{t}) = 0$, which shows that $\bar{t}$ is the unique maximum solution of H over $\left(0, \frac{\sqrt{3}}{L}\right)^N$. □

The step length $\frac{1}{L}$ commonly is regarded as the optimal step length in the literature; see [Nes03, Chapter 1]. Due to the example introduced in (4.14), we see that the worst-case convergence rate for the step length $\frac{1}{L}$ cannot be better than $\left(\frac{4L(f(x^0)-f^\star)}{3N+2}\right)^{1/2}$. By our analysis, it follows that, for the step length $\frac{\sqrt{\frac{4}{3}}}{L}$, we get the convergence rate (4.11), which is better than $\left(\frac{4L(f(x^0)-f^\star)}{3N+2}\right)^{1/2}$, since the constant in the bound improves from ca. $\frac{4}{3} \approx 1.333$ to $\frac{6\sqrt{3}}{8} \approx 1.299$.

In the next section we provide an upperbound on the module of smoothness of a function that is the extension of an L-smooth function over a convex set $\mathbb{D} \subseteq \mathbb{R}^n$.

4.4 Extension of L-smooth functions

In this section, we try to extend any L-smooth function on $\mathbb{D} \subseteq \mathbb{R}^n$ to an $\bar{L}$-smooth function on $\mathbb{R}^n$. Results of this type are usually called *extension theorems*, with the most famous extension theorem being due to Whitney [Whi34], known as *Whitney extension theorem*. This allows us to extend our results to any $\mathbb{D} \subseteq \mathbb{R}^n$ rather than $\mathbb{R}^n$.

Let $\mathbb{D} \subseteq \mathbb{R}^n$ be a given set and let $\alpha : \mathbb{D} \to \mathbb{R}$ and $\nu : \mathbb{D} \to \mathbb{R}^n$. In this section, we list some key results concerning the existence of an L-smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with

$$f(x) = \alpha(x),\ \nabla f(x) = \nu(x),\ x \in \mathbb{D}.$$

Before we get to the results, we need to introduce the constant $\Gamma(\mathbb{D}, (\alpha, \nu))$, which is defined as

$$\Gamma(\mathbb{D}, (\alpha, \nu)) = \sup_{s_1, s_2 \in \mathbb{D}, s_1 \neq s_2} \left(\sqrt{A_{s_1 s_2}^2 + B_{s_1 s_2}^2} + \left| A_{s_1 s_2} \right| \right) \tag{4.15}$$

where

$$A_{s_1 s_2} = \frac{2(\alpha(s_1) - \alpha(s_2)) + \langle \nu(s_1) + \nu(s_2), s_2 - s_1 \rangle}{\|s_1 - s_2\|^2},\ B_{s_1 s_2} = \frac{\|\nu(s_1) - \nu(s_2)\|}{\|s_1 - s_2\|}.$$

The following is an extension theorem for L-smooth functions.

Theorem 4.6. *[Gru09, Theorem 2.6] If* $\bar{L} := \Gamma(\mathbb{D}, (\alpha, \nu)) < \infty$, *then there exists an* $\bar{L}$*-smooth function* $f : \mathbb{R}^n \to \mathbb{R}$ *with*

$$f(x) = \alpha(x),\ \nabla f(x) = \nu(x)\ \ x \in \mathbb{D}.$$

One may wonder if there exists an L-smooth function $g : \mathbb{R}^n \to \mathbb{R}$ with $L < \Gamma(\mathbb{D}, (\alpha, \nu))$ such that

$$g(x) = \alpha(x),\ \nabla g(x) = \nu(x),\ \ x \in \mathbb{D},$$

if α is an L-smooth function on $\mathbb{D}$. The answer is negative when $\mathbb{D}$ is compact; see [Gru09, Theorem 3.2].

Under the assumptions of Theorem 4.6, Daniilidis et al. [DHLGL18] introduce an L-smooth function $g : \mathbb{R}^n \to \mathbb{R}$ with explicit formula such that

$$g(x) = \alpha(x),\ \nabla g(x) = \nu(x),\ x \in \mathbb{D}.$$

Moreover, $\bar{L} \leq \left(\frac{5+\sqrt{29}}{2}\right) L$; see [DHLGL18, Theorem 3.1].

In the following proposition, we give a bound for $\Gamma(\mathbb{D}, (\alpha, \nu))$ when $\mathbb{D}$ is an open convex set and we study the extension of an L-smooth function on $\mathbb{D}$.

Corollary 4.7. *Let $\mathbb{D}$ be an open convex set. If $f : \mathbb{D} \to \mathbb{R}$ is an L-smooth function on $\mathbb{D}$ then*

$$\Gamma(\mathbb{D}, (f, \nabla f)) \leq (\sqrt{2}+1)L$$

Proof. Let $s_1, s_2 \in \mathbb{D}$. First we show that $B_{s_1 s_2} \leq L$. To this end, note that

$$B_{s_1 s_2} = \frac{\|\nabla f(s_1) - \nabla f(s_2)\|}{\|s_1 - s_2\|} \leq \frac{L\,\|s_1 - s_2\|}{\|s_1 - s_2\|} \leq L,$$

where the last inequality is due to the L-smooth property of f. On the other hand, by the fundamental theorem of calculus,

$$f(s_1) - f(s_2) = \int_0^1 \langle \nabla f(\lambda s_1 + (1-\lambda)s_2), s_1 - s_2 \rangle \, d\lambda.$$

It follows that

$$\begin{aligned}
\left|A_{s_1 s_2}\right| &= \frac{\left|2\left(\int_0^1 \langle \nabla f(\lambda s_1 + (1-\lambda)s_2), s_1 - s_2\rangle \, d\lambda\right) + \langle \nabla f(s_1) + \nabla f(s_2), s_2 - s_1\rangle\right|}{\|s_1 - s_2\|^2} \\
&= \frac{\left|\begin{array}{l} \int_0^1 \langle \nabla f(\lambda s_1 + (1-\lambda)s_2) - \nabla f(s_1), s_1 - s_2\rangle \, d\lambda + \\ \qquad \int_0^1 \langle \nabla f(\lambda s_1 + (1-\lambda)s_2) - \nabla f(s_2), s_1 - s_2\rangle \, d\lambda \end{array}\right|}{\|s_1 - s_2\|^2} \\
&\leq \frac{\int_0^1 L(1-\lambda)\,\|s_1 - s_2\|^2 \, d\lambda + \int_0^1 L\lambda \,\|s_1 - s_2\|^2 \, d\lambda}{\|s_1 - s_2\|^2} = L.
\end{aligned}$$

Hence, in view of (4.15), $\Gamma(\mathbb{D}, (f, \nabla f)) \leq (\sqrt{2}+1)L$, and the proof is complete. □

Using Corollary 4.7, we can extend our convergence rate results for L-smooth functions on open convex sets. In particular, one can derive the following corollary.

Corollary 4.8. *If the function f is L-smooth on the open convex set $\mathbb{D}$ and the level set $\{x : f(x) < f(x_0)\}$ is contained in the set $\mathbb{D}$, then the gradient method with step lengths $\frac{1}{L}$ has convergence rate of*

$$\min_{0 \leq k \leq N} \left\|\nabla f\left(x^k\right)\right\| \leq \left(\frac{4\left(\sqrt{2}+1\right)L\left(f\left(x^0\right) - f^\star\right)}{3N+2}\right)^{1/2}.$$

Proof. By the descent lemma 2.22, the points generated by the gradient descent method for step length $1/L$ lies on the level set $\{x : f(x) < f(x_0)\}$. By Corollary 4.7, f can be extended to a $\left(\sqrt{2}+1\right)L$-smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. Using Theorem 4.3 the desired results now follow. □

We end this chapter by some concluding remarks and a conjecture.

4.5 Concluding remarks

In this chapter, we studied the convergence rate of the gradient method for L-smooth functions and we provided a new convergence rate when the step lengths belong to the interval $(0, \frac{\sqrt{3}}{L})$. Moreover, we have shown that this convergence rate is tight for step lengths in the interval $(0, \frac{1}{L}]$.

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.

Carl Friedrich Gauss

5

Conditions for linear convergence of the gradient method for non-convex optimization

Preamble

In the previous chapter, we studied convergence rate of the gradient method for L-smooth functions and derived sub-linear convergence rate for this class of functions. In this chapter, we derive a new linear convergence rate for the gradient method with fixed step lengths for non-convex smooth optimization problems satisfying the Polyak-Łojasiewicz (PŁ) inequality. We establish that the PŁ inequality is a necessary and sufficient condition for linear convergence to the optimal value for this class of problems. We list some related classes of functions for which the gradient method may enjoy linear convergence rate. Moreover, we investigate their relationship with the PŁ inequality. This chapter is based on the paper [AdKZ23a].

5.1 Introduction

We consider the gradient method for the unconstrained optimization problem

$$f^\star := \inf_{x\in\mathbb{R}^n} f(x), \tag{5.1}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable, and $f^\star$ is finite. The gradient method with fixed step lengths may be described as follows, which is the same as Algorithm 4.1.

Algorithm 5.1 Gradient method with fixed step lengths

Set N and $\{t_k\}_{k=1}^N$ (step lengths) and pick $x^0\in\mathbb{R}^n$.
For $k=1,2,\ldots,N$ perform the following step:
1. $x^k = x^{k-1} - t_k\nabla f(x^{k-1})$

In addition, we assume that f has a maximum curvature $L\in(0,\infty)$ and a minimum curvature $\mu\in(-\infty,L)$. Recall that f has a *maximum curvature* L if $\frac{L}{2}\|.\|^2 - f$ is convex. Similarly, f has a *minimum curvature* μ if $f-\frac{\mu}{2}\|.\|^2$ is convex. We denote smooth functions with curvature belonging to the interval $[\mu,L]$ by $\mathcal{H}_{\mu,L}(\mathbb{R}^n)$. The class $\mathcal{H}_{\mu,L}(\mathbb{R}^n)$ includes all smooth functions with Lipschitz gradient (note that $\mu\geq 0$ corresponds to convexity). Indeed, f is L-smooth on $\mathbb{R}^n$ if and only if f has a maximum and minimum curvature $\bar{L}>0$ and $\bar{\mu}$, respectively, with $\max(\bar{L},|\bar{\mu}|)\leq L$; recall the discussion before Theorem 2.42. This class of functions is broad and appears naturally in many models in machine learning, see [DD19] and the references therein.

For $f\in\mathcal{H}_{\mu,L}(\mathbb{R}^n)$, we have the following inequalities for $x,y\in\mathbb{R}^n$

$$f(y)\leq f(x)+\langle\nabla f(x), y-x\rangle + \tfrac{L}{2}\|y-x\|^2, \tag{5.2}$$

$$f(y)\geq f(x)+\langle\nabla f(x), y-x\rangle + \tfrac{\mu}{2}\|y-x\|^2; \tag{5.3}$$

see Lemma 2.5 in [RGP22], where (5.2) and (5.3) are similar to Lemma 2.22 and Definition 2.25, respectively.

It is known that the number of iterations of first-order methods, which is needed to be performed for obtaining an ϵ-stationary[1] point, is of the order $\Omega\left(\epsilon^{-2}\right)$ for L-smooth functions [CDHS20]. Hence, it is of interest to investigate the classes of functions for which the gradient method enjoys linear convergence rate. This subject has been investigated by some scholars and some classes of functions

[1] A point x is called ϵ-stationary if $\|\nabla f(x)\|\leq\epsilon$.

have been introduced where linear convergence is possible; see [HSS20, KNS16, HSL21, DDG⁺22] and the references therein. This includes the class of functions satisfying the Polyak-Łojasiewicz (PŁ) inequality [KNS16, Pol63].

Definition 5.1. A function f is said to satisfy the *PŁ inequality* on $\mathbb{X} \subseteq \mathbb{R}^n$, where $x^\star \in \mathbb{X}$ is a minimizer of f over $\mathbb{R}^n$, if there exists $\mu_p > 0$ such that

$$f(x) - f^\star \leq \tfrac{1}{2\mu_p} \|\nabla f(x)\|^2, \quad \forall x \in \mathbb{X}. \tag{5.4}$$

Note that the PŁ inequality is also known as *gradient dominated*; see [Nes18, Definition 4.1.3]. Strongly convex functions satisfy the PŁ inequality, but some classes of non-convex functions also fulfill this inequality. For instance, the following proposition provides an example, which is a slightly more general case than the example presented by [Nes18, Example 4.1.3].

Proposition 5.2. *Consider a differentiable function $G : \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$ and let $J_G(x)$ be the Jacobian matrix of G at x. If*

$$\inf_{x \in \mathbb{X}} \lambda_{\min} \left(J_G(x) J_G(x)^T \right) = \alpha > 0,$$

for some $\mathbb{X} \subseteq \mathbb{R}^n$, then the function $f(x) = \|G(x)\|^2$ fulfils the PŁ inequality (5.4) *with constant $\mu_p = 2\alpha$.*

Proof. Let $f(x) = \sum_{j=1}^m G_j^2(x)$ and $\left(J_G^T(x)\right)_{ij} = \frac{\partial}{\partial x_i} G_j(x)$ where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Moreover,

$$\nabla f(x) = 2 J_G^T(x) G(x) \in \mathbb{R}^n.$$

Assume that $J_G(x) J_G^T(x) \succeq \alpha I_{m \times m}$ for all $x \in \mathbb{X}$ and some $\alpha > 0$. Then,

$$\frac{1}{4} \|\nabla f(x)\|^2 = G^T(x) J_G(x) J_G^T(x) G(x) \geq \alpha \|J_G(x)\|^2 = \alpha f(x).$$

Thus,

$$f(x) - f^\star \leq \frac{1}{4\alpha} \|\nabla f(x)\|^2 - f^\star \leq \frac{1}{4\alpha} \|\nabla f(x)\|^2,$$

where the last inequality is due to $f^\star \geq 0$. □

In other words, nonlinear least squares problems sometimes correspond to instances of (5.1) where the objective satisfies the PŁ inequality.

The following classical theorem provides a linear convergence rate for Algorithm 5.1 under the PŁ inequality.

Theorem 5.3. *[Pol63, Theorem 4] Let f be L-smooth and let f satisfy PŁ inequality on $\mathbb{X} = \{x : f(x) \leq f(x^0)\}$. If $t_1 \in (0, \frac{2}{L})$ and x^1 is generated by Algorithm 5.1, then*

$$f(x^1) - f^\star \leq \left(1 - t_1\mu_p(2 - t_1 L)\right)\left(f(x^0) - f^\star\right). \tag{5.5}$$

In particular, if $t_1 = \frac{1}{L}$, we have

$$f(x^1) - f^\star \leq (1 - \tfrac{\mu_p}{L})\left(f(x^0) - f^\star\right). \tag{5.6}$$

In this chapter we will sharpen this bound; see Theorem 5.4. Under the assumptions of Theorem 5.3, Karimi et al. [KNS16] established linear convergence rates for some other methods including the randomized coordinate descent. We refer the interested reader to the recent survey [DDG$^+$22] for more details on the convergence of algorithms under the PŁ inequality.

In this chapter, we study the convergence rate of Algorithm 5.1 by using performance estimation; see Chapter 3.

The rest of the chapter is organized as follows. In Section 5.2, we consider problem (5.1) when f satisfies the PŁ inequality. We derive a new linear convergence rate for Algorithm 5.1 by using performance estimation. Furthermore, we provide an optimal step length with respect to the given bound. We also show that the PŁ inequality is necessary and sufficient for linear convergence, in a well-defined sense. Section 5.3 lists some other situations where Algorithm 5.1 is linearly convergent. Moreover, we study the relationships between these situations. Finally, we conclude the chapter with some remarks and questions for future research.

5.2 Linear convergence under the PŁ inequality

In this section we study linear convergence of the gradient descent for $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ under the PŁ inequality. It is readily seen that the PŁ inequality implies that every stationary point is a global minimum on X. By virtue of the descent lemma [Nes18, Page 29] and Lemma 2.22, we have

$$f(x) - f^\star \geq \tfrac{1}{2L}\|\nabla f(x)\|^2, \ \forall x \in \mathbb{R}^n. \tag{5.7}$$

Hence, μ_p can only take values in $(0, L]$ by (5.7). On the other hand, if f in $\mathcal{H}_{\mu,L}(\mathbb{R}^n)$ we may assume without loss of generality $\mu \leq \mu_p$ where μ is the minimum curvature parameter. The inequality is trivial if $\mu \leq 0$, and we therefore assume that $\mu > 0$. By taking the minimum with respect to y from both side of inequality (5.3), we get

$$f(x) - f^\star \leq \tfrac{1}{2\mu}\|\nabla f(x)\|^2.$$

Hence, one may assume without loss of generality $\mu_p = \max\{\mu, \mu_p\}$ in inequality (5.4).

In what follows, we employ performance estimation to get a new bound under the assumptions of Theorem 5.3. In this setting, the worst-case convergence rate of Algorithm 5.1 may be cast as the following optimization problem,

$$\begin{aligned}
\max\ & \frac{f(x^1) - f^\star}{f(x^0) - f^\star} \\
& x^1 \text{ is generated by Algorithm 5.1 w.r.t. } f, x^0 \\
& f(x) \geq f^\star\ \forall x \in \mathbb{R}^n \\
& f(x) - f^\star \leq \tfrac{1}{2\mu_p}\|\nabla f(x)\|^2, \quad \forall x \in X \\
& f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n) \\
& x^0 \in \mathbb{R}^n.
\end{aligned} \tag{5.8}$$

In problem (5.8), f and x^0 are decision variables and $X = \{x : f(x) \leq f(x^0)\}$. We may replace the infinite dimensional condition $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ by a finite set of constraints, by using interpolation. Theorem 2.42 gives some necessary and sufficient conditions for the interpolation of given data by some $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$.

It is worth noting that Theorem 2.42 addresses non-smooth functions as well. In fact, $L = \infty$ covers non-smooth functions. Note that we only investigate the smooth case in this chapter, that is $L \in (0, \infty)$ and $\mu \in (-\infty, 0]$.

By Theorem 2.42, problem (5.8) may be relaxed as follows,

$$\begin{aligned}
\max\ & \frac{f^1 - f^\star}{f^0 - f^\star} \\
\text{s.t.}\ & \tfrac{1}{2(1-\frac{\mu}{L})}\left(\tfrac{1}{L}\left\|g^i - g^j\right\|^2 + \mu\left\|x^i - x^j\right\|^2 - \tfrac{2\mu}{L}\left\langle g^j - g^i, x^j - x^i\right\rangle\right) \leq \\
& f^i - f^j - \left\langle g^j, x^i - x^j\right\rangle \ \ i, j \in \{0, 1\} \\
& x^1 = x^0 - t_1 g^0 \\
& f^k \geq f^\star \ \ k \in \{0, 1\} \\
& f^k - f^\star \leq \tfrac{1}{2\mu_p}\|g^k\|^2, \ \ k \in \{0, 1\}.
\end{aligned} \tag{5.9}$$

As we replace the constraint $f(x) - f^\star \leq \frac{1}{2\mu_p}\|\nabla f(x)\|^2$ for each $x \in X$ by $f^0 - f^\star \leq \frac{1}{2\mu_p}\|g^0\|^2$ and $f^1 - f^\star \leq \frac{1}{2\mu_p}\|g^1\|^2$, problem (5.9) is a relaxation of problem (5.8). By using the constraint $x^1 = x^0 - t_1 g^0$, problem (5.9) may be reformulated as,

$$\begin{aligned}
&\max \frac{f^1 - f^\star}{f^0 - f^\star} \\
&\text{s. t. } \frac{1}{2(L-\mu)}\left(\|g^1\|^2 + (1 + \mu L t_1^2 - 2\mu t_1)\|g^0\|^2 + 2(\mu t_1 - 1)\langle g^0, g^1\rangle\right) - \\
&\quad f^1 + f^0 - \left\langle g^0, t_1 g^0 \right\rangle \leq 0 \\
&\quad \frac{1}{2(L-\mu)}\left(\|g^1\|^2 + (1 + \mu L t_1^2 - 2\mu t_1)\|g^0\|^2 + 2(\mu t_1 - 1)\langle g^0, g^1\rangle\right) - \\
&\quad f^0 + f^1 + \left\langle g^1, t_1 g^0 \right\rangle \leq 0 \\
&\quad f^\star - f^k \leq 0 \quad k \in \{0,1\} \\
&\quad f^k - f^\star - \tfrac{1}{2\mu_p}\|g^k\|^2 \leq 0, \quad k \in \{0,1\}\,.
\end{aligned} \tag{5.10}$$

By using the Gram matrix,

$$X = \begin{pmatrix} (g^0)^T \\ (g^1)^T \end{pmatrix} \begin{pmatrix} g^0 & g^1 \end{pmatrix} = \begin{pmatrix} \|g^0\|^1 & \langle g^0, g^1\rangle \\ \langle g^0, g^1\rangle & \|g^1\|^2 \end{pmatrix},$$

problem (5.10) can be relaxed as follows,

$$\begin{aligned}
&\max \frac{f^1 - f^\star}{f^0 - f^\star} \\
&\text{s. t. } \operatorname{tr}(A_1 X) - f^1 + f^0 \leq 0 \\
&\quad \operatorname{tr}(A_2 X) - f^0 + f^1 \leq 0 \\
&\quad f^0 - f^\star + \operatorname{tr}(A_3 X) \leq 0 \\
&\quad f^1 - f^\star + \operatorname{tr}(A_4 X) \leq 0 \\
&\quad f^0, f^1 \geq f^\star, X \succeq 0,
\end{aligned} \tag{5.11}$$

where

$$A_1 = \begin{pmatrix} \frac{1+\mu L t_1^2 - 2\mu t_1}{2(L-\mu)} - t_1 & \frac{\mu t_1 - 1}{2(L-\mu)} \\ \frac{\mu t_1 - 1}{2(L-\mu)} & \frac{1}{2(L-\mu)} \end{pmatrix} \qquad A_2 = \begin{pmatrix} \frac{1+\mu L t_1^2 - 2\mu t_1}{2(L-\mu)} & \frac{\mu t_1 - 1}{2(L-\mu)} + \frac{t_1}{2} \\ \frac{\mu t_1 - 1}{2(L-\mu)} + \frac{t_1}{2} & \frac{1}{2(L-\mu)} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} \frac{-1}{\mu_p^2} & 0 \\ 0 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{\mu_p^2} \end{pmatrix}.$$

In addition, X, f^0, f^1 are decision variables in this formulation. In the next theorem, we obtain an upper bound for problem (5.10) by using weak duality. This bound gives a new convergence rate for Algorithm 5.1 for a wide variety of functions.

Theorem 5.4. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ with $L \in (0,\infty), \mu \in (-\infty,0]$ and let f satisfy the PŁ inequality on $X = \{x : f(x) \le f(x^0)\}$. Suppose that x^1 is generated by Algorithm 5.1.*

i) If $t_1 \in \left(0, \frac{1}{L}\right)$, then

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star} \le \left(\frac{\mu_p(1-Lt_1)+\sqrt{(L-\mu)\left(\mu-\mu_p\right)(2-Lt_1)\mu_p t_1+(L-\mu)^2}}{L-\mu+\mu_p}\right)^2.$$

ii) If $t_1 \in \left[\frac{1}{L}, \frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$, then

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star} \le \left(\frac{(Lt_1-2)(\mu t_1-2)\mu_p t_1}{\left(L+\mu-\mu_p\right)t_1-2}+1\right).$$

iii) If $t_1 \in \left(\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}, \frac{2}{L}\right)$, then

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star} \le \frac{(Lt_1-1)^2}{(Lt_1-1)^2+\mu_p t_1(2-Lt_1)}.$$

In particular, if $t_1 = \frac{1}{L}$ and $\mu = -L$, we have

$$f(x^1)-f^\star \le \left(\frac{2L-2\mu_p}{2L+\mu_p}\right)\left(f(x^0)-f^\star\right). \tag{5.12}$$

Proof. First we consider $t_1 \in \left(0, \frac{1}{L}\right)$. Let

$$b_1 = \frac{(L-\mu)\left(\alpha+\mu_p(1-Lt_1)\right)}{\alpha\left(L-\mu+\mu_p\right)}$$

$$b_2 = b_1 - \left(\frac{\alpha}{L-\mu}b_1\right)^2,$$

where

$$\alpha = \sqrt{(L-\mu)\left(\mu_p t_1\left(\mu_p-\mu\right)(Lt_1-2)+(L-\mu)\right)}.$$

It is readily seen that $b_1, b_2 \geq 0$. Furthermore,

$$f^1 - f^\star - (b_1-b_2)\left(f^0-f^\star\right) - b_2\left(-\frac{1}{2\mu_p}\left\|g^0\right\|^2 + f^0 - f^\star\right)$$
$$-(1-b_1)\left(-\frac{1}{2\mu_p}\left\|g^1\right\|^2 + f^1 - f^\star\right) - b_1\left(\frac{1}{2(L-\mu)}\left(\|g^1\|^2 +\right.\right.$$
$$\left.\left.(1+\mu L t_1^2 - 2\mu t_1)\|g^0\|^2 + 2(\mu t_1 - 1)\langle g^0, g^1\rangle\right) - f^0 + f^1 + \left\langle g^1, t_1 g^0\right\rangle\right) =$$
$$-\frac{1-Lt_1}{2\alpha}\left\|\frac{\alpha b_1}{L-\mu} g^0 - g^1\right\|^2 \leq 0.$$

Therefore, for any feasible solution of problem (5.10), we have

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star} \leq$$
$$\left(\frac{\mu_p(1-Lt_1)+\sqrt{(L-\mu)\left(\mu-\mu_p\right)(2-Lt_1)\mu_p t_1 + (L-\mu)^2}}{L-\mu+\mu_p}\right)^2,$$

and the proof of this part is complete. Now, we consider the case that $t_1 \in \left[\frac{1}{L}, \frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$. Suppose that

$$a_1 = \frac{\mu t_1 - 1}{\left(L+\mu-\mu_p\right)t_1 - 2}, \qquad a_2 = \frac{1-Lt_1}{\left(L+\mu-\mu_p\right)t_1 - 2},$$
$$a_3 = -\frac{((Lt_1-2)(\mu t_1-2)-1)\mu_p t_1}{\left(L+\mu-\mu_p\right)t_1 - 2}, \qquad a_4 = -\frac{\mu_p t_1}{\left(L+\mu-\mu_p\right)t_1 - 2}.$$

It is readily seen that $a_1, a_2, a_3, a_4 \geq 0$. Furthermore,

$$f^1 - f^\star - \left(1-a_3-a_4\right)\left(f^0-f^\star\right) - a_3\left(-\frac{1}{2\mu_p}\left\|g^0\right\|^2 + f^0 - f^\star\right) -$$
$$a_4\left(-\frac{1}{2\mu_p}\left\|g^1\right\|^2 + f^1 - f^\star\right) - a_1\left(\frac{1}{2(L-\mu)}\left(\|g^1\|^2 + (1+\mu L t_1^2 - 2\mu t_1)\|g^0\|^2 +\right.\right.$$
$$\left.\left.2(\mu t_1 - 1)\langle g^0, g^1\rangle\right) - f^0 + f^1 + \left\langle g^1, t_1 g^0\right\rangle\right) - a_2\left(\frac{1}{2(L-\mu)}\left(\|g^1\|^2 +\right.\right.$$
$$\left.\left.(1+\mu L t_1^2 - 2\mu t_1)\|g^0\|^2 + 2(\mu t_1 - 1)\langle g^0, g^1\rangle\right) - f^1 + f^0 - \left\langle g^0, t_1 g^0\right\rangle\right) = 0.$$

Therefore, for any feasible solution of problem (5.10), we have

$$f(x^1)-f^\star-\left(\frac{L\mu_p\mu t_1^3-2\mu_p\,(L+\mu)\,t_1^2+4\mu_p t_1}{\left(L+\mu-\mu_p\right)t_1-2}+1\right)\left(f(x^0)-f^\star\right)\leq 0.$$

Now, we prove the last part. Assume that $t_1\in\left(\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}},\frac{2}{L}\right)$. With some algebra, one can show

$$f^1-f^\star-\left(\frac{(Lt_1-1)^2}{\beta}\right)\left(f^0-f^\star\right)-\left(\frac{\mu_p t_1(2-Lt_1)}{\beta}\right)\left(-\frac{1}{2\mu_p}\left\|g^1\right\|^2+f^1-f^\star\right)-$$
$$\left(\frac{(Lt_1-1)(2-Lt_1)}{\beta}\right)\left(\frac{1}{2(L-\mu)}\left(\|g^1\|^2+(1+\mu Lt_1^2-2\mu t_1)\|g^0\|^2+2(\mu t_1-1)\langle g^0,g^1\rangle\right)-\right.$$
$$\left.f^1+f^0-\left\langle g^0,t_1g^0\right\rangle\right)-\left(\frac{Lt_1-1}{\beta}\right)\left(\frac{1}{2(L-\mu)}\left(\|g^1\|^2+(1+\mu Lt_1^2-2\mu t_1)\|g^0\|^2+\right.\right.$$
$$\left.\left.2(\mu t_1-1)\langle g^0,g^1\rangle\right)-f^0+f^1+\left\langle g^1,t_1g^0\right\rangle\right)=$$
$$-\frac{(1-Lt_1)\left(L\mu t^2-2(\mu+L)t+3\right)}{2\beta(L-\mu)}\left\|\sqrt{Lt_1-1}g^0+\frac{1}{\sqrt{Lt_1-1}}g^1\right\|^2\leq 0,$$

where

$$\beta=(Lt_1-1)^2+\mu_p t_1(2-Lt_1).$$

The rest of the proof is similar to that of the former cases. □

As the expressions provided in Theorem 5.4 are complicated here we provide an example for some parameters. Let $L=10$, $\mu=-L=-10$ and $\mu_p=1$. Then part i) of Theorem 5.4 is given by $t_1\in(0,0.1)$ and

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star}\leq\frac{1}{441}\left(1-10t_1+\sqrt{440t_1(5t_1-1)+400}\right)^2.$$

One may wonder how we obtain Lagrange multipliers (dual variables) in Theorem 5.4. The multipliers are computed by solving the dual of problem (5.11) by hand; see Chapter 3 for more discussion on solving PEP. Furthermore, Theorem 5.4 provides a tighter bound in comparison with the convergence rate given in Theorem 5.3 for L-smooth functions with $t_1\in(0,\frac{2}{L})$. To show this, we need investigate three subintervals:

i) Suppose that $t_1\in\left(0,\frac{1}{L}\right)$. As $1-Lt_1\leq 0$,

$$\left(\frac{\mu_p(1-Lt_1)+\sqrt{2L\left(-L-\mu_p\right)(2-Lt_1)\mu_p t_1+4L^2}}{2L+\mu_p}\right)^2\leq$$
$$\frac{4L^2+2L\mu_p t_1(L+\mu_p)(Lt_1-2)+(\mu_p-L\mu_p t_1)^2}{(2L+\mu_p)^2}\leq 1-t_1\mu_p(2-t_1L),$$

where the last inequality follows from non-positivity of the quadratic function $T_1(t_1) = -Lt_1^2\left(2L^2 + L\mu_p + \mu_p^2\right) + 2t_1\left(2L^2 + L\mu_p + \mu_p^2\right) - 4L$ on the given interval.

ii) Let $t_1 \in \left[\frac{1}{L}, \frac{\sqrt{3}}{L}\right]$. Since $\mu_p \leq L$ and $(2 - Lt_1) > 0$, we have

$$1 \leq \frac{Lt_1+2}{\mu_p t_1+2} \Rightarrow 1 - \frac{(2-Lt_1)(Lt_1+2)\mu_p t_1}{\mu_p t_1+2} \leq 1 - t_1\mu_p(2 - Lt_1).$$

iii) Assume that $t_1 \in (\frac{\sqrt{3}}{L}, \frac{2}{L})$. It is readily verified that the quadratic function $T_2(t_1) = (Lt_1 - 1)^2 + \mu_p t_1(2 - Lt_1) - 1$ is non-positive on the given interval. Hence,

$$\frac{(Lt_1-1)^2}{(Lt_1-1)^2+\mu_p t_1(2-Lt_1)} = 1 - \frac{\mu_p t_1(2-Lt_1)}{(Lt_1-1)^2+\mu_p t_1(2-Lt_1)} \leq 1 - t_1\mu_p(2 - Lt_1).$$

Therefore, for $t_1 \in \left(0, \frac{2}{L}\right)$ the bound provided by Theorem 5.4 is tighter than that given by Theorem 5.3.

In most problems, the smoothness constant, L, is unknown. By using (5.2), any estimation of the smoothness constant L, say $\tilde{L}$, should satisfy the following inequality,

$$f\left(x - \tfrac{1}{\tilde{L}}\nabla f(x)\right) \leq f(x) - \tfrac{1}{2\tilde{L}}\|\nabla f(x)\|^2.$$

Thus one may try to obtain a suitable estimate by searching for a sufficiently large value of $\tilde{L}$ that satisfies this inequality. This technique is due to Nesterov; see [Nes13, Section 3] for details.

The next proposition gives the optimal step length with respect to the worst-case convergence rate.

Proposition 5.5. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ with $L \in (0, \infty)$, $\mu \in (-\infty, 0]$ and let f satisfy the PŁ inequality on $X = \{x : f(x) \leq f(x^0)\}$. Suppose that $r(t) = L\mu(L + \mu - \mu_p)t^3 - \left(L^2 - \mu_p(L + \mu) + 5L\mu + \mu^2\right)t^2 + 4(L + \mu)t - 4$ and $\bar{t}$ is the unique root of r in $\left[\frac{1}{L}, \frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$ if it exists. Then $t^\star$ given by*

$$t^\star = \begin{cases} \bar{t} & \text{if } \bar{t} \text{ exists} \\ \frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}} & \text{otherwise,} \end{cases}$$

is the optimal step length for Algorithm 5.1 with respect to the worst-case convergence rate.

Proof. To obtain an optimal step length, we need to solve the optimization problem

$$\min_{t\in(0,\frac{2}{L})} h(t),$$

where h is given by

$$h(t)=\begin{cases}\left(\frac{\mu_p(1-Lt)+\sqrt{(L-\mu)\left(\mu-\mu_p\right)(2-Lt)\mu_p t+(L-\mu)^2}}{L-\mu+\mu_p}\right)^2 & t\in\left(0,\frac{1}{L}\right)\\ \frac{(Lt-2)(\mu t-2)\mu_p t}{\left(L+\mu-\mu_p\right)t-2}+1 & t\in\left[\frac{1}{L},\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]\\ \frac{(Lt-1)^2}{(Lt-1)^2+(2-Lt)\mu_p t} & t\in\left(\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}},\frac{2}{L}\right).\end{cases}$$

It is easily seen that h is decreasing on $\left(0,\frac{1}{L}\right)$ and is increasing on $\left(\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}},\frac{2}{L}\right)$. Hence, we need investigate the closed interval $\left[\frac{1}{L},\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$. We will show that h is convex on the interval in question. First, we consider the case $L+\mu-\mu_p\leq 0$. Let $p(t)=\frac{\mu t-2}{\left(L+\mu-\mu_p\right)t-2}$ and $q(t)=(Lt-2)\mu_p t$. By some algebra, one can show the following inequalities for $t\in\left[\frac{1}{L},\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$:

$$\begin{aligned} &p(t)\geq 0 && q(t)\leq 0\\ &p'(t)\geq 0 && q'(t)\geq 0\\ &p''(t)\leq 0 && q''(t)\geq 0. \end{aligned}$$

Hence, the convexity of h follows from $h''=p''q+2p'q'+pq''$. Now, we investigate the case that $L+\mu-\mu_p>0$. Suppose that $p(t)=\frac{\mu_p t}{\left(L+\mu-\mu_p\right)t-2}$ and $q(t)=(Lt-2)(\mu t-2)$. For these functions, we have the following inequalities

$$\begin{aligned} &p(t)\leq 0 && q(t)\geq 0\\ &p'(t)\leq 0 && q'(t)\leq 0\\ &p''(t)\geq 0 && q''(t)\leq 0, \end{aligned}$$

which analogous to the former case one can infer the convexity of h on the given interval. Hence, if h has a root in $\left[\frac{1}{L},\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}\right]$, it will be the minimum. Otherwise, the point $t^\star=\frac{3}{\mu+L+\sqrt{\mu^2-L\mu+L^2}}$ will be the minimum. This follows from the point that $h'(\frac{1}{L})=\frac{2L\mu_p(\mu_p-L)}{(L+\mu_p-\mu)^2}\leq 0$ and the convexity of h on the interval in question. □

Thanks to Proposition 5.5, the following corollary gives the optimal step length for L-smooth convex functions satisfying the PŁ inequality.

Corollary 5.6. *If f is an L-smooth convex function satisfying the PŁ inequality, then the optimal step length with respect to the worst-case convergence rate given by Theorem 5.4 is* $\min\left\{\frac{2}{L+\sqrt{L\mu_p}}, \frac{3}{2L}\right\}$.

The constant $\frac{2}{L+\sqrt{L\mu_p}}$ also appears in the the *fast gradient algorithm* introduced in [NNG19] for L-smooth convex functions which are $(1,\mu_s)$-quasar-convex, see Definition 5.14. By Theorem 5.16, $(1,\mu_s)$-quasar-convexity implies the PŁ inequality with the same constant. Algorithm 5.2 describes the method in question.

Algorithm 5.2 Fast gradient method

Pick $x^0 \in \mathbb{R}^n$, set N and $y^0 = x^0$.
For $k = 1, 2, \ldots, N$ perform the following step:
1. $y^k = x^{k-1} - \frac{1}{L}\nabla f(x^{k-1})$
2. $x^k = y^k + \frac{\sqrt{L}-\sqrt{\mu_p}}{\sqrt{L}+\sqrt{\mu_p}}\left(y^k - y^{k-1}\right)$

One can verify that Algorithm 5.2, at the first iteration, generates $x^1 = x^0 - \frac{2}{L+\sqrt{L\mu_p}}\nabla f(x^0)$.

A more general form of the PŁ inequality, called the *Łojasiewicz inequality*, may be written as

$$(f(x) - f^\star)^{2\theta} \leq \frac{1}{2\mu_p}\|\nabla f(x)\|^2, \quad \forall x \in X, \tag{5.13}$$

where $\theta \in (0,1)$. As an example for a function with $\theta > \frac{1}{2}$ one can easily see that the function $f(x) = x^4$ has $\theta = \frac{3}{4}$ with $\mu_p = 8$. It is known that when $\theta \in (0, \frac{1}{2}]$ some algorithms, including Algorithm 5.1, are linearly convergent; see [AB09, ABRS10]. In the next theorem, we show that for functions with finite maximum and minimum curvature the Łojasiewicz inequality cannot hold for $\theta \in (0, \frac{1}{2})$.

Theorem 5.7. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ be a non-constant function. If f satisfies the Łojasiewicz inequality on $X = \{x : f(x) \leq f(x^0)\}$, then $\theta \geq \frac{1}{2}$.*

Proof. To the contrary, assume that $\theta \in (0, \frac{1}{2})$. Without loss of generality, we may assume that $\mu = -L$. It is known that Algorithm 5.1 generates a decreasing sequence $\{f(x^k)\}$ and it is convergent, that is $\|\nabla f(x^k)\| \to 0$; see [Nes18, page 28]. Furthermore, (5.13) implies that $f(x^k) \to f^\star$. Without loss of generality, we may assume that $f^\star = 0$. First, we investigate the case that $f(x^0) = 1$. The semidefinite programming problem corresponding to performance estimation in this case may be formulated as follows,

$$\begin{aligned} \max\ & f^1 \\ \text{s.t. } & \operatorname{tr}(A_1X) - f^1 + 1 \leq 0 \\ & \operatorname{tr}(A_2X) - 1 + f^1 \leq 0 \\ & 1 + \operatorname{tr}(A_3X) \leq 0 \\ & (f^1)^{2\theta} + \operatorname{tr}(A_4X) \leq 0 \\ & f^1 \geq 0, X \succeq 0. \end{aligned} \tag{5.14}$$

Since Algorithm 5.1 is a monotone method, f^1 can take value in $[0,1]$. In addition, we have $f^1 \leq (f^1)^{2\theta}$ on this interval. Hence, by using Theorem 5.4, we get the following bound,

$$f^1 \leq \frac{2L - 2\mu_p}{2L + \mu_p}.$$

Now, suppose that $f(x^0) = f^0 > 0$. Consider the function $h : \mathbb{R}^n \to \mathbb{R}$ given by $h(x) = \frac{f(x)}{f^0}$. It is seen that h is $\frac{L}{f^0}$-smooth and

$$h(x)^{2\theta} \leq \tfrac{1}{2\mu_p (f^0)^{2\theta-2}} \|\nabla h(x)\|^2, \quad \forall x \in X.$$

As Algorithm 5.1 generates the same x^1 for both functions, by using the first part, we obtain

$$\frac{f(x^1)}{f(x^0)} \leq \frac{2L(f^0)^{-1} - 2\mu_p(f^0)^{2\theta-2}}{2L(f^0)^{-1} + \mu_p(f^0)^{2\theta-2}} = \frac{2L - 2\mu_p(f^0)^{2\theta-1}}{2L + \mu_p(f^0)^{2\theta-1}}.$$

For f^0 sufficiently small, we have $\frac{2L-2\mu_p(f^0)^{2\theta-1}}{2L+\mu_p(f^0)^{2\theta-1}} < 0$, which contradicts $f^\star \geq 0$ and the proof is complete. □

Necoara et al. gave necessary and sufficient conditions for linear convergence of the gradient method with constant step lengths when f is a smooth convex function; see [NNG19, Theorem 13]. Indeed, the theorem says that Algorithm 5.1 is linearly convergent if and only if f has a quadratic functional growth on $\{x : f(x) \leq f(x^0)\}$; see Definition 5.11. However, this theorem does not hold necessarily for non-convex functions. The next theorem provides necessary and sufficient conditions for linear convergence of Algorithm 5.1.

Theorem 5.8. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$. Algorithm 5.1 is linearly convergent to the optimal value if and only if f satisfies PŁ inequality on $\{x : f(x) \leq f(x^0)\}$.*

Proof. Let $\bar{x} \in \{x : f(x) \leq f(x^0)\}$. Linear convergence implies the existence of $\gamma \in [0,1)$ with

$$f(\hat{x}) - f^\star \leq \gamma(f(\bar{x}) - f^\star), \tag{5.15}$$

where $\hat{x} = \bar{x} - \frac{1}{L}\nabla f(\bar{x})$. By (5.3), we have $f(\bar{x}) - f(\hat{x}) \leq \frac{2L-\mu}{2L^2}\|\nabla f(\bar{x})\|^2$. By using this inequality with (5.15), we get

$$f(\bar{x}) - f^\star \leq \frac{1}{1-\gamma}(f(\bar{x}) - f(\hat{x})) \leq \frac{2L-\mu}{2L^2(1-\gamma)}\|\nabla f(\bar{x})\|^2,$$

which shows that f satisfies PŁ inequality on $\{x : f(x) \leq f(x^0)\}$. The other implication follows from Theorem 5.4. □

5.3 The PŁ inequality: relation to some classes of functions

In this section, we study some classes of functions for which Algorithm 5.1 may be linearly convergent. We establish that these classes of functions satisfy the PŁ inequality under mild assumptions, and we infer the linear convergence by using Theorem 5.4. Moreover, one can get convergence rates by applying performance estimation.

Throughout the section, we denote the optimal solution set of problem (5.1) by $X^\star$ and we assume that $X^\star$ is non-empty. We denote the distance function to $X^\star$ by $d_{X^\star}(x) := \inf_{y \in X^\star} \|y - x\|$. The set-valued mapping $\Pi_{X^\star}(x)$ stands for the projection of x on $X^\star$, that is, $\Pi_{X^\star}(x) = \{y : \|y - x\| = d_{X^\star}(x)\}$. Note that, as $X^\star$ is non-empty closed set, $\Pi_{X^\star}(x)$ exists and is well-defined.

Definition 5.9. Let $\mu_g > 0$. A function f has a *quadratic gradient growth* on $X \subseteq \mathbb{R}^n$ if

$$\langle \nabla f(x), x - x^\star \rangle \geq \mu_g d_{X^\star}^2(x), \quad \forall x \in X, \tag{5.16}$$

for some $x^\star \in \Pi_{X^\star}(x)$.

Note that inequality (5.2) implies that $\mu_g \leq L$. Hu et al. [HSL21] investigated the convergence rate $\{x^k\}$ when f satisfies (5.16) and $X^\star$ is singleton. To the best knowledge of me, there is no convergence rate result in terms of $\{f(x^k)\}$ for functions with a quadratic gradient growth. The next proposition states that quadratic gradient growth property implies the PŁ inequality.

Proposition 5.10. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$. If f has a quadratic gradient growth on $X \subseteq \mathbb{R}^n$ with $\mu_g > 0$, then f satisfies the PŁ inequality with $\mu_p = \frac{\mu_g^2}{L}$.*

Proof. Suppose that $x^\star \in \Pi_{X^\star}(x)$ satisfies (5.16). By the Cauchy-Schwarz inequality, we have

$$\mu_g \|x - x^\star\| \leq \|\nabla f(x)\|. \tag{5.17}$$

On the other hand, (5.2) implies that

$$f(x) \leq f(x^\star) + \tfrac{L}{2}\|x - x^\star\|^2. \tag{5.18}$$

The PŁ inequality follows from (5.17) and (5.18). □

By Proposition 5.10 and Theorem 5.4, one can infer the linear convergence of Algorithm 5.1 when f has a quadratic gradient growth on $X = \{x : f(x) \leq f(x^0)\}$. Indeed, one can derive the following bound if $t_1 = \frac{1}{L}$ and $\mu = -L$,

$$f(x^1) - f^\star \leq \left(\frac{2L^2 - 2\mu_g^2}{2L^2 + \mu_g^2}\right)\left(f(x^0) - f^\star\right). \tag{5.19}$$

Nevertheless, by using the performance estimation method, one can derive a better bound than the bound given by (5.19). The performance estimation problem for $t_1 = \frac{1}{L}$ in this case may be formulated as

$$\begin{aligned}
&\max \ \frac{f^1 - f^\star}{f^0 - f^\star} \\
&\text{s.t. } \{x^k, g^k, f^k\} \cup \{y^k, 0, f^\star\} \text{ satisfy interpolation constraints (2.8) for } k \in \{0,1\} \\
&\qquad x^1 = x^0 - \tfrac{1}{L} g^0 \\
&\qquad f^k \geq f^\star \quad k \in \{0,1\} \\
&\qquad \langle g^k, x^k - y^k\rangle \geq \mu_g \|y^k - x^k\|^2, \quad k \in \{0,1\} \\
&\qquad \|x^0 - y^0\|^2 \leq \|x^0 - y^1\|^2 \\
&\qquad \|x^1 - y^1\|^2 \leq \|x^1 - y^0\|^2.
\end{aligned} \tag{5.20}$$

Analogous to Section 5.2, one can obtain an upper bound for problem (5.20) by solving a semidefinite program. Our numerical results show that the bounds

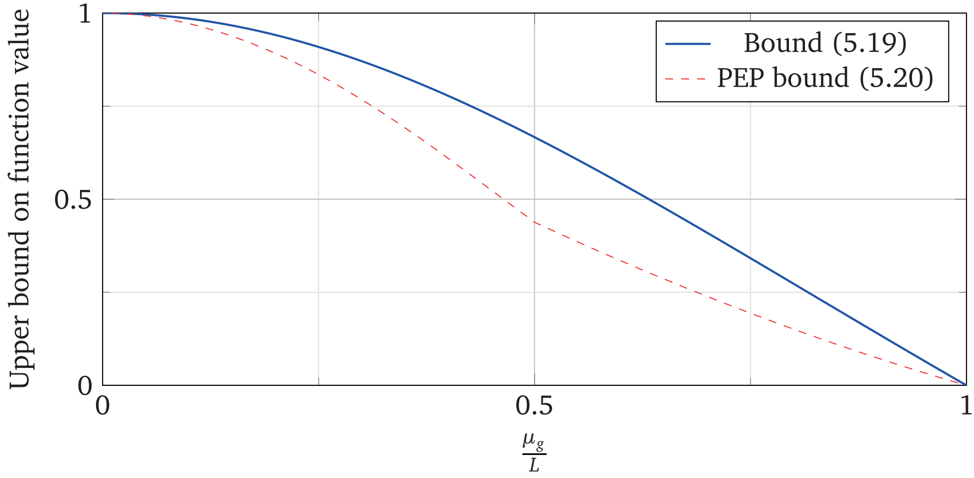


Figure 5.1: Convergence rate computed by performance estimation (red line) and the bound given by (5.19) (blue line) for $\frac{\mu_g}{L} \in (0,1)$.

generated by performance estimation are tighter than bound (5.19); see Figure 5.1. We do not have a closed-form bound on the optimal value of (5.20), though.

Definition 5.11. [NNG19, Definition 4], [Nes18, Definition 4.1.2] Let $\mu_q > 0$. A function f has a *quadratic functional growth* on $X \subseteq \mathbb{R}^n$ if

$$\frac{\mu_q}{2} d^2_{X^\star}(x) \le f(x) - f^\star, \quad \forall x \in X. \tag{5.21}$$

It is readily seen that, contrary to the previous situations, the quadratic functional growth property does not necessarily imply that each stationary point is a global optimal solution. The next theorem investigates the relationship between quadratic functional growth property and other notions.

Theorem 5.12. *Let $f \in \mathcal{H}_{\mu,L}(\mathbb{R}^n)$ and let $X = \{x : f(x) \le f(x^0)\}$. We have the following implications:*

i) (5.4) $\Rightarrow$ (5.21) *with* $\mu_q = \mu_p$.

ii) *If* $\mu_q > \frac{-\mu L}{L-\mu}$, *then* (5.21) $\Rightarrow$ (5.16) *with* $\mu_g = \frac{\mu_q}{2}(1 - \frac{\mu}{L}) + \frac{\mu}{2}$.

iii) *If*

$$f(x) - f(x^\star) \le \langle \nabla f(x), x - x^\star \rangle, \ \forall x \in X,$$

for some $x^\star \in \Pi_{X^\star}(x)$ *then* (5.21) $\Rightarrow$ (5.16) *with* $\mu_g = \frac{\mu_q}{2}$.

Proof. One can establish $i)$ similarly to the proof of [KNS16, Theorem 2]. Consider part $ii)$. Let $x \in X$ and $x^\star \in \Pi_{X^\star}(x)$ with $d_{X^\star}(x) = \|x - x^\star\|$. By (2.7), we have

$$f(x) - f(x^\star) \leq \tfrac{-1}{2(L-\mu)}\|\nabla f(x)\|^2 - \tfrac{\mu L}{2(L-\mu)}\|x - x^\star\|^2 + \tfrac{L}{L-\mu}\langle \nabla f(x), x - x^\star\rangle.$$

As $\frac{\mu_q}{2}\|x - x^\star\|^2 \leq f(x) - f(x^\star)$, we get

$$\left(\tfrac{\mu_q}{2}(1 - \tfrac{\mu}{L}) + \tfrac{\mu}{2}\right)\|x - x^\star\|^2 \leq \langle \nabla f(x), x - x^\star\rangle,$$

which establishes the desired inequality. Part $iii)$ is proved similarly to the former case. □

By Theorem 5.4, it is clear that Algorithm 5.1 enjoys linear convergence rate if f has a quadratic gradient growth on $X = \{x : f(x) \leq f(x^0)\}$ and if f satisfies assumptions $ii)$ or $iii)$ in Theorem 5.12. For instance, if $\mu = -L$ and $\mu_q \in (\frac{L}{2}, L)$, one can derive the following convergence rate for Algorithm 5.1 for fixed step length $t_k = \frac{1}{L}$, $k \in \{1, ..., N\}$,

$$f(x^N) - f(x^0) \leq \left(\frac{2L^2 - 2(\mu_q - \frac{L}{2})^2}{2L^2 + (\mu_q - \frac{L}{2})^2}\right)^N \left(f(x^0) - f^\star\right). \tag{5.22}$$

It is interesting to compare the convergence rate (5.22) to the convergence rate obtained by using the performance estimation framework. In this case, the performance estimation problem may be cast as follows,

$$\begin{aligned}
&\max \frac{f^N - f^\star}{f^0 - f^\star}\\
&\text{s.t. } \{x^k, g^k, f^k\} \cup \{y^k, 0, f^\star\} \text{ satisfy inequality (2.8) for } k \in \{0, ..., N\}\\
&\qquad x^k = x^{k-1} - \tfrac{1}{L} g^{k-1}, \quad k \in \{1, ..., N\}\\
&\qquad f^k \geq f^\star \quad k \in \{0, ..., N\}\\
&\qquad f^k - f^\star \geq \tfrac{\mu_q}{2}\|x^k - y^k\|^2, \quad k \in \{0, ..., N\}\\
&\qquad \|x^k - y^k\|^2 \leq \|x^k - y^{k'}\|^2, \quad k \in \{0, ..., N\}, k' \in \{0, ..., N\}.
\end{aligned} \tag{5.23}$$

Since $x^k = x^{k-1} - \frac{1}{L} g^{k-1}$, we get $x^k = x^0 - \frac{1}{L}\sum_{l=0}^{k-1} g^l$. Hence, problem (5.23) may be reformulated as follows,

$$
\begin{aligned}
&\max \frac{f^N - f^\star}{f^0 - f^\star}\\
&\text{s. t. } \{x^0 - \tfrac{1}{L}\sum_{l=0}^{k-2} g^l, g^k, f^k\} \cup \{y^k, 0, f^\star\} \text{ satisfy interpolation constraints (2.8)}\\
&\quad f^k \geq f^\star \quad k \in \{1, \ldots, N\} \qquad (5.24)\\
&\quad f^k - f^\star \geq \tfrac{\mu_q}{2}\|x^0 - \tfrac{1}{L}\sum_{l=1}^{k-2} g^l - y^k\|^2, \quad k \in \{0, \ldots, N\}\\
&\quad \|x^0 - \tfrac{1}{L}\sum_{l=0}^{k-2} g^l - y^k\|^2 \leq \|x^0 - \tfrac{1}{L}\sum_{l=0}^{k-2} g^l - y^{k'}\|^2, \quad k, k' \in \{0, \ldots, N\}.
\end{aligned}
$$

The next theorem provides an upper bound for problem (5.24) by using weak duality.

Theorem 5.13. *Let $f \in \mathcal{H}_{-L,L}(\mathbb{R}^n)$ and let f have a quadratic functional growth on $X = \{x : f(x) \leq f(x^0)\}$ with $\mu_q \in (\frac{L}{2}, L)$. If $t_k = \frac{1}{L}$, $k \in \{1, \ldots, N\}$, then we have the following convergence rate for Algorithm 5.1,*

$$
f(x^N) - f^\star \leq \frac{L}{\mu_q}\left(2 - \frac{2\mu_q}{L}\right)^N \left(f(x^0) - f^\star\right). \qquad (5.25)
$$

Proof. The proof is analogous to that of Theorem 5.4. Without loss of generality, we may assume that $f^\star = 0$. By some algebra, one can show that

$$f^N - f^\star - \tfrac{L}{\mu_q}\left(2-\tfrac{2\mu_q}{L}\right)^N\left(f^0-f^\star\right)+\sum_{j=0}^{N}\left(2^{N-j}\left(1-\tfrac{\mu_q}{L}\right)^{N-1}\right)\times$$
$$\left(f^\star - f^j - \left\langle g^j, y^0 - x^0 + \tfrac{1}{L}\sum_{l=0}^{j-1} g^l\right\rangle - \tfrac{1}{2L}\left\|g^j\right\|^2 + \tfrac{L}{4}\left\|y^0 - x^0 + \tfrac{1}{L}\sum_{l=0}^{j-1} g^l + \tfrac{1}{L}g^j\right\|^2\right)+$$
$$\sum_{i=1}^{N-1}\sum_{j=i}^{N}\left(2^{N-j}\left(\tfrac{\mu_q}{L}\right)\left(1-\tfrac{\mu_q}{L}\right)^{N-i-1}\right)\left(f^\star - f^j - \left\langle g^j, y^i - x^0 + \tfrac{1}{L}\sum_{l=0}^{j-1} g^l\right\rangle - \right.$$
$$\left.\tfrac{1}{2L}\left\|g^j\right\|^2 + \tfrac{L}{4}\left\|y^i - x^0 + \tfrac{1}{L}\sum_{l=0}^{j-1} g^l + \tfrac{1}{L}g^j\right\|^2\right)+\sum_{j=1}^{N-1}\left(2^{N-j}\left(1-\tfrac{\mu_q}{L}\right)^{N-j-1}\right)\times$$
$$\left(f^j - f^\star - \tfrac{\mu_q}{2}\left\|y^j - x^0 + \tfrac{1}{L}\sum_{l=0}^{j-1} g^l\right\|^2\right)+\left(2^N\left(1-\tfrac{\mu_q}{L}\right)^{N-1}+\tfrac{L}{\mu_q}\left(2-\tfrac{2\mu_q}{L}\right)^N\right)\times$$
$$\left(f^0 - f^\star - \tfrac{\mu_q}{2}\left\|y^0-x^0\right\|^2\right) = -\left(\tfrac{L}{4}(1-\tfrac{\mu_q}{L})^{N-1}\left\|y^0 - x^0 + \tfrac{1}{L}\sum_{l=0}^{N} g^l\right\|^2\right)-$$
$$\sum_{i=1}^{N-1}\left(\tfrac{\mu_q}{4}\left(1-\tfrac{\mu_q}{L}\right)^{N-i-1}\left\|y^i - x^0 + \tfrac{1}{L}\sum_{l=0}^{N} g^l\right\|^2\right) \leq 0.$$

By using the above inequality, we get

$$f^N - f^\star \leq \tfrac{L}{\mu_q}\left(2-\tfrac{2\mu_q}{L}\right)^N\left(f^0 - f^\star\right),$$

for any feasible point of (5.24), and the proof is complete. □

By doing some calculus, one can verify the following inequality

$$\frac{2L^2 - 2\left(\mu_q - \frac{L}{2}\right)^2}{2L^2 + \left(\mu_q - \frac{L}{2}\right)^2} \geq \left(2-\tfrac{2\mu_q}{L}\right), \quad \mu_q \in \left(\tfrac{L}{2}, L\right).$$

Hence, Theorem 5.13 provides a tighter bound than (5.22).

Definition 5.14. [HSS20, Definition 1] Let $\gamma \in (0,1]$ and $\mu_s \geq 0$. A function f is called (γ, μ_s)-*quasar-convex* on $X \subseteq \mathbb{R}^n$ with respect to $x^\star \in \operatorname{argmin}_{x\in\mathbb{R}^n} f(x)$ if

$$f(x) + \tfrac{1}{\gamma}\langle \nabla f(x), x^\star - x\rangle + \tfrac{\mu_s}{2}\|x^\star - x\|^2 \leq f^\star, \quad \forall x \in X. \tag{5.26}$$

The class of quasar-convex functions is large. For instance, non-negative homogeneous functions are $(1,0)$-quasar-convex on $\mathbb{R}^n$. (Recall that a function

$f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree k if $f(\alpha x) = \alpha^k f(x)$ for all $x \in \mathbb{R}^n, \alpha \in \mathbb{R}$.) Indeed, if f is non-negative homogeneous of degree $k \geq 1$, by the Euler identity, we have

$$f(x) + \langle \nabla f(x), x^\star - x \rangle = (1-k)f(x) \leq 0, \ \forall x \in \mathbb{R}^n,$$

where $x^\star = 0$. In what follows, we list some convergence results concerning quasar-convex functions for Algorithm 5.1.

Theorem 5.15. *[BM20, Remark 4.3] Let f be L-smooth and let f be (γ, μ_s)-quasar-convex on $X = \{x : f(x) \leq f(x^0)\}$. If $t_1 = \frac{1}{L}$ and if x^1 is from Algorithm 5.1, then*

$$f(x^1) - f^\star \leq \left(1 - \frac{\gamma^2 \mu_s}{L}\right)\left(f(x^0) - f^\star\right). \tag{5.27}$$

In the following theorem, we state the relationship between quasar-convexity and other concepts. Before we get to the theorem, we recall star convexity. A set X is called star convex at $x^\star$ if

$$\lambda x + (1-\lambda)x^\star \in X, \ \forall x \in X, \forall \lambda \in [0,1].$$

Theorem 5.16. *Let $x^\star$ be the unique solution of problem* (5.1) *and let $X = \{x : f(x) \leq f(x^0)\}$. If X is star convex at $x^\star$, then we have the following implications:*

i) (5.26) $\Rightarrow$ (5.16) *with* $\mu_g = \frac{\mu_s \gamma}{2} + \frac{\mu_s \gamma^2}{4}$.

ii) (5.16) $\Rightarrow$ (5.26) *with* $\mu_s = \ell - \frac{L}{2}$ *and* $\gamma = \frac{\mu_g}{\ell}$ *for each* $\ell \in (\max(\frac{L}{2}, \mu_g), \infty)$.

iii) (5.26) $\Rightarrow$ (5.4) *with* $\mu_p = \mu_s \gamma^2$.

Proof. The proof of i) is similar in spirit to the proof of Theorem 1 in [NNG19]. Let $x \in X$. By the fundamental theorem of calculus and (5.26),we have

$$\begin{aligned} f(x) - f(x^\star) &= \int_0^1 \tfrac{1}{\lambda} \langle \nabla f(\lambda x + (1-\lambda)x^\star), \lambda x + (1-\lambda)x^\star - x^\star \rangle d\lambda \\ &\geq \int_0^1 \tfrac{\gamma}{\lambda} \left(f(\lambda x + (1-\lambda)x^\star) - f(x^\star) + \tfrac{\mu_s \lambda^2}{2} \|x - x^\star\|^2 \right) d\lambda \\ &\geq \tfrac{\gamma \mu_s}{4} \|x - x^\star\|^2, \end{aligned}$$

where the last inequality follows from the global optimality of $x^\star$. By summing $f(x)-f(x^\star) \geq \frac{\gamma\mu_s}{4}\|x-x^\star\|^2$ and (5.26), we get the desired inequality. Now, we prove part ii). Let $x \in \mathbb{R}^n$ and $\ell \in (\max(\frac{L}{2}, \mu_g), \infty)$. By (5.2), we have

$$f(x) \leq f(x^\star) + \tfrac{L}{2}\|x-x^\star\|^2. \tag{5.28}$$

By using (5.28) and (5.16), we get

$$f(x) + (\tfrac{\ell}{\mu_g})\langle \nabla f(x), x^\star - x\rangle + (\ell - \tfrac{L}{2})\|x-x^\star\|^2 \leq f(x^\star).$$

For the proof of iii), we refer the reader to [BM20, Lemma 3.2]. □

By combining Theorem 5.4 and Theorem 5.16, under the assumptions of Theorem 5.15, one can get the following convergence rate for Algorithm 5.1 with $t_1 = \frac{1}{L}$,

$$f(x^1) - f^\star \leq \left(\frac{2L - 2\mu_s\gamma^2}{2L + \mu_s\gamma^2}\right)\left(f(x^0) - f^\star\right),$$

which is tighter than the bound given in Theorem 5.15.

5.4 Concluding remarks

In this chapter we studied the convergence rate of the gradient method with fixed step lengths for smooth functions satisfying the PŁ inequality. We gave a new linear convergence rate, which is sharper than known bounds in the literature. One important question which remains to be addressed is the computation of the tightest convergence rate bound for Algorithm 5.1. Moreover, the performance analysis of fast gradient methods, like Algorithm 5.2, for these classes of functions that are discussed in this chapter may also be of interest.

We only studied the linear convergence in terms of the convergence of objective values. However, one can also infer the linear convergence in terms of distance to the solution set or the norm of the gradient by using our results. For instance, under the assumption of Theorem 5.4, we have

$$\tfrac{\mu_p}{2} d^2_{X^\star}(x^k) \leq f(x^k) - f^\star \leq \gamma^k\left(f(x^0) - f^\star\right) \leq \tfrac{L\gamma^k}{2} d^2_{X^\star}(x^0),$$

where the first inequality follows from Theorem 5.12, γ is the linear convergence rate given in Theorem 5.4, and the last inequality resulted from (5.2). Hence,

$$d^2_{X^\star}(x^k) \leq \tfrac{L\gamma^k}{\mu_p} d^2_{X^\star}(x^0).$$

Moreover, the quadratic gradient growth is a necessary and sufficient conditions for the linear convergence in terms of distance to the solution set; see [ZAdK24, Theorem 3.4] and Theorem 7.10.

The object of mathematics is the honor of the human spirit.

Carl Gustav Jacob Jacobi

6

Convergence rate analysis of the randomized and cyclic coordinate descent methods for convex optimization through semidefinite programming

Preamble

In line with the previous two chapters, in this chapter we study a derivative of the gradient descent method, known as the coordinate descent method. We study randomized and cyclic coordinate descent for convex unconstrained optimization problems. We improve the known convergence rates in some cases by using the numerical semidefinite programming performance estimation method. As a spin-off we provide a method to analyse the worst-case performance of the Gauss–Seidel iterative method for linear systems where the coefficient matrix is positive semidefinite with a positive diagonal. Moreover, we study weighted Jacobi method for solving quadratic programming problems and revisit some well-known results in the literature. This chapter is based on the paper [AdKZ23b], except for Section 6.3.2 which deals with the Jacobi method for solving linear system of equations.

6.1 Introduction

We consider the unconstrained optimization problem

$$f^\star = \inf_{x \in \mathbb{R}^n} f(x), \tag{6.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex. We assume that f attains its infimum and $f^\star$ denotes the optimal value. In addition, we assume that f is an L-smooth function, that is,

$$\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|, \quad \forall y, x \in \mathbb{R}^n.$$

Moreover, we denote the component Lipschitz constants by ℓ_i ($i \in \{1, \ldots, n\}$), i.e.,

$$|[\nabla f(x + te_i)]_i - [\nabla f(x)]_i| \leq \ell_i |t|, \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{6.2}$$

where e_i is the ith standard unit vector. Let $\ell_{\max} := \max_{1 \leq i \leq n} \ell_i$, and note that $\ell_{\max} \leq L \leq n\ell_{\max}$ [Wri15].

Due to the simplicity and small per-iteration cost, *coordinate descent methods* have been employed extensively for large-scale optimization problems [Nes12, Wri15].

The generic coordinate descent method is shown in Algorithm 6.1.

Algorithm 6.1 Generic coordinate descent

Set N and $\{t_k\}_{k=0}^{N-1}$ (step lengths) and pick $x^0 \in \mathbb{R}^n$.
For $k = 0, 1, \ldots, N - 1$ perform the following step:
1. Choose an index i_k from $\{1, 2, ..., n\}$.
2. $x^{k+1} = x^k - t_k [\nabla f(x^k)]_{i_k} e_{i_k}$.

In this chapter, we revisit the worst-case convergence rate analysis for Algorithm 6.1 for two of the best known variants, namely *randomized coordinate descent*, and *cyclic coordinate descent*. In the former, the index i_k is chosen uniformly at random from $\{1, 2, ..., n\}$, and in the latter, the cyclic ordering, $(1, 2, \ldots, n, 1, 2, \ldots, n, \ldots)$, is used.

We will improve the best-known convergence rates from the literature for some specific values of the parameters n, L, N, t_k for $k \in \{0, 1, \ldots, N - 1\}$ and ℓ_i for $i \in \{1, \ldots, n\}$. Finally, the Gauss–Seidel iterative method for positive semidefinite linear systems is a special case cyclic coordinate descent for convex quadratic functions, and we will investigate the implications of our analysis for this classical method as well.

Recently, Taylor and Bach [TB19, Appendix I] and Kamri et al. [KHG23] studied the convergence of the coordinate descent algorithm using the semidefinite programming (SDP) performance estimation method. We will also use SDP performance estimation in our analysis, but in a different way than Taylor and Bach [TB19, Appendix I], and our main contribution may be seen as the extension and refinement of the approach by Kamri et al. [KHG23]. For general background information on SDP, see e.g. [WSV12].

6.2 Convergence rate of randomized coordinate descent

The *randomized coordinate descent method* is shown in Algorithm 6.2 for easy reference.

Algorithm 6.2 Randomized coordinate descent

Set N and $\{t_k\}_{k=0}^{N-1}$ (step lengths) and pick $x^0 \in \mathbb{R}^n$.
For $k = 0, 1, \ldots, N-1$ perform the following step:

1. Choose index i_k with uniform probability from $\{1, 2, ..., n\}$.
2. $x^{k+1} = x^k - t_k [\nabla f(x^k)]_{i_k} e_{i_k}$.

We proceed to revisit its worst-case convergence rate for three classes for function, namely convex L-smooth functions, convex quadratic functions, and strongly convex, L-smooth functions.

6.2.1 The case of L-smooth functions

Regarding the convergence of Algorithm 6.2 for L-smooth convex functions, the following is known. (We state the result as in the survey [Wri15, Theorem 1], but it is originally due to Nesterov [Nes12]).

Theorem 6.1. *[Wri15, Theorem 1] Let $f : \mathbb{R}^n \to \mathbb{R}$ is an L-smooth convex function for some $L > 0$. If $t_k = \frac{1}{\ell_{\max}}$ for all k, then, for each $k > 0$,*

$$\mathbb{E}\left(f(x^k)\right) - f^\star \leq \left(\frac{2n\ell_{\max}}{k}\right) R_0^2, \tag{6.3}$$

where R_0 satisfies $\max_{x^\star \in \mathbb{S}} \max_x \{\|x - x^\star\| : f(x) \leq f(x^0)\} \leq R_0$ and $\mathbb{S}$ denotes the optimal solution set.

In this section, we study the behaviour of the randomized coordinate descent method for L-smooth convex functions. The worst-case convergence rate of the Algorithm 6.2 can be formulated as follows.

$$
\begin{aligned}
&\max \mathbb{E}[f(x^N)] - f(x^\star) \\
&\quad \text{s.t. } f \in \mathcal{F}_{0,L}(\mathbb{R}^n) \\
&\qquad f \text{ satisfies (6.2) for every } x \in \mathbb{R}^n \text{ for some } \ell_i,\ i \in \{1,\dots,n\} \\
&\qquad \|x^0 - x^\star\|^2 \le \Delta \\
&\qquad x^k,\ k \in \{1,2,\dots,N\}, \text{ are generated by Algorithm 6.2 with respect to } x^0 \\
&\qquad \text{and step length } t_k \\
&\qquad x^0 \in \mathbb{R}^n,\ \nabla f(x^\star) = 0,
\end{aligned}
\tag{6.4}
$$

where $f, x^k, x^\star$ are decision variables and t, L, n and $\ell_i, i \in \{1,\dots,n\}$, are the given parameters. Problem (6.4) in general is intractable. Moreover, note that x^k depends on the index i_k which is chosen uniformly at random from the set $\{1,\dots,n\}$ therefore (6.4) is a stochastic programming problem. To deal with this we introduce a random variable d^k which depends on the index i_k and is defined by $d^k := [\nabla f(x^k)]_{i_k} e_{i_k}$. Note that d^k has the following properties:

$$
\begin{aligned}
&\mathbb{E}\left[\|d^k\|^2\right] = \tfrac{1}{n}\mathbb{E}\left[\|\nabla f(x^k)\|^2\right] \\
&\mathbb{E}\left[\left\langle d^k, \nabla f(x^k)\right\rangle\right] = \tfrac{1}{n}\mathbb{E}\left[\|\nabla f(x^k)\|^2\right] \\
&\mathbb{E}\left[\left\langle d^k, x^k\right\rangle\right] = \tfrac{1}{n}\mathbb{E}\left[\left\langle \nabla f(x^k), x^k\right\rangle\right],
\end{aligned}
\tag{6.5}
$$

where the expectation again refers to the joint distribution of all the random variables d^k for $k \in \{0,1,\dots,N\}$ and $x^k, \nabla f(x^k), f(x^k)$ for $k \in \{0,1,\dots,N\}$. By Taylor's theorem and (6.2), we have

$$
\begin{aligned}
&f(x^{k+1}) \le f(x^k) + \left\langle \nabla f(x^k), x^{k+1} - x^k\right\rangle + \tfrac{\ell_{\max}}{2}\|x^{k+1} - x^k\|^2 \\
&f(x^k) \le f(x^{k+1}) + \left\langle \nabla f(x^{k+1}), x^k - x^{k+1}\right\rangle + \tfrac{\ell_{\max}}{2}\|x^k - x^{k+1}\|^2,
\end{aligned}
\tag{6.6}
$$

where $\ell_{\max} = \max_{i\in\{1,\dots,n\}} \ell_i$ as before. Therefore, the relaxation of problem (6.4) is given by

$$
\begin{aligned}
&\max\ \mathbb{E}[f(x^N)] - f(x^\star)\\
&\quad \text{s.t. } \{(x^k;\nabla f(x^k); f(x^k))\} \text{ satisfy (2.4) for } k\in\{0,1,\cdots,N,\star\} \text{ w.r.t. } \mu=0,L\\
&\qquad \{(x^k;\nabla f(x^k); f(x^k))\} \text{ satisfy (6.6) for } k\in\{0,1,\cdots,N\} \text{ w.r.t. } \ell_{\max}\\
&\qquad \{x^k;\nabla f(x^k); d^k\} \text{ satisfies (6.5)} \quad (k\in\{0,\cdots,N\})\\
&\qquad \|x^0 - x^\star\|^2 \le \Delta\\
&\qquad x^{k+1} = x^k - t_k d^k\\
&\qquad x^0\in\mathbb{R}^n,\ \nabla f(x^\star)=0,
\end{aligned}
\tag{6.7}
$$

where $f(x^k), x^k, x^\star, \nabla f(x^k)$ and d^k are decision variables. Note that because the problem (6.1) is invariant under translation, without loss of generality we may assume that $x^\star$ is the zero vector. Since $x^{k+1} = x^k - t_k d^k$ is a recursive relation, x^k can be written as linear combination of x^0 and d^is. In this way, all the unknowns appear as entries in the following matrix:

$$
\begin{aligned}
G &= \mathbb{E}\left[\mathrm{Gram}\left(x^0,\nabla f(x^0),\cdots,\nabla f(x^0), d^0,\cdots,d^N\right)\right]\\
&= \begin{pmatrix}
\mathbb{E}[\|x^0\|^2] & \mathbb{E}[\langle x^0,\nabla f(x^0)\rangle] & \cdots & \mathbb{E}[\langle x^0, d^N\rangle]\\
\mathbb{E}[\langle \nabla f(x^0), x^0\rangle] & \mathbb{E}[\|\nabla f(x^0)\|^2] & \cdots & \mathbb{E}[\langle \nabla f(x^0), d^N\rangle]\\
\vdots & \vdots & \ddots & \vdots\\
\mathbb{E}[\langle \nabla f(x^N), x^0\rangle] & \mathbb{E}[\langle \nabla f(x^N), \nabla f(x^0)\rangle] & \cdots & \mathbb{E}[\langle \nabla f(x^N), d^N\rangle]\\
\mathbb{E}[\langle d^0, x^0\rangle] & \mathbb{E}[\langle d^0, \nabla f(x^0)\rangle] & \cdots & \mathbb{E}[\langle d^0, d^N\rangle]\\
\vdots & \vdots & \ddots & \vdots\\
\mathbb{E}[\langle d^N, x^0\rangle] & \mathbb{E}[\langle d^N, \nabla f(x^0)\rangle] & \cdots & \mathbb{E}[\|d^N\|^2]
\end{pmatrix}.
\end{aligned}
$$

Note that G is the expectation of a random Gram matrix. Since every realization of this random matrix is positive semidefinite, and the expectation preserves positive semidefiniteness, it follows that G is positive semidefinite as well. Therefore, problem (6.7) can be written as an SDP problem, where the variables are G and $\mathbb{E}[f(x^i)]$.

In what follows we compare the convergence rate derived by solving the problem (6.7) and the bound by Wright (6.3) for some specific values of the parameters n, L, N, t_k for $k\in\{0,1,\cdots,N\}$ and ℓ_i for $i\in\{1,\cdots,n\}$. All the figures in this chapter were obtained by solving the SDP problems with the solver Mosek [ApS19], using the Yalmip [Löf04] Matlab interface.

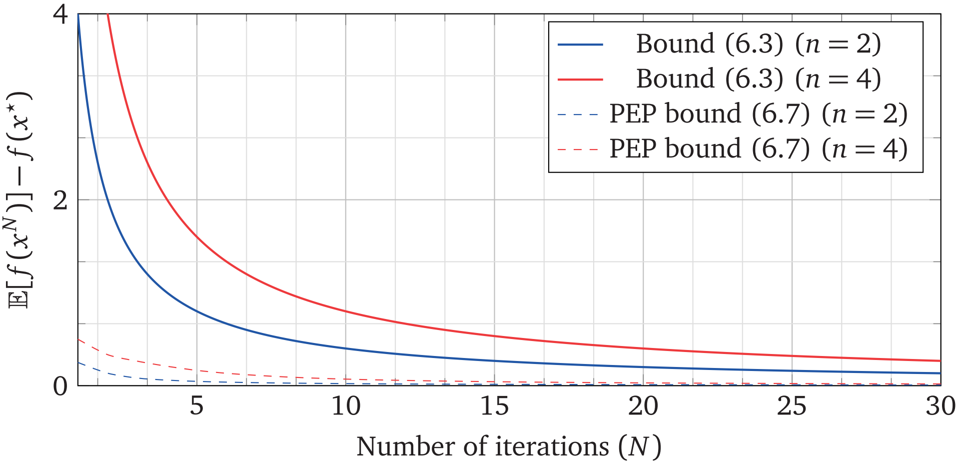


Figure 6.1: Convergence rate for Algorithm 6.2 computed by performance estimation problem (6.7) (dashed lines) and the bound given by (6.3) (thick lines) for $L = 2, l_{\max} = 1, t = 1, \Delta = 1$ and different n

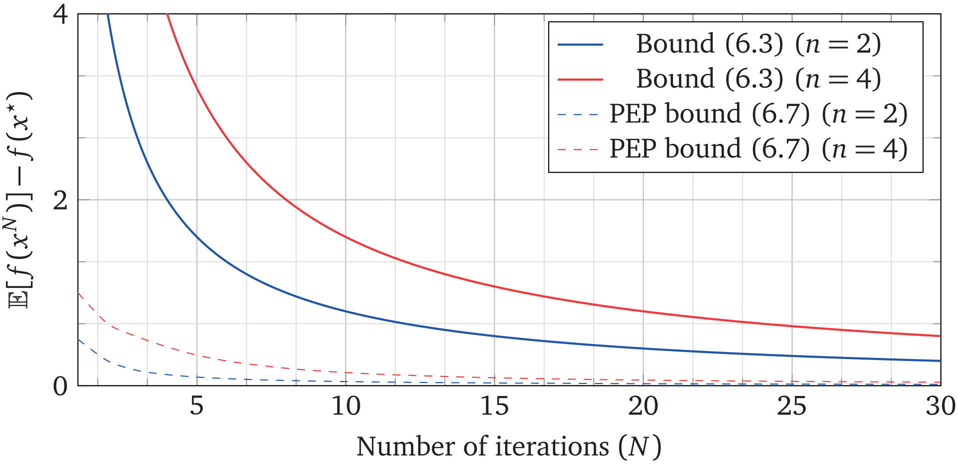


Figure 6.2: Convergence rate for Algorithm 6.2 computed by performance estimation problem (6.7) (dashed lines) and the bound given by (6.3) (thick lines) for $L = 4, l_{\max} = 2, t = 0.5, \Delta = 1$ and different n

Note that the convergence rate provided by solving performance estimation is strictly better than the bound given by Wright. In other words, the bound (6.3) is not tight for the values of the parameters that we considered. Moreover, the bound given by performance estimation can also be calculated for different step lengths than the fixed step lengths $1/\ell_{\max}$ in the bound (6.3).

6.2.2 The case of convex quadratic functions

In this section we study the convergence rate of the randomized coordinate descent method in case that the objective function is a *quadratic function* of the form

$$\min_{x\in\mathbb{R}^n} f(x) := \tfrac{1}{2}x^\top Ax - b^\top x, \tag{6.8}$$

where A is a symmetric positive semidefinite matrix. To study this case we need to add an additional constraint to restrict our model to quadratic functions. The following necessary condition for f to be a quadratic function can be verified easily, and has been used in SDP performance analysis by Drori et al [DS20]

$$\tfrac{1}{2}\langle\nabla f(x)-\nabla f(y), x-y\rangle = f(x)-f(y)-\langle\nabla f(y), x-y\rangle.$$

Since this constraint holds for every point in the domain we just consider the relaxed constraint that only holds for the point generated by the method in addition to the initial point and the optimal point. In this case we add the following constraint to the problem (6.7):

$$\tfrac{1}{2}\langle\nabla f(x^i)-\nabla f(x^j), x^i-x^j\rangle = f(x^i)-f(x^j)-\langle\nabla f(x^j), x^i-x^j\rangle \quad \forall i,j\in\{0,\dots,N,*\}. \tag{6.9}$$

In what follows we compare the convergence rate of the randomized coordinate descent method for the general problem (6.7) to the convergence rate for quadratic problems.

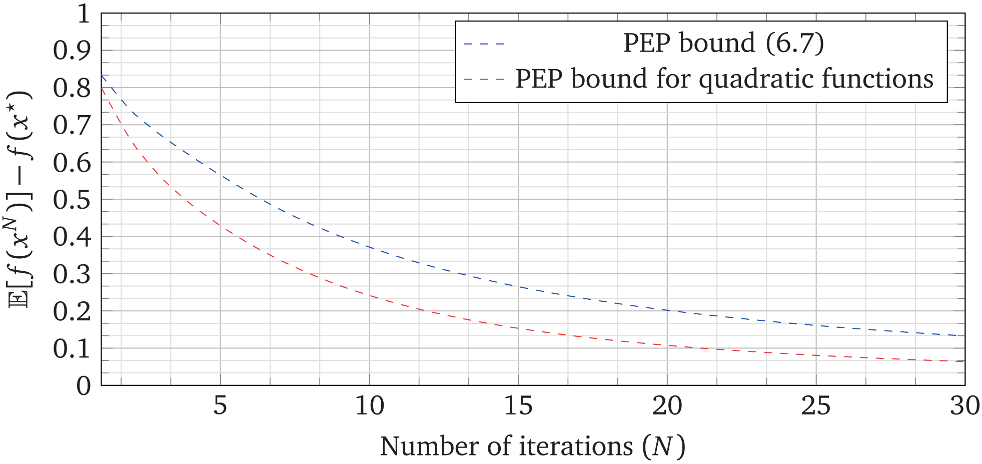


Figure 6.3: Convergence rate for Algorithm 6.2 computed by performance estimation problem for quadratic functions (red line) and the bound given by (6.7) (blue line) for $n = 10, L = 2, l_{\max} = 1, t = 1, \Delta = 1$.

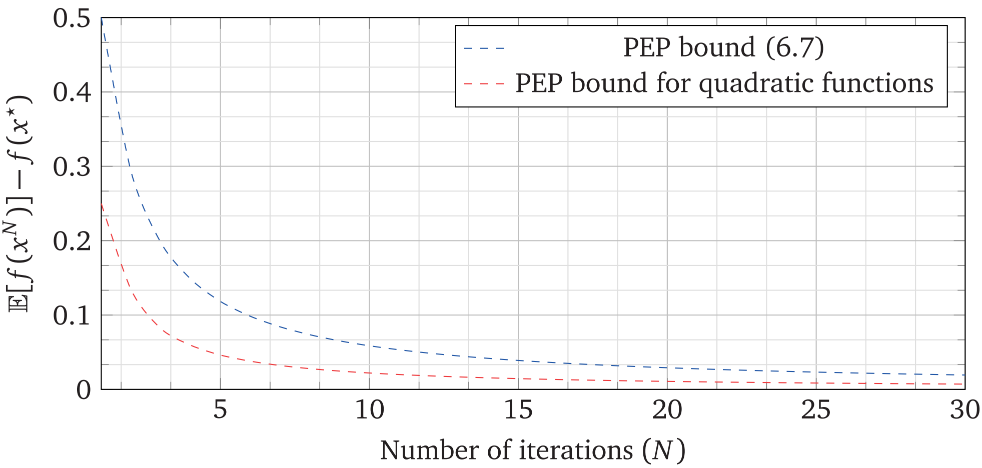


Figure 6.4: Convergence rate for Algorithm 6.2 computed by performance estimation problem for quadratic functions (red line) and the bound given by (6.7) (blue line) for $n = 2, L = 2, l_{\max} = 1, t = 1, \Delta = 1$.

Note that the convergence rate for the quadratic problem is sightly better than that of the general case.

6.2.3 The case of μ-strongly convex L-smooth functions

In this section, we study the convergence rate of the μ-strongly convex L-smooth functions. If $\mu > 0$ the optimal value of problem (6.7) for one iteration of Algorithm 6.2, i.e. $N = 1$, appears to be the same as the following bound (6.10) given by Wright [Wri15].

Theorem 6.2. *[Wri15, Theorem 1] Let $f \in \mathcal{F}_{\mu,L}(\mathbb{R}^n)$. If $t_k = \frac{1}{\ell_{\max}}$ for each k, and $\mu > 0$, then, for all $N > 0$,*

$$\mathbb{E}\left[f(x^N)\right] - f^\star \leq \left(1 - \frac{\mu}{n\ell_{\max}}\right)^N \left(f(x^0) - f^\star\right). \tag{6.10}$$

The fact that the two bounds seem to coincide does not suggest that (6.10) is tight, since the SDP bound (6.7) is a relaxation, and not exact.

6.3 Cyclic coordinate descent

Cyclic coordinate descent is one of the most important coordinate descent algorithms due to its simplicity. The convergence rate of *cyclic coordinate descent method* for the class of L-smooth convex functions is studied by Kamri et al using the performance estimation method [KHG23]. This method is described in Algorithm 6.3.

Algorithm 6.3 Cyclic coordinate descent

Set number of cycles K, $\{t_k\}_{k=0}^{N-1}$ (step lengths), pick $x^0 \in \mathbb{R}^n$ and set $N = nK$.
For $k = 0, 1, 2, \ldots, N-1$ perform the following step:

1. Set $i = k \pmod{n} + 1$
2. $x^{k+1} = x^k - t_k[\nabla f(x^k)]_i e_i$.

In each iteration the method updates the current point over one of the coordinates in cyclic order.

The following result is known about the rate of convergence. We present it as in [Wri15], but it is originally due to Beck and Tetruashvili [BT13].

Theorem 6.3. *[Wri15, Theorem 3] Let $f : \mathbb{R}^n \to \mathbb{R}$ be an L-smooth convex function for some $L > 0$. If $t_k = \frac{1}{\ell_{\max}}$ for all k, then, for $N = n, 2n, 3n, \ldots$,*

$$f(x^N) - f^\star \leq \left(\frac{4nR_0^2\ell_{\max}(1 + nL^2/\ell_{\max}^2)}{N+8}\right), \tag{6.11}$$

where R_0 satisfies $\max_{x^\star\in\mathbb{S}}\max_x\{\|x-x^\star\| : f(x)\le f(x^0)\}\le R_0$ and $\mathbb{S}$ denotes the optimal solution set. If f is also strongly convex with parameter $\mu>0$, then one has, for $k=n,2n,3n,\ldots$,

$$f(x^k)-f^\star \le \left(1-\frac{\mu}{2\ell_{\max}(1+nL^2/\ell_{\max}^2)}\right)^{k/n}\left(f(x^0)-f^\star\right).$$

For easy reference, we recall the interpolation conditions from Theorem 2.8 in the case that $\mu=0$: The set $\{x^i,\nabla f(x^i),f(x^i)\}$ for $i\in\{0,1,\cdots,N,\star\}$ is $\mathcal{F}_{0,L}-$ interpolable if and only if

$$f(x^i)\ge f(x^j)+\langle\nabla f(x^j),x^i-x^j\rangle+\tfrac{1}{2L}\left\|\nabla f(x^i)-\nabla f(x^j)\right\|^2,\quad\forall i,j\in\{0,1,\cdots,N,\star\}. \tag{6.12}$$

Using these conditions, we may formulate the worst-case convergence rate as performance estimation problem:

$$\begin{aligned}
&\max f(x^N)-f(x^\star)\\
&\ \text{s.t. } \{(x^i;\nabla f(x^i);f(x^i))\} \text{ satisfy (6.12) for } i\in\{0,1,\cdots,N,\star\} \text{ w.r.t. } L\\
&\qquad \|x^0-x^\star\|^2\le\Delta\\
&\qquad x^k\ \ (k\in\{1,2,\cdots,N\}) \text{ are generated using Algorithm 6.3}\\
&\qquad x^0\in\mathbb{R}^n,\ \nabla f(x^\star)=0.
\end{aligned} \tag{6.13}$$

Problem (6.13) can be formulated as a semidefinite programming problem, and this is precisely what was done by Kamri et al. [KHG23].

Since the univariate function $t\mapsto f(x^k+te_i)$ is convex and ℓ_i-smooth, it follows from (6.12) that, for every two consecutive points x^k and x^{k+1} generated by Algorithm 6.3, the following inequalities hold if $i=k\pmod n+1$:

$$\begin{aligned}
f(x^k)&\ge f(x^{k+1})+\nabla f(x^{k+1})_i(x_i^k-x_i^{k+1})+\tfrac{1}{2\ell_i}(\nabla f(x^k)_i-\nabla f(x^{k+1})_i)^2 \quad (6.14)\\
f(x^{k+1})&\ge f(x^k)+\nabla f(x^k)_i(x_i^{k+1}-x_i^k)+\tfrac{1}{2\ell_i}(\nabla f(x^{k+1})_i-\nabla f(x^k)_i)^2.
\end{aligned}$$

By adding the above inequalities to (6.13) one can get a better upper bound for the worst-case convergence rate, i.e.

$$\begin{aligned}
&\max f(x^N)-f(x^\star)\\
&\ \text{s.t. } \{(x^i;\nabla f(x^i);f(x^i))\} \text{ satisfy (6.12) for } i\in\{1,\cdots,N,\star\} \text{ w.r.t. } L\\
&\qquad \{(x^i;\nabla f(x^i);f(x^i))\} \text{ satisfy (6.14) for } i\in\{1,\cdots,N,\star\} \text{ w.r.t. } \{\ell_1,\cdots,\ell_n\}\\
&\qquad \|x^0-x^\star\|^2\le\Delta\\
&\qquad x^k\ \ (k\in\{1,\cdots,N\}) \text{ are generated using Algorithm 6.3}\\
&\qquad x^0\in\mathbb{R}^n,\ \nabla f(x^\star)=0.
\end{aligned} \tag{6.15}$$

In order to obtain an SDP relaxation to (6.15), we proceed in the same way as Kamri et al. [KHG23]. We view (6.15) as a quadratically constrained quadratic program (QCQP) in variables corresponding to the unknowns

$$x_i^k,\ \frac{\partial f(x^k)}{\partial x_i},\ f(x^k) \quad k \in \{0,1,2,\cdots,N\},\ i \in \{1,\ldots,n\},$$

Next we use the following relations to eliminate variables:

$$x_i^{k+1} - x_i^k = \begin{cases} -t_k \frac{\partial f(x_i^k)}{\partial x_i} & \text{if } i = k \pmod{n} + 1 \\ 0 & else \end{cases}$$

which hold for all $k \in \{0,1,2,\cdots,N-1\}$, and $i \in \{1,\ldots,n\}$. Subsequently we form the standard Shor SDP relaxation (see e.g. [WKK20]) of the resulting QCQP. Note that this is different to the approach we followed for randomized coordinate descent. In particular, the size of the SDP relaxation now depends on n, which was not the case before. This also limits the parameter values for which we may solve the SDP relaxations.

In Figure 6.5 we compare the SDP upper bounds from (6.13) and (6.15) for various parameter values.

The figure shows that the bound can be improved slightly by adding the set of constraints (6.14) to the model provided by Kamri et al. [KHG23]. Moreover, we add the constraint which correspond to the quadratic functions (6.9) to the model (6.15) which provides us with a better bound for quadratic functions. The computed values are much better that the theoretical bound (6.11), to the extent that we do not include this bound in the plot. Indeed, Kamri et al. [KHG23] already mentioned in their paper that the computed values for their model are much better than the theoretical bound (6.11).

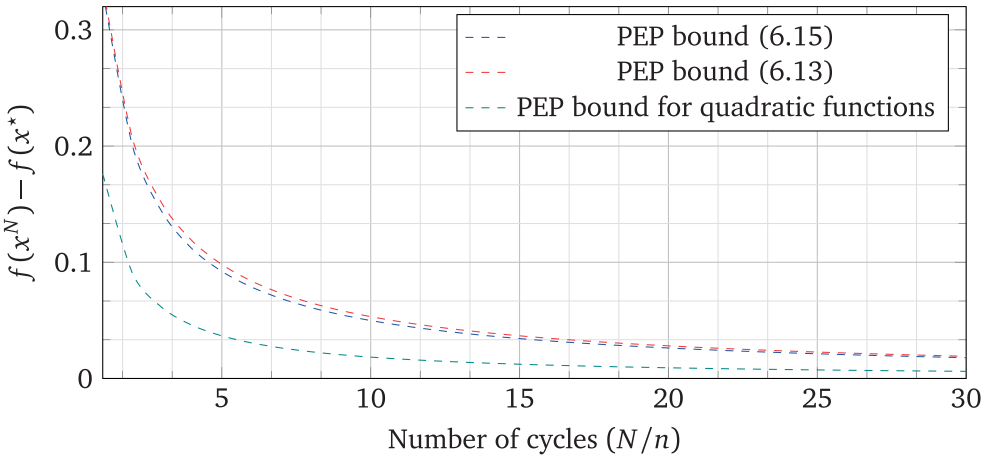


Figure 6.5: Convergence rate Algorithm 6.3 computed by performance estimation problem (6.15) (blue), the bound given by (6.13) (red) and the bound for quadratic functions (green) for $n = 2, L = 2, \ell_1 = 1, \ell_2 = 1, t = 0.5, \Delta = 1$.

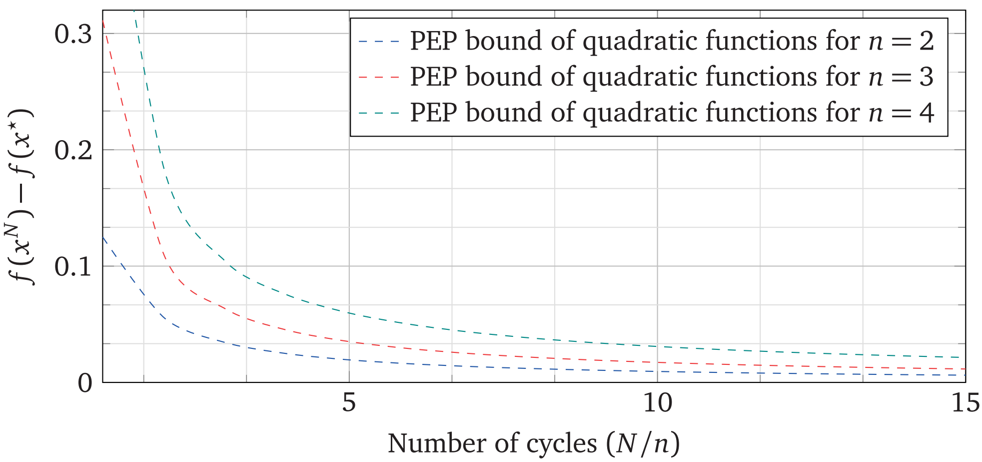


Figure 6.6: Convergence rate for Algorithm 6.3 computed by performance estimation problem of quadratic functions for $\ell_i = 1$ $i \in \{1, \cdots, n\}, L = \sum_{i=1}^{n} \ell_i, t = 1, \Delta = 1$ and different n.

Note that our discussion for coordinate-wise cyclic coordinate descent in this section could be extended to block-wise cyclic coordinate descent in a similar way as was done by Kamri et al. [KHG23].

6.3.1 Relation to the Gauss–Seidel method

The minimization of the convex quadratic function in (6.8) is equivalent to the solution of the linear system $Ax = b$. Here, we may assume w.l.o.g. that A has a positive diagonal. Cyclic coordinate decent for problem (6.8) is closely related to the iterative Gauss–Seidel method for solving this linear system. For this reason, cyclic coordinate descent is sometimes also referred to as *nonlinear Gauss–Seidel*. It is therefore an interesting question whether the SDP performance estimation framework yields any new insights on the performance of the *Gauss–Seidel method*.

Denoting $A = (a_{ij})$, the iterative Gauss–Seidel method may be described as follows.

Algorithm 6.4 Gauss–Seidel method

Set N and pick $x^0 \in \mathbb{R}^n$.

For $k = 0, 1, \ldots, N-1$ perform the following:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^{k}\right) \qquad (i = 1, \ldots, n).$$

This is exactly cyclic coordinate descent with unit step lengths if the gradient at a point x is replaced by $D^{-1}\nabla f(x)$, where $f(x) = \frac{1}{2}x^\top Ax - b^\top x$ as before, and D is the diagonal matrix with the same diagonal entries as A. To see this, recall that the Fréchet derivative of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ is the unique linear operator, say $D_f(x) : \mathbb{R}^n \to \mathbb{R}$, such that

$$\lim_{\|h\|\to 0} \frac{f(x+h) - f(x) - D_f(x)h}{\|h\|} = 0.$$

Once an inner product on $\mathbb{R}^n$ is fixed, say $\langle\cdot,\cdot\rangle$, one may, by the Riesz representation theorem, express $D_f(x)h = \langle g(x), h\rangle$, where $g(x)$ is called the gradient vector of f at x with respect to $\langle\cdot,\cdot\rangle$. In particular, if $\langle\cdot,\cdot\rangle$ is the Euclidean dot product, then $g(x) = \nabla f(x)$. If one changes to the inner product $\langle\cdot,\cdot\rangle_D$ defined by

$$\langle u, v\rangle_D = \sum_{i=1}^{n} a_{ii}u_i v_i \qquad (u, v \in \mathbb{R}^n), \tag{6.16}$$

then the gradient vector at x becomes $D^{-1}\nabla f(x)$, by the uniqueness of the Fréchet derivative.

It was shown in [DKGT20] that the interpolation condition in Theorem 2.37 holds for any reference inner product $\langle\cdot,\cdot\rangle$, provided that the gradient vector is interpreted accordingly.

In other words, the following SDP performance estimation problem gives a bound on the worst-case performance of the Gauss–Sidel method after N iterations, when A is a symmetric positive semidefinite matrix with a positive diagonal.

$$\begin{aligned}
\max\ & f(x^N) - f(x^\star) \\
\text{s.t. } & \{(x^i; \nabla f(x^i); f(x^i))\} \text{ satisfy (6.12) and (6.9) for } i \in \{0, 1, \cdots, N, \star\} \\
& \quad \text{w.r.t. } L = \lambda_{\max}(D^{-1}A) \\
& \quad \{(x^i; \nabla f(x^i); f(x^i))\} \text{ satisfy (6.14) for } i \in \{0, 1, \cdots, N, \star\} \\
& \quad \text{w.r.t. } \ell_1 = \ldots = \ell_n = 1 \\
& \quad \|x^0 - x^\star\|^2 \le \Delta \\
& \quad x^k \ (k \in \{1, 2, \cdots, N\}) \text{ is generated using Algorithm 6.3} \\
& \quad x^0 \in \mathbb{R}^n, \ \nabla f(x^\star) = 0,
\end{aligned} \tag{6.17}$$

where the inner product is now understood to be the one in (6.16), and the norm the induced norm for this inner product, and $\lambda_{\max}(D^{-1}A)$ denotes the largest eigenvalue of $D^{-1}A$. (Note that the eigenvalues of $D^{-1}A$ are real.) Importantly, the reference inner product is not visible in the SDP performance estimation problem reformulation of (6.17), since only a Gram matrix for this inner product appears. It is therefore equally valid, for any inner product, provided that the inner product and norm are interpreted accordingly. Of course, the Lipschitz constants like (6.2) depend on the norm as well. It is easy to verify that, for the inner product (6.16), and $f(x) = \frac{1}{2}x^\top Ax - b^\top x$, one has $\ell_1 = \ldots = \ell_n = 1$ and $L = \lambda_{\max}(D^{-1}A)$ as is used in (6.17).

In summary, we have shown the following.

Theorem 6.4. *Consider a solvable system of linear equations $Ax = b$ where A is a symmetric positive semidefinite matrix with positive diagonal, and let $x^\star$ denote a solution. Letting $f(x) = \frac{1}{2}x^\top Ax - b^\top x$, after N iterations of the Gauss–Seidel method, an upper bound on $f(x^N) - f(x^\star)$ is given by the optimal value of the SDP problem* (6.17)*, provided that the starting point x^0 satisfies $\|x^0 - x^\star\| \le \Delta$ for a given Δ, where the norm is the induced norm of the inner product* (6.16).

The Gauss–Seidel method is known to be convergent when A is symmetric positive-definite, e.g. [GVL13, Theorem 10.1.2], or strictly or irreducibly diagonally dominant, e.g. [Bag95]. The case when A is only positive semidefinite (with positive diagonal) seems to be less well-understood, and our approach sheds more light on this case. In particular, numerical results of the type shown in Figure 6.6 apply here.

6.3.2 Weighted-Jacobi method

This section is dedicated to the studying of the quadratic optimization problem presented in (6.8), where the matrix A is both symmetric and positive definite. The challenge of solving (6.8) is inherently similar to the task of solving the subsequent linear equation system $Ax = b$. As a natural progression from our previous section where we delved into the Gauss–Seidel method, we now turn our attention to the analysis of an algorithm commonly employed for the general solution of linear equation systems.

The matrix $A = [a_{ij}]$ can be expressed as the sum of its components, $A = D + L + L^T$, with D representing the diagonal, and L denoting the lower triangular part of the matrix A. The Jacobi method is a well-known and widely-used approach for solving linear equation systems [Jac46]. In this section, we delve into a comprehensive examination of the Jacobi method and conduct a thorough review of its convergence analysis. The Jacobi method solves the following system in each iteration,

$$x^k = \left(I - D^{-1}A\right)x^{k-1} + D^{-1}b. \tag{6.18}$$

It is worth noting that there are certain similarities between the Gauss–Seidel and Jacobi methods. However, the key distinction between them lies in their updates of values from previous generated points. In the Jacobi method (6.18), values from the previous step are used, whereas the Gauss–Seidel method, as demonstrated in Algorithm 6.4, always utilizes the most recent updated values in every iteration, i.e. the Gauss–Seidel method runs the following updates at iteration $k+1$,

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_j^k\right) \quad \forall i,$$

whereas the Jacobi method performs the following updates at iteration $k+1$,

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i}^{n} a_{ij}x_j^k\right) \quad \forall i.$$

It is a well-known that the iteration $x^k = Rx^{k-1} + c$ converges if and only if the spectral radius $\rho(R) < 1$ [Dem97, Chapter 6.5]. In our specific case, where we consider matrix A as symmetric and positive semidefinite, it is sufficient to ensure that $\lambda_{\max}(R) < 1$ since the spectral radius $\rho(R)$ is equivalent to $\lambda_{\max}(R)$. For the Jacobi method, this condition translates to $0 < \rho(D^{-1}A) < 2$, which is a somewhat limiting requirement. To address this issue, the weighted-Jacobi method is introduced, which is detailed in Algorithm 6.5, with the specific choice of $q^k = 0$ for $0 \leq k \leq N$ [Ric11].

In cases that each step of the *weighted-Jacobi method* is performed inexactly, an adaptation known as the *inexact weighted-Jacobi method* has been introduced; see [GO88, GO82] for more discussion. To be more specific, for a given parameter δ, the value of q_k is determined in each iteration as follows

$$\|q^k\| \leq \delta \|b - Ax^k\|. \tag{6.19}$$

Algorithm 6.5 Inexact Weighted-Jacobi method

Set N and α (weight) and pick $x^0 \in \mathbb{R}^n$.
For $k = 1, 2, \ldots, N$ perform the following step:

1. select q_k that satisfies (6.19)
2. $x^k = \left(I - \alpha D^{-1} A\right) x^{k-1} + \alpha D^{-1} q^{k-1} + \alpha D^{-1} b$

The convergence of the weighted-Jacobi method and its various adaptations has been extensively explored in the literature. You can find a concise overview in [Saa03, Chapter 4]. The following theorems represent well-established convergence results for the Jacobi method.

Theorem 6.5. *If A is strictly diagonally dominant ($a_{ii} > \sum_{j=1}^{n} |a_{ij}|$ for all i), the Jacobi method converges.*

Theorem 6.6. *[E.g. Che] The weighted-Jacobi method converges if and only if* $0 < \alpha < \frac{2}{\lambda_{\max}(D^{-1}A)}$. *Moreover, by considering $\alpha^\star = \frac{2}{\lambda_{\min}(D^{-1}A) + \lambda_{\max}(D^{-1}A)}$ one get the optimal convergence rate*

$$\rho^\star = \frac{1 - \kappa_{D^{-1}A}}{1 + \kappa_{D^{-1}A}},$$

where $\kappa_{D^{-1}A} = \frac{\lambda_{\min}(D^{-1}A)}{\lambda_{\max}(D^{-1}A)}$

Now, let's delve into an examination of the convergence rate for the inexact weighted-Jacobi method.

Theorem 6.7. *The inexact weighted-Jacobi method with $\delta \geq 0$ has the following convergent rate for a suitable choice of α*

$$\|e^{k+1}\| \leq (\|I - \alpha D^{-1}A\| + \alpha\delta\|D^{-1}\|\|A\|)\|e^k\|.$$

where $e^k = x^\star - x^k$.

Proof. We define $r^k = b - Ax^k$. By some calculation one can show that $r^k = Ae^k$. By this definition we have

$$e^{k+1} = Ge^k + p^k,$$

where $p^k = -\alpha D^{-1}q^k$, and $G = I - \alpha D^{-1}A$. It can be easily seen that

$$\|p^k\| = \|\alpha D^{-1}q^k\| \leq \alpha\delta\|D^{-1}\|\|b - Ax^k\| = \alpha\delta\|D^{-1}\|\|Ae^k\|.$$

Therefore,

$$\|e^{k+1}\| = \|Ge^k + p^k\| \leq (\|G\| + \alpha\delta\|D^{-1}\|\|A\|)\|e^k\|,$$

which completes the proof. □

It is evident that when $\delta = 0$, Theorem 6.7 simplifies to the weighted-Jacobi method, and the convergence rate mirrors that of Theorem 6.6. In other words, by maintaining the induced norm as two, you can establish $\|I - \alpha D^{-1}A\| = 1 - \alpha\lambda_{\min}(D^{-1}A)$. Setting α as defined in Theorem 6.6 yields an equivalent convergence rate, while the performance estimation method offers the following convergence rate.

Theorem 6.8 (Based on Theorem 5.3. [DKGT20])**.** *Consider the inexact Jacobi method with fixed step size α where $\mu = \lambda_{\min}(D^{-1}A)$ and $L = \lambda_{\max}(D^{-1}A)$. If $\delta \in [0, \frac{2\mu}{L+\mu}]$, and $\alpha \in [0, \frac{2\mu - \delta(L+\mu)}{(1-\delta)\mu(L+\mu)}]$, one has*

$$\|x^1 - x^\star\|_D \leq (1 - (1-\delta)\mu\alpha)\,\|x^0 - x^\star\|_D. \tag{6.20}$$

Golub and Overton in [GO82] investigate the convergence of the inexact second-order *Richardson method*. Notably, in their analysis by setting $\omega = 1$ and $M = D$, one can derive the inexact Jacobi method. In what follows, we aim to derive the convergence rate of the inexact Jacobi method based on their analysis. Note that here, $\|\cdot\|$ represents the Euclidean norm for vectors and the *spectral norm* for matrices.

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}, \quad \|A\| = \max_{\|x\|=1} \|Ax\|.$$

Let us introduce the following definitions $e^k = x^\star - x^k$ and $r^k = b - Ax^k$. Just as in the previous case, we have $r^k = Ae^k$ and $e^{k+1} = Ge^k + p^k$, where $p^k = -\alpha D^{-1} q^k$ and $G = I - \alpha D^{-1}A$. The eigenvector decomposition (SVD-decomposition) of the matrix G is as follows

$$G = D^{-1/2}(I - \alpha D^{-1/2}AD^{-1/2})D^{1/2} = D^{-1/2}V\Lambda V^T D^{1/2},$$

where Λ is a diagonal matrix with elements representing the eigenvalues λ_i of the matrix $I - \alpha D^{-1/2}AD^{-1/2}$, and V is an orthogonal matrix. Now we introduce

$$\hat{e}^k = V^T D^{1/2} e^k \quad \text{and} \quad \hat{p}^k = -\alpha V^T D^{-1/2} q^k.$$

These result in

$$\hat{e}^{k+1} = \Lambda \hat{e}^k + \hat{p}^k.$$

By the Lemma 1 in [GO82], one gets

$$\hat{e}^k = S^k \hat{e}^1 + \sum_{l=1}^{k-1} S^{k-l} \hat{p}^l,$$

where S^k is diagonal matrix with diagonal elements

$$S^k_{jj} = \lambda_j^{k-1}.$$

Define $\rho = \lambda_{\max}$ which is equal to $1 - \alpha\lambda_{\min}(D^{-1}A)$ if $\alpha \in [0, \frac{2}{\lambda_{\max}(D^{-1}A)+\lambda_{\min}(D^{-1}A)}]$. It is easily seen that $\|S^k\| \leq \rho^{k-1}$. Therefore,

$$\|\hat{e}^k\| \leq \rho^{k-1}\|\hat{e}^1\| + \sum_{l=1}^{k-1} \rho^{k-l-1}\|\hat{p}^l\|.$$

By the fact that $\|\hat{p}^k\| \leq \epsilon\|\hat{e}^k\|$ where $\epsilon = \delta\alpha\|D^{-1/2}\|\|AD^{-1/2}\|$ and δ is given by (6.19), the above inequality yields to

$$\|\hat{e}^k\| \leq \rho^{k-1}\|\hat{e}^1\| + \epsilon \sum_{l=1}^{k-1} \rho^{k-l-1}\|\hat{e}^l\|.$$

Now, we introduce a new variable, $\tau^k = \rho^{k-1}\|\hat{e}^1\| + \epsilon \sum_{l=1}^{k-1} \rho^{k-l-1}\tau^l$. Through some algebraic manipulation, we find that $\tau^1 = \|\hat{e}^1\|$ by definition. Furthermore, τ^k satisfies the following homogeneous equation

$$\tau^{k+1} = (\rho + \epsilon)\tau^k.$$

Therefore by Theorem 1 in [GO82] one may have the following convergence result.

Theorem 6.9. *[GO82] The error norm $\|\hat{e}^k\|$ associated with the kth iterate of the inexact Jacobi method is bounded by*

$$\|\hat{e}^k\| \leq (\rho + \epsilon)^k \|\hat{e}^0\|. \tag{6.21}$$

Note that, based on norm properties, we have $\|\hat{e}^k\| = \|x^k - x^\star\|_D$. To compare the result from the last theorem with Theorem 6.8, one can set $k = 1$ in the last theorem, yielding

$$\|\hat{e}^1\| \leq \left(\|I - \alpha D^{-1}A\| + \alpha\delta\|D^{-1/2}\|\|AD^{-1/2}\|\right)\|\hat{e}^0\|.$$

As per the assumptions of Theorem 6.8, if we set $\delta = 0$, both bounds (6.20) and (6.21) become equal. However, when $\delta \neq 0$, one can demonstrate that

$$\alpha\delta\mu \leq \alpha\delta\|AD^{-1}\| \leq \alpha\delta\|D^{-\frac{1}{2}}\|\|AD^{-\frac{1}{2}}\|. \tag{6.22}$$

This inequality shows that the bound provided by Theorem 6.8 is better when compared to the bound given by Theorem 6.9. The example below illustrates that the inequality 6.22 can be strict in some cases.

Example 6.10. *Let us define* $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. *It is easily seen that* $\mu = 1.3820$ *and* $\|D^{-\frac{1}{2}}\|\|AD^{-\frac{1}{2}}\| = 1.5811$ *which shows that for any positive* α *and* δ *inequality* (6.22) *is strict.*

6.4 Conclusion

We have studied SDP performance estimation approaches to analyse randomized and cyclic coordinate descent, thereby complementing recent results in [KHG23]. For randomized coordinate descent, we have given the first known SDP performance estimation bound. For cyclic coordinate descent, we were able to improve slightly on the numerical values given in [KHG23]. Of course, to obtain new rates of convergence in general, it is necessary to solve the SDP performance estimation problems analytically, as opposed to numerically, but we have been unable to obtain analytic solutions for the SDP problems presented in this chapter. In this chapter, we also discussed the link with the Gauss–Seidel method in the case of convex quadratic functions. Moreover, we studied the weighted Jacobi method for solving a linear system of equations which is closely related to the Gauss–Seidel method.

Obvious is the most dangerous word in mathematics.

E.T. Bell (1883–1960)

7

Convergence rate analysis of the gradient descent-ascent method for convex-concave saddle-point problems

Preamble

In this chapter, we study the gradient descent-ascent method for convex-concave saddle-point problems. We derive a new non-asymptotic global convergence rate in terms of distance to the solution set by using the semidefinite programming performance estimation method. The given convergence rate incorporates most parameters of the problem and it is exact for a large class of strongly convex-strongly concave saddle-point problems for one iteration. We also investigate the algorithm without strong convexity and we provide some necessary and sufficient conditions under which the gradient descent-ascent method enjoys linear convergence. This chapter is based on the paper [ZAdK24].

7.1 Introduction

We consider the *convex-concave saddle point problem*

$$\min_{x\in\mathbb{R}^n}\max_{y\in\mathbb{R}^m} F(x,y), \tag{7.1}$$

where $F:\mathbb{R}^n\times\mathbb{R}^m\to(-\infty,\infty)$, and $F(\cdot,y)$ and $F(x,\cdot)$ are convex and concave, respectively, for any fixed $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$. We assume that problem (7.1) has some solution, that is, there exists a saddle point $(x^\star,y^\star)\in\mathbb{R}^n\times\mathbb{R}^m$ with

$$F(x^\star,y)\le F(x^\star,y^\star)\le F(x,y^\star),\quad \forall x\in\mathbb{R}^n,\forall y\in\mathbb{R}^m.$$

We denote the solution set of problem (7.1) with $S^\star$. We call F smooth if for some L_x,L_y,L_{xy}, we have

$$\begin{array}{lll}
i) & \|\nabla_x F(x_2,y)-\nabla_x F(x_1,y)\|\le L_x\|x_2-x_1\| & \forall x_1,x_2,y\\
ii) & \|\nabla_y F(x,y_2)-\nabla_y F(x,y_1)\|\le L_y\|y_2-y_1\| & \forall x,y_1,y_2\\
iii) & \|\nabla_x F(x,y_2)-\nabla_x F(x,y_1)\|\le L_{xy}\|y_2-y_1\| & \forall x,y_1,y_2\\
iv) & \|\nabla_y F(x_2,y)-\nabla_y F(x_1,y)\|\le L_{xy}\|x_2-x_1\| & \forall x_1,x_2,y.
\end{array}$$

The function F is said to be strongly convex-strongly concave if

$$\begin{array}{l}
i)\ F(\cdot,y)-\frac{\mu_x}{2}\|\cdot\|^2 \text{ is convex for any fixed } y\\
ii)\ F(x,\cdot)+\frac{\mu_y}{2}\|\cdot\|^2 \text{ is concave for any fixed } x,
\end{array}$$

for some $\mu_x,\mu_y>0$. Note that strong convexitystrong concavity implies that problem (7.1) has a unique solution $(x^\star,y^\star)$. We denote the set of smooth strongly convex-strongly concave functions by $\mathcal{F}(L_x,L_y,L_{xy},\mu_x,\mu_y)$.

Problem (7.1) has applications in game theory [BO98], robust optimization [BTEGN09], adversarial training [GPAM+20], and reinforcement learning [YND+20], to name but a few. In addition, various other algorithms have been developed for solving saddle point problems; see e.g. [HA21, JM22, LJJ20, NYZ21, SPPMD19, WL20, XZXL23].

One of the simplest approaches for handling problem (7.1) introduced in [AAH+58, Chapter 6] is the gradient-descent-ascent method, which may be regarded as a generalization of the gradient method to saddle point problems. The *gradient descent-ascent method* is described in Algorithm 7.1.

The local and global linear convergence of Algorithm 7.1 have been investigated in the literature; see [FOP20, LS19, ZWLG22] and the references therein. As

Algorithm 7.1 The gradient descent-ascent method

Set N and $t>0$ (step length), pick x^0 and y^0.
For $k=1,2,\ldots,N$ perform the following simultaneous steps:
1. $x^k = x^{k-1} - t\nabla_x F(x^{k-1}, y^{k-1})$.
2. $y^k = y^{k-1} + t\nabla_y F(x^{k-1}, y^{k-1})$.

we investigate the global linear convergence rate of Algorithm 7.1, we mention one known global convergence result, which is derived by using variational inequality techniques. Suppose that $z=(x,y)$. Let the function $\phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ given by $\phi(z) = \begin{pmatrix}\nabla_x F(z) & -\nabla_y F(z)\end{pmatrix}^T$. It is shown that, see e.g. [MOP20b],

$$\|\phi(\bar z) - \phi(\hat z)\| \le 2L\|\bar z - \hat z\|,$$
$$\langle \phi(\bar z) - \phi(\hat z), \bar z - \hat z\rangle \ge \mu\|\bar z - \hat z\|^2,$$

where $L = \max\{L_x, L_y, L_{xy}\}$ and $\mu = \min\{\mu_x, \mu_y\}$. Indeed, ϕ is Lipschitz continuous and strongly monotone. By [FP03, Theorem 12.1.2], for $t \in (0, \frac{\mu}{2L^2})$, we have

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \le (1 + 4L^2t^2 - 2\mu t)\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right). \quad (7.2)$$

In this study, we revisit Algorithm 7.1 and improve the convergence rate (7.2). Indeed, we derive a new convergence rate involving most parameters of problem (7.1). It is worth noting that if one sets $L = \max\{L_x, L_y, L_{xy}\}$ and $\mu = \min\{\mu_x, \mu_y\}$, the new bound dominates the convergence rate (7.2) for any step length $t \in \left(0, \frac{\mu}{2L^2}\right)$. Furthermore, by setting $t = \frac{\mu}{4L^2}$, one can infer that Algorithm 7.1 has a complexity of $\mathcal{O}\left(\frac{L^2}{\mu^2}\ln\left(\frac{1}{\epsilon}\right)\right)$ to compute iterates (x^k, y^k) such that $\|x^k - x^\star\|^2 + \|y^k - y^\star\|^2 \le \epsilon\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right)$, which is the known iteration complexity bound in the literature; see e.g. [BPG+23, ZBLG21]. In this study, thanks to the new convergence rate given in Theorem 7.2, the order of complexity of $\mathcal{O}\left(\left(\frac{L}{\mu} + \frac{L_{xy}^2}{\mu^2}\right)\ln\left(\frac{1}{\epsilon}\right)\right)$ is obtained when $L = \max\{L_x, L_y\}$ and $\mu = \min\{\mu_x, \mu_y\}$, which is more informative in comparison with the above-mentioned one. Moreover, by providing some example, we show that the given convergence rate is exact for one iteration.

The goal of this work is not to achieve the optimal algorithmic complexity for the class of saddle point problems introduced above. Rather, we have the more modest goal of giving the best possible worst-case complexity analysis of the gradient descent-ascent method (Algorithm 7.1). It is important to note that there are accelerated gradient descent-ascent methods with better worst-case complexity than Algorithm 7.1; see e.g. [LJJ20, WL20]. In particular, the accelerated methods may be shown to have a worst-case complexity $\mathcal{O}\left(\sqrt{\frac{L_x^2}{\mu_x^2}+\frac{L_{xy}^2}{\mu_x\mu_y}+\frac{L_y^2}{\mu_y^2}}\cdot\ln\left(\frac{1}{\epsilon}\right)\right)$, which may be compared to the best-known lower complexity bound $\mathcal{O}\left(\sqrt{\frac{L_x}{\mu_x}+\frac{L_{xy}^2}{\mu_x\mu_y}+\frac{L_y}{\mu_y}}\cdot\ln\left(\frac{1}{\epsilon}\right)\right)$ for the class of pure first-order algorithms [ZHZ22].

The chapter is organized as follows. First, we present basic definitions and preliminaries used to establish the results. Section 7.2 is devoted to the study of the linear convergence of Algorithm 7.1. In Section 7.3, we study the linear convergence of the gradient descent-ascent method without strong convexity. Indeed, we let $F\in\mathcal{F}(L_x,L_y,L_{xy},0,0)$ and give some necessary and sufficient conditions for the linear convergence. Moreover, we derive a convergence rate under this setting.

Notation

Let $X\subseteq\mathbb{R}^n$. We denote the distance function to X by $d_X(x):=\inf_{\bar{x}\in X}\|x-\bar{x}\|$ and the set-valued mapping $\Pi_X(x)$ stands for the projection of x on X, i.e., $\Pi_X(x):=\{y\in X:\|x-y\|=d_X(x)\}$; see also Definition 2.5.

It is worth mentioning that, under the assumptions of Theorem 2.37, the set $\left\{(x^i;g^i;f^i)\right\}_{i\in\mathcal{I}}$ is interpolable with an L-smooth μ-strongly concave function if and only if for any $i,j\in\mathcal{I}$, we have

$$\frac{1}{2(1-\frac{\mu}{L})}\left(\frac{1}{L}\left\|g^i-g^j\right\|^2+\mu\left\|x^i-x^j\right\|^2+\frac{2\mu}{L}\left\langle g^j-g^i,x^j-x^i\right\rangle\right)\leq -f^i+f^j+\left\langle g^j,x^i-x^j\right\rangle. \tag{7.3}$$

7.2 The gradient descent-ascent method

In this section, we study the convergence rate of gradient descent-ascent method when $F\in\mathcal{F}(L_x,L_y,L_{xy},\mu_x,\mu_y)$ with $\min\{\mu_x,\mu_y\}>0$. Indeed, we investigate the worst-case behavior of one step of Algorithm 7.1 in terms of distance to the unique saddle point $(x^\star,y^\star)$. Let (x^1,y^1) be generated by the algorithm using the starting point (x^0,y^0). The worst-cast convergence rate of Algorithm 7.1 is given by the solution of the following abstract optimization problem:

$$
\begin{aligned}
\max\ & \frac{\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2}{\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2} \\
\text{s.t. } & (x^1, y^1) \text{ is generated by Algorithm 7.1 w.r.t. } F, x^0, y^0 \\
& (x^\star, y^\star) \text{ is the unique saddle point of problem (7.1)} \\
& F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y) \\
& x^0 \in \mathbb{R}^n, y^0 \in \mathbb{R}^m.
\end{aligned}
\tag{7.4}
$$

In problem (7.4), $F, x^0, x^1, x^\star, y^0, y^1, y^\star$ are decision variables and $\mu_x, L_x, \mu_y, L_y, L_{xy}, t$ are fixed parameters. As it is mentioned in Chapter 3, problem (7.4) seems completely intractable, but its solution may in fact be approximated using a suitable semidefinite programming (SDP) problem, as shown below. Suppose that

$$
\begin{aligned}
F^{i,j} &= F(x^i, y^j) && i, j \in \{0, 1, \star\}, \\
G_x^{i,j} &= \nabla_x F(x^i, y^j) && i, j \in \{0, 1, \star\}, \\
G_y^{i,j} &= \nabla_y F(x^i, y^j) && i, j \in \{0, 1, \star\},
\end{aligned}
$$

where indices $\{0, 1, \star\}$ refers to the starting point, the point generated by Algorithm 7.1 and the saddle point of the problem, respectively. Note that due to the the necessary and sufficient conditions for convex-concave saddle point problems, we have

$$
G_x^{\star,\star} = 0, \qquad G_y^{\star,\star} = 0.
$$

By using Theorem 2.37, problem (7.4) may be relaxed as a finite dimensional optimization problem,

$$
\begin{aligned}
\max\ & \frac{\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2}{\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2} \\
\text{s.t. } & \{(x^0; G_x^{0,k}; F^{0,k}), (x^1; G_x^{1,k}; F^{1,k}), (x^\star; G_x^{\star,k}; F^{\star,k})\} \text{ satisfy (2.4) for} \\
& \qquad k \in \{0, 1, \star\} \text{ w.r.t. } \mu_x, L_x \\
& \{(y^0; G_y^{k,0}; F^{k,0}), (y^1; G_y^{k,1}; F^{k,1}), (y^\star; G_y^{k,\star}; F^{k,\star})\} \text{ satisfy (7.3) for} \\
& \qquad k \in \{0, 1, \star\} \text{ w.r.t. } \mu_y, L_y \\
& \|G_x^{k,i} - G_x^{k,j}\| \le L_{xy}\|y^i - y^j\|, \quad i, j, k \in \{0, 1, \star\} \\
& \|G_y^{i,k} - G_y^{j,k}\| \le L_{xy}\|x^i - x^j\|, \quad i, j, k \in \{0, 1, \star\} \\
& x^1 = x^0 - tG_x^{0,0} \\
& y^1 = y^0 + tG_y^{0,0}, \\
& G_x^{\star,\star} = 0, \ G_y^{\star,\star} = 0.
\end{aligned}
\tag{7.5}
$$

In problem (7.5), $\{(x^i; G_x^{i,j}; F^{i,j})\}$ and $\{(y^i; G_y^{j,i}; F^{j,i})\}$ $(i, j \in \{0, 1, \star\})$ are decision variables. We may assume that $x^\star = 0$ and $y^\star = 0$ as Algorithm 7.1 is invariant under translation. By elimination, problem (7.5) may be reformulated as follows,

$$
\begin{aligned}
\max\ & \frac{\|x^0 - tG_x^{0,0}\|^2 + \|y^0 + tG_y^{0,0}\|^2}{\|x^0\|^2 + \|y^0\|^2} \\
\text{s.t. } & \{(x^0; G_x^{0,k}; F^{0,k}), (x^0 - tG_x^{0,0}; G_x^{1,k}; F^{1,k}), (0; G_x^{\star,k}; F^{\star,k})\} \text{ satisfy (2.4) for} \\
& \qquad k \in \{0, 1, \star\} \text{ w.r.t. } \mu_x, L_x \\
& \{(y^0; G_y^{k,0}; F^{k,0}), (y^0 + tG_y^{0,0}; G_y^{k,1}; F^{k,1}), (0; G_y^{k,\star}; F^{k,\star})\} \text{ satisfy (7.3) for} \\
& \qquad k \in \{0, 1, \star\} \text{ w.r.t. } \mu_y, L_y \\
& \|G_x^{k,0} - G_x^{k,1}\| \le L_{xy}\|tG_y^{0,0}\|, \quad k \in \{0, 1, \star\} \\
& \|G_y^{0,k} - G_y^{1,k}\| \le L_{xy}\|tG_x^{0,0}\|, \quad k \in \{0, 1, \star\} \\
& \|G_x^{k,0} - G_x^{k,\star}\| \le L_{xy}\|y^0 - y^\star\|, \quad k \in \{0, 1, \star\} \\
& \|G_y^{0,k} - G_y^{\star,k}\| \le L_{xy}\|x^0 - x^\star\|, \quad k \in \{0, 1, \star\} \\
& \|G_x^{k,1} - G_x^{k,\star}\| \le L_{xy}\|y^0 + tG_y^{0,0} - y^\star\|, \quad k \in \{0, 1, \star\} \\
& \|G_y^{1,k} - G_y^{\star,k}\| \le L_{xy}\|x^0 - tG_x^{0,0} - x^\star\|, \quad k \in \{0, 1, \star\} \\
& G_x^{\star,\star} = 0, \ G_y^{\star,\star} = 0.
\end{aligned}
\tag{7.6}
$$

To approximate the solution of problem (7.6), we reformulate it as a semidefinite program by using the Gram matrix of the unknown vectors in the problem. Indeed, we form the Gram matrices X and Y corresponding to $\{(x^i; G_x^{i,j})\}$ and $\{(y^i; G_y^{j,i})\}$ $(i,j \in \{0,1,\star\})$, respectively. This results in an SDP problem, as long as we view the value $\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2$, that appears in the denominator of the objective of problem (7.6), as a fixed parameter. For this reason we may indeed view problem (7.6) as an SDP problem in the positive semidefinite matrix variables X and Y.

For the convenience of the analysis, we investigate the linear convergence of Algorithm 7.1 in terms of $L = \max\{L_x, L_y\}$ and $\mu = \min\{\mu_x, \mu_y\}$. Before we present the main theorem in this section, we need to present a lemma.

Lemma 7.1. *Let $0 < \mu \leq L$, $c \geq 0$ and let $I = \left(0, \frac{2\mu}{\mu L + c^2}\right)$. Suppose that the function $u : I \to \mathbb{R}$ given by*

$$u(t) = \tfrac{1}{2}\left(L^2 + \mu^2 + 2c^2\right)t^2 - (L+\mu)t + \tfrac{1}{2}(L-\mu)t\sqrt{(Lt + \mu t - 2)^2 + 4c^2t^2}.$$

Then u is convex on I and $u(I) \subseteq [-1, 0)$.

Proof. Consider the function $v : I \to \mathbb{R}$ given by

$$v(t) = \left(L^2 + \mu^2 + 2c^2\right)t + (L-\mu)\sqrt{(Lt + \mu t - 2)^2 + 4c^2t^2}.$$

The function v is convex and positive on I. By elementary calculus, one can show that $v'(0) > 0$. So v is increasing on I due to the convexity. As the product of positive monotone convex functions is a convex function, the function $t \mapsto tv(t)$ is also convex, which implies the convexity of u. Indeed, u is strictly convex on I. Since strictly convex functions attain their maximum on endpoints of a given interval, $u(t) < \max\{u(0), u(\frac{2\mu}{\mu L + c^2})\} = 0$ for $t \in I$. It remains to show that $\min_{t \in I} u(t) \geq -1$. This follows from the point that

$$u(t) \geq \tfrac{1}{2}\left(L^2 + \mu^2\right)t^2 - (L+\mu)t \geq \tfrac{-1}{2}\left(1 + \tfrac{2L\mu}{L^2 + \mu^2}\right) \geq -1,$$

and the proof is complete. □

By the weak duality theorem for SDP, one may demonstrate an upper bound for the optimal value of the SDP problem (7.6), by constructing a feasible solution to its dual problem, i.e. feasible dual multipliers for the constraints of problem (7.6). This is done in the next theorem. In the proof, the correct value of the dual multipliers are simply given, and their correctness is verified. The correct values were obtained by solving the SDP problem (7.6) repeatedly for different numerical values of the parameters, and noting the (numerical) optimal dual multiplier values. Based on these values, it was possible to deduce the analytical expressions for the multipliers. For this reason, the proof was found in a computer-assisted way, but it does not rely on any numerical calculations. Having said that, the proof involves a long identity, given in full in Appendix A.2 to this chapter, that is so long that it could only be obtained in a computer-assisted way.

Theorem 7.2. *Let $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. Suppose that $L = \max\{L_x, L_y\}$ and $\mu = \min\{\mu_x, \mu_y\} > 0$. If $t \in \left(0, \frac{2\mu}{\mu L + L_{xy}^2}\right)$, then Algorithm 7.1 generates (x^1, y^1) such that*

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \leq \alpha\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right), \tag{7.7}$$

where

$$\alpha = 1 + \tfrac{1}{2}\left(L^2 + \mu^2 + 2L_{xy}^2\right)t^2 - (L+\mu)t + \tfrac{1}{2}(L-\mu)t\sqrt{(Lt + \mu t - 2)^2 + 4L_{xy}^2 t^2}.$$

Proof. As mentioned earlier, we may assume without loss of generality that $x^\star = 0$ and $y^\star = 0$. By assumption, $F(\cdot, y) \in \mathcal{F}_{\mu,L}(\mathbb{R}^n)$ and $F(x, \cdot) \in \mathcal{F}_{\mu,L}(\mathbb{R}^m)$ for any fixed x, y. Suppose that $L_{xy} \neq 0$. Without loss of generality, we may also assume that $L_{xy} = 1$, by replacing F by $\frac{1}{L_{xy}}F$. This follows from the observation that Algorithm 7.1 generates the same point (x^1, y^1) for the problem

$$\min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^m} \frac{1}{L_{xy}} F(x, y),$$

with the step length $L_{xy}t$. Moreover, one has $\frac{1}{L_{xy}}F \in \mathcal{F}(\frac{L_x}{L_{xy}}, \frac{L_y}{L_{xy}}, 1, \frac{\mu_x}{L_{xy}}, \frac{\mu_y}{L_{xy}})$ if and only if $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. Now let $t \in \left(0, \frac{2\mu}{\mu L + 1}\right)$ and define (the multipliers):

$$\bar{\alpha} = 1 + \tfrac{1}{2}\left(L^2 + \mu^2 + 2\right)t^2 - (L+\mu)t + \tfrac{1}{2}(L-\mu)t\sqrt{(Lt + \mu t - 2)^2 + 4t^2},$$

$$\beta = \sqrt{(Lt + \mu t - 2)^2 + 4t^2}, \qquad \gamma_1 = \frac{t\left(t^2\left(2 + L^2 + L\mu\right) - t(3L + \mu) + (Lt - 1)\beta + 2\right)}{\beta},$$

$$\gamma_2 = \frac{t\left(t^2\left(2 + \mu^2 + L\mu\right) - t(3\mu + L) + (1 - \mu t)\beta + 2\right)}{\beta}, \quad \gamma_3 = \frac{t^2(\beta + Lt - \mu t)}{2\beta}.$$

It is readily verified that $\gamma_1, \gamma_2, \gamma_3 \geq 0$, but since this calculation is somewhat tedious we present it in Appendix A.1. Moreover, Lemma 7.1 implies that $\bar{\alpha} \in [0, 1)$.

The idea of the proof is now as follows: we first establish that, for any feasible solution of the SDP problem (7.6), it holds that

$$\left\|x^0 - tG_x^{0,0}\right\|^2 + \left\|y^0 + tG_y^{0,0}\right\|^2 - \bar{\alpha}\left(\left\|x^0\right\|^2 + \left\|y^0\right\|^2\right) \le 0. \tag{7.8}$$

We do this by establishing an algebraic identity for the left-hand side of the inequality (7.8). The first and last terms of this identity (shown in full in Appendix A.2 to this chapter) are as follows:

$$\begin{aligned}
& \left\|x^0 - tG_x^{0,0}\right\|^2 + \left\|y^0 + tG_y^{0,0}\right\|^2 - \bar{\alpha}\left(\left\|x^0\right\|^2 + \left\|y^0\right\|^2\right) \\
= \; & -\gamma_1\Big(F^{0,0} - F^{\star,0} - \left\langle G_x^{\star,0}, x^0\right\rangle - \tfrac{L}{2(L-\mu)}\Big(\tfrac{1}{L}\left\|G_x^{0,0} - G_x^{\star,0}\right\|^2 + \mu\left\|x^0\right\|^2 - \tfrac{2\mu}{L}\left\langle G_x^{\star,0} - G_x^{0,0}, -x^0\right\rangle\Big)\Big) \\
& \vdots \\
& -\tfrac{t(\beta+Lt-\mu t)^2}{4(L-\mu)\beta}\left\|G_y^{0,0} - G_y^{\star,0} - G_y^{0,\star}\right\|^2.
\end{aligned}$$

Note that the first term on the right-hand-side is indeed nonpositive, since $\gamma_1 \ge 0$, and the expression in brackets is nonnegative at any feasible solution of the SDP problem (7.6), since it corresponds to one of the constraints in (7.6). The last term is nonpositive as well, since it is the product of a nonpositive multiplier with a squared expression. The remaining terms in the identity are similarly nonpositive (see Appendix A.2), proving the inequality (7.8). All that remains is to recognize that, in (7.8), $G_x^{0,0}$ corresponds to $\nabla_x F(x^0, y^0)$, so that $x^0 - tG_x^{0,0}$ corresponds to x^1, etc. This yields the statement of the theorem, after rescaling to remove the assumption $L_{xy} = 1$. □

One may wonder how we obtained the (analytical) expression for α in Theorem 7.2. Consider the optimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{7.9}$$

where $f \in \mathcal{F}_{\mu,L}$. It is known that the quadratic function $q(x) = x^T Q x$ with $\lambda_{\max}(Q) = L$ and $\lambda_{\min}(Q) = \mu$ attains the worst-case convergence rate for the gradient method; see e.g. [dKGT17]. We guessed that this property may hold for problem (7.1) and we investigated the bilinear saddle point problem

$$\min_{x\in\mathbb{R}^2}\max_{y\in\mathbb{R}^2} \tfrac{1}{2}x^T\begin{pmatrix} L_x & 0 \\ 0 & \mu_x \end{pmatrix}x + x^T\begin{pmatrix} 0 & L_{xy} \\ L_{xy} & 0 \end{pmatrix}y - \tfrac{1}{2}y^T\begin{pmatrix} L_y & 0 \\ 0 & \mu_y \end{pmatrix}y, \tag{7.10}$$

where $L_x \geq \mu_x > 0$, $L_y \geq \mu_y > 0$ and L_{xy} are fixed parameters and we derived the worst case convergence of Algorithm 7.1 with respect to this problem. Our numerical experiments showed that the derived convergence rate is the same as the optimal value of the semidefinite programming problem corresponding to problem (7.6). Moreover, as a by-product, we exhibit that the convergence rate (7.7) is exact for one iteration by using problem (7.10); see Proposition 7.4.

Theorem 7.2 provides some new information concerning Algorithm 7.1. Firstly, Theorem 7.2 improves the known convergence factor in the literature; see our discussion in Introduction. In addition, it investigates the convergence rate for a step length in a larger interval. Secondly, it does not assume the second order continuous differentiability of F, which is commonly used for deriving a local convergence rate; see [LS19, MNG17, ZWLG22]. Finally, the given convergence rate incorporates three parameter $\mu = \min\{\mu_x, \mu_y\}$, $L = \max\{L_x, L_y\}$ and L_{xy}, which is more informative in comparison with the results in the literature mostly given in terms of $\mu = \min\{\mu_x, \mu_y\}$ and $L = \max\{L_x, L_y, L_{xy}\}$; see [MOP20a, ZWLG22, ZHZ22] and references therein. Even if one considers $L = \max\{L_x, L_y, L_{xy}\}$ and $\mu = \min\{\mu_x, \mu_y\}$, convergence rate (7.7) dominates (7.2). This follows from that for $t \in \left(0, \frac{\mu}{2L^2}\right)$, one has

$$\begin{aligned}&\left(1+4L^2t^2-2\mu t\right)-\left(1+\tfrac{1}{2}\left(3L^2+\mu^2\right)t^2-(L+\mu)t+\tfrac{1}{2}(L-\mu)t\sqrt{(Lt+\mu t-2)^2+4L^2t^2}\right)\\ &\geq (2L^2+L\mu-\mu^2)t^2 \geq 2L^2t^2,\end{aligned}$$

where the first inequality results from $\sqrt{(Lt+\mu t-2)^2+4L^2t^2} \leq (2-Lt-\mu t)+2Lt$. In addition, in this case, the step length can take value in a larger interval as $\left(0, \frac{\mu}{2L^2}\right) \subseteq \left(0, \frac{2\mu}{L(L+\mu)}\right)$. Moreover, Conjecture 7.6 discusses the convergence rate in terms of $L_x, L_y, L_{xy}, \mu_x, \mu_y$.

The next proposition gives the optimal step length with respect to the worst case convergence rate.

Proposition 7.3. *Let $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. If $L = \max\{L_x, L_y\}$ and $\mu = \min\{\mu_x, \mu_y\} > 0$, then the optimal step length for Algorithm 7.1 with respect to the bound (7.7) is*

$$t^\star = \frac{2\left((L+\mu)\sqrt{L_{xy}^2+L\mu}+L_{xy}(\mu-L)\right)}{\left(4L_{xy}^2+(L+\mu)^2\right)\sqrt{L_{xy}^2+L\mu}}. \tag{7.11}$$

Moreover, the convergence rate with respect to $t^\star$ is

$$\alpha^\star = \frac{8L_{xy}\left(L^2-\mu^2\right)\sqrt{L\mu+L_{xy}^2}+\left(L^2-\mu^2\right)^2+16L_{xy}^2\left(L\mu+L_{xy}^2\right)}{\left((L+\mu)^2+4L_{xy}^2\right)^2}. \tag{7.12}$$

Proof. Let $\alpha : \left[0, \frac{2\mu}{\mu L + L_{xy}^2}\right] \to \mathbb{R}$ given by

$$\alpha(t) = 1 + \tfrac{1}{2}\left(L^2 + \mu^2 + 2L_{xy}^2\right)t^2 - (L+\mu)t + \tfrac{1}{2}(L-\mu)t\sqrt{(Lt+\mu t-2)^2 + 4L_{xy}^2 t^2}.$$

By Lemma 7.1, α is a strictly convex function on its domain. By doing some algebra, one can verify that $\alpha'(t^\star) = 0$, which implies that $t^\star$ is the minimum. □

If $L_{xy} = 0$, problem (7.1) reduces to a separable optimization problem. Indeed, the variables x and y are independent. Under this assumption, the optimal step length given by Proposition 7.3 is $t^\star = \frac{2}{L+\mu}$, which is the well-known optimal step length for the optimization problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where $f \in \mathcal{F}_{\mu,L}$; see [Nes18, Theorem 2.1.15]. Moreover, the convergence rate corresponding to $t^\star$ is $\alpha^\star = \left(\frac{L-\mu}{L+\mu}\right)^2$. By some algebra, one can show that under the assumptions of Proposition (7.3), Algorithm 7.1 has a complexity of $\mathcal{O}\left(\left(\frac{L}{\mu} + \frac{L_{xy}^2}{\mu^2}\right)\ln\left(\frac{1}{\epsilon}\right)\right)$. Note that the lower iteration complexity bound for first-order methods with $L = \max\{L_x, L_y\}$ and $\mu = \min\{\mu_x, \mu_y\}$ is $\Omega\left(\sqrt{\frac{L}{\mu} + \frac{L_{xy}^2}{\mu^2}}\ln\left(\frac{1}{\epsilon}\right)\right)$; see [ZHZ22].

As mentioned earlier, we calculated the convergence rate by using problem (7.10). The next proposition states that the bound (7.7) is tight for some class of bilinear saddle point problems.

Proposition 7.4. *Let $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. Suppose that $L_x = L_y$ and $\mu = \min\{\mu_x, \mu_y\} > 0$. If $t \in \left(0, \frac{2\mu}{\mu L + L_{xy}^2}\right)$, then convergence rate (7.7) is exact for one iteration.*

Proof. To establish the proposition, it suffices to introduce a problem for which Algorithm 7.1 generates (x^1, y^1) with respect to the initial point (x^0, y^0) such that

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 = \alpha\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right),$$

where α is the convergence rate factor given in Theorem 7.2. Consider problem (7.10). Due to the symmetry of Algorithm 7.1 and the class of problems, we may assume $\mu_x \geq \mu_y$. Moreover, without loss of generality, we can take $L_{xy} = 1$; see our discussion in the proof of Theorem 7.2. Suppose $L = L_x$, $\mu = \mu_y$ and

$\beta = \sqrt{(Lt + \mu t - 2)^2 + 4t^2}$. One can verify that Algorithm 7.1 with the initial point

$$x_1^0 = 0, \qquad x_2^0 = \sqrt{\tfrac{2-t(L+\mu)+\beta}{2\beta}},$$
$$y_1^0 = -t\sqrt{\tfrac{2}{\beta(2-t(L+\mu)+\beta)}}, \qquad y_2^0 = 0.$$

generates (x^1, y^1) with the desired equality. □

One may wonder why we stress on one iteration in Proposition 7.4. Based on our numerical results if $L_{xy} > 0$, under the setting of Theorem 7.2, we observed that

$$\|x^k - x^\star\|^2 + \|y^k - y^\star\|^2 < \alpha^k \left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right), \quad k \geq 2,$$

for some $t \in \left(0, \frac{2\mu}{\mu L + L_{xy}^2}\right)$. The reason may be related to the fact that the vector field $\left(\nabla_x F(x,y) \quad -\nabla_y F(x,y)\right)^T$ is not conservative.

It may be of interest whether inequality (7.7) may hold without strong convexity. By removing strong convexity, the solution set may not be singleton. Hence, we investigate distance to the solution set, that is, if there exists $0 \leq \alpha < 1$ with

$$d_{S^\star}^2((x^1, y^1)) \leq \alpha d_{S^\star}^2((x^0, y^0)).$$

The next proposition says in general the answer is negative. Indeed, it gives an example with $\min\{\mu_x, \mu_y\} = 0$ and a unique saddle point for which

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \geq \alpha\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right),$$

for some $\alpha \geq 1$, no matter how close (x^0, y^0) is to the unique saddle point and which positive step length t is taken. In the next proposition, we may assume without loss of generality $\mu_x = 0$ and make an example analogous to that given in Proposition 7.4.

Proposition 7.5. *Let $L, L_{xy}, \mu_y, t, r > 0$ be given. Then there exist $\alpha \geq 1$ and a function $F \in \mathcal{F}(L, L, L_{xy}, 0, \mu_y)$ with the unique saddle point $(x^\star, y^\star)$ and (x^1, y^1) such that, for (x^2, y^2) generated by Algorithm 7.1, we have*

$$\|x^2 - x^\star\|^2 + \|y^2 - y^\star\|^2 \geq \alpha\left(\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2\right),$$

and $\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 = r^2$.

Proof. As discussed before, we may assume $L_{xy} = 1$. Consider the bilinear saddle point problem,

$$\min_{x\in\mathbb{R}^2}\max_{y\in\mathbb{R}^2} F(x,y) = \tfrac{1}{2}x^T\begin{pmatrix} L & 0 \\ 0 & 0\end{pmatrix}x + x^T\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}y - \tfrac{1}{2}y^T\begin{pmatrix} L & 0 \\ 0 & \mu_y\end{pmatrix}y.$$

It is clear that $F \in \mathcal{F}(L, L, L_{xy}, 0, \mu_y)$, and the unique saddle point is $(x^\star, y^\star) = (0,0)$. Suppose that

$$x_1^1 = 0, \qquad x_2^1 = r\sqrt{\tfrac{2-tL+\beta}{2\beta}},$$

$$y_1^1 = -rt\sqrt{\tfrac{2}{\beta(2-tL+\beta)}}, \qquad y_2^1 = 0,$$

where $\beta = \sqrt{(Lt-2)^2+4t^2}$. One can verify Algorithm 7.1 generates (x^2, y^2) with

$$\|x^2 - x^\star\|^2 + \|y^2 - y^\star\|^2 \geq \alpha\left(\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2\right) = \alpha r^2,$$

where $\alpha = 1 + \frac{1}{2}\left(L^2+2\right)t^2 - Lt + \frac{1}{2}Lt\sqrt{(Lt-2)^2+4t^2}$. By Proposition 7.3, one can infer that $\alpha \geq 1$. □

Note that in Proposition 7.5 r can take any positive value. By Proposition 7.4, one can infer that the convergence rate factor for bilinear saddle point problems may not be improved for one iteration since the given example is a bilinear saddle point problem. Furthermore, the given convergence rate factor is tight whether $L_x = L_y$. As discussed in [WL20], the function $H(x,y) = F\left(\sqrt[4]{\frac{L_y}{L_x}}x, \sqrt[4]{\frac{L_x}{L_y}}y\right)$ shares the same smoothness constants with respect to x and y, that is, $\nabla_x H(\cdot, y)$ and $\nabla_y H(x, \cdot)$ are Lipschitz continuous with the same modulus $\sqrt{L_x L_y}$. However, the gradient methods are not invariant under scaling; see [BV04, Chapter 9]. Hence, we may lose the generality of our discussion by assuming this condition.

Based on our numerical results and analysis of problem (7.10), we conjecture the following (exact) convergence rate of Algorithm 7.1 in terms of $L_x, L_y, L_{xy}, \mu_x, \mu_y$. Due to the symmetry of Algorithm 7.1, we may assume that $L_x \geq L_y$. Moreover, Proposition 7.4 implies that bound (7.7) is tight when $\mu_y \leq \mu_x$. Hence, we need only consider $\mu_y > \mu_x$.

Conjecture 7.6. *Let $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. Suppose that $\mu_y > \mu_x > 0$, $\max\{L_x, L_y\} = L_x$ and*

$$c = \tfrac{1}{2}(L_y^2+\mu_x^2)t - (L_y+\mu_x) + \tfrac{1}{2}(L_y-\mu_x)\sqrt{(L_y t + \mu_x t - 2)^2 + 4L_{xy}^2 t^2},$$

$$\bar{\mu} = \frac{c + 2L_x - L_x^2 t + L_x L_{xy}^2 t^2 - (c + L_x(2 - L_x t))\sqrt{1 + t(c + tL_{xy}^2)}}{tL_{xy}(c + tL_{xy}^2 + L_x(2 - L_x t))},$$

$$\alpha(\mu, L, L_{xy}, t) = 1 + \tfrac{1}{2}\left(L^2 + \mu^2 + 2L_{xy}^2\right)t^2 - (L+\mu)t + \tfrac{1}{2}(L-\mu)t\sqrt{(Lt+\mu t-2)^2 + 4L_{xy}^2 t^2}.$$

Then, one of the following scenarios holds.

a) *Assume that* $\mu_x\mu_y(L_x - L_y) \geq L_{xy}^2\left(\mu_y - \mu_x\right)$ *and* $t \in \left(0, \frac{2\mu_y}{L_x\mu_y + L_{xy}^2}\right)$.

i) *If* $\mu_y \leq \bar{\mu}$, *then*

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \leq \alpha(\mu_y, L_x, L_{xy}, t)\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right).$$

ii) *If* $\mu_y \geq \bar{\mu}$, *then*

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \leq \alpha(\mu_x, L_y, L_{xy}, t)\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right).$$

b) *Assume that* $\mu_x\mu_y(L_x - L_y) \leq L_{xy}^2\left(\mu_y - \mu_x\right)$ *and* $t \in \left(0, \frac{2\mu_x}{L_y\mu_x + L_{xy}^2}\right)$.

i) *If* $\mu_y \leq \bar{\mu}$, *then*

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \leq \alpha(\mu_y, L_x, L_{xy}, t)\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right).$$

ii) *If* $\mu_y \geq \bar{\mu}$, *then*

$$\|x^1 - x^\star\|^2 + \|y^1 - y^\star\|^2 \leq \alpha(\mu_x, L_y, L_{xy}, t)\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right).$$

Although we have extensive numerical evidence supporting Conjecture 7.6, we have been unable to prove either part a) or part b). To be more precise, we have verified numerically that the optimal value of the SDP problem (7.6) and problem (7.10) corresponds to the expressions in Conjecture 7.6 for many different numerical values of the parameters L_x, L_y, L_{xy}, μ_x, and μ_y, but we were unable to derive analytical expressions for the dual multipliers of the SDP problem (7.6) that would prove the conjecture.

7.2.1 Numerical illustration

In this section we provide randomly generated examples to compare the optimal step length (7.11) given in this chapter to the known step length $t = \mu/(4L^2)$ for the bilinear problem

$$\min_{x \in \mathbb{R}^5} \max_{y \in \mathbb{R}^4} \tfrac{1}{2}x^T A_x x + x^T A_{xy} y - \tfrac{1}{2}y^T A_y y,$$

where A_x and A_y are symmetric positive definite matrices. Moreover, the instances are constructed such that the spectra of A_x and A_y are contained in the interval $[0.5, 5]$. For this class of instances, one has $L = \max\{\lambda_{\max}(A_x), \lambda_{\max}(A_y)\} \in$

$[0.5, 5]$ and $\mu = \min\{\lambda_{\min}(A_x), \lambda_{\min}(A_y)\} \in [0.5, L]$. The matrix $A_{xy} \in \mathbb{R}^{5\times 4}$ has entries chosen uniformly at random from $[0, 1]$, and subsequently we set $L_{xy} = \|A_{xy}\|_2$. By construction, the solution (saddle point) is $(x^\star, y^\star) = (0, 0)$. The starting points x^0 and y^0 are randomly drawn unit vectors so that the initial condition $\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2 = 2$ is satisfied.

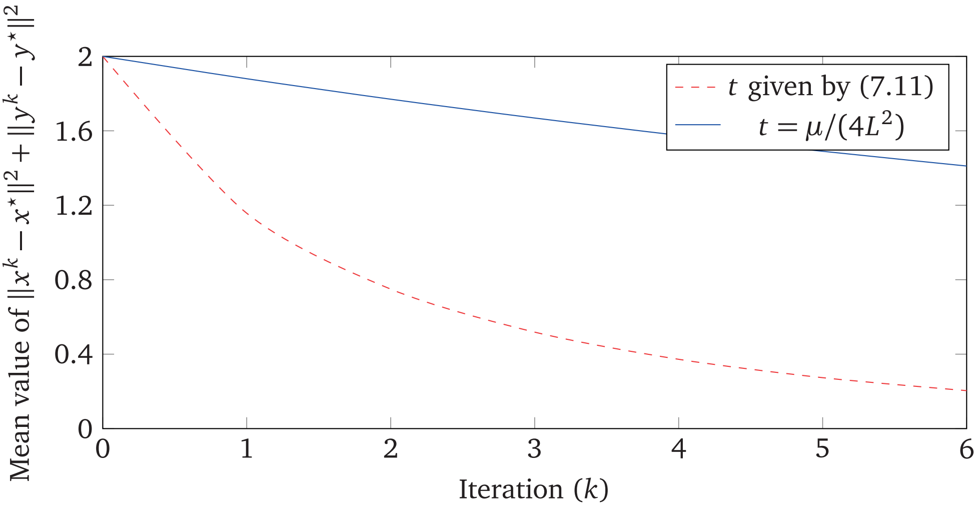


Figure 7.1: Mean values of $\|x^k - x^\star\|^2 + \|y^k - y^\star\|^2$ for 100 randomly generated instances for each iteration k using the two different step lengths t

In Figure 7.1 we show average values (over 100 randomly generated instances) for the convergence indicator $\|x^k - x^\star\|^2 + \|y^k - y^\star\|^2$ after k iterations, for the two step lengths.[1] Note that our new step length (7.11) gives a clear improvement over the known step length $\mu/(4L^2)$.

7.3 Linear convergence without strong convexity

In this section, we study the linear convergence of Algorithm 7.1 without assuming strong convexity. Indeed, we suppose that $F \in \mathcal{F}(L_x, L_y, L_{xy}, 0, 0)$ and we propose some necessary and sufficient conditions for the linear convergence. This subject has received some attention in recent years and some sufficient conditions have

[1] The 100 random instances and starting points that we generated to produce Figure 7.1 may be found on GitHub; see: `https://github.com/molsemzamani/Bilinear-Minimax`

been proposed in [DH19, ZWLG22] under which Algorithm 7.1 enjoys local linear convergence rate or it is linearly convergent for bilinear saddle point problems. This topic has been investigated extensively in the context of optimization. The interested reader can refer to [AdKZ23a, BNPS17, LT93, NNG19] alongside with Chapter 5 and references therein. In this study, we extend the quadratic gradient growth property introduced in [LT93] for saddle point problems.

Recall that we denote the nonempty solution set of problem (7.1) by $S^\star$. As we do not assume the strong convexity (concavity), $S^\star$ may not be singleton. Note that $S^\star$ is a closed convex set under our assumptions. Recall that $\Pi_{S^\star}((x,y))$ denotes the projection of (x,y) onto $S^\star$.

Definition 7.7. Let $\mu_F > 0$. A function F has a quadratic gradient growth if for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$,

$$\langle \nabla_x F(x,y), x - x^\star \rangle - \langle \nabla_y F(x,y), y - y^\star \rangle \geq \mu_F d^2_{S^\star}((x,y)), \tag{7.13}$$

where $(x^\star, y^\star) = \Pi_{S^\star}((x,y))$.

Note that if we set $y = y^\star$ in (7.13), we have

$$\langle \nabla_x F(x,y^\star), x - x^\star \rangle \geq \mu_F \|x - x^\star\|^2.$$

Hence, L_x-smoothness implies that $\mu_F \leq L_x$. Consequently, due to the symmetry, we have $\mu_F \leq \min\{L_x, L_y\}$. The next proposition states that the quadratic gradient growth condition is weaker than the strong convexity-strong concavity. Indeed, the strong convexity-strong concavity implies the quadratic gradient growth property.

Proposition 7.8. *Let $F \in \mathcal{F}(L_x, L_y, L_{xy}, \mu_x, \mu_y)$. If* $\min\{\mu_x, \mu_y\} > 0$*, then F has a quadratic gradient growth with $\mu_F = \min\{\mu_x, \mu_y\}$.*

Proof. Under the assumptions, problem (7.1) has a unique solution $(x^\star, y^\star)$ and $\nabla_x F(x^\star, y^\star) = 0$ and $\nabla_y F(x^\star, y^\star) = 0$. Let $\mu = \min\{\mu_x, \mu_y\}$ and $L = \max\{L_x, L_y\}$.

Suppose that $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$. By Theorem 2.37, we have

$$
\begin{aligned}
0 \le & \Big(F(x^\star,y)-F(x,y)+\langle\nabla_x F(x,y),x-x^\star\rangle-\tfrac{L}{2(L-\mu)}\Big(\tfrac{1}{L}\|\nabla_x F(x^\star,y)-\nabla_x F(x,y)\|^2 \\
& +\mu\|x-x^\star\|^2-\tfrac{2\mu}{L}\langle\nabla_x F(x,y)-\nabla_x F(x^\star,y),x-x^\star\rangle\Big)\Big)+\Big(F(x,y^\star)-F(x^\star,y^\star)- \\
& \tfrac{L}{2(L-\mu)}\Big(\tfrac{1}{L}\|\nabla_x F(x,y^\star)\|^2+\mu\|x-x^\star\|^2-\tfrac{2\mu}{L}\langle\nabla_x F(x,y^\star),x-x^\star\rangle\Big)\Big)+\Big(F(x,y)- \\
& F(x,y^\star)-\left\langle\nabla_y F(x,y),y-y^\star\right\rangle-\tfrac{L}{2(L-\mu)}\Big(\tfrac{1}{L}\left\|\nabla_y F(x,y^\star)-\nabla_y F(x,y)\right\|^2+\mu\|y-y^\star\|^2 \\
& -\tfrac{2\mu}{L}\left\langle\nabla_y F(x,y^\star)-\nabla_y F(x,y),y-y^\star\right\rangle\Big)\Big)+\Big(F(x^\star,y^\star)-F(x^\star,y)-\tfrac{L}{2(L-\mu)} \\
& \Big(\tfrac{1}{L}\left\|\nabla_y F(x^\star,y)\right\|^2+\mu\|y-y^\star\|^2-\tfrac{2\mu}{L}\left\langle\nabla_y F(x^\star,y),y^\star-y\right\rangle\Big)\Big) \\
= & \tfrac{-\mu^2}{L-\mu}\left\|(x-x^\star)-\tfrac{1}{2\mu}\left(\nabla_x F(x,y)+\nabla_x F(x,y^\star)-\nabla_x F(x^\star,y)\right)\right\|^2- \\
& \tfrac{1}{4(L-\mu)}\|\nabla_x F(x,y)-\nabla_x F(x,y^\star)-\nabla_x F(x^\star,y)\|^2- \\
& \tfrac{\mu^2}{L-\mu}\left\|(y-y^\star)+\tfrac{1}{2\mu}\left(\nabla_y F(x,y)-\nabla_y F(x,y^\star)+\nabla_y F(x^\star,y)\right)\right\|^2- \\
& \tfrac{1}{4(L-\mu)}\left\|\nabla_y F(x,y)-\nabla_y F(x,y^\star)-\nabla_y F(x^\star,y)\right\|^2- \\
& \mu\left(\|x-x^\star\|^2+\|y-y^\star\|^2\right)+\langle\nabla_x F(x,y),x-x^\star\rangle-\left\langle\nabla_y F(x,y),y-y^\star\right\rangle.
\end{aligned}
$$

Hence,

$$
\mu\left(\|x-x^\star\|^2+\|y-y^\star\|^2\right)\le\langle\nabla_x F(x,y),x-x^\star\rangle-\left\langle\nabla_y F(x,y),y-y^\star\right\rangle,
$$

and the proof is complete. □

Note that the converse of Proposition 7.8 does not hold necessarily. Consider the following saddle point problem

$$
\min_{x\in\mathbb{R}}\max_{y\in\mathbb{R}} F(x,y):=f(x+y)-2y^2, \tag{7.14}
$$

where

$$
f(s)=\begin{cases}0 & |s|\le 1\\ (s-1)^2 & s>1\\ (s+1)^2 & s<-1.\end{cases}
$$

It is seen that F is not strongly convex-strongly concave and the solution set of problem (7.14) is $\{(x,0):|x|\le 1\}$. By doing some algebra, one can check that F has a quadratic gradient growth with $\mu_F=1$ while it is not strongly convex with respect to the first component. For the case that $F(\cdot,y)$ is neither strongly convex nor is $F(x,\cdot)$ strongly concave, one may consider uncoupled problem $\min_{x\in\mathbb{R}}\max_{y\in\mathbb{R}} f(x)-f(y)$.

In what follows, by using performance estimation, we establish that Algorithm 7.1 enjoys the linear convergence whether $F\in\mathcal{F}(L_x,L_y,L_{xy},0,0)$ has a quadratic gradient growth. Without loss of generality, we may assume that $(0,0)=\Pi_{S^\star}\left((x^0,y^0)\right)$. To establish the linear convergence, it suffices to show that

$$d^2_{S^\star}((x^1,y^1))\le\|x^1\|^2+\|y^1\|^2\le\alpha d^2_{S^\star}((x^0,y^0)),$$

for some $\alpha\in[0,1)$. Similarly to Section 7.2, we formulate the following optimization problem

$$\begin{aligned}
\max\ & \frac{\|x^0-tG_x^{0,0}\|^2+\|y^0+tG_y^{0,0}\|^2}{\|x^0\|^2+\|y^0\|^2}\\
\text{s. t. }& \{(x^0;G_x^{0,k};F^{0,k}),(x^0-tG_x^{0,0};G_x^{1,k};F^{1,k}),(0;G_x^{\star,k};F^{\star,k})\}\text{ satisfy (2.4) for}\\
&\quad k\in\{0,1,\star\}\text{ w.r.t. }\mu_x=0,L_x\\
&\{(y^0;G_y^{k,0};F^{k,0}),(y^0+tG_y^{0,0};G_y^{k,1};F^{k,1}),(0;G_y^{k,\star};F^{k,\star})\}\text{ satisfy (7.3) for}\\
&\quad k\in\{0,1,\star,*\}\text{ w.r.t. }\mu_y=0,L_y \qquad (7.15)\\
&\|G_x^{k,i}-G_x^{k,j}\|\le L_{xy}\|y^i-y^j\|,\quad i,j,k\in\{0,1,\star\}\\
&\|G_y^{i,k}-G_y^{j,k}\|\le L_{xy}\|x^i-x^j\|,\quad i,j,k\in\{0,1,\star\}\\
&\mu_F\left(\|x^0\|^2+\|y^0\|^2\right)\le\langle G_x^{0,0},x^0\rangle-\langle G_y^{0,0},y^0\rangle,\\
&G_x^{\star,\star}=0,\ G_y^{\star,\star}=0.
\end{aligned}$$

Note that in the formulation (7.15), we only use a subset of constraints for the performance estimation. In the next theorem, we prove the linear convergence of Algorithm 7.1 when F has a quadratic gradient growth.

Theorem 7.9. *Let $F\in\mathcal{F}(L_x,L_y,L_{xy},0,0)$ and $L=\max\{L_x,L_y\}$. Assume that F has a quadratic gradient growth with $\mu_F>0$. If $t\in\left(0,\frac{2\mu_F}{L\mu_F+2L_{xy}\sqrt{\mu_F(L-\mu_F)}+L_{xy}^2}\right)$, then Algorithm 7.1 generates (x^1,y^1) such that*

$$d^2_{S^\star}((x^1,y^1))\le\alpha d^2_{S^\star}((x^0,y^0)), \qquad (7.16)$$

where

$$\alpha=t\left(2tL_{xy}\sqrt{\mu_F(L-\mu_F)}+\mu_F(Lt-2)+tL_{xy}^2\right)+1.$$

Proof. The argument is similar to that of Theorem 7.2. It is seen that for any step length t in the given interval, $\alpha \in [0,1)$. We may assume without loss of generality $L_{xy} = 1$. By the assumptions, $F(\cdot, y) \in \mathcal{F}_{0,L}(\mathbb{R}^n)$ and $F(x, \cdot) \in \mathcal{F}_{0,L}(\mathbb{R}^m)$ for any fixed x, y. Suppose that

$$\bar{\alpha} = t\left(2t\sqrt{\mu_F(L-\mu_F)} + \mu_F(Lt-2) + t\right) + 1, \quad \beta = t^2\left(\mu_F\sqrt{L-\mu_F} + \sqrt{\mu_F}\right),$$
$$\gamma_1 = t^2\left(\frac{\mu_F}{\sqrt{\mu_F(L-\mu_F)}} + \mu_F\right), \qquad \gamma_2 = \frac{t^2\left(\mu_F(L-\mu_F) + \sqrt{\mu_F(L-\mu_F)}\right)}{\mu_F},$$
$$\gamma_3 = -\frac{t^2\left(\mu_F(L+\mu_F) + \sqrt{\mu_F(L-\mu_F)}\right)}{\mu_F} + \frac{\beta}{\sqrt{L-\mu_F}} + 2t, \quad \gamma_4 = \tfrac{1}{2}t^2\left(\sqrt{\mu_F(L-\mu_F)} + 1\right).$$

One may readily verify that $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0$. By doing some algebra, one can show that

$$\begin{aligned}
&\left\|x^0 - tG_x^{0,0}\right\|^2 + \left\|y^0 + tG_y^{0,0}\right\|^2 - \bar{\alpha}\left(\left\|x^0\right\|^2 + \left\|y^0\right\|^2\right) + \gamma_1\Big(F^{0,0} - F^{\star,0} - \left\langle G_x^{\star,0}, x^0\right\rangle - \\
&\tfrac{1}{2L}\left\|G_x^{0,0} - G_x^{\star,0}\right\|^2\Big) + \gamma_2\Big(F^{\star,0} - F^{0,0} + \left\langle G_x^{0,0}, x^0\right\rangle - \tfrac{1}{2L}\left\|G_x^{\star,0} - G_x^{0,0}\right\|^2\Big) + \gamma_2\Big(F^{0,\star} - \\
&F^{\star,\star} - \tfrac{1}{2L}\left\|G_x^{0,\star}\right\|^2\Big) + \gamma_1\Big(F^{\star,\star} - F^{0,\star} + \left\langle G_x^{0,\star}, x^0\right\rangle - \tfrac{1}{2L}\left\|G_x^{0,\star}\right\|^2\Big) + \gamma_1\Big(F^{0,\star} - F^{0,0} + \\
&\left\langle G_y^{0,\star}, y^0\right\rangle - \tfrac{1}{2L}\left\|G_y^{0,0} - G_y^{0,\star}\right\|^2\Big) + \gamma_2\Big(F^{0,0} - F^{0,\star} - \left\langle G_y^{0,0}, y^0\right\rangle - \tfrac{1}{2L}\left\|G_y^{0,\star} - G_y^{0,0}\right\|^2\Big) + \\
&\gamma_2\Big(-F^{\star,0} + F^{\star,\star} - \tfrac{1}{2L}\left\|G_y^{\star,0}\right\|^2\Big) + \gamma_1\Big(-F^{\star,\star} + F^{\star,0} + \left\langle G_y^{\star,0}, -y^0\right\rangle - \tfrac{1}{2L}\left\|G_y^{\star,0}\right\|^2\Big) + \\
&\gamma_3\Big(\left\langle G_x^{0,0}, x^0\right\rangle - \left\langle G_y^{0,0}, y^0\right\rangle - \mu_F\Big(\left\|x^0\right\|^2 + \left\|y^0\right\|^2\Big)\Big) + \gamma_4\left(\left\|x^0\right\|^2 - \left\|G_y^{0,0} - G_y^{\star,0}\right\|^2\right) + \\
&\gamma_4\left(\left\|x^0\right\|^2 - \left\|G_y^{0,\star}\right\|^2\right) + \gamma_4\left(\left\|y^0\right\|^2 - \left\|G_x^{0,0} - G_x^{0,\star}\right\|^2\right) + \gamma_4\left(\left\|y^0\right\|^2 - \left\|G_x^{\star,0}\right\|^2\right) \\
=& -\zeta_1\left\|x^0 + \zeta_2 G_x^{0,0} - \zeta_3(G_x^{0,\star} - G_x^{\star,0})\right\|^2 - \zeta_4\left\|G_x^{0,0} - G_x^{0,\star} - G_x^{\star,0}\right\|^2 - \\
&\zeta_1\left\|y^0 - \zeta_2 G_y^{0,0} - \zeta_3(G_y^{0,\star} - G_y^{\star,0})\right\|^2 - \zeta_4\left\|G_y^{0,0} - G_y^{\star,0} - G_y^{0,\star}\right\|^2 \leq 0,
\end{aligned}$$

where the multipliers $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are given as follows

$$\zeta_1 = \mu_F\left(\frac{\beta}{\sqrt{L-\mu_F}} - \mu_F t^2\right), \ \zeta_2 = \frac{\beta}{2\mu_F\sqrt{\mu_F}t^2} - \frac{1}{\mu_F}, \ \zeta_3 = \frac{t^2\left(\sqrt{\mu_F(L-\mu_F)}+1\right)}{2\mu_F t^2},$$
$$\zeta_4 = \frac{1}{4}\left(\frac{2t^2(\mu_F(L-\mu_F)+1)}{\sqrt{\mu_F(L-\mu_F)}} - \frac{\left(2\mu_F t^2(\mu_F - L) + \beta\sqrt{L-\mu_F}\right)^2}{\mu_F(L-\mu_F)t^2\sqrt{\mu_F(L-\mu_F)}}\right).$$

One can show by some algebra that $\zeta_1, \zeta_4 \geq 0$. Hence, for any feasible solution of problem (7.15), we have

$$\frac{\left\|x^0 - tG_x^{0,0}\right\|^2 + \left\|y^0 + tG_y^{0,0}\right\|^2}{\|x^0\|^2 + \|y^0\|^2} \leq \bar{\alpha},$$

and the proof is complete. □

We obtained the linear convergence by using quadratic gradient growth in Theorem 7.9. The next theorem states that quadratic gradient growth property is also a necessary condition for the linear convergence.

Theorem 7.10. *If Algorithm 7.1 is linearly convergent for any initial point, then F has a quadratic gradient growth for some $\mu_F > 0$.*

Proof. Let $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^m$ and (x^1, y^1) be generated by Algorithm 7.1. Suppose that $(x^\star, y^\star) = \Pi_{S^\star}\left((x^1, y^1)\right)$. As Algorithm 7.1 is linearly convergent, there exist $\alpha \in [0, 1)$ with

$$d_{S^\star}^2((x^1, y^1)) \leq \alpha d_{S^\star}^2((x^0, y^0)) \leq \alpha\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right). \tag{7.17}$$

By setting $x^1 = x^0 - t\nabla_x F(x^0, y^0)$ and $y^1 = y^0 + t\nabla_y F(x^0, y^0)$ in inequality (7.17), we get

$$\tfrac{1-\alpha}{2t}\left(\|x^0 - x^\star\|^2 + \|y^0 - y^\star\|^2\right) \leq \left\langle \nabla_x F(x^0, y^0), x^0 - x^\star\right\rangle - \left\langle \nabla_y F(x^0, y^0), y^0 - y^\star\right\rangle,$$

which implies that

$$\mu_F d_{S^\star}^2(x^0, y^0) \leq \left\langle \nabla_x F(x^0, y^0), x^0 - x^\star\right\rangle - \left\langle \nabla_y F(x^0, y^0), y^0 - y^\star\right\rangle,$$

for $\mu_F = \frac{1-\alpha}{2t}$ and the proof is complete. □

7.4 Concluding remarks

In this chapter, we provided a new convergence rate for the gradient descent-ascent method for saddle point problems. Furthermore, we gave some necessary and sufficient conditions for the linear convergence without strong convexity. We employed the performance estimation method for proving the results. For future work, it would be interesting to consider the case where the variables x and y in the saddle point problem are constrained to lie in given, compact convex sets, since many saddle point problems fall in this category. In this case, one could use the performance estimation framework to analyze other methods, e.g. proximal type algorithms.

Be less curious about people and more curious about ideas.

Marie Curie

8 On the convergence rate of the difference-of-convex algorithm (DCA)

Preamble

In this chapter, we study the non-asymptotic convergence rate of the DCA (difference-of-convex algorithm), also known as the convex–concave procedure, with two different termination criteria that are suitable for smooth and non-smooth decompositions, respectively. The DCA is a popular algorithm for difference-of-convex (DC) problems and known to converge to a stationary point of the objective under some assumptions. We derive a worst-case convergence rate of $\mathcal{O}(1/\sqrt{N})$ after N iterations of the objective gradient norm for certain classes of DC problems, without assuming strong convexity in the DC decomposition and give an example which shows the convergence rate is exact. We also provide a new convergence rate of $\mathcal{O}(1/N)$ for the DCA with the second termination criterion. Furthermore, we study the convergence rate for the proximal gradient method. Additionally, we study the impact of using regularization in DCA. Moreover, we derive a new linear convergence rate result for the DCA under the assumption of the Polyak–Łojasiewicz inequality. The novel aspect of our analysis is that it employs semidefinite programming performance estimation. This chapter is based on the paper [AdKZ23c], except for Section 8.2.2, 8.3.3, 8.3.4 and 8.5 that deals with an example for DCA, studying the gradient descent method, studying the proximal gradient method, and DCA with regularization, respectively.

8.1 Introduction

We consider the general *difference-of-convex (DC) optimization problem*,

$$\begin{aligned} &\inf f(x) := f_1(x) - f_2(x) \\ &\text{s.t. } x \in \mathbb{R}^n, \end{aligned} \tag{8.1}$$

where $f_1, f_2 : \mathbb{R}^n \to (-\infty, \infty]$ are convex functions and f is an lower-semicontinuous function on $\mathbb{R}^n$ to $(-\infty, \infty]$. Throughout the chapter, we assume that the infimum in problem (8.1) is finite, and denote by $f^\star$ a lower bound of f on $\mathbb{R}^n$.

DC problems appear naturally in many applications, e.g. power allocation in digital communication systems [ASP14], production-transportation planning [HT99a], location planning [CHJT98], image processing [LZOX15], sparse signal recovering [GRC09], cluster analysis [BU18, BTU16], and supervised data classification [AFG12, LTN17], to name but a few.

This wide range of applications is to be expected, since some important classes of nonconvex functions may be represented as DC functions. For instance, twice continuously differentiable functions on any convex subset of $\mathbb{R}^n$ [Har59], and continuous piece-wise linear functions [Mel86] may be written as DC functions. Furthermore, every continuous function on a compact and convex set can be approximated by a DC function [HT99b, THH$^+$98]. We refer the interested reader to [HU85, THH$^+$98] for more information on DC representable functions.

The celebrated Difference-of-Convex Algorithm (DCA), also known as the convex-concave procedure, has been applied extensively to problem (8.1); see [LTPD18, LB16, TA97] and the references therein. Algorithm 8.1 presents the basic form of the *DCA*.

In the description of the DCA in Algorithm 8.1, (sub)gradients of f_1 and f_2 are assumed to be available at given points, the so-called black-box formulation. The DCA is sometimes also presented as a primal-dual method, where a dual subproblem is solved to obtain the required (sub)gradients; see [LTPD18, LB16] for further discussions of this topic. In recent years, some scholars have also extended

Algorithm 8.1 DCA

Pick $x^0 \in \mathbb{R}^n$.

For $k = 0, 1, \dots$ perform the following steps:

1. Choose $g_2^k \in \partial f_2(x^k)$.
2. Choose

$$x^{k+1} \in \operatorname{argmin}_{x \in \mathbb{R}^n} f_1(x) - f_2(x^k) - \langle g_2^k, x - x^k \rangle. \tag{8.2}$$

3. If the termination criteria are satisfied, then stop.

---

the DCA and proposed some new variations; see [GTT18, LZ19, LZS19, PRA17, SS22].

The first convergence results for Algorithm 8.1 were given in [TA97, Theorem 3(iv)]. The authors showed that, if the sequence of iterates $\{x^k\}$ is bounded, then each accumulation point of this sequence is a critical point of f.

Le Thi et al. [LTHPD18] established an asymptotic linear convergence rate of $\{x^k\}$ under some conditions, in particular under the assumption that f satisfies the *Łojasiewicz gradient inequality* at all stationary points. Recall that a differentiable function f is said to satisfy this inequality at a stationary point a ($\nabla f(a) = 0$), if there exist constants $\theta \in (0, 1)$, $C > 0$ and $\epsilon > 0$ such that

$$|f(x) - f(a)|^\theta \leq C\|\nabla f(x)\| \text{ if } \|x - a\| \leq \epsilon, \tag{8.3}$$

where the constant θ is called the *Łojasiewicz exponent*. This inequality is known to hold, for example, for real analytic functions, but has been extended to include classes of non-smooth functions as well by considering general sub-differentials instead of gradients; see [BDL07, BST14], and the references therein.

The convergence rates established by Le Thi et al. [LTHPD18] depend on the value of the Łojasiewicz exponent, as the following theorem shows. The theorem stated here is a special case of Theorems 3.4 and 3.5 in [LTHPD18], to give a flavor of the convergence results in [LTHPD18].

Theorem 8.1 (Theorems 3.4 and 3.5 in Le Thi et al. [LTHPD18]). *Let f_1 and f_2 be proper convex functions and let the domain of f be closed. Also assume that at least one of f_1 and f_2 is strongly convex, and f_1 or f_2 is differentiable with locally Lipschitz gradient in every critical point of the DC problem. Finally, assume the sequence $\{x^k\}$*

is bounded, and let x^∞ *be a limit point of* $\{x^k\}$. *Then* x^∞ *is also a stationary point. Moreover, if* f *satisfies the Łojasiewicz gradient inequality* (8.3) *at all stationary points, then*

1. *if* $\theta \in (1/2, 1)$, *then* $\|x^k - x^\infty\| \le c k^{\frac{1-\theta}{1-2\theta}}$ *for some* $c > 0$.

2. *if* $\theta \in (0, 1/2]$, *then* $\|x^k - x^\infty\| \le c q^k$ *for some* $c > 0$ *and* $q \in (0, 1)$.

In particular, item 2 shows a linear convergence rate when $\theta \in (0, 1/2]$. Yen et al. [YPWL12] had already shown linear convergence earlier for a much smaller class of DC functions. We will present a complementary result to this theorem (see Theorem 8.20 below), for the case $\theta = 1/2$, where we show linear convergence of the objective function values, and give explicit expressions for the constants that determine the linear convergence rate. Moreover, we will relax the assumption of a bounded sequence of iterates, and the assumption of strong convexity.

In the absence of conditions like the Łojasiewicz gradient inequality (8.3), only weaker convergence rates are known for the DCA. In particular, Tao and An [TA97, Proposition 2] and Le Thi et al. [LTPD21, Corollary 1] have shown an $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ convergence rate after N iterations under suitable assumptions, as given in the next theorem.

Theorem 8.2 (Corollary 1 in [LTPD21], Proposition 2 in [TA97])**.** *If* x^∞ *is a limit point of the iteration sequence generated by the DCA, and at least one of* f_1 *and* f_2 *is strongly convex, i.e. , for some* $\mu_1, \mu_2 \ge 0$ *such that* $\mu_1 + \mu_2 > 0$,

$$x \mapsto f_i(x) - \frac{\mu_i}{2}\|x\|^2 \text{ is convex for } i \in \{1, 2\},$$

then the series $\|x^k - x^{k-1}\|$ *converges, and, after* $N + 1$ *iterations,*

$$\sum_{k=0}^{N} \|x^k - x^{k-1}\|^2 \le \frac{2(f(x^0) - f(x^N))}{\mu_1 + \mu_2},$$

and, consequently,

$$\min_{0 \le k \le N} \|x^k - x^{k-1}\| \le \sqrt{\frac{2(f(x^0) - f^\star)}{(\mu_1 + \mu_2)N}} = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

We will derive some variants on this $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ convergence result in Corollary 8.7 and in Section 8.3.2, where we improve the constants in the $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ bounds. We also show that we obtain the best possible constants, by demonstrating an example where our bound in Corollary 8.7 is tight.

Outline and further contributions of this chapter

The novel aspect of the analysis in this chapter is that we will apply performance estimation to derive convergence rates. This chapter is organized as follows. In Section 8.2 we review some definitions and notions from convex analysis, which will be used in the following sections. We continue with a simple example to show how DCA works in Section 8.2.2. We study the DCA for sufficiently smooth DC decompositions in Section 8.3. By using performance estimation, we give a convergence rate of $\mathcal{O}(1/\sqrt{N})$ in Corollary 8.7, without any strong convexity assumption, thus extending and complementing Le Thi et al. [LTPD21, Corollary 1]. We construct an example that shows this $\mathcal{O}(1/\sqrt{N})$ bound is tight. Since the first termination criterion is not suitable for the analysis of nonsmooth DC compositions, we investigate the DCA with another stopping criterion in Section 8.4, and we show a convergence rate of $\mathcal{O}(1/N)$. This result is completely new to the best of our knowledge. Furthermore, we investigate DCA with regularization and derive a convergence rate for this version. Moreover, we discuss the best choice of regularization parameter from worst-case complexity perspective. In Section 8.6 we study the DCA when the objective function satisfies the Polyak-Łojasiewicz inequality, and we derive a linear convergence rate in Theorem 8.20, thereby refining some linear convergence results in Le Thi et al. [LTHPD18] as described above.

8.2 Basic Definitions and Preliminaries

$I_{\mathbb{R}_+}$ stands for the indicator function on $\mathbb{R}_+ \cup \{\infty\}$, i.e.,

$$I_{\mathbb{R}_+}(x) = \begin{cases} 1 & x \geq 0 \cup \{\infty\} \\ 0 & x < 0 \cup \{-\infty\}. \end{cases} \tag{8.4}$$

We denote the *convex hull* of $X \subseteq \mathbb{R}^n$ by $\text{co}(X)$. We adopt the conventions that, for $a, b, c, d \in \mathbb{R}$ with $c \neq d$ and $a \neq 0$, $\frac{b}{\infty} = 0, 0 \times \infty = 0$ and $\frac{a\infty + b}{c\infty - d\infty} = \frac{a}{c-d}$.

Let $L \in (0, \infty]$ and $\mu \in (0, \infty)$. We call an extended convex function $f : \mathbb{R}^n \to (-\infty, \infty]$ L-smooth if for any $x_1, x_2 \in \mathbb{R}^n$,

$$\|g_1 - g_2\| \leq L\|x_1 - x_2\| \ \ \forall g_1 \in \partial f(x_1),\ g_2 \in \partial f(x_2).$$

Note that if $L < \infty$, then f must be differentiable on $\mathbb{R}^n$. In addition, any extended convex function is ∞-smooth.

Lemma 8.3. *If $-\eta \leq \mu$, then $\mathcal{F}_{\mu,L}(\mathbb{R}^n) + \frac{\eta}{2}\|\cdot\|^2 = \mathcal{F}_{\mu+\eta,L+\eta}(\mathbb{R}^n)$.*

Proof. First, we show the inclusion $\mathcal{F}_{\mu,L}(\mathbb{R}^n) + \frac{\eta}{2}\|\cdot\|^2 \subseteq \mathcal{F}_{\mu+\eta,L+\eta}(\mathbb{R}^n)$. Let $f \in \mathcal{F}_{\mu,L}$. The $(\mu+\eta)$-strong convexity of the function $f + \frac{\eta}{2}\|\cdot\|^2$ is immediate from the definition. It is easily seen the inclusion holds when $L = \infty$. Hence, we investigate $L < \infty$. By *L-smoothness*, we have

$$f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \tfrac{L}{2}\|y - x\|^2.$$

By adding $\frac{\eta}{2}\|y\|^2 = \frac{\eta}{2}\|x\|^2 + \langle \eta x, y - x\rangle + \frac{\eta}{2}\|y - x\|^2$ to the above inequality, we get

$$f(y) + \tfrac{\eta}{2}\|y\|^2 \leq f(x) + \tfrac{\eta}{2}\|x\|^2 + \langle \nabla f(x) + \eta x, y - x\rangle + \tfrac{L+\eta}{2}\|y - x\|^2,$$

which establishes $(L+\eta)$-smoothness $f + \frac{\eta}{2}\|\cdot\|^2$; see Theorem 2.15 in [Nes18]. Now, we establish the converse inclusion. Suppose that $f \in \mathcal{F}_{\mu+\eta,L+\eta}(\mathbb{R}^n)$. By the definition, it follows that $f - \frac{\eta}{2}\|\cdot\|^2$ is μ-strongly convex. The L-smoothness of f is proved similar to the former case and the proof is complete. □

In the next lemma, we extend the descent lemma for DCA when L_1 or L_2 is finite.

Lemma 8.4. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$ and let $f = f_1 - f_2$. If $g_1 \in \partial f_1(x)$ and $g_2 \in \partial f_2(x)$, then*

$$f^\star \leq f(x) - \tfrac{1}{2(L_1-\mu_2)}\|g_1 - g_2\|^2.$$

Proof. If $L_1 = \infty$, the proof is immediate. Let $L_1 < \infty$. By L-smoothness and strong convexity, we have

$$f_1(y) \leq f_1(x) + \langle g_1, y - x\rangle + \tfrac{L_1}{2}\|y - x\|^2,$$
$$f_2(y) \geq f_2(x) + \langle g_2, y - x\rangle + \tfrac{\mu_2}{2}\|y - x\|^2,$$

for $y \in \mathbb{R}^n$. By the above inequalities, we get

$$f(y) \leq f(x) + \langle g_1 - g_2, y - x\rangle + \tfrac{L_1-\mu_2}{2}\|y - x\|^2.$$

Hence, by taking minimum on both sides of the last inequality with respect to y for fixed x, we get

$$f^\star \leq f(x) - \tfrac{1}{2(L_1-\mu_2)}\|g_1 - g_2\|^2.$$

□

Since the DC optimization problem (8.1) may have a non-convex and non-smooth objective function f, we will also need a more general notion of subgradients than in the convex case which is defined in Section 2.1.6.

Definition 8.5. Let f_1, f_2 be closed proper convex functions, and let f be lower semi-continuous.

- The point $\bar{x} \in \text{dom}(f)$ is called a *critical point* of problem (8.1) if

$$\partial f_1(\bar{x}) \cap \partial f_2(\bar{x}) \neq \emptyset. \tag{8.5}$$

- The point $\bar{x} \in \text{dom}(f)$ is called a *stationary point* of problem (8.1) if

$$0 \in \partial_L f(\bar{x}), \tag{8.6}$$

  where ∂_L stands for general subdifferential, see Section 2.1.6.

Obviously, the stationarity condition is stronger than criticality. We recall that a convex function will be locally Lipschitz around $\bar{x}$ providing it takes finite values in a neighborhood of $\bar{x}$; see Theorem 35.1 in [Roc97]. Consequently, if f_1 or f_2 takes finite values around a neighborhood of a stationary point $\bar{x}$, then $\bar{x}$ is a critical point; see Corollary 10.9 in [RW09]. However, its converse does not hold in general. For instance, consider $f : \mathbb{R} \to \mathbb{R}$ given as $f(x) = x$. The function f may be written as $f = f_1 - f_2$ where $f_1(x) = \max(x, 0)$ and $f_2(x) = \max(-x, 0)$. Suppose that $\bar{x} = 0$. It is readily seen that $\partial f_1(\bar{x}) \cap \partial f_2(\bar{x}) \neq \emptyset$, but $\bar{x} = 0$ is not a stationary point of f. It is worth noting that, if f_2 is strictly differentiable at $\bar{x}$, these definitions are equivalent; see Example 10.10 in [RW09]. Recall that function f is strictly differentiable at $\bar{x}$, if

$$\lim_{\substack{(x,x') \to (\bar{x},\bar{x}) \\ x \neq x'}} \frac{f(x) - f(x') - \langle \nabla f(\bar{x}), x - x' \rangle}{\|x - x'\|} = 0.$$

We refer the interested reader to [AT05, JBK+18, PRA17] and references therein for more discussions on optimality conditions for DC problems.

8.2.1 The DC problem

In this section, we consider

$$\begin{aligned} &\min f(x) = f_1(x) - f_2(x) \\ &\text{s.t. } x \in \mathbb{R}^n, \end{aligned} \tag{8.7}$$

where $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$. Here, we assume that $L_1, L_2 \in (0, \infty]$ and $\mu_1, \mu_2 \in [0, \infty)$, and consequently f may be non-differentiable. We may assume without loss of generality that f_1 and f_2 satisfy the following assumptions:

$$L_1 > \mu_2, \quad L_2 > \mu_1. \tag{8.8}$$

Indeed, if $L_1 \leq \mu_2$, then for $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
\lambda f_1(x) + (1-\lambda) f_1(y) &\leq f_1(\lambda x + (1-\lambda)y) + \lambda(1-\lambda)\tfrac{L_1}{2}\|x-y\|^2 \\
-\lambda f_2(x) - (1-\lambda) f_2(y) &\leq -f_2(\lambda x + (1-\lambda)y) - \lambda(1-\lambda)\tfrac{\mu_2}{2}\|x-y\|^2;
\end{aligned}$$

see Theorem 2.15 and Theorem 2.19 in [Nes18]. By summing the above inequalities, we obtain

$$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda)y) + \lambda(1-\lambda)\tfrac{L_1-\mu_2}{2}\|x-y\|^2,$$

which implies concavity of f on $\mathbb{R}^n$. In this case, problem (8.7) will be unbounded from below. This follows from the fact that a concave function on $\mathbb{R}^n$ is unbounded from below unless it is constant. Likewise, one can show that problem (8.7) will be convex providing $L_2 \leq \mu_1$.

The *Toland dual* [Tol79] of problem (8.7) may be written as

$$\begin{aligned}
&\min\ f_2^*(x) - f_1^*(x) \\
&\text{s.t. } x \in \mathbb{R}^n.
\end{aligned} \tag{8.9}$$

It is known that problems (8.7) and (8.9) share the same optimal value [Tol79].

In what follows, we investigate the convergence rate of Algorithm 8.1 with the termination criterion $\|g_1^k - g_2^k\| \leq \epsilon$. As a motivation of this criterion, recall that $\|g_1^k - g_2^k\| = 0$ implies that x^k is a critical point of (8.1) in the non-smooth case, and a stationary point of f if f_2 is differentiable; see our discussion following Definition 8.5. In Section 8.3 we will derive results for the case that at least one of f_1 or f_2 is differentiable, and we will consider the more general situation in Section 8.4.

For well-definedness of the DCA (Algorithm 8.1), throughout the chapter, we assume that

$$x^k \in \operatorname{dom}(\partial f_1) \cap \operatorname{dom}(\partial f_2) \quad k = 0, 1, \ldots,$$

where $\operatorname{dom}(\partial f_1) = \{x : \partial f_1(x) \neq \emptyset\}$. It is worth noting that similar algorithm has been developed for the dual problem in [LTPD18], and (8.2) is equivalent to $x^{k+1} \in \partial f_1^*(g_2^k)$.

8.2.2 One iteration of DCA on an example

Consider the following unconstrained optimization problem (see Figure 8.1).

$$\min_{x\in\mathbb{R}} f(x) := \tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + 2x, \quad x^0 = 3. \tag{8.10}$$

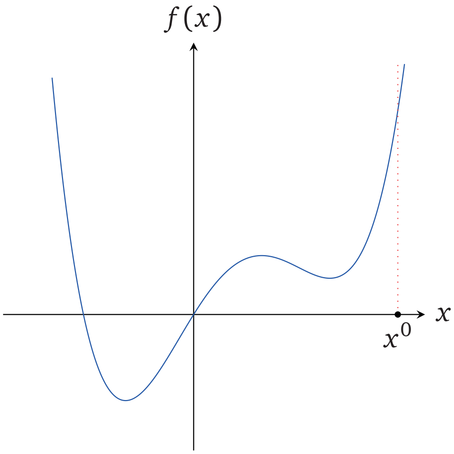


Figure 8.1: Illustration of the function (8.10)

It is easily seen that the problem can be written in the following form.

$$f(x) = \underbrace{\left(\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + 2x^2\right)}_{f_1} - \underbrace{\left(\tfrac{1}{2}x^2 - 2x + 2x^2\right)}_{f_2}, \tag{8.11}$$

where f_1 and f_2 are convex functions (see Figure 8.2).

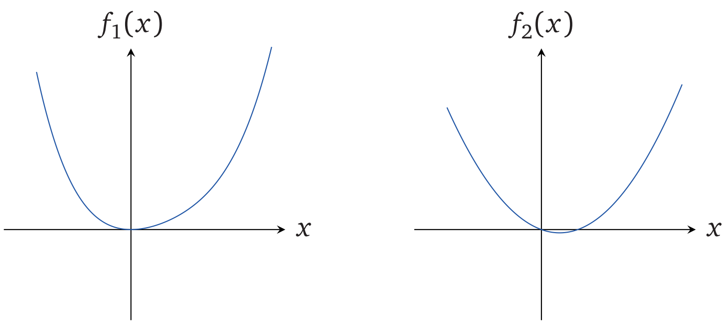


Figure 8.2: Illustration of the functions f_1 and f_2 of problem (8.11)

By taking $x^0 = 3$ as the starting point of the algorithm, the following subproblem of the algorithm should be solved (see Figure 8.3).

$$\min_{x\in\mathbb{R}} u(x) := \underbrace{\left(\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + 2x^2\right)}_{f_1} - \underbrace{(16.5 + 13(x-3))}_{f_2(x^0)+f_2'(x^0)(x-x^0)} \tag{8.12}$$

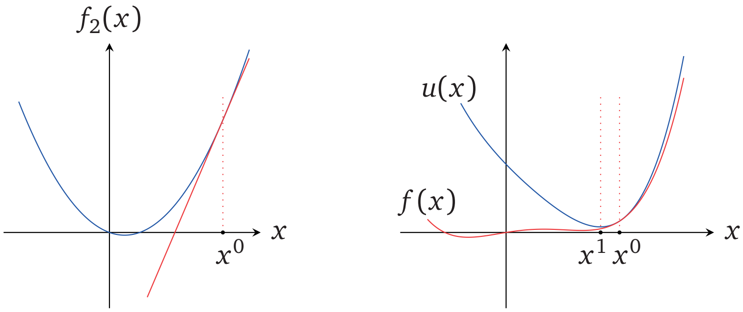


Figure 8.3: Illustration of the subproblem (8.12)

By solving subproblem (8.12), DCA generates $x^1 = 2.5$ which is illustrated in Figure 8.4.

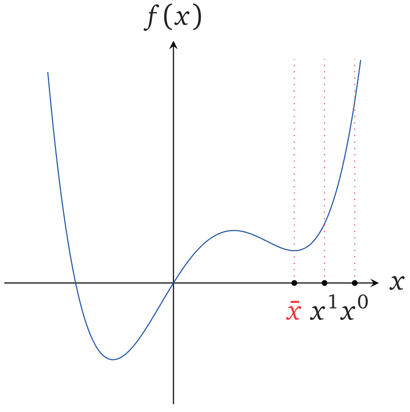


Figure 8.4: Illustration of the first iteration of the Algorithm 8.1 on the function (8.10)

For this example, in the next iteration DCA generates $x^2 \approx 2.2728$, $x^3 \approx 2.1576$ and so on that converges to the local minimizer at $\bar{x} = 2$.

8.3 Performance analysis of the DCA for smooth f_1 or f_2

In this subsection, we apply performance estimation for the analysis of Algorithm 8.1 for the case that at least one of f_1 or f_2 is L-smooth for some finite $L > 0$. The worst-case convergence rate of Algorithm 8.1 can be obtained by solving the following abstract optimization problem:

$$\begin{aligned}
\max \ & \left(\min_{0\le k\le N} \left\| g_1^k - g_2^k \right\|^2 \right) \\
& g_1^N, g_2^N, x^N, \ldots, x^1 \text{ are generated by Algorithm 8.1 w.r.t. } f_1, f_2, x^0 \\
& f(x) \ge f^\star \quad \forall x \in \mathbb{R}^n \\
& f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n), f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n) \\
& f_1(x^0) - f_2(x^0) - f^\star \le \Delta \\
& x^0 \in \mathbb{R}^n,
\end{aligned} \tag{8.13}$$

where $\Delta \ge 0$ denote the difference between the optimal value and the value of f at the starting point. Here, f_1, f_2 and x^k, g_1^k and g_2^k ($k \in \{0, ..., N\}$) are decision variables, and $\Delta, \mu_1, L_1, \mu_2, L_2$ and N are fixed parameters.

Problem (8.13) is an intractable *infinite-dimensional optimization problem* with an infinite number of constraints. In what follows, we provide a *semidefinite programming* relaxation of the problem.

By Theorem 2.37, problem (8.13) can be written as,

$$\begin{aligned}
\max \ & \left(\min_{0\le k\le N} \left\| g_1^k - g_2^k \right\|^2 \right) \\
\text{s.t. } & \tfrac{1}{2(1-\frac{\mu_1}{L_1})} \left(\tfrac{1}{L_1} \left\| g_1^i - g_1^j \right\|^2 + \mu_1 \left\| x^i - x^j \right\|^2 - \tfrac{2\mu_1}{L_1} \left\langle g_1^j - g_1^i, x^j - x^i \right\rangle \right) \le \\
& \quad f_1^i - f_1^j - \left\langle g_1^j, x^i - x^j \right\rangle \quad i, j \in \{0, \ldots, N\} \\
& \tfrac{1}{2(1-\frac{\mu_2}{L_2})} \left(\tfrac{1}{L_2} \left\| g_2^i - g_2^j \right\|^2 + \mu_2 \left\| x^i - x^j \right\|^2 - \tfrac{2\mu_2}{L_2} \left\langle g_2^j - g_2^i, x^j - x^i \right\rangle \right) \le \\
& \quad f_2^i - f_2^j - \left\langle g_2^j, x^i - x^j \right\rangle \quad i, j \in \{0, \ldots, N\} \\
& g_1^{k+1} = g_2^k \quad k \in \{0, \ldots, N-1\} \\
& f_1^k - f_2^k - \frac{1}{2(L_1 - \mu_2)} \| g_1^k - g_2^k \|^2 \ge f^\star \quad k \in \{0, \ldots, N\} \\
& f_1^0 - f_2^0 - f^\star \le \Delta.
\end{aligned} \tag{8.14}$$

In problem (8.14), $f^\star$ and x^k, g_1^k, g_2^k, f_1^k, f_2^k, $k \in \{0, \ldots, N\}$, are decision variables. By virtue of Lemma 8.4, constraints $f(x) \ge f^\star$ for each $x \in \mathbb{R}^n$

is replaced by $f_1^k - f_2^k - \frac{1}{2(L_1-\mu_2)}\|g_1^k - g_2^k\|^2 \geq f^\star, \quad k \in \{0,\ldots,N\}$. Due to the necessary and sufficient optimality conditions for convex problems, $x^{k+1} \in \mathrm{argmin}_{x\in\mathbb{R}^n} f_1(x) - f_2(x^k) - \langle g_2^k, x - x^k\rangle$, $k \in \{0,\ldots,N-1\}$ implies $g_1^{k+1} = g_2^k$ for some $g_1^{k+1} \in \partial f(x^{k+1})$; see Theorem 3.63 in [Bec17]. By substituting $g_2^k = g_1^{k+1}$, $k \in \{0,\ldots,N-1\}$, the above formulation may be written as:

$$
\begin{aligned}
&\max \ell \\
&\text{s.t. } \left\|g_1^i - g_1^{i+1}\right\|^2 \geq \ell \quad i \in \{0,\ldots,N-1\} \\
&\quad \left\|g_1^N - g_2^N\right\|^2 \geq \ell \\
&\quad \tfrac{1}{2(1-\frac{\mu_1}{L_1})}\left(\tfrac{1}{L_1}\left\|g_1^i - g_1^j\right\|^2 + \mu_1\left\|x^i - x^j\right\|^2 - \tfrac{2\mu_1}{L_1}\left\langle g_1^j - g_1^i, x^j - x^i\right\rangle\right) \leq \\
&\qquad f_1^i - f_1^j - \left\langle g_1^j, x^i - x^j\right\rangle \quad i,j \in \{0,\ldots,N\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_1^{i+1} - g_1^{j+1}\right\|^2 + \mu_2\left\|x^i - x^j\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_1^{j+1} - g_1^{i+1}, x^j - x^i\right\rangle\right) \leq \\
&\qquad f_2^i - f_2^j - \left\langle g_1^{j+1}, x^i - x^j\right\rangle \quad i,j \in \{0,\ldots,N-1\} \qquad (8.15)\\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_2^N - g_1^{j+1}\right\|^2 + \mu_2\left\|x^N - x^j\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_1^{j+1} - g_2^N, x^j - x^N\right\rangle\right) \\
&\qquad \leq f_2^N - f_2^j - \left\langle g_1^{j+1}, x^N - x^j\right\rangle \quad j \in \{0,\ldots,N-1\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_1^{i+1} - g_2^N\right\|^2 + \mu_2\left\|x^i - x^N\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_2^N - g_1^{i+1}, x^N - x^i\right\rangle\right) \\
&\qquad \leq f_2^i - f_2^N - \left\langle g_2^N, x^i - x^N\right\rangle \quad i \in \{0,\ldots,N-1\} \\
&\quad f_1^k - f_2^k - \frac{1}{2(L_1-\mu_2)}\|g_1^k - g_1^{k+1}\|^2 \geq f^\star \quad k \in \{0,\ldots,N-1\} \\
&\quad f_1^N - f_2^N - \frac{1}{2(L_1-\mu_2)}\|g_1^N - g_2^N\|^2 \geq f^\star \\
&\quad f_1^0 - f_2^0 - f^\star \leq \Delta.
\end{aligned}
$$

By using this formulation, the next result (Theorem 8.6) provides a convergence rate for Algorithm 8.1. Since the proof is quite technical, a few remarks are in order. The proof uses the performance estimation technique of Drori and Teboulle [DT14], that consists of the following steps:

1. Observe that problem (8.15) may be rewritten as a semidefinite programming (SDP) problem (for sufficiently large N) by replacing all inner products by the entries of an unknown Gram matrix.

2. Use weak duality of SDP to bound the optimal value of (8.15) by construct-

ing a dual feasible solution.

3. The dual feasible solution is constructed empirically, by first doing numerical experiments with fixed values of the parameters $\Delta, N, \mu_1, L_1, \mu_2, L_2$, and noting the dual multipliers.

4. Subsequently, the analytical expressions of the dual multipliers are guessed, based on the numerical values, and the guess is verified analytically.

5. In the proof of Theorem 8.6, the conjectured dual multipliers are simply stated, and then shown to provide the required bound on the optimal value of (8.15) through the corresponding aggregation of the constraints of (8.15).

Theorem 8.6. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$ and let $f(x^0) - f^\star = \Delta$. Suppose that L_1 or L_2 is finite. Then after N iterations of Algorithm 8.1, one has:*

$$\min_{0\le k\le N} \left\| g_1^k - g_2^k \right\| \le \sqrt{\frac{\mathcal{A}\Delta}{\mathcal{B}N + \mathcal{C}}}, \tag{8.16}$$

where

$$\mathcal{A} = 2\left(L_1L_2 - \mu_1 L_2 I_{\mathbb{R}_+}(L_1 - L_2) - \mu_2 L_1 I_{\mathbb{R}_+}(L_2 - L_1)\right),$$
$$\mathcal{B} = L_1 + L_2 + \mu_1\left(\tfrac{L_1}{L_2} - 3\right) I_{\mathbb{R}_+}(L_1 - L_2) + \mu_2\left(\tfrac{L_2}{L_1} - 3\right) I_{\mathbb{R}_+}(L_2 - L_1),$$

and

$$\mathcal{C} = \frac{L_1L_2 - \mu_1 L_2 I_{\mathbb{R}_+}(L_1 - L_2) - \mu_2 L_1 I_{\mathbb{R}_+}(L_2 - L_1)}{L_1 - \mu_2},$$

where $I_{\mathbb{R}_+}$ stands for indicator function defined by (8.4).

Proof. We investigate two cases $L_1 \ge L_2$ and $L_1 < L_2$. Suppose that U denotes the square of the right-side of inequality (8.16) and let $B = \frac{U}{\Delta}$. To prove this bound, we show that U is an upper bound for problem (8.15). First, we consider $L_1 \ge L_2$.

Let

$$\bar{\lambda} = \frac{2\left(L_1L_2 - \mu_1(2L_2 - L_1)\right)}{N\left(L_1 + L_2 + \mu_1\left(\frac{L_1}{L_2} - 3\right)\right) + \frac{L_2(L_1-\mu_1)}{L_1-\mu_2}}$$

$$\bar{\eta}_1 = \frac{L_2 - \mu_1}{\left(L_1 + L_2 + \mu_1(\frac{L_1}{L_2} - 3)\right)N + \frac{L_2(L_1-\mu_1)}{L_1-\mu_2}}$$

$$\bar{\eta}_k = \frac{\frac{L_1\mu_1}{L_2} + (L_1 + L_2 - 3\mu_1)}{\left(L_1 + L_2 + \mu_1(\frac{L_1}{L_2} - 3)\right)N + \frac{L_2(L_1-\mu_1)}{L_1-\mu_2}}, \quad k \in \{2, \ldots, N\}$$

$$\bar{\eta}_{N+1} = 1 - \bar{\eta}_1 - \sum_{k=2}^{N} \bar{\eta}_k = \frac{\frac{L_1\mu_1}{L_2} + L_1 - 2\mu_1 + \frac{L_2(L_1-\mu_1)}{L_1-\mu_2}}{\left(L_1 + L_2 + \mu_1(\frac{L_1}{L_2} - 3)\right)N + \frac{L_2(L_1-\mu_1)}{L_1-\mu_2}}.$$

By direct calculation, one can verify that

$$\begin{aligned}
&\ell - U + \bar{\eta}_1\left(\left\|g_1^0 - g_1^1\right\|^2 - \ell\right) + \sum_{k=2}^{N}\bar{\eta}_k\left(\left\|g_1^{k-1} - g_1^k\right\|^2 - \ell\right) + \bar{\eta}_{N+1}\left(\left\|g_1^N - g_2^N\right\|^2 - \ell\right)\\
&+ B\left(f^\star - f_1^0 + f_2^0 + \Delta\right) + B\left(f_1^N - f_2^N - \frac{1}{2(L_1-\mu_2)}\|g_1^N - g_2^N\|^2 - f^\star\right) +\\
&B\sum_{k=1}^{N}\left(f_1^{k-1} - f_1^k - \left\langle g_1^k, x^{k-1} - x^k\right\rangle - \frac{1}{2(1-\frac{\mu_1}{L_1})}\left(\frac{1}{L_1}\left\|g_1^{k-1} - g_1^k\right\|^2 + \mu_1\left\|x^{k-1} - x^k\right\|^2 -\right.\right.\\
&\left.\left.\frac{2\mu_1}{L_1}\left\langle g_1^k - g_1^{k-1}, x^k - x^{k-1}\right\rangle\right)\right) + \bar{\lambda}\sum_{k=1}^{N-1}\left(f_2^k - f_2^{k-1} - \left\langle g_1^k, x^k - x^{k-1}\right\rangle -\right.\\
&\left.\frac{1}{2(1-\frac{\mu_2}{L_2})}\left(\frac{1}{L_2}\left\|g_1^k - g_1^{k+1}\right\|^2 + \mu_2\left\|x^{k-1} - x^k\right\|^2 - \frac{2\mu_2}{L_2}\langle g_1^{k+1} - g_1^k, x^k - x^{k-1}\rangle\right)\right)\\
&+ (\bar{\lambda} - B)\sum_{k=1}^{N-1}\left(f_2^{k-1} - f_2^k - \left\langle g_1^{k+1}, x^{k-1} - x^k\right\rangle - \frac{1}{2(1-\frac{\mu_2}{L_2})}(\frac{1}{L_2}\left\|g_1^k - g_1^{k+1}\right\|^2 +\right.\\
&\left.\mu_2\left\|x^{k-1} - x^k\right\|^2 - \frac{2\mu_2}{L_2}\langle g_1^{k+1} - g_1^k, x^k - x^{k-1}\rangle\right) + (\bar{\lambda} - B)\left(f_2^{N-1} - f_2^N - \left\langle g_2^N, x^{N-1} - x^N\right\rangle\right.\\
&\left.- \frac{1}{2(1-\frac{\mu_2}{L_2})}\left(\frac{1}{L_2}\left\|g_1^N - g_2^N\right\|^2 + \mu_2\left\|x^{N-1} - x^N\right\|^2 - \frac{2\mu_2}{L_2}\langle g_2^N - g_1^N, x^N - x^{N-1}\rangle\right)\right)\\
&+ \bar{\lambda}\left(f_2^N - f_2^{N-1} - \left\langle g_1^N, x^N - x^{N-1}\right\rangle - \frac{1}{2(1-\frac{\mu_2}{L_2})}\left(\frac{1}{L_2}\left\|g_1^N - g_2^N\right\|^2 + \mu_2\left\|x^{N-1} - x^N\right\|^2\right.\right.\\
&\left.\left.- \frac{2\mu_2}{L_2}\langle g_2^N - g_1^N, x^N - x^{N-1}\rangle\right)\right) =\\
&-\bar{\beta}_1^{-1}\sum_{i=1}^{N}\left\|\bar{\beta}_1 g_1^{i-1} - \bar{\beta}_1 g_1^i - \bar{\alpha}_1 x^{i-1} + \bar{\alpha}_1 x^i\right\|^2 - \bar{\alpha}_2^{-1}\sum_{i=1}^{N-1}\left\|\bar{\alpha}_2 x^{i-1} - \bar{\alpha}_2 x^i - \bar{\beta}_2 g_1^i + \bar{\beta}_2 g_1^{i+1}\right\|^2\\
&- \bar{\alpha}_2^{-1}\left\|\bar{\alpha}_2 x^{N-1} - \bar{\alpha}_2 x^N - \bar{\beta}_2 g_1^N + \bar{\beta}_2 g_2^N\right\|^2 \le 0,
\end{aligned}$$

where

$$\bar{\alpha}_1 = \frac{\mu_1 B}{2(L_1-\mu_1)}, \quad \bar{\beta}_1 = \frac{\mu_1 B}{2L_2(L_1-\mu_1)},$$
$$\bar{\alpha}_2 = \frac{(-\mu_1 L_2^2 - 2\mu_1\mu_2 L_2 + \mu_1 L_1 L_2 + \mu_1\mu_2 L_1 + \mu_2 L_1 L_2)B}{2(L_1-\mu_1)(L_2-\mu_2)},$$
$$\bar{\beta}_2 = \frac{(L_1 L_2 \mu_2 - 2\mu_1\mu_2 L_2 + \mu_1\mu_2 L_1 - \mu_1 L_2^2 + \mu_1 L_1 L_2)B}{2L_2(L_1-\mu_1)(L_2-\mu_2)}.$$

It is readily seen that $\bar{\lambda}, \bar{\eta}_k$ $(k \in \{1,\dots,N+1\}), \bar{\lambda}-B, \bar{\beta}_1, \bar{\alpha}_2 \geq 0$. Thus we have $\ell \leq U$ for any feasible point of problem (8.15). Now, we consider $L_1 < L_2$. In this case, because bound (8.16) does not depend on μ_1, we may assume $\mu_1 = 0$ in problem (8.15). Let

$$\hat{\lambda} = \frac{2\left(L_1 L_2 - \mu_2(2L_1 - L_2)\right)}{\left(L_1 + L_2 + \mu_2\left(\frac{L_2}{L_1} - 3\right)\right)N + \frac{L_1(L_2-\mu_2)}{L_1-\mu_2}}$$
$$\hat{\eta}_1 = \frac{\frac{L_2(L_1+\mu_2)}{L_1} - 2\mu_2}{\left(L_1 + L_2 + \mu_2(\frac{L_2}{L_1} - 3)\right)N + \frac{L_1(L_2-\mu_2)}{L_1-\mu_2}}$$
$$\hat{\eta}_k = \frac{\frac{L_2(L_1+\mu_2)}{L_1} + (L_1 - 3\mu_2)}{\left(L_1 + L_2 + \mu_2(\frac{L_2}{L_1} - 3)\right)N + \frac{L_1(L_2-\mu_2)}{L_1-\mu_2}}, \quad k \in \{2,\dots,N\}$$
$$\hat{\eta}_{N+1} = 1 - \hat{\eta}_1 - \sum_{k=2}^{N} \hat{\eta}_k = \frac{\frac{L_1(L_2-\mu_2)}{L_1-\mu_2} + L_1 - \mu_2}{\left(L_1 + L_2 + \mu_2(\frac{L_2}{L_1} - 3)\right)N + \frac{L_1(L_2-\mu_2)}{L_1-\mu_2}}.$$

With some calculation, one can establish that

$$
\begin{aligned}
&\ell - U + \hat{\eta}_1\left(\left\|g_1^0 - g_1^1\right\|^2 - \ell\right) + \sum_{k=2}^{N} \hat{\eta}_k\left(\left\|g_1^{k-1} - g_1^k\right\|^2 - \ell\right) + \hat{\eta}_{N+1}\left(\left\|g_1^N - g_2^N\right\|^2 - \ell\right)\\
&+ B\left(f^\star - f_1^0 + f_2^0 + \Delta\right) + B\left(f_1^N - f_2^N - \frac{1}{2(L_1-\mu_2)}\|g_1^N - g_2^N\|^2 - f^\star\right)\\
&+ (\hat{\lambda} - B)\sum_{k=1}^{N}\left(f_1^k - f_1^{k-1} - \left\langle g_1^{k-1}, x^k - x^{k-1}\right\rangle - \tfrac{1}{2L_1}\left\|g_1^k - g_1^{k-1}\right\|^2\right)\\
&+ \hat{\lambda}\sum_{k=1}^{N}\left(f_1^{k-1} - f_1^k - \left\langle g_1^k, x^{k-1} - x^k\right\rangle - \tfrac{1}{2L_1}\left\|g_1^{k-1} - g_1^k\right\|^2\right)\\
&+ B\sum_{k=1}^{N-1}\Bigg(f_2^k - f_2^{k-1} - \left\langle g_1^k, x^k - x^{k-1}\right\rangle -\\
&\tfrac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_1^k - g_1^{k+1}\right\|^2 + \mu_2\left\|x^{k-1} - x^k\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_1^{k+1} - g_1^k, x^k - x^{k-1}\right\rangle\right)\Bigg)\\
&+ B\Bigg(f_2^N - f_2^{N-1} - \left\langle g_1^N, x^N - x^{N-1}\right\rangle -\\
&\tfrac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_1^N - g_2^N\right\|^2 + \mu_2\left\|x^{N-1} - x^N\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_2^N - g_1^N, x^N - x^{N-1}\right\rangle\right)\Bigg)\\
&= -\hat{\beta}_1^{-1}\sum_{i=1}^{N}\left\|\hat{\beta}_1 g_1^{i-1} - \hat{\beta}_1 g_1^i - \hat{\alpha}_1 x_1^{i-1} + \hat{\alpha}_1 x^i\right\|^2 - \hat{\alpha}_2^{-1}\sum_{i=1}^{N-1}\left\|\hat{\alpha}_2 x^{i-1} - \hat{\alpha}_2 x^i - \hat{\beta}_2 g_1^i + \hat{\beta}_2 g_1^{i+1}\right\|^2\\
&- \hat{\alpha}_2^{-1}\left\|\hat{\alpha}_2 x^{N-1} - \hat{\alpha}_2 x^N - \hat{\beta}_2 g_1^N + \hat{\beta}_2 g_2^N\right\|^2 \le 0,
\end{aligned}
$$

where

$$
\hat{\alpha}_1 = \frac{\mu_2 B(1-\frac{L_1}{L_2})}{2L_1(1-\frac{\mu_2}{L_2})},\quad \hat{\alpha}_2 = \frac{\mu_2 L_1 B}{2(L_2-\mu_2)},\quad \hat{\beta}_1 = \frac{\mu_2 B(1-\frac{L_1}{L_2})}{2L_1^2(1-\frac{\mu_2}{L_2})},\quad \hat{\beta}_2 = \frac{\mu_2 B}{2(L_2-\mu_2)}.
$$

It is readily seen that $\hat{\lambda}, \hat{\eta}_k$ $(k \in \{1, \ldots, N+1\}), \hat{\lambda} - B, \hat{\beta}_1, \hat{\alpha}_2 \ge 0$. The rest of proof is similar to that of the former case, and the proof is complete. □

The theorem implies that Algorithm 8.1 is convergent when at least one of the Lipschitz constants is finite. In the following corollary, we simplify the inequality (8.16) for some special cases of L_1, L_2, μ_1, and μ_2.

Corollary 8.7. *Suppose that $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$. Then, after N iterations of Algorithm 8.1, one has:*

i) If $L_1 = \infty$, $L_2 < \infty$, then

$$
\min_{0\le k\le N}\left\|g_1^k - g_2^k\right\| \le \sqrt{\frac{2L_2^2(f(x^0) - f^\star)}{N(L_2 + \mu_1)}}.
$$

ii) If $L_2 = \infty$, $L_1 < \infty$, then

$$\min_{0\le k\le N} \left\|g_1^k - g_2^k\right\| \le \sqrt{\frac{2L_1^2(L_1-\mu_2)(f(x^0)-f^\star)}{\left(L_1^2-\mu_2^2\right)N + L_1^2}}. \tag{8.17}$$

iii) If $L_1, L_2 < \infty$, and $\mu_1 = \mu_2 = 0$ then

$$\min_{0\le k\le N} \left\|g_1^k - g_2^k\right\| \le \sqrt{\frac{2L_1L_2(f(x^0)-f^\star)}{(L_1+L_2)N + L_2}}.$$

One can compare the results in Corollary 8.7 to that of Le Thi et al. [LTPD21] as reviewed earlier in Theorem 8.2. First of all, Corollary 8.7 part iii) does not assume strict convexity of f_1 or f_2, and in this sense it is more general than the result in Theorem 8.2. If we do assume $\mu_1+\mu_2 > 0$, then, for example, if $L_1 < \infty$, Theorem 8.2 implies,

$$\min_{0\le k\le N} \left\|g_1^k - g_2^k\right\| \le L_1\sqrt{\frac{2(f(x^0)-f^\star)}{(\mu_1+\mu_2)N}},$$

which is weaker than our bound (8.17) since $\mu_1 \le L_1$, although the $\mathcal{O}(1/\sqrt{N})$ dependence on N is the same. We will do a further, more direct, comparison of Theorem 8.2 and Corollary 8.7 in Section 8.3.2, where we consider the convergence rate of the sequence $\|x^{k+1} - x^k\|$.

8.3.1 An example to prove tightness

In what follows, we give a class of functions for which the bound in Corollary 8.7, part ii), is attained, implying that the $\mathcal{O}(1/\sqrt{N})$ convergence rate is tight. This result is new to the best of our knowledge.

Example 8.8. *Let $L_1 \in (0,\infty)$. Suppose that N is selected such that $U := \sqrt{\frac{2}{L_1(N+1)}} <$ 1. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be given as follows,*

$$f_1(x) = \begin{cases} \frac{L_1}{2}\left(x - i(1-U)\right)^2 + \frac{L_1Ui(i-1)(1-U)}{2} & x \in [\alpha_i, \beta_{i+1}) \\ L_1U\beta_i(x-\beta_i) + \frac{\beta_iL_1U^2}{2} + \frac{\beta_i(\beta_i-1)L_1U}{2} & x \in [\beta_i, \alpha_i) \\ \frac{L_1}{2}x^2 & x \in (-\infty, 0), \end{cases}$$

where, for $i \in \{1,\ldots,N+1\}$, $\alpha_i = i - U$, $\beta_i = i-1$, and $\beta_{N+2} = \infty$. Note that $f_1 \in \mathcal{F}_{0,L_1}(\mathbb{R})$. Suppose that $f_2 : \mathbb{R} \to \mathbb{R}$ is given by

$$f_2(x) = \max_{1\le i\le N+1}\left\{L_1U(i-1)(x-i) + \tfrac{i(i-1)L_1U}{2}\right\}.$$

An easy computation shows that

$$\begin{cases} \partial f_2(i) = [L_1U(i-1), L_1Ui] & i \in \{1,\dots,N,\} \\ \partial f_2(N+1) = L_1UN. \end{cases}$$

Note that $f_2 \in \mathcal{F}_{0,\infty}(\mathbb{R})$*. One can check that, at* $x^0 = N+1$*, one has* $f_1(x^0) - f_2(x^0) = 1$*,* $\min_{x\in\mathbb{R}} f_1(x) - f_2(x) = 0$ *and* $\operatorname{argmin}_{x\in\mathbb{R}} f_1(x) - f_2(x) = [0, 1-U]$*. By taking* x^0 *as a starting point, Algorithm 8.1 can generate the following iterates:*

$$x^k = N+1-k, \quad k \in \{0,\dots,N\}.$$

Here at iteration, $k \in \{0,\dots,N\}$*, we set* $g_2^k = L_1U(N-k)$*. It follows that* $|\nabla f_1(x_k) - g_2^k| = \sqrt{\frac{2L_1}{N+1}}$*,* $k \in \{0,\dots,N\}$*. Hence,*

$$\min_{0\le k\le N} \left\|g_1^k - g_2^k\right\| = \sqrt{\tfrac{2L_1}{N+1}},$$

which shows bound (8.17) *in Corollary 8.7 is exact for this example.*

8.3.2 Convergence rates for the iterates

In this section we investigate the implications of our results so far on convergence rates of the iterates $\{x^k\}$.

Proposition 8.9. *Let* $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ *and* $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$ *and let* $f(x^0) - f^\star \le \Delta$*. If* μ_1 *or* μ_2 *is strictly positive, then after* N *iterations of Algorithm 8.1, one has:*

$$\min_{0\le k\le N-1} \left\|x^{k+1} - x^k\right\| \le \left(\frac{\mathcal{A}}{\mathcal{B}N + \mathcal{C}} \cdot \Delta\right)^{\frac{1}{2}},$$

where

$$\mathcal{A} = 2\left(\mu_2^{-1}\mu_1^{-1} - L_2^{-1}\mu_1^{-1} I_{\mathbb{R}_+}(\mu_2^{-1} - \mu_1^{-1}) - L_1^{-1}\mu_2^{-1} I_{\mathbb{R}_+}(\mu_1^{-1} - \mu_2^{-1})\right),$$

$$\mathcal{B} = \mu_2^{-1} + \mu_1^{-1} + L_2^{-1}\left(\tfrac{\mu_1}{\mu_2} - 3\right) I_{\mathbb{R}_+}\left(\mu_2^{-1} - \mu_1^{-1}\right) + L_1^{-1}\left(\tfrac{\mu_2}{\mu_1} - 3\right) I_{\mathbb{R}_+}\left(\mu_1^{-1} - \mu_2^{-1}\right),$$

and

$$\mathcal{C} = \frac{\mu_2^{-1}\mu_1^{-1} - L_2^{-1}\mu_1^{-1} I_{\mathbb{R}_+}\left(\mu_2^{-1} - \mu_1^{-1}\right) - L_1^{-1}\mu_2^{-1} I_{\mathbb{R}_+}\left(\mu_1^{-1} - \mu_2^{-1}\right)}{\mu_2^{-1} - L_1^{-1}}.$$

Proof. The proof is based on the computation of the worst case convergence rate of DCA for problem (8.9) by applying Theorem 8.6. By Toland duality, $f^\star$ is also a lower bound of problem (8.9). By virtue of conjugate function properties, it follows that $f_2^*(g_2^0)-f_1^*(g_2^0)-f^\star \le \Delta$ and $f_2^* \in \mathcal{F}_{L_2^{-1},\mu_2^{-1}}(\mathbb{R}^n)$ and $f_1^* \in \mathcal{F}_{L_1^{-1},\mu_1^{-1}}(\mathbb{R}^n)$. In addition, $x^{k+1} \in \partial f_1^*(g_2^k)$ and $x^k \in \partial f_2^*(g_2^k)$ for $k \in \{0,\dots,N-1\}$. Hence, all assumptions of Theorem 8.6 hold, and subsequently the bound follows from Theorem 8.6. □

Recall the known result from Theorem 8.2:

$$\min_{0\le k\le N-1} \left\|x^{k+1}-x^k\right\| \le \left(\frac{2(f(x^0)-f^\star)}{N(\mu_1+\mu_2)}\right)^{\frac{1}{2}}. \tag{8.18}$$

By employing Theorem 8.9, we get

$$\min_{0\le k\le N-1} \left\|x^{k+1}-x^k\right\| \le \left(\frac{2(f(x^0)-f^\star)}{N(\mu_1+\mu_2)+\mu_1}\right)^{\frac{1}{2}},$$

which is tighter than the bound (8.18). Moreover, the bound given in Proposition 8.9 provides more information concerning the worst-case convergence rate of the DCA when $L_1 < \infty$ or $L_2 < \infty$.

8.3.3 The gradient descent method

In this section, we study the relationship between the gradient descent method and DCA given by Algorithm 4.1 and 8.1, respectively. Using the convergence rate provided by Theorem 8.6 we derive the same convergence rate for the gradient descent given by Theorem 4.3 when the step length $t_k = t$ lies in $(0, \frac{1}{L}]$.

Consider the following optimization problem

$$\inf_{x\in\mathbb{R}^n} f(x)$$

where f is lower-bounded by $f^\star$ and is an L-smooth function on $\mathbb{R}^n$ with $L < \infty$, that is $f \in \mathcal{F}_{-L,L}(\mathbb{R}^n)$. It is easily seen that the function f can be written as

$$f(x) := \frac{1}{2t}\|x\|^2 - \left(\frac{1}{2t}\|x\|^2 - f(x)\right),$$

where t is in $(0, \frac{1}{L}]$. We define $f_1 := \frac{1}{2t}\|x\|^2$ and $f_2 := \frac{1}{2t}\|x\|^2 - f(x)$. Both functions f_1 and f_2 are convex since t is in $(0, \frac{1}{L}]$. If we solve the subproblem of the DCA at iteration k given by

$$x^{k+1} = \operatorname{argmin}_x \ \frac{1}{2t}\|x\|^2 - \left(\frac{1}{2t}\left\|x^k\right\|^2 - f(x^k)\right) - \left\langle \frac{1}{t}x^k - \nabla f(x^k), x - x^k \right\rangle,$$

we get

$$x^{k+1} = x^k - t\nabla f(x^k),$$

which is exactly the steps of the gradient descent method; see Algorithm 4.1. Note that f_1 and f_2 are $\frac{1}{t}$-smooth $\frac{1}{t}$-strongly convex and $\left(\frac{1}{t}+L\right)$-smooth $\left(\frac{1}{t}-L\right)$-strongly convex functions, respectively. Using the Theorem 8.6, one can derive the following convergence rate for the gradient descent method.

Proposition 8.10. *Suppose that $f(x)$ is an L-smooth function. If N iterations of the gradient descent method runs with $t \in (0, \frac{1}{L}]$, then*

$$\min_{0\le k\le N} \|g^k\| \le \sqrt{\frac{4(f(x^0)-f^\star)}{(4t-Lt^2)N+\frac{2}{L}}}.$$

As the given bound with Proposition 8.10 matches the bound given by Theorem 4.3 when $t_k = t \in (0, \frac{1}{L}]$, the given bound by Theorem 8.6 is tight for some cases.

8.3.4 Proximal gradient method

As a by-product of our analysis, we conclude the section by giving a convergence rate for the *proximal gradient method* (known as forward-backward algorithm) with constant step length for non-convex problems. Consider the non-convex optimization problem,

$$\begin{aligned} &\inf \phi(x) := g(x)+h(x) \qquad (8.19)\\ &\text{s.t. } x \in \mathbb{R}^n, \end{aligned}$$

where g is an L-smooth function on $\mathbb{R}^n$ and $h \in \mathcal{F}_{0,\infty}(\mathbb{R}^n)$. Algorithm 8.2 below describes the proximal gradient method with constant step.

Remark that Algorithm 8.2 reduces to the gradient method when $h = 0$. To the best knowledge of the authors, the proximal gradient method for non-convex problems, first is developed in [FM81].

We use the first-order optimality condition for computing the convergence rate. Note that $\bar{x}$ satisfies the first order optimality condition with accuracy $\epsilon \ge 0$ if

$$\min_{\xi\in\partial_L\phi(\bar{x})} \|\xi\| \le \epsilon. \qquad (8.21)$$

Algorithm 8.2 Proximal gradient method with constant step length

Pick $x^0 \in \mathbb{R}^n$ and the step length $t > 0$.
For $k = 0, 1, \dots$ perform the following steps:

1. Compute the *proximal operator* of th at $x^k - t\nabla g(x^k)$, that is,

$$x^{k+1} = \text{argmin}_{x\in\mathbb{R}^n} th(x) + \tfrac{1}{2}\left\|x - x^k + t\nabla g(x^k)\right\|^2. \tag{8.20}$$

2. If the termination criterion is satisfied, then stop.

It is worth noting that (8.21) implies that

$$\phi'(\bar{x}; d) \geq -\epsilon, \quad \|d\| = 1,$$

where $\phi'(\bar{x}; d)$ denotes the directional derivative of ϕ at $\bar{x}$ in the direction of d; see [Nes13].

Suppose that $t \leq \frac{1}{L}$ and the sequence $\{x^k\}$ is generated by Algorithm 8.2. We define $f_1 = h + \frac{1}{2t}\|.\|^2$ and $f_2 = \frac{1}{2t}\|.\|^2 - g$. Clearly, $f_1 \in \mathcal{F}_{\frac{1}{t},\infty}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\frac{1}{t}-L,\frac{1}{t}+L}(\mathbb{R}^n)$. Consider the functions f_1 and f_2 as just defined above. The subproblem (8.2) is

$$x^{k+1} = \text{argmin}_{x\in\mathbb{R}^n} h(x) + \frac{1}{2t}\|x\|^2 - \frac{1}{2t}\|x^k\|^2 + g(x^k) - \langle \frac{1}{t}x^k - \nabla g(x^k), x - x^k\rangle.$$

The optimality conditions imply

$$x^{k+1} = x^k - t\left(\nabla h(x^{k+1} + \nabla g(x^k)\right).$$

The last expression is the same as x^{k+1} generated by considering optimality conditions of subproblem (8.20). This shows that Algorithm 8.2 and Algorithm 8.1 share the same sequence $\{x^k\}$. Hence, one can obtain the following convergence rate by using Corollary 8.7.

Proposition 8.11. *Let g be a L-smooth function on $\mathbb{R}^n$ and $h \in \mathcal{F}_{0,\infty}(\mathbb{R}^n)$ and let $-\infty < \phi^\star = \min_{x\in\mathbb{R}^n} \phi(x)$. If $t \leq \frac{1}{L}$, then after N iterations of Algorithm 8.2, we have*

$$\min_{0\leq k\leq N}\left(\min_{\xi\in\partial_L\phi(x^k)} \|\xi\|\right) \leq \sqrt{\frac{2(L+\frac{1}{t})^2(\phi(x^1) - \phi^\star)}{(L+\frac{2}{t})N}}.$$

If $h = 0$, in the same line by using Theorem 8.6, one can infer the following convergence rate for the gradient descent method with fixed step length $\frac{1}{L}$,

$$\min_{0\le k\le N} \|\nabla \phi(x^k)\| \le \sqrt{\frac{4L\big(\phi(x^0)-\phi^\star\big)}{4N+3}},$$

which is established by authors recently; see [AdKZ22, Theorem 2] and Theorem 4.3.

8.4 Performance estimation using a convergence criterion for critical points in the nonsmooth case

Theorem 8.6 addresses the case that f_1 or f_2 is L-smooth with $L < \infty$. In what follows, we investigate the case that f_1 and f_2 are proper convex functions and where both may be non-smooth. For this general case, we need to adopt a different termination criterion to obtain results, since the termination criterion $\|g_1^k - g_2^k\| \le \epsilon$ may be of no use in this case. For example, suppose that a DC function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is given by

$$f(x) = \begin{cases} f_1(x) - f_2(x) & x \ge 0 \\ \infty & x < 0, \end{cases}$$

where

$$\begin{aligned} f_1(x) &= \max_{n\in\mathbb{N}\cup\{0\}} \{-n(x-2^{-n}) + 2 - 2^{1-n} - n2^{-n}\}, \\ f_2(x) &= \max_{n\in\mathbb{N}\cup\{0\}} \{-(n+1)(x-2^{-n}) + 2 - 3(2^{-n}) - n2^{-n}\}. \end{aligned}$$

With $x^0 = 1$ and the given DC decomposition, Algorithm 8.1 may generate

$$x^k = 2^{-k}, \quad g_1^k = -(k-1), \quad g_2^k = -k, \quad k \in \{1, 2, \ldots\}.$$

As $|g_1^k - g_2^k| = 1$, Algorithm 8.1 never stops by employing the given termination criterion while it is convergent to global minimum $\bar{x} = 0$. We therefore will use the termination criterion of the following value being sufficiently small:

$$\begin{aligned} T(x^{k+1}) &:= f_1(x^k) - f_2(x^k) - \min_{x\in\mathbb{R}^n} \big(f_1(x) - f_2(x^k) - \big\langle g_2^k, x - x^k \big\rangle\big) \\ &= f_1(x^k) - f_1(x^{k+1}) - \big\langle g_2^k, x^k - x^{k+1} \big\rangle. \end{aligned} \tag{8.22}$$

Note that $T(x^{k+1}) \geq 0$. It follows that if $T(x^{k+1}) = 0$ then $f(x^k) = f(x^{k+1})$, and $x^k \in \text{argmin}_{x \in \mathbb{R}^n} f_1(x) - f_2(x^k) - \langle g_2^k, x - x^k \rangle$. Indeed, by the optimality conditions for convex problems, we have $\partial f_1(x^k) \cap \partial f_2(x^k) \neq \emptyset$. Consequently, $T(x^{k+1}) = 0$ implies that x^k is a critical point of problem (8.7). The aforementioned stopping criterion has also been employed for the analysis of the Frank-Wolfe method for nonconvex problems; see equation (2.6) in [Gha19].

In what follows, we investigate Algorithm 8.1 with the termination criterion $T(x^{k+1}) < \epsilon$ for the given accuracy $\epsilon > 0$. The performance estimation problem with termination criterion (8.22) may be written as follows,

$$
\begin{aligned}
&\max \ \ell \\
&\text{s.t. } f_1(x^k) - f_1(x^{k+1}) - \left\langle g_1^{k+1}, x^k - x^{k+1} \right\rangle \geq \ell \quad i \in \{0, \dots, N-1\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_1}{L_1})} \left(\tfrac{1}{L_1} \left\| g_1^i - g_1^j \right\|^2 + \mu_1 \left\| x^i - x^j \right\|^2 - \tfrac{2\mu_1}{L_1} \left\langle g_1^j - g_1^i, x^j - x^i \right\rangle \right) \\
&\quad \leq f_1^i - f_1^j - \left\langle g_1^j, x^i - x^j \right\rangle \quad i, j \in \{0, \dots, N\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})} \left(\tfrac{1}{L_2} \left\| g_1^{i+1} - g_1^{j+1} \right\|^2 + \mu_2 \left\| x^i - x^j \right\|^2 - \tfrac{2\mu_2}{L_2} \left\langle g_1^{j+1} - g_1^{i+1}, x^j - x^i \right\rangle \right) \\
&\quad \leq f_2^i - f_2^j - \left\langle g_1^{j+1}, x^i - x^j \right\rangle \quad i, j \in \{0, \dots, N-1\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})} \left(\tfrac{1}{L_2} \left\| g_2^N - g_1^{j+1} \right\|^2 + \mu_2 \left\| x^N - x^j \right\|^2 - \tfrac{2\mu_2}{L_2} \left\langle g_1^{j+1} - g_2^N, x^j - x^N \right\rangle \right) \\
&\quad \leq f_2^N - f_2^j - \left\langle g_1^{j+1}, x^N - x^j \right\rangle \quad j \in \{0, \dots, N-1\} \\
&\quad \tfrac{1}{2(1-\frac{\mu_2}{L_2})} \left(\tfrac{1}{L_2} \left\| g_1^{i+1} - g_2^N \right\|^2 + \mu_2 \left\| x^i - x^N \right\|^2 - \tfrac{2\mu_2}{L_2} \left\langle g_2^N - g_1^{i+1}, x^N - x^i \right\rangle \right) \\
&\quad \leq f_2^i - f_2^N - \left\langle g_2^N, x^i - x^j \right\rangle \quad i \in \{0, \dots, N-1\} \\
&\quad f_1^k - f_2^k \geq f^\star \quad k \in \{0, \dots, N\} \\
&\quad f_1^0 - f_2^0 - f^\star \leq \Delta.
\end{aligned} \tag{8.23}
$$

Note that we do not employ Lemma 8.4 in this formulation because we consider a general DC problem. Using the performance estimation procedure as described before the proof of Theorem 8.6 once more, we obtain the following result.

Theorem 8.12. *Let $f_1 \in \mathcal{F}_{\mu_1, L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2, L_2}(\mathbb{R}^n)$. Then, after N iterations of Algorithm 8.1, one has*

$$
\begin{aligned}
&\min_{0 \leq k \leq N-1} f_1(x^k) - f_1(x^{k+1}) - \langle g_2^k, x^k - x^{k+1} \rangle \leq \\
&\min \left\{ \frac{L_1}{N(L_1 + \mu_2)}, \frac{L_2}{N(L_2 + \mu_1) - \mu_1} \right\} \left(f(x^0) - f^\star \right).
\end{aligned} \tag{8.24}
$$

Proof. We show separately that $\frac{L_1(f(x^0)-f^\star)}{N(L_1+\mu_2)}$ and $\frac{L_2(f(x^0)-f^\star)}{N(L_2+\mu_1)-\mu_1}$ are upper bounds for problem (8.23). The proof is analogous to that of Theorem 8.6. First, consider the bound $\frac{L_1(f(x^0)-f^\star)}{N(L_1+\mu_2)}$. Since the given bound does not depend on μ_1 and L_2, we may assume without loss of generality that $L_2=\infty$ and $\mu_1=0$. Suppose that $B_1=\frac{L_1}{N(L_1+\mu_2)}$. With some algebra, one can show that

$$
\begin{aligned}
&\ell - B_1\Delta + \tfrac{1}{N}\sum_{k=1}^{N}\left(f_1^{k-1}-f_1^k-\langle g_1^k, x^{k-1}-x^k\rangle-\ell\right)+B_1\left(f_1^N-f_2^N-f^\star\right)+\\
&B_1\left(f^\star-f_1^0+f_2^0+\Delta\right)+(\tfrac{1}{N}-B_1)\sum_{k=1}^{N}\left(f_1^k-f_1^{k-1}-\left\langle g_1^{k-1}, x^k-x^{k-1}\right\rangle-\tfrac{1}{2L_1}\left\|g_1^k-g_1^{k-1}\right\|^2\right)\\
&+B_1\sum_{k=1}^{N}\left(f_2^k-f_2^{k-1}-\left\langle g_1^k, x^k-x^{k-1}\right\rangle-\tfrac{\mu_2}{2}\left\|x^k-x^{k-1}\right\|^2\right)\\
&=-\tfrac{B_1\mu_2}{2}\sum_{k=1}^{N}\left\|x^{k-1}-x^k-\tfrac{1}{L_1}(g_1^{k-1}-g_1^k)\right\|^2\leq 0.
\end{aligned}
$$

The rest of proof is similar to that of Theorem 8.6. Now, we consider the bound $\frac{L_2(f(x^0)-f^\star)}{N(L_2+\mu_1)-\mu_1}$. Without loss generality, we may assume that $L_1=\infty$ and $\mu_2=0$. By doing some calculus, one can show that

$$
\begin{aligned}
&\ell - B_2\Delta + B_2\left(f_1^0-f_1^1-\left\langle g_1^1, x^0-x^1\right\rangle-\ell\right)+B_2\left(f_1^N-f_2^N-f^\star\right)\\
&+B_2\left(f^\star-f_1^0+f_2^0+\Delta\right)+\tfrac{1-B_2}{N-1}\sum_{k=2}^{N}\left(f_1^{k-1}-f_1^k-\left\langle g_1^k, x^{k-1}-x^k\right\rangle-\ell\right)\\
&+\alpha\sum_{k=2}^{N}\left(f_1^k-f_1^{k-1}-\left\langle g_1^{k-1}, x^k-x^{k-1}\right\rangle-\tfrac{\mu_1}{2}\left\|x^k-x^{k-1}\right\|^2\right)\\
&+B_2\sum_{k=1}^{N}\left(f_2^k-f_2^{k-1}-\left\langle g_1^k, x^k-x^{k-1}\right\rangle-\tfrac{1}{2L_2}\left\|g_1^{k+1}-g_1^k\right\|^2\right)\\
&+B_2\left(f_2^N-f_2^{N-1}-\left\langle g_1^N, x^N-x^{N-1}\right\rangle-\tfrac{1}{2L_2}\left\|g_2^N-g_1^N\right\|^2\right)\\
&=-\tfrac{B_2}{2L_2}\left\|g_2^N-g_1^N\right\|^2-\tfrac{B_2}{2L_2}\sum_{k=2}^{N}\left\|g_1^{k-1}-g_1^k-\tfrac{\alpha L_2}{B_2}(x^{k-1}-x^k)\right\|^2\leq 0,
\end{aligned}
$$

where $B_2=\frac{L_2}{N(L_2+\mu_1)-\mu_1}$ and $\alpha=\frac{1-B_2}{N-1}-B_2$. Since we assume $L_2>\mu_1$, we have $B_2,\ \alpha\geq 0$. The rest of the proof runs as before. □

The important point is that the last result provides a rate of convergence even if neither L_1 nor L_2 is finite, and we therefore state it as a corollary.

Corollary 8.13. *Let $f_1 \in \mathcal{F}_{\mu_1,\infty}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,\infty}(\mathbb{R}^n)$, i.e. consider any DC decomposition in problem* (8.1)*. Then, after N iterations of Algorithm 8.1, one has*

$$\min_{0\le k\le N-1} f_1(x^k) - f_1(x^{k+1}) - \langle g_2^k, x^k - x^{k+1}\rangle \le \frac{1}{N}\left(f(x^0) - f^\star\right).$$

This result is new to the best of our knowledge.

8.5 Performance analysis of DCA with regularization

Pang et al. [PRA17] developed a version of Algorithm 8.1 with regularization. The DCA with a *regularization term* is described in Algorithm 8.3. As $r \ge -\min(\mu_1, \mu_2)$,

Algorithm 8.3 DCA with regularization

Pick $x^0 \in \mathbb{R}^n$, $N \in \mathbb{N}$, $r \ge -\min(\mu_1, \mu_2)$ and $\epsilon > 0$.
For $k = 0, 1, \ldots, N-1$ perform the following steps:

1. Choose $g_1^k \in \partial f_1(x^k)$ and $g_2^k \in \partial f_2(x^k)$. If $\left\|g_1^k - g_2^k\right\| \le \epsilon$, then stop.
2. Choose

$$x^{k+1} \in \operatorname{argmin}_{x\in\mathbb{R}^n} f_1(x) - f_2(x^k) - \left\langle g_2^k, x - x^k\right\rangle + \tfrac{r}{2}\left\|x - x^k\right\|^2. \quad (8.25)$$

the regularization parameter may take negative values. Our aim for the investigation of regularization parameter on this large interval is to have a comprehensive analysis. One can check that subproblem (8.25) is convex; see Lemma 8.3. It is worth noting that they [PRA17] consider only the fixed regularization parameter $r = 1$. Subproblem (8.25) may be reformulated as follows:

$$x^{k+1} \in \operatorname{argmin}_{x\in\mathbb{R}^n} f_1(x) + \tfrac{r}{2}\|x\|^2 - f_2(x^k) - \tfrac{r}{2}\|x_k\|^2 - \left\langle g_2^k + r x^k, x - x^k\right\rangle.$$

Corollary 8.14. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$, $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$ and $f = f_1 - f_2$. If $r \ge -\min(\mu_1, \mu_2)$, then f has a DC decomposition of form*

$$f = \bar{f}_1 - \bar{f}_2,$$

where $\bar{f}_1 \in \mathcal{F}_{\mu_1+r,L_1+r}(\mathbb{R}^n)$ and $\bar{f}_2 \in \mathcal{F}_{\mu_2+r,L_2+r}(\mathbb{R}^n)$ given by $\bar{f}_1(x) = f_1(x) + \tfrac{r}{2}\|x\|^2$ and $\bar{f}_2(x) = f_2(x) + \tfrac{r}{2}\|x\|^2$.

Proof. The proof is immediate from Lemma 8.3. □

By Corollary 8.14, it is readily seen that Algorithm 8.3 is the same as Algorithm 8.1, but with a different decomposition of f. It is clear that $\partial f_1(x) = \partial \bar{f}_1(x) + rx$ and $\partial f_2(x) = \partial \bar{f}_2(x) + rx$. Hence, by using Theorem 8.6, the following corollary gives a convergence rate for Algorithm 8.3.

Corollary 8.15. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$ and let L_1 or L_2 be finite. Suppose that $f(x^0) - f^\star \le \Delta$ and $r \ge -\min(\mu_1, \mu_2)$. Then after N iterations of Algorithm 8.3, one has:*

$$\min_{0\le k\le N} \left\|g_1^k - g_2^k\right\| \le \left(\frac{\mathcal{A}}{\mathcal{B}N + \mathcal{C}} \cdot \Delta\right)^{\frac{1}{2}},$$

where

$$\mathcal{A} = 2\big((L_1+r)(L_2+r) - (\mu_1+r)(L_2+r)I_{\mathbb{R}_+}(L_1-L_2) - (\mu_2+r)(L_1+r)I_{\mathbb{R}_+}(L_2-L_1)\big),$$

$$\mathcal{B} = (L_1+r) + (L_2+r) + (\mu_1+r)\left(\tfrac{L_1+r}{L_2+r} - 3\right)I_{\mathbb{R}_+}(L_1-L_2) + (\mu_2+r)\left(\tfrac{L_2+r}{L_1+r} - 3\right)I_{\mathbb{R}_+}(L_2-L_1),$$

and

$$\mathcal{C} = \frac{(L_1+r)(L_2+r) - I_{\mathbb{R}_+}(L_1-L_2)(\mu_1+r)(L_2+r) - I_{\mathbb{R}_+}(L_2-L_1)(\mu_2+r)(L_1+r)}{L_1-\mu_2}.$$

One interesting question concerning Algorithm 8.3 is the choice of suitable regularization parameter, r. By minimizing the right hand side of bound given in Corollary 8.15 with respect to r, the next proposition provides the optimal regularization parameter in terms of the worst-case convergence rate.

Proposition 8.16. *Let the assumptions of Corollary 8.15 hold. Then, the optimal regularization parameter is $\bar{r} = -\min(\mu_1, \mu_2)$.*

Proof. By Corollary 8.15, the following optimization problem gives the optimal regularization parameter,

$$\min_{r\ge -\min(\mu_1,\mu_2)} h(r) := \frac{(L_1+r)(L_2+r) - (\mu_1+r)(L_2+r)}{\Big((L_1+r)+(L_2+r)+(\mu_1+r)(\frac{L_1+r}{L_2+r}-3)\Big)N + L_1 - \mu_1}.$$

It is readily seen that h is positive on $[-\min(\mu_1, \mu_2), \infty)$. In addition, we have

$$\begin{aligned}\frac{dh(r)}{dr} &= \frac{(L_1-\mu_1)(L_2+r)\big(2L_2^2N - L_2(\mu_1+4\mu_1N) - \mu_1(1+2N)r + L_1(L_2+2\mu_1N+r+2Nr)\big)}{\big(L_2^2N - L_2(\mu_1+3\mu_1N) - \mu_1(1+2N)r + L_1(L_2+L_2N+\mu_1N+r+2Nr)\big)^2}\\ &= \frac{(L_1-\mu_1)(L_2+r)\big(r(2N+1)(L_1-\mu_1) + 2N\left(\mu_1(L_1-L_2) + L_2(L_2-\mu_1)\right) + L_2(L_1-\mu_1)\big)}{\big(L_2^2N - L_2(\mu_1+3\mu_1N) - \mu_1(1+2N)r + L_1(L_2+L_2N+\mu_1N+r+2Nr)\big)^2}.\end{aligned}$$

For $r \geq -\min(\mu_1, \mu_2) \geq -\mu_1$, we have

$$r(L_1 - \mu_1) + L_2^2 + L_1\mu_1 - 2L_2\mu_1 \geq -\mu_1(L_1 - \mu_1) + L_2^2 + L_1\mu_1 - 2L_2\mu_1 = (L_2 - \mu_1)^2 \geq 0,$$

which implies positivity of $\frac{dh(r)}{dr}$ on the given interval. Hence, the aforementioned problem attains its minimum at $\bar{r} = -\min(\mu_1, \mu_2)$. □

The result of Proposition 8.16 is unexpected. Due to Lemma 8.3, the Lipschitz modulus and strongly convex constant alter equally by the change of r. The underlying reason for this result may be that Lipschitz modulus plays a more important role than the strongly convex constant in the worst-case convergence rate.

Remark 8.17. *For undominated D.C. decompositions, one has* $\min(\mu_1, \mu_2) = 0$. *(Recall that a D.C. decomposition* $f = f_1 - f_2$ *is undominated if there is no other D.C. decomposition, say* $f = \hat{f}_1 - \hat{f}_2$*, such that* $f_1 - \hat{f}_1$ *is convex but not affine; see e.g. [AH18] for more details on undominated D.C. decompositions, including their construction for polynomials.)*

Thus, if we only consider given instances with undominated decompositions $f = f_1 - f_2$*, the optimal regularization parameter (with respect to the worst-case convergence rate) is zero.*

In the last part of the section, we investigate the nonsmooth case with regularization. First, we need to adopt an appropriate termination criterion. Suppose that $r \geq -\min(\mu_1, \mu_2)$. Similar to the termination criterion (8.22), we stop the algorithm if the following value is sufficiently small:

$$\begin{aligned} T_r(x^{k+1}) &= f_1(x^k) - f_2(x^k) - \min_{x \in \mathbb{R}^n} \left(f_1(x) - f_2(x^k) - \left\langle g_2^k, x - x^k \right\rangle + \tfrac{r}{2} \left\| x - x^k \right\|^2 \right) \\ &= f_1(x^k) - f_1(x^{k+1}) - \left\langle g_2^k, x^k - x^{k+1} \right\rangle - \tfrac{r}{2} \left\| x^k - x^{k+1} \right\|^2. \end{aligned} \tag{8.26}$$

One has $T_r(x^{k+1}) \geq 0$ and one can show that $\partial f_1(x^k) \cap \partial f_2(x^k) \neq \emptyset$ providing $T(x^{k+1}) = 0$.

By defining convex functions $\bar{f}_1(x) = f_1(x) + \frac{r}{2}\|x\|^2$ and $\bar{f}_2(x) = f_2(x) + \frac{r}{2}\|x\|^2$, it is readily seen that

$$T_r(x^{k+1}) = \bar{f}_1(x^k) - \bar{f}_1(x^{k+1}) - \left\langle g_2^k + r x^k, x^k - x^{k+1} \right\rangle.$$

By analysis similar to that of Corollary 8.15, the next remark provides a convergence rate.

Algorithm 8.4

Pick $x^0 \in \mathbb{R}^n$, $N \in \mathbb{N}$, and $\epsilon > 0$.
For $k = 0, 1, \ldots, N-1$ perform the following steps:

1. Choose $g_2^k \in \partial f_2(x^k)$ and

$$x^{k+1} \in \operatorname{argmin}_{x\in\mathbb{R}^n} f_1(x) - f_2(x^k) - \left\langle g_2^k, x - x^k \right\rangle + \tfrac{r}{2} \left\| x - x^k \right\|^2. \quad (8.27)$$

2. If $f_1(x^k) - f_1(x^{k+1}) - \left\langle g_2^k, x^k - x^{k+1} \right\rangle - \tfrac{r}{2} \left\| x^k - x^{k+1} \right\|^2 \le \epsilon$, then stop.

Remark 8.18. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$. Then, after N iterations of Algorithm 8.4, one has*

$$\min_{0\le k\le N-1} f_1(x^k) - f_1(x^{k+1}) - \left\langle g_2^k, x^k - x^{k+1} \right\rangle - \tfrac{r}{2} \left\| x^k - x^{k+1} \right\|^2 \le \quad (8.28)$$
$$\min\left\{ \frac{L_1 + r}{N(L_1 + \mu_2 + 2r)}, \frac{L_2 + r}{N(L_2 + \mu_1 + 2r) - \mu_1 - r} \right\} \left(f(x^0) - f^\star \right).$$

Note that this result holds even if L_1 and L_2 are not finite. In this case one has the bound

$$\min_{0\le k\le N-1} f_1(x^k) - f_1(x^{k+1}) - \left\langle g_2^k, x^k - x^{k+1} \right\rangle - \tfrac{r}{2} \left\| x^k - x^{k+1} \right\|^2 \le \frac{1}{N} \left(f(x^0) - f^\star \right).$$

8.6 Linear convergence of the DCA under the Polyak-Łojasiewicz inequality

In the section, we provide some sufficient conditions under which the DCA is linearly convergent. Similar to the former sections, we employ the performance estimation for obtaining convergence rate.

In recent years, the linear convergence of some optimization methods for non-convex problems have been investigated under the Polyak-Łojasiewicz (PŁ) inequality; see [AdKZ23a, BNPS17, KNS16] alongside with Chapter 5 and the reference therein. We say that f satisfies PŁ inequality on X if there exists $\eta > 0$ such that

$$f(x) - f^\star \le \tfrac{1}{2\eta} \|\xi\|^2, \quad \forall x \in X, \forall \xi \in \operatorname{co}(\partial_L f(x)). \quad (8.29)$$

Note that when f is differentiable inequality (8.29) is a special case of (8.3) with $\theta = \frac{1}{2}$ and different ground set. If f_2 is *strictly differentiable*, we have have

$\mathrm{co}(\partial_L f) = \partial f_1 - \partial f_2$; see Example 10.10 in [RW09]. Hence, the performance estimation problem with the PŁ inequality may be formulated as follows:

$$\begin{aligned}
\max\ & \frac{(f_1^1 - f_2^1) - f^\star}{(f_1^0 - f_2^0) - f^\star} \\
\text{s.t. } & \frac{1}{2(1-\frac{\mu_1}{L_1})}\left(\tfrac{1}{L_1}\left\|g_1^i - g_1^j\right\|^2 + \mu_1\left\|x^i - x^j\right\|^2 - \tfrac{2\mu_1}{L_1}\left\langle g_1^j - g_1^i, x^j - x^i\right\rangle\right) \\
& \le f_1^i - f_1^j - \left\langle g_1^j, x^i - x^j\right\rangle \quad i,j \in \{0,1\} \\
& \frac{1}{2(1-\frac{\mu_2}{L_2})}\left(\tfrac{1}{L_2}\left\|g_2^i - g_2^j\right\|^2 + \mu_2\left\|x^i - x^j\right\|^2 - \tfrac{2\mu_2}{L_2}\left\langle g_2^j - g_2^i, x^j - x^i\right\rangle\right) \\
& \le f_2^i - f_2^j - \left\langle g_2^j, x^i - x^j\right\rangle \quad i,j \in \{0,1\} \\
& f_1^k - f_2^k \ge f^\star \quad k \in \{0,1\} \\
& g_2^0 = g_1^1 \\
& \left(f_1^k - f_2^k\right) - f^\star \le \tfrac{1}{2\eta}\|g_1^k - g_2^k\|^2, \quad k \in \{0,1\}.
\end{aligned} \tag{8.30}$$

The following lemma shows that $\eta \le L_1$.

Lemma 8.19. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$, where L_1 is finite. If f satisfies the PŁ inequality on $X = \{x : f(x) \le f(x^0)\}$ with modulus $\eta > 0$, then $\eta \le L_1$.*

Proof. Without loss of generality, assume that $\mu_1 = \mu_2 = 0$. Let $g_1 \in \partial f_1$ and $g_2 \in \partial f_2$. According to the PŁ inequality,

$$f(x) - f^\star \le \frac{1}{2\eta}\|g_1 - g_2\|^2, \quad \forall x \in X,$$

and using Lemma 8.4, we obtain

$$\frac{1}{2L_1}\|g_1 - g_2\|^2 \le f(x) - f^\star, \quad \forall x \in X,$$

which implies $\eta \le L_1$. □

By doing constraint aggregation in problem (8.30) as before (i.e. demonstrating a dual feasible solution and using weak duality), we obtain the following linear convergence rate for the DCA under the PŁ inequality.

Theorem 8.20. *Let $f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$. If L_1 is finite and if f satisfies PŁ inequality on $X = \{x : f(x) \leq f(x^0)\}$ with modulus $0 < \eta \leq L_1$, then for x^1 from Algorithm 8.1, we have*

$$\frac{f(x^1)-f^\star}{f(x^0)-f^\star} \leq \left(\frac{1-\frac{\eta}{L_1}}{1+\frac{\eta}{L_2}}\right). \tag{8.31}$$

Proof. Since the given bound is independent of μ_1 and μ_2, without loss of generality, we assume that $\mu_1 = \mu_2 = 0$. In addition, we assume that $f^\star = 0$. Direct calculation shows that

$$\begin{aligned}
&\left(f_1^1-f_2^1\right)-f^\star-\left(\frac{1-\frac{\eta}{L_1}}{1+\frac{\eta}{L_2}}\right)\left(\left(f_1^0-f_2^0\right)-f^\star\right)+\left(\frac{1}{1+\frac{\eta}{L_2}}\right)\times\\
&\left(f_1^0-f_1^1-\left\langle g_1^1, x^0-x^1\right\rangle-\tfrac{1}{2L_1}\left\|g_1^0-g_1^1\right\|^2\right)\\
&+\left(\frac{1}{1+\frac{\eta}{L_2}}\right)\left(f_2^1-f_2^0-\left\langle g_1^1, x^1-x^0\right\rangle-\tfrac{1}{2L_2}\left\|g_1^1-g_2^1\right\|^2\right)+\left(\frac{\frac{\eta}{L_1}}{1+\frac{\eta}{L_2}}\right)\times\\
&\left(\frac{1}{2\eta}\left\|g_1^0-g_1^1\right\|^2-f_1^0+f_2^0\right)+\left(\frac{\frac{\eta}{L_2}}{1+\frac{\eta}{L_2}}\right)\left(\frac{1}{2\eta}\left\|g_1^1-g_2^1\right\|^2-f_1^1+f_2^1\right)=0.
\end{aligned}$$

As all the multipliers in the last expression are non-negative, for any feasible solution of problem (8.15), we have

$$f(x^1)-f^\star-\left(\frac{1-\frac{\eta}{L_1}}{1+\frac{\eta}{L_2}}\right)\left(f(x^0)-f^\star\right) \leq 0,$$

completing the proof. □

Note that Theorem 8.1 by Le Thi et al. [LTHPD18] does not imply Theorem 8.20 if inequality (8.3) holds on $\{x : f(x) \leq f(x^0)\}$ with $\theta = \frac{1}{2}$, since we assume neither strong convexity of f_1 or f_2, nor boundedness of the sequence of iterates. Moreover, we give explicit expressions for the constants that determine the linear convergence rate of the sequence of objective values.

8.7 Conclusion

We have shown that the performance estimation framework of Drori and Teboulle [DT14] yields new insights into the convergence behavior of the Difference-of-convex algorithm (DCA). As future work, one may also consider the convergence of the DCA on more restricted classes of DC problems, e.g. where f_1 and f_2 are convex polynomials, as studied in [AH18]. For constrained problems, even the case where f_1 and f_2 are quadratic polynomials is of interest, e.g. in the study of (extended) trust region problems.

Nothing contributes so much to tranquilize the mind as a steady purpose - a point on which the soul may fix its intellectual eyes.

Mary W Shelley

9

The exact worst-case convergence rate of the alternating direction method of multipliers

Preamble

Recently, semidefinite programming performance estimation has been employed as a strong tool for the worst-case performance analysis of first-order methods. In this chapter, we derive new non-ergodic convergence rates for the alternating direction method of multipliers (ADMM) by using performance estimation. We give some examples which show the exactness of the given bounds. We also study the linear and R-linear convergence of ADMM under some assumptions. We establish that ADMM enjoys a global linear convergence rate if and only if the dual objective satisfies the Polyak-Łojasiewicz (PŁ) inequality in the presence of strong convexity. In addition, we give an explicit formula for the linear convergence rate factor. Moreover, we study the *R*-linear convergence of ADMM under two new scenarios. This chapter is based on the paper [ZAdK23].

9.1 Introduction

We consider the optimization problem

$$\min_{(x,z)\in\mathbb{R}^n\times\mathbb{R}^m} f(x)+g(z), \quad \text{s.t. } Ax+Bz=b, \tag{9.1}$$

where $f:\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ and $g:\mathbb{R}^m\to\mathbb{R}\cup\{\infty\}$ are closed proper convex functions, $0\neq A\in\mathbb{R}^{r\times n}$, $0\neq B\in\mathbb{R}^{r\times m}$ and $b\in\mathbb{R}^r$. Moreover, we assume that $(x^\star,z^\star)$ is an optimal solution of problem (9.1) and $\lambda^\star$ is its corresponding Lagrange multipliers. Moreover, we denote the value of f and g at $x^\star$ and $z^\star$ with $f^\star$ and $g^\star$, respectively.

Problem (9.1) appears naturally (or after variable splitting) in many applications in statistics, machine learning and image processing to name but a few [BPC+11, ROF92, HTW15, LLF22]. The most common method for solving problem (9.1) is the *alternating direction method of multipliers* (ADMM). ADMM is a dual based approach that exploits separable structure and it may be described as follows.

Algorithm 9.1 ADMM

Set N and $t>0$ (step length), pick λ^0,z^0.
For $k=1,2,\dots,N$ perform the following step:

1. $x^k\in\operatorname{argmin} f(x)+\langle\lambda^{k-1},Ax\rangle+\frac{t}{2}\|Ax+Bz^{k-1}-b\|^2$
2. $z^k\in\operatorname{argmin} g(z)+\langle\lambda^{k-1},Bz\rangle+\frac{t}{2}\|Ax^k+Bz-b\|^2$
3. $\lambda^k=\lambda^{k-1}+t(Ax^k+Bz^k-b)$.

ADMM was first proposed in [GM76, GM75] for solving nonlinear variational problems. We refer the interested reader to [GOY17] for a historical review of ADMM. The popularity of ADMM is due to its capability to be implemented parallelly and hence can handle large-scale problems [BPC+11, Han22, MKL15, SBG+20]. For example, it is used for solving inverse problems governed by partial differential equation forward models [LV21], and distributed energy resource coordinations [LSW+22], to mention but a few.

The convergence of ADMM has been investigated extensively in the literature and there exist many convergence results. However, different performance measures have been used for the computation of convergence rate; see [FRV18,

GOSB14, ST22, GMS13, HY12, MS13, LLF22, LL19]. In this chapter, we consider the dual objective value as a performance measure.

Throughout the chapter, we assume that each subproblem in steps 1 and 2 of Algorithm 9.1 attains its minimum. The Lagrangian function of problem (9.1) may be written as

$$L(x,z,\lambda) = f(x) + g(z) + \langle \lambda, Ax + Bz - b \rangle, \tag{9.2}$$

and the dual objective of problem (9.1) is also defined as

$$D(\lambda) = \min_{(x,z)\in\mathbb{R}^n\times\mathbb{R}^m} f(x) + g(z) + \langle \lambda, Ax + Bz - b \rangle.$$

We assume throughout the chapter that strong duality holds for problem (9.1), that is

$$\max_{\lambda\in\mathbb{R}^r} D(\lambda) = \min_{Ax+Bz=b} f(x) + g(z).$$

Note that we have strong duality when both functions f and g are real-valued. For extended convex functions, strong duality holds under some mild conditions; see e.g. [Bec17, Chapter 15].

Some common performance measures for the analysis of ADMM are as follows,

- Objective value: $\left| f(x^N) + g(z^N) - f^\star - g^\star \right|$;
- Primal and dual feasibility: $\left\| Ax^N + Bz^N - b \right\|$ and $\left\| A^T B(z^N - z^{N-1}) \right\|$;
- Dual objective value: $D(\lambda^\star) - D(\lambda^N)$;
- Distance between (x^N, z^N, λ^N) and a saddle point of problem (9.2).

Note that the mathematical expressions are written in a non-ergodic sense for convenience. Each measure is useful in monitoring the progress and convergence of ADMM. The objective value is the most commonly used performance measure for the analysis of algorithms in convex optimization [Ber15, Bec17, Nes03]. As mentioned earlier, ADMM is a dual based method and it may be interpreted as a proximal method applied to the dual problem; see [Ber15, LLF22] for further discussions and insights. Thus, a natural performance measure for ADMM would be dual objective value. In this study, we investigate the convergence rate of ADMM in terms of dual objective value and feasibility. It worth noting that most performance measures may be analyzed through the framework developed in Section 9.2.

Regarding the dual objective value, the following convergence rate is known in the literature. This theorem holds for strongly convex functions f and g; recall that f is called strongly convex with modulus $\mu \geq 0$ if the function $f - \frac{\mu}{2}\|\cdot\|^2$ is convex.

Theorem 9.1. *[GOSB14, Theorem 1] Let f and g be strongly convex with moduli $\mu_1 > 0$ and $\mu_2 > 0$, respectively. If $t \leq \sqrt[3]{\frac{\mu_1 \mu_2^2}{\lambda_{\max}(A^T A)\lambda_{\max}^2(B^T B)}}$, then*

$$D(\lambda^\star) - D(\lambda^N) \leq \frac{\|\lambda^1 - \lambda^\star\|^2}{2t(N-1)}. \tag{9.3}$$

In this study we establish that Algorithm 9.1 has the convergence rate of $\mathcal{O}(\frac{1}{N})$ in terms of dual objective value without assuming the strong convexity of g. Under this setting, we also prove that Algorithm 9.1 has the convergence rate of $\mathcal{O}(\frac{1}{N})$ in terms of primal and dual residuals. Moreover, we show that the given bounds are exact. Furthermore, we study the linear and R-linear convergence.

Outline of the chapter

The chapter is structured as follows. We present the semidefinite programming (SDP) performance estimation method of ADMM in Section 9.2, and we develop the performance estimation to handle dual based methods including ADMM. In Section 9.3, we derive some new non-asymptotic convergence rates by using performance estimation for ADMM in terms of dual function, primal and dual residuals. Furthermore, we show that the given bounds are tight by providing some examples. In Section 9.4 we proceed with the study of the linear convergence of ADMM. We establish that ADMM enjoys a linear convergence if and only if the dual function satisfies the PŁ inequality when the objective function is strongly convex. Furthermore, we investigate the relation between the PŁ inequality and common conditions used by scholars to prove the linear convergence. Section 9.5 is devoted to the R-linear convergence. We prove that ADMM is R-linear convergent under two new scenarios which are weaker than the existing ones in the literature.

Terminology and notation

We have the following identity

$$\xi \in \partial f(x) \quad \Leftrightarrow \quad x \in \partial f^*(\xi). \tag{9.4}$$

By using conjugate functions, the dual of problem (9.1) may be written as

$$D(\lambda) = \min_{(x,z)\in\mathbb{R}^n\times\mathbb{R}^m} f(x) + g(z) + \langle \lambda, Ax + Bz - b\rangle$$
$$= -\langle \lambda, b\rangle - f^*(-A^T\lambda) - g^*(-B^T\lambda). \tag{9.5}$$

By the optimality conditions for the dual problem, we get

$$b - Ax^\star - Bz^\star = 0, \tag{9.6}$$

for some $x^\star \in \partial f^*(-A^T\lambda^\star)$ and $z^\star \in \partial g^*(-B^T\lambda^\star)$. Equation (9.6) with (9.4) imply that $(x^\star, z^\star)$ is an optimal solution to problem (9.1).

The optimality conditions for the subproblems of Algorithm 9.1 may be written as

$$0 \in \partial f(x^k) + A^T\lambda^{k-1} + tA^T\left(Ax^k + Bz^{k-1} - b\right),$$
$$0 \in \partial g(z^k) + B^T\lambda^{k-1} + tB^T\left(Ax^k + Bz^k - b\right). \tag{9.7}$$

As $\lambda^k = \lambda^{k-1} + t(Ax^k + Bz^k - b)$, we get

$$0 \in \partial f(x^k) + A^T\lambda^k + tA^TB\left(z^{k-1} - z^k\right),\ \ 0 \in \partial g(z^k) + B^T\lambda^k. \tag{9.8}$$

So, (x^k, z^k) is optimal for dual objective at λ^k if and only if $A^TB\left(z^{k-1} - z^k\right) = 0$. We call $A^TB\left(z^{k-1} - z^k\right)$ *dual residual*.

9.2 Performance estimation

In this section, we develop the performance estimation for ADMM. Gu and Yang [GY20] employed performance estimation to study the extension of the dual step length for ADMM. Note that while there are some similarities between our work and [GY20] in using performance estimation, the formulations and results are different.

The worst-case convergence rate of Algorithm 9.1 with respect to dual objective value may be cast as the following abstract optimization problem,

$$\begin{aligned}
\max\ & D(\lambda^\star) - D(\lambda^N)\\
\text{s. t. } & \{x^k, z^k, \lambda^k\}_1^N \text{ is generated by Algorithm 9.1 w.r.t. } f, g, A, B, b, \lambda^0, z^0, t\\
& (x^\star, z^\star) \text{ is an optimal solution with Lagrangian multipliers } \lambda^\star\\
& \|\lambda^0 - \lambda^\star\|^2 + t^2\left\|z^0 - z^\star\right\|_B^2 = \Delta\\
& f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n),\ g \in \mathcal{F}^B_{c_2,\infty}(\mathbb{R}^m)\\
& \lambda^0 \in \mathbb{R}^r, z^0 \in \mathbb{R}^m, A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}, b \in \mathbb{R}^r,
\end{aligned} \tag{9.9}$$

where $f, g, A, B, b, z^0, \lambda^0, x^\star, z^\star, \lambda^\star$ are decision variables and N, t, c_1, c_2, Δ are the given parameters. Note that problem (9.9) will be unbounded unless we impose some initial condition. We regard boundedness of $\|\lambda^0-\lambda^\star\|^2+t^2\left\|z^0-z^\star\right\|_B^2$ as an initial condition. The boundedness of $t^{-1}\|\lambda^0-\lambda^\star\|^2+t\left\|z^0-z^\star\right\|_B^2$ is commonly used for the convergence analysis of ADMM; see e.g. [BPC+11, LLF22]. We opt to utilize the positive multiplication of this criterion for notational convenience as t is a fixed positive constant in Algorithm 9.1. Moreover, we use this measure to establish R-linear convergence in terms of dual objective; see Section 9.5 for more discussion.

Note that $D(\lambda^\star)=f^\star+g^\star$ and $(\tilde{x},\tilde{z})\in \operatorname{argmin} f(x)+g(z)+\langle\lambda^N, Ax+Bz-b\rangle$ if and only if

$$\tilde{\xi}+A^T\lambda^N=0, \quad \tilde{\eta}+B^T\lambda^N=0, \tag{9.10}$$

for some $\tilde{\xi}\in\partial f(\tilde{x})$ and $\tilde{\eta}\in\partial g(\tilde{z})$. It is worth noting that a point $\tilde{x}$ satisfying these conditions exists, as function f is strongly convex relative to A. In addition, one may consider $\tilde{z}=z^N$ by virtue of (9.8). For the sake of notational convenience, we introduce $x^{N+1}=\tilde{x}$ and $\xi^{N+1}=\tilde{\xi}$. The reader should bear in mind that x^{N+1} is not generated by Algorithm 9.1. Therefore,

$$D(\lambda^N)=f(x^{N+1})+g(z^N)+\left\langle\lambda^N, Ax^{N+1}+Bz^N-b\right\rangle$$

for some x^{N+1} with $-A^T\lambda^N\in\partial f(x^{N+1})$.

By using Theorem 2.40 to replace the conditions $f\in\mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$, and $g\in\mathcal{F}^B_{c_2,\infty}(\mathbb{R}^m)$ by finite interpolation conditions, and by using the optimality conditions (9.7), problem (9.9) may be reformulated as a finite dimensional optimization problem, through the performance estimation technique:

$$
\begin{aligned}
&\max f^\star + g^\star - \left(f^{N+1} + g^N + \left\langle \lambda^N, Ax^{N+1} + Bz^N - b\right\rangle\right)\\
&\text{s.t. } \{(x^k;\xi^k;f^k)\}_1^{N+1} \cup \{(x^\star;\xi^\star;f^\star)\} \text{ satisfy interpolation constraints (2.5)}\\
&\quad \{(z^k;\eta^k;g^k)\}_0^{N} \cup \{(z^\star;\eta^\star;g^\star)\} \text{ satisfy interpolation constraints (2.5)}\\
&\quad (x^\star, z^\star) \text{ is an optimal solution with Lagrangian multipliers } \lambda^\star\\
&\quad \|\lambda^0 - \lambda^\star\|^2 + t^2\left\|z^0 - z^\star\right\|_B^2 = \Delta \qquad (9.11)\\
&\quad \xi^k = tA^Tb - tA^TAx^k - tA^TBz^{k-1} - A^T\lambda^{k-1}, \quad k \in \{1,\dots,N\}\\
&\quad \eta^k = tB^Tb - tB^TAx^k - tB^TBz^k - B^T\lambda^{k-1}, \quad k \in \{1,\dots,N\}\\
&\quad \lambda^k = \lambda^{k-1} + t(Ax^k + Bz^k - b), \quad k \in \{1,\dots,N\}\\
&\quad \xi^{N+1} + A^T\lambda^N = 0\\
&\quad \lambda^0 \in \mathbb{R}^r, z^0 \in \mathbb{R}^m, A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}, b \in \mathbb{R}^r.
\end{aligned}
$$

In problem (9.11), $A, B, \{x^k;\xi^k;f^k\}_1^{N+1}, \{(x^\star;\xi^\star;f^\star)\}, \{\lambda^k\}_0^N, \{z^k;\eta^k;g^k\}_0^N$, $\{(z^\star;\eta^\star;g^\star)\}, \lambda^\star, b$ are decision variables. To handle problem (9.11), without loss of generality, we assume that the matrix $\begin{pmatrix}A & B\end{pmatrix}$ has full row rank. Note this assumption does not appear in our arguments in the following sections. In addition, we introduce some new variables. As problem (9.1) is invariant under translation of (x, z), we may assume without loss of generality that $b = 0$ and $(x^\star, z^\star) = (0, 0)$. In addition, due to the full row rank of the matrix $\begin{pmatrix}A & B\end{pmatrix}$, we may assume that $\lambda^0 = \begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}x^\dagger\\ z^\dagger\end{pmatrix}$ and $\lambda^\star = \begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{z}\end{pmatrix}$ for some $\bar{x}, x^\dagger, \bar{z}, z^\dagger$. So,

$$
\xi^\star = -A^TA\bar{x} - A^TB\bar{z} \in \partial f(0), \quad \eta^\star = -B^TA\bar{x} - B^TB\bar{z} \in \partial g(0),
$$

and $D(\lambda^\star) = f^\star + g^\star$.

By using equality constraints of problem (9.11) and the newly introduced variables, we have for $k \in \{1,\dots,N\}$

$$
\begin{aligned}
&\lambda^k = (Ax^\dagger + Bz^\dagger) + \sum_{i=1}^{k} t(Ax^i + Bz^i), \qquad (9.12)\\
&-(A^TAx^\dagger + A^TBz^\dagger) - \sum_{i=1}^{k-1} t(A^TAx^i + A^TBz^i) - tA^TAx^k - tA^TBz^{k-1} \in \partial f(x^k),\\
&-(B^TAx^\dagger + B^TBz^\dagger) - \sum_{i=1}^{k} t(B^TAx^i + B^TBz^i) \in \partial g(z^k).
\end{aligned}
$$

Hence, problem (9.11) may be written as

$$
\begin{aligned}
\max\ & f^\star + g^\star - f^{N+1} - g^N - \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{N} t(Ax^i + Bz^i), Ax^{N+1} + Bz^N \right\rangle \\
\text{s.t.}\ & \tfrac{c_1}{2}\left\|x^k - x^j\right\|_A^2 \le \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{k-1} t(Ax^i + Bz^i) + tAx^k + tBz^{k-1}, A(x^j - x^k) \right\rangle + \\
& \qquad f^j - f^k, \quad k \in \{1, \dots, N\}, \quad j \in \{1, \dots, N+1\}, \\
& \tfrac{c_1}{2}\left\|x^{N+1} - x^j\right\|_A^2 \le \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{N} t(Ax^i + Bz^i), A\left(x^j - x^{N+1}\right) \right\rangle + \\
& \qquad f^j - f^{N+1}, \quad j \in \{1, \dots, N\}, \\
& \tfrac{c_2}{2}\left\|z^k - z^j\right\|_B^2 \le \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{k} t(Ax^i + Bz^i), B\left(z^j - z^k\right) \right\rangle + \\
& \qquad g^j - g^k, \quad j, k \in \{1, \dots, N\}, \\
& \tfrac{c_1}{2}\left\|x^k\right\|_A^2 \le f^k - f^\star + \left\langle A\bar{x} + B\bar{z}, Ax^k \right\rangle, \quad k \in \{1, \dots, N+1\}, \\
& \tfrac{c_1}{2}\left\|x^k\right\|_A^2 \le -\left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{k-1} t(Ax^i + Bz^i) + tAx^k + tBz^{k-1}, Ax^k \right\rangle + \\
& \qquad f^\star - f^k, \quad k \in \{1, \dots, N\}, \\
& \tfrac{c_1}{2}\left\|x^{N+1}\right\|_A^2 \le f^\star - f^{N+1} - \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{N} t(Ax^i + Bz^i), Ax^{N+1} \right\rangle, \\
& \tfrac{c_2}{2}\left\|z^k\right\|_B^2 \le g^k - g^\star + \left\langle A\bar{x} + B\bar{z}, Bz^k \right\rangle, \quad k \in \{1, \dots, N\}, \\
& \tfrac{c_2}{2}\left\|z^k\right\|_B^2 \le g^\star - g^k - \left\langle Ax^\dagger + Bz^\dagger + \sum_{i=1}^{k} t(Ax^i + Bz^i), Bz^k \right\rangle, \quad k \in \{1, \dots, N\}, \\
& \left\|Ax^\dagger + Bz^\dagger - (A\bar{x} + B\bar{z})\right\|^2 + t^2\left\|z^0\right\|_B^2 = \Delta, \\
& x^\dagger \in \mathbb{R}^n, z^0, z^\dagger \in \mathbb{R}^m, A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}.
\end{aligned}
\tag{9.13}
$$

In problem (9.13), $A, B, \{x^k, f^k\}_1^{N+1}, \{z^k, g^k\}_1^N, x^\dagger, z^\dagger, \bar{x}, f^\star, \bar{z}, g^\star, z^0$ are decision variables. By using the Gram matrix method, problem (9.13) may be relaxed as a semidefinite program as follows. Let

$$U = \begin{pmatrix} x^\dagger & x^1 & \dots & x^{N+1} & \bar{x} \end{pmatrix}, \qquad V = \begin{pmatrix} z^\dagger & z^0 & \dots & z^N & \bar{z} \end{pmatrix}.$$

By introducing matrix variable

$$Y = \begin{pmatrix} AU & BV \end{pmatrix}^T \begin{pmatrix} AU & BV \end{pmatrix},$$

problem (9.13) may be relaxed as the following SDP,

$$
\begin{aligned}
\max\ & f^\star + g^\star - f^{N+1} - g^N - \operatorname{tr}(L_o Y) \\
\text{s.t. } & \operatorname{tr}(L^f_{i,j} Y) \le f^i - f^j, \quad i,j \in \{1,\dots,N+1,\star\} \\
& \operatorname{tr}(L^g_{i,j} Y) \le g^i - g^j, \quad i,j \in \{1,\dots,N,\star\} \\
& \operatorname{tr}(L_0 Y) = \Delta \\
& Y \succeq 0,
\end{aligned}
\tag{9.14}
$$

where the constant matrices $L^f_{i,j}, L^g_{i,j}, L_o, L_0$ are determined according to the constraints of problem (9.13). In the following sections, we present some new convergence results that are derived by solving this kind of formulation.

9.3 Worst-case convergence rate

In this section, we provide new convergence rates for ADMM with respect to some performance measures. Before we get to the theorems we need to present some lemmas.

Lemma 9.2. *Let $N \ge 4$ and $t, c \in \mathbb{R}$. Let $E(t,c)$ be $(N+1) \times (N+1)$ symmetric matrix given by*

$$
E(t,c) = \begin{pmatrix}
2c & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & t-c \\
0 & \alpha_2 & \beta_2 & 0 & \dots & 0 & 0 & \dots & 0 & -t \\
0 & \beta_2 & \alpha_3 & \beta_3 & \dots & 0 & 0 & \dots & 0 & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \alpha_k & \beta_k & \dots & 0 & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \alpha_N & \beta_N \\
t-c & -t & t & t & \dots & t & t & \dots & \beta_N & \alpha_{N+1}
\end{pmatrix},
$$

where

$$
\alpha_k = \begin{cases}
6c - 5t, & k = 2 \\
2\left(2k^2 - 3k + 1\right)c - (4k-1)t, & 3 \le k \le N-1 \\
2N(N-1)c - (2N+1)t, & k = N \\
2Nc - (N+1)t, & k = N+1,
\end{cases}
$$

$$
\beta_k = \begin{cases}
2kt - (2k^2 - k - 1)c, & 2 \le k \le N-1 \\
3t - 2(N-1)c, & k = N,
\end{cases}
$$

and k denotes row number. If $c > 0$ is given, then

$$[0, c] \subseteq \{t : E(t, c) \succeq 0\}.$$

Proof. As $\{t : E(t, c) \succeq 0\}$ is a convex set, it suffices to prove the positive semidefiniteness of $E(0, c)$ and $E(c, c)$. Since $E(0, c)$ is diagonally dominant, it is positive semidefinite. Now, we establish that the matrix $K = E(1, 1)$ is positive definite. To this end, we show that all leading principal minors of K are positive. To compute the leading principal minors, we perform the following elementary row operations on K:

i) Add the second row to the third row;

ii) Add the second row to the last row;

iii) Add the third row to the forth row;

iv) For $i = 4 : N - 1$

- Add $i - th$ row to $(i + 1) - th$ row;
- Add $\frac{3-i}{2i^2-3i-1}$ times of $i - th$ row to the last row;

v) Add $\frac{N-1}{3N-5}$ times of $N - th$ row to $(N + 1) - th$ row.

It is seen that $K_{k-1,k} + K_{k,k} = -K_{k+1,k}$ for $2 \leq k \leq N - 1$. Hence, by performing these operations, we get an upper triangular matrix J with diagonal

$$J_{k,k} = \begin{cases} 2, & k = 1 \\ 2k^2 - 3k - 1, & 2 \leq k \leq N - 1 \\ 3N - 5, & k = N \\ N - 2 - \frac{(N-1)^2}{3N-5} - \sum_{i=4}^{N-1} \frac{(i-3)^2}{2i^2-3i-1}, & k = N + 1. \end{cases}$$

It is seen all first N diagonal elements of J are positive. We show that $J_{N+1,N+1}$ is also positive. For $i \geq 4$ we have

$$\frac{(i-3)^2}{2i^2-3i-1} \leq \frac{(i-1)^2+4}{2(i-1)^2} \leq \frac{1}{2} + \frac{2}{(i-1)(i-2)}. \tag{9.15}$$

So,

$$\frac{2N^2-9N+9}{3N-5} - \sum_{i=4}^{N-1} \frac{(i-3)^2}{2i^2-3i-1} \geq \frac{(N-2)(N^2-5N+10)}{2N(3N-5)} > 0,$$

which implies $J_{N+1,N+1} > 0$. Since we add a factor of $i-th$ row to $j-th$ row with $i < j$, all leading principal minors of matrices K and J are the same. Hence K is positive definite. As $E(c,c) = cK$, one can infer the positive definiteness of $E(c,c)$ and the proof is complete. □

In the upcoming lemma, we establish a valid inequality for ADMM that will be utilized in all the subsequent results presented in this section.

Lemma 9.3. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$, $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ and $x^\star = 0$, $z^\star = 0$. Suppose that ADMM with the starting points λ^0 and z^0 generates $\{(x^k; z^k; \lambda^k)\}$. If $N \geq 4$ and $v \in \mathbb{R}^r$, then*

$$\begin{aligned}
&N\langle \lambda^N, Ax^N + Bz^N\rangle - \langle \lambda^N + tAx^N + tBz^{N-1}, Ax^N - v\rangle + \langle \lambda^0 + tAx^1 + tBz^0, Ax^1 - v\rangle + \\
&\tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 - \tfrac{1}{2t}\left\|\lambda^N - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2 - t\left\langle Ax^1 - Ax^2 + (N+1)Ax^N + Bz^N, v\right\rangle - \\
&t\sum_{k=3}^{N}\langle Ax^k, v\rangle + \tfrac{t(N-1)}{2}\|v\|^2 - \tfrac{c_1}{2}\left\|x^1\right\|_A^2 + \sum_{k=2}^{N}\tfrac{\alpha_k}{2}\left\|x^k\right\|_A^2 + \sum_{k=2}^{N-1}\beta_k\langle Ax^k, Ax^{k+1}\rangle + \\
&tN\langle Bz^{N-1}, Ax^N - v\rangle + t\langle Ax^N, Bz^N\rangle - \tfrac{t(N-1)^2}{2}\left\|z^N - z^{N-1}\right\|_B^2 - \tfrac{tN^2}{2}\left\|Ax^N + Bz^N\right\|^2 - \\
&t\left\|x^2\right\|_A^2 + f(x^1) - f(x^N) + N\left(f(x^N) - f^\star + g(x^N) - g^\star\right) \geq 0, \qquad (9.16)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_k &= \begin{cases} (4k-1)t - 2\left(2k^2 - 3k + 1\right)c_1, & 2 \leq k \leq N-1, \\ (4N+1)t - \left(2N^2 - 5N + 3\right)c_1, & k = N, \end{cases} \\
\beta_k &= \left(2k^2 - k - 1\right)c_1 - 2kt.
\end{aligned}$$

Proof. To establish the desired inequality, we demonstrate its validity by summing a series of valid inequalities. To simplify the notation, let $f^k = f(x^k)$ and $g^k = g(z^k)$ for $k \in \{1, \ldots, N\}$. Note that $b = 0$ because $x^\star = 0, z^\star = 0$. By (2.5) and (9.7), we get the following inequality

$$
\begin{aligned}
&\sum_{k=1}^{N-1}(k^2-1)\Big(f^{k+1}-f^{k}+\big\langle \lambda^{k-1}+tAx^{k}+tBz^{k-1}, A(x^{k+1}-x^{k})\big\rangle - \tfrac{c_1}{2}\left\|x^{k+1}-x^{k}\right\|_A^2\Big)\\
&+\sum_{k=1}^{N-1}(k^2-k)\Big(f^{k}-f^{k+1}+\big\langle \lambda^{k}+tAx^{k+1}+tBz^{k}, A(x^{k}-x^{k+1})\big\rangle - \tfrac{c_1}{2}\left\|x^{k+1}-x^{k}\right\|_A^2\Big)\\
&+\sum_{k=1}^{N}\Big(f^{k}-f^{\star}+\big\langle \lambda^{\star}, Ax^{k}\big\rangle - \tfrac{c_1}{2}\left\|x^{k}\right\|_A^2\Big)+\sum_{k=1}^{N-1}k^2\left(g^{k}-g^{k+1}+\big\langle \lambda^{k+1}, B(z^{k}-z^{k+1})\big\rangle\right)\\
&+\sum_{k=1}^{N-1}(k^2+k)\left(g^{k+1}-g^{k}+\big\langle \lambda^{k}, B(z^{k+1}-z^{k})\big\rangle\right)+\sum_{k=1}^{N}\left(g^{k}-g^{\star}+\big\langle \lambda^{\star}, Bz^{k}\big\rangle\right)\\
&+\tfrac{t}{2}\left\|Ax^{1}+Bz^{0}-v\right\|^2 \geq 0.
\end{aligned}
$$

As $\lambda^k = \lambda^{k-1} + tAx^k + tBz^k$, the inequality can be expressed as

$$
\begin{aligned}
&\sum_{k=1}^{N-1}(k^2-1)\Big(\big\langle tAx^{k}+tBz^{k-1}, A(x^{k+1}-x^{k})\big\rangle - \tfrac{c_1}{2}\left\|x^{k+1}-x^{k}\right\|_A^2\Big)+\\
&\sum_{k=1}^{N-1}(k^2-1)\left(\big\langle \lambda^{k}, Ax^{k+1}\big\rangle - \big\langle \lambda^{k-1}, Ax^{k}\big\rangle - \big\langle tAx^{k}+tBz^{k}, Ax^{k+1}\big\rangle\right)+\\
&\sum_{k=1}^{N-1}(k^2-k)\Big(\big\langle tAx^{k+1}+tBz^{k}, A(x^{k}-x^{k+1})\big\rangle - \tfrac{c_1}{2}\left\|x^{k+1}-x^{k}\right\|_A^2\Big)+\\
&\sum_{k=1}^{N-1}(k^2-k)\left(\big\langle \lambda^{k-1}, Ax^{k}\big\rangle - \big\langle \lambda^{k}, Ax^{k+1}\big\rangle + \big\langle tAx^{k}+tBz^{k}, Ax^{k}\big\rangle\right)+\\
&\sum_{k=1}^{N-1}(k^2+k)\left(\big\langle \lambda^{k}, Bz^{k+1}\big\rangle - \big\langle \lambda^{k-1}, Bz^{k}\big\rangle - \big\langle tAx^{k}+tBz^{k}, Bz^{k}\big\rangle\right)+\\
&\sum_{k=1}^{N-1}k^2\Bigg(\big\langle \lambda^{k-1}, Bz^{k}\big\rangle - \big\langle \lambda^{k}, Bz^{k+1}\big\rangle + \big\langle tAx^{k}+tBz^{k}+tAx^{k+1}+tBz^{k+1}, Bz^{k}\big\rangle -\\
&\quad \big\langle tAx^{k+1}+tBz^{k+1}, Bz^{k+1}\big\rangle\Bigg)+\sum_{k=1}^{N}\Big(\big\langle \lambda^{\star}, Ax^{k}+Bz^{k}\big\rangle - \tfrac{c_1}{2}\left\|x^{k}\right\|_A^2\Big)+\tfrac{t}{2}\left\|Bz^{0}\right\|^2+\\
&\tfrac{t}{2}\left\|Ax^{1}-v\right\|^2+t\big\langle Ax^{1}-v, Bz^{0}\big\rangle+f^{1}-f^{N}+N(f^{N}-f^{\star}+g^{N}-g^{\star})\geq 0.
\end{aligned}
$$

After performing some algebraic manipulations, we obtain

$$N\langle\lambda^{N-1},Ax^N+Bz^N\rangle-\langle\lambda^{N-1},Ax^N\rangle+\langle\lambda^0,Ax^1\rangle-\sum_{k=0}^{N-1}\langle\lambda^k-\lambda^\star,Ax^{k+1}+Bz^{k+1}\rangle+$$
$$\tfrac{t}{2}\left\|Ax^1-v\right\|^2+\tfrac{t}{2}\left\|Bz^0\right\|^2+t\left\langle Ax^1-v,Bz^0\right\rangle-t(N^2-3N+1)\langle Ax^N,Bz^{N-1}\rangle-$$
$$t\sum_{k=1}^{N-1}\Big((k-1)^2\|Ax^k\|^2-(k^2-k)\langle Ax^k,Ax^{k+1}\rangle-(k^2-1)\langle Ax^{k+1},Bz^{k-1}\rangle\Big)-$$
$$t\sum_{k=1}^{N-1}\Big((k^2-k+1)\|Bz^k\|^2+(-k^2+k+1)\langle Ax^k,Bz^k\rangle-k^2\langle Bz^k,Bz^{k+1}\rangle\Big)-$$
$$t\sum_{k=2}^{N-1}\Big((2k^2-3k)\langle Ax^k,Bz^{k-1}\rangle\Big)-t(N-1)^2\|Bz^N\|^2-t(N^2-3N+2)\|Ax^N\|^2-$$
$$t(N-1)^2\langle Ax^N,Bz^N\rangle-\sum_{k=1}^{N-1}\Big((2k^2-k-1)\tfrac{c_1}{2}\left\|x^{k+1}-x^k\right\|_A^2+\tfrac{c_1}{2}\left\|x^{k+1}\right\|_A^2\Big)-$$
$$\tfrac{c_1}{2}\left\|x^1\right\|_A^2+f^1-f^N+N(f^N-f^\star+g^N-g^\star)\geq 0.$$

By using $\lambda^{N-1}=\lambda^N-tAx^N-tBz^N$ and

$$2\langle\lambda^k-\lambda^\star,Ax^{k+1}+Bz^{k+1}\rangle=\tfrac{1}{t}\|\lambda^{k+1}-\lambda^\star\|^2-\tfrac{1}{t}\|\lambda^k-\lambda^\star\|^2-t\|Ax^{k+1}+Bz^{k+1}\|^2,$$

we get

$$N\langle\lambda^N,Ax^N+Bz^N\rangle-\langle\lambda^N+tAx^N+tBz^{N-1},Ax^N-v\rangle+\langle\lambda^0+tAx^1+tBz^0,Ax^1-v\rangle$$
$$+\tfrac{1}{2t}\left\|\lambda^0-\lambda^\star\right\|^2-\tfrac{1}{2t}\left\|\lambda^N-\lambda^\star\right\|^2+\tfrac{t}{2}\left\|z^0\right\|_B^2-t\left\langle Ax^1-Ax^2+(N+1)Ax^N+Bz^N,v\right\rangle$$
$$-t\sum_{k=3}^{N}\left\langle Ax^k,v\right\rangle-\tfrac{t}{2}\sum_{k=2}^{N-1}\left\|(k-1)Bz^{k-1}-(k-1)Bz^k+kAx^k-(k+1)Ax^{k+1}+v\right\|^2$$
$$+\tfrac{t(N-1)}{2}\|v\|^2-\frac{c_1}{2}\left\|x^1\right\|_A^2-2t\left\|x^2\right\|_A^2+\tfrac{1}{2}\sum_{k=2}^{N-1}\left((4k-1)\,t-2\left(2k^2-3k+1\right)c_1\right)\left\|x^k\right\|_A^2$$
$$+\sum_{k=2}^{N-1}\left(\left(2k^2-k-1\right)c_1-2kt\right)\langle Ax^k,Ax^{k+1}\rangle+\left(\left(2N+\tfrac{1}{2}\right)t-\left(N^2-\tfrac{5}{2}N+\tfrac{3}{2}\right)c_1\right)\left\|x^N\right\|_A^2$$
$$+tN\left\langle Bz^{N-1},Ax^N-v\right\rangle+t\left\langle Ax^N,Bz^N\right\rangle-\tfrac{t(N-1)^2}{2}\left\|z^N-z^{N-1}\right\|_B^2$$
$$-\tfrac{tN^2}{2}\left\|Ax^N+Bz^N\right\|^2+f^1-f^N+N\left(f^N-f^\star+g^N-g^\star\right)\geq 0,$$

which implies the desired inequality.

□

We may now prove the main result of this section.

Theorem 9.4. *Let $f \in \mathcal{F}^{A}_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ with $c_1 > 0$. If $t \leq c_1$ and $N \geq 4$, then*

$$D(\lambda^\star) - D(\lambda^N) \leq \frac{\|\lambda^0 - \lambda^\star\|^2 + t^2 \left\|z^0 - z^\star\right\|_B^2}{4Nt}. \tag{9.17}$$

Proof. As discussed in Section 9.2, we may assume that $x^\star = 0$ and $z^\star = 0$. By (9.10), we have $D(\lambda^N) = f(\hat{x}^N) + g(z^N) + \left\langle \lambda^N, A\hat{x}^N + Bz^N \right\rangle$ for some $\hat{x}^N$ with $-A^T\lambda^N \in \partial f(\hat{x}^N)$. By employing (2.5) and (9.7), we obtain

$$\begin{aligned}
&N\left(g(x^N) - g^\star + \langle \lambda^\star, Bz^N\rangle\right) + (N-1)\left(f(x^N) - f^\star + \langle \lambda^\star, Ax^N\rangle - \tfrac{c_1}{2}\left\|x^N\right\|_A^2\right) + \\
&\left(f(\hat{x}^N) - f(x^1) + \left\langle \lambda^0 + tAx^1 + tBz^0, A\hat{x}^N - Ax^1\right\rangle - \tfrac{c_1}{2}\left\|\hat{x}^N - x^1\right\|_A^2\right) + \\
&(2N-2)\Bigg(f(\hat{x}^N) - f(x^N) + \left\langle \lambda^N - tBz^N + tBz^{N-1}, A\hat{x}^N - Ax^N\right\rangle - \\
&\quad \tfrac{c_1}{2}\left\|\hat{x}^N - x^N\right\|_A^2\Bigg) + \left(f(\hat{x}^N) - f^\star + \langle \lambda^\star, A\hat{x}^N\rangle - \tfrac{c_1}{2}\left\|\hat{x}^N\right\|_A^2\right) \geq 0.
\end{aligned} \tag{9.18}$$

By substituting v with $A\hat{x}^N$ in inequality (9.16) and summing it with (9.18), we get the following inequality after performing some algebraic manipulations

$$\begin{aligned}
&2N\left(f(\hat{x}^N) + g(x^N) + \left\langle \lambda^N, A\hat{x}^N + Bz^N\right\rangle - f^\star - g^\star\right) + \tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2 - \\
&\tfrac{1}{2t}\left\|\lambda^N - \lambda^\star + t(N-1)Ax^N + tA\hat{x}^N + tNBz^N\right\|^2 - \\
&\tfrac{t}{2}\left\|(N-1)(Bz^{N-1} - Bz^N) + tAx^N - tA\hat{x}^N\right\|^2 - \\
&\tfrac{1}{2}\operatorname{tr}\left(E(t,c_1)\begin{pmatrix}Ax^1 & \dots & A\hat{x}^N\end{pmatrix}^T\begin{pmatrix}Ax^1 & \dots & A\hat{x}^N\end{pmatrix}\right) \geq 0,
\end{aligned} \tag{9.19}$$

where the positive semidefinite matrix $E(t, c_1)$ is given in Lemma 9.2. As the inner product of positive semidefinite matrices is non-negative, inequality (9.19) implies that

$$2N\left(D(\lambda^\star) - D(\lambda^N)\right) \leq \tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2,$$

and the proof is complete.

□

In comparison with Theorem 9.1, we get a new convergence rate when only f is strongly convex, i.e. g does not need to be strongly convex. Also, the constant does not depend on λ^1. One important question concerning bound (9.17) is its tightness, that is, if there is an optimization problem which attains the given convergence rate. It turns out that the bound (9.17) is exact. The following example demonstrates this point.

Example 9.5. *Suppose that $c_1 > 0$, $N \geq 4$ and $t \in (0, c_1]$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given as follows,*

$$f(x) = \tfrac{1}{2}|x| + \tfrac{c_1}{2}x^2, \quad g(z) = \tfrac{1}{2}\max\{\tfrac{N-1}{N}(z - \tfrac{1}{2Nt}) - \tfrac{1}{2Nt}, -z\}.$$

Consider the optimization problem

$$\min_{(x,z)\in\mathbb{R}\times\mathbb{R}} f(x) + g(z), \quad \text{s.t. } x + z = 0,$$

It is seen that $A = B = I$ in this problem. Note that $(x^\star, z^\star) = (0,0)$ with Lagrangian multiplier $\lambda^\star = \frac{1}{2}$ is an optimal solution and the optimal value is zero. One can check that Algorithm 9.1 with initial point $\lambda^0 = \frac{-1}{2}$ and $z^0 = 0$ generates the following points,

$$\begin{aligned} x^k &= 0 && k \in \{1, \ldots, N\} \\ z^k &= \tfrac{1}{2Nt} && k \in \{1, \ldots, N\} \\ \lambda^k &= \tfrac{-1}{2} + \tfrac{k}{2N} && k \in \{1, \ldots, N\}. \end{aligned}$$

At λ^N, we have $D(\lambda^N) = \frac{-1}{4Nt} = -\frac{\|\lambda^0 - \lambda^\star\|^2 + t^2\|z^0 - z^\star\|_B^2}{4Nt}$, which shows the tightness of bound (9.17).

One important factor concerning dual-based methods that determines the efficiency of an algorithm is primal and dual feasibility (residual) convergence rates. In what follows, we study this subject under the setting of Theorem 9.4. The next theorem gives a convergence rate in terms of primal residual under the setting of Theorem 9.4.

Theorem 9.6. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ with $c_1 > 0$. If $t \leq c_1$ and $N \geq 4$, then*

$$\left\|Ax^N + Bz^N - b\right\| \leq \frac{\sqrt{\|\lambda^0 - \lambda^\star\|^2 + t^2\left\|z^0 - z^\star\right\|_B^2}}{tN}. \tag{9.20}$$

Proof. The argument is similar to that used in the proof of Theorem 9.4. By setting $v = Ax^N$ in (9.16), one can infer the following inequality

$$N\left\langle \lambda^N, Ax^N + Bz^N \right\rangle + \left\langle \lambda^0 + tAx^1 + tBz^0, Ax^1 - Ax^N \right\rangle + \tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2 -$$
$$t\left\langle Ax^1 - Ax^2, Ax^N \right\rangle + \tfrac{t(N-1)}{2}\left\|Ax^N\right\|^2 - t\sum_{k=3}^{N}\left\langle Ax^k, Ax^N\right\rangle - \frac{c_1}{2}\left\|x^1\right\|_A^2 - t\left\|x^2\right\|_A^2 +$$
$$\sum_{k=2}^{N-1}\left(\left(2k - \tfrac{1}{2}\right)t - \left(2k^2 - 3k + 1\right)c_1\right)\left\|x^k\right\|_A^2 + \left(\left(\tfrac{3}{2}N - \tfrac{3}{2}\right)t - \left(N^2 - \tfrac{5}{2}N + \tfrac{3}{2}\right)c_1\right)\left\|x^N\right\|_A^2 +$$
$$\sum_{k=2}^{N-1}\left(\left(2k^2 - k - 1\right)c_1 - 2kt\right)\langle Ax^k, Ax^{k+1}\rangle - \tfrac{t(N-1)^2}{2}\left\|z^N - z^{N-1}\right\|_B^2 -$$
$$\tfrac{tN^2}{2}\left\|Ax^N + Bz^N\right\|^2 + f(x^1) - f(x^N) + N\left(f(x^N) - f^\star + g(x^N) - g^\star\right) \geq 0. \tag{9.21}$$

By employing (2.5) and (9.7), we have

$$N\left(f^\star - f(x^N) - \langle \lambda^N + Bz^{N-1} - Bz^N, Ax^N\rangle - \tfrac{c_1}{2}\left\|x^N\right\|_A^2\right) +$$
$$\left(f(x^N) - f^1 + \left\langle \lambda^0 + tAx^1 + tBz^0, Ax^N - Ax^1\right\rangle - \tfrac{c_1}{2}\left\|x^N - x^1\right\|_A^2\right) + \tag{9.22}$$
$$N\left(g^\star - g(x^N) - \langle \lambda^N, Bz^N\rangle\right) \geq 0.$$

By summing (9.21) and (9.22), we obtain

$$\tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2 - \tfrac{t(N-1)^2}{2}\left\|z^{N-1} - z^N + \tfrac{N}{(N-1)^2}x^N\right\|_B^2 -$$
$$\tfrac{tN^2}{2}\left\|Ax^N + Bz^N\right\|^2 - \tfrac{1}{2}\operatorname{tr}\left(D(t,c_1)\begin{pmatrix}Ax^1 & \dots & Ax^N\end{pmatrix}^T\begin{pmatrix}Ax^1 & \dots & Ax^N\end{pmatrix}\right) \geq 0, \tag{9.23}$$

where the matrix $D(t, c_1)$ is as follows,

$$D(t,c_1) = \begin{pmatrix}
2c_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & t - c_1 \\
0 & \alpha_2 & \beta_2 & 0 & \dots & 0 & 0 & \dots & 0 & -t \\
0 & \beta_2 & \alpha_3 & \beta_3 & \dots & 0 & 0 & \dots & 0 & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \alpha_k & \beta_k & \dots & 0 & t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} \\
t - c_1 & -t & t & t & \dots & t & t & \dots & \beta_{N-1} & \alpha_N
\end{pmatrix},$$

and

$$\alpha_k = \begin{cases} 6c_1 - 5t, & k = 2 \\ 2\left(2k^2 - 3k + 1\right)c_1 - (4k-1)\,t, & 3 \le k \le N-1, \\ \left(2N^2 - 4N + 4\right)c_1 - \left(3N - 5 + \frac{N^2}{(N-1)^2}\right)t, & k = N, \end{cases}$$
$$\beta_k = 2kt - \left(2k^2 - k - 1\right)c_1, \quad 2 \le k \le N-1$$

As the matrix $D(t, c_1)$ is positive semidefinite, see Appendix A.3 Lemma A.1, inequality (9.23) implies that

$$\tfrac{tN^2}{2}\left\|Ax^N + Bz^N\right\|^2 \le \tfrac{1}{2t}\left\|\lambda^0 - \lambda^\star\right\|^2 + \tfrac{t}{2}\left\|z^0\right\|_B^2,$$

and the proof is complete. □

The following example shows the exactness of bound (9.20).

Example 9.7. *Let $c_1 > 0$, $N \ge 4$ and $t \in (0, c_1]$. Consider functions $f, g : \mathbb{R} \to \mathbb{R}$ given by the formulae follows,*

$$f(x) = \tfrac{1}{2}|x| + \tfrac{c_1}{2}x^2,$$
$$g(z) = \max\{\left(\tfrac{1}{2} - \tfrac{1}{N}\right)\left(z - \tfrac{1}{Nt}\right), \tfrac{1}{2}\left(\tfrac{1}{Nt} - z\right)\}.$$

We formulate the following optimization problem,

$$\min_{(x,z)\in\mathbb{R}\times\mathbb{R}} f(x) + g(z), \quad \text{s.t. } Ax + Bz = 0,$$

where $A = B = I$. One can verify that $(x^\star, z^\star) = (0, 0)$ with Lagrangian multiplier $\lambda^\star = \frac{1}{2}$ is an optimal solution. Algorithm 9.1 with initial point $\lambda^0 = \frac{-1}{2}$ and $z^0 = 0$ generates the following points,

$$x^k = 0 \qquad k \in \{1, \ldots, N\}$$
$$z^k = \tfrac{1}{Nt} \qquad k \in \{1, \ldots, N\}$$
$$\lambda^k = \tfrac{2k-N}{2N} \qquad k \in \{1, \ldots, N\}.$$

At iteration N, we have $\|Ax^N + Bz^N\| = \frac{1}{tN} = \frac{\sqrt{\|\lambda^0-\lambda^\star\|^2 + t^2\|z^0 - z^\star\|_B^2}}{tN}$, which shows the tightness of bound (9.20).

In what follows, we study the convergence rate of ADMM in terms of residual dual. To this end, we investigate the convergence rate of $\{B\left(z^{k-1}-z^k\right)\}$ as $\left\|A^T B\left(z^{k-1}-z^k\right)\right\| \leq \|A\| \left\|z^{k-1}-z^k\right\|_B$. The next theorem provides a convergence rate for the aforementioned sequence.

Theorem 9.8. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ with $c_1 > 0$. If $t \leq c_1$ and $N \geq 4$, then*

$$\left\|z^N - z^{N-1}\right\|_B \leq \frac{\sqrt{\|\lambda^0 - \lambda^\star\|^2 + t^2 \|z^0 - z^\star\|_B^2}}{(N-1)t}. \tag{9.24}$$

Proof. Similar to the proof of Theorem 9.4, by setting $v = Ax^N$ in (9.16) for $N-1$ iterations, one can infer the following inequality

$$\begin{aligned}
&(N-1)\langle \lambda^{N-1}, Ax^{N-1} + Bz^{N-1}\rangle + \tfrac{1}{2t}\|\lambda^0 - \lambda^\star\|^2 - \tfrac{1}{2t}\|\lambda^{N-1} - \lambda^\star\|^2 + \\
&\tfrac{t}{2}\left\|z^0\right\|_B^2 - \langle \lambda^{N-1} + tAx^{N-1} + tBz^{N-2}, Ax^{N-1} - Ax^N\rangle + \tfrac{t(N-2)}{2}\|x^N\|_A^2 + \\
&\langle \lambda^0 + tAx^1 + tBz^0, Ax^1 - Ax^N\rangle - t\left\langle Ax^1 - Ax^2 + NAx^{N-1} + Bz^{N-1}, Ax^N\right\rangle \\
&+ \frac{1}{2}\sum_{k=2}^{N-2}\left((4k-1)t - 2\left(2k^2 - 3k + 1\right)c_1\right)\left\|x^k\right\|_A^2 + t\langle Ax^{N-1}, Bz^{N-1}\rangle + \\
&\sum_{k=2}^{N-2}\left(\left(2k^2 - k - 1\right)c_1 - 2kt\right)\langle Ax^k, Ax^{k+1}\rangle + t(N-1)\langle Bz^{N-2}, Ax^{N-1} - Ax^N\rangle \\
&+ \frac{1}{2}\left((4N-3)t - \left(2N^2 - 9N + 10\right)c_1\right)\left\|x^{N-1}\right\|_A^2 - t\left\|x^2\right\|_A^2 - \tfrac{c_1}{2}\left\|x^1\right\|_A^2 - \\
&\tfrac{t(N-2)^2}{2}\left\|z^{N-1} - z^{N-2}\right\|_B^2 - \tfrac{t(N-1)^2}{2}\|Ax^{N-1} + Bz^{N-1}\|^2 - t\sum_{k=3}^{N-1}\langle Ax^k, Ax^N\rangle + \\
&f(x^1) - f(x^{N-1}) + (N-1)(f(x^{N-1}) - f^\star + g(x^{N-1}) - g^\star) \geq 0.
\end{aligned} \tag{9.25}$$

By using (2.5) and (9.7), we have

$$
\begin{aligned}
&(N^2-3N+2)\Big(f(x^{N-1})-f(x^N)+\left\langle\lambda^{N-1}+tAx^N+tBz^{N-1},A\left(x^{N-1}-x^N\right)\right\rangle-\\
&\tfrac{c_1}{2}\left\|x^N-x^{N-1}\right\|_A^2\Big)+\Big(f(x^N)-f(x^1)+\left\langle\lambda^0+tAx^1+tBz^0,A\left(x^N-x^1\right)\right\rangle-\\
&\tfrac{c_1}{2}\|x^N-x^1\|_A^2\Big)+N(N-1)\left(g(z^N)-g(z^{N-1})+\left\langle\lambda^{N-1},B\left(z^N-z^{N-1}\right)\right\rangle\right)+\\
&(N^2-3N+1)\Big(f(x^N)-f(x^{N-1})+\left\langle\lambda^{N-1}-tBz^{N-1}+tBz^{N-2},A\left(x^N-x^{N-1}\right)\right\rangle\\
&-\tfrac{c_1}{2}\|x^N-x^{N-1}\|_A^2\Big)+(N-1)\left(g^\star-g(z^N)-\left\langle\lambda^{N-1}+tAx^N+tBz^N,Bz^N\right\rangle\right)+\\
&(N-1)\Big(f^\star-f(x^{N-1})-\left\langle\lambda^{N-1}-tBz^{N-1}+tBz^{N-2},Ax^{N-1}\right\rangle-\tfrac{c_1}{2}\|x^{N-1}\|_A^2\Big)+\\
&(N-1)^2\left(g(z^{N-1})-g(z^N)+\left\langle\lambda^{N-1}+tAx^N+Bz^N,B\left(z^{N-1}-z^N\right)\right\rangle\right)\geq 0.
\end{aligned}
\tag{9.26}
$$

By summing (9.25) and (9.26), we obtain

$$
\begin{aligned}
&\tfrac{1}{2t}\left\|\lambda^0-\lambda^\star\right\|^2+\tfrac{t}{2}\left\|z^0\right\|_B^2-\tfrac{(N^2-1)t}{2}\left\|\tfrac{N}{N+1}Ax^N+Bz^N\right\|^2-\frac{t(N-1)^2}{2}\left\|z^N-z^{N-1}\right\|_B^2-\\
&\tfrac{(N-2)^2t}{2}\left\|Bz^{N-2}-Bz^{N-1}+\tfrac{N-1}{N-2}Ax^{N-1}-\left(1-\tfrac{1}{(N-2)^2}\right)Ax^N\right\|^2-\\
&\tfrac{1}{2}\operatorname{tr}\left(F(t,c_1)\begin{pmatrix}Ax^1 & \dots & Ax^N\end{pmatrix}^T\begin{pmatrix}Ax^1 & \dots & Ax^N\end{pmatrix}\right)\geq 0,
\end{aligned}
$$

where the matrix $F(t,c_1)$ is as follows,

$$
F(t,c_1)=\begin{pmatrix}
2c_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & t-c_1\\
0 & \alpha_2 & \beta_2 & 0 & \dots & 0 & 0 & \dots & 0 & -t\\
0 & \beta_2 & \alpha_3 & \beta_3 & \dots & 0 & 0 & \dots & 0 & t\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & \alpha_k & \beta_k & \dots & 0 & t\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1}\\
t-c_1 & -t & t & t & \dots & t & t & \dots & \beta_{N-1} & \alpha_N
\end{pmatrix},
$$

and

$$
\alpha_k=\begin{cases}
6c_1-5t, & k=2\\
2\left(2k^2-3k+1\right)c_1-(4k-1)t, & 3\leq k\leq N-1,\\
\left(2N^2-6N+4\right)c_1-2\left(N+\frac{1}{(N-2)^2}-\frac{2}{N+1}-3\right)t, & k=N,
\end{cases}
$$

$$
\beta_k=\begin{cases}
2kt-\left(2k^2-k-1\right)c_1, & 2\leq k\leq N-2,\\
(N+\frac{1}{2-N}-1)t-(2N^2-6N+3)c_1, & k=N-1,
\end{cases}
$$

The rest of the proof proceeds analogously to the proof of Theorem 9.6. □

The following example shows the tightness of this bound.

Example 9.9. *Assume that* $c_1 > 0$, $N \geq 4$ *and* $t \in (0, c_1]$ *are given, and* $f, g : \mathbb{R} \to \mathbb{R}$ *are defined by,*

$$f(x) = \tfrac{1}{2} \max\left\{-\tfrac{N+1}{N-1}x, x\right\} + \tfrac{c_1}{2}x^2,$$
$$g(z) = \tfrac{1}{2} \max\left\{\tfrac{1}{t(N-1)} - z, \tfrac{N-3}{N-1}\left(z - \tfrac{1}{t(N-1)}\right)\right\}.$$

Consider the optimization problem

$$\min_{(x,z)\in\mathbb{R}\times\mathbb{R}} f(x) + g(z),$$
$$\text{s.t. } Ax + Bz = 0.$$

where $A = B = I$. *The point* $(x^\star, z^\star) = (0, 0)$ *with Lagrangian multiplier* $\lambda^\star = \frac{1}{2}$ *is an optimal solution. After performing* N *iterations of Algorithm 9.1 with setting* $\lambda^0 = \frac{-1}{2}$ *and* $z^0 = 0$, *we have*

$$x^k = 0, \qquad k \in \{1, \dots, N\},$$
$$z^k = \begin{cases} \frac{1}{t(N-1)}, & k \in \{1, \dots, N-1\}, \\ 0, & k = N, \end{cases}$$
$$\lambda^k = \begin{cases} \frac{2k+1-N}{2(N-1)}, & k \in \{1, \dots, N-1\}, \\ \frac{1}{2}, & k = N. \end{cases}$$

It can be seen that $\left\|A^T B\left(z^N - z^{N-1}\right)\right\| = \frac{1}{(N-1)t} = \frac{\sqrt{\|\lambda^0 - \lambda^\star\|^2 + t^2\|z^0 - z^\star\|_B^2}}{(N-1)t}$, *which shows that the bound is tight.*

Theorem 9.4 and 9.6 address the case that f is strongly convex relative to $\|.\|_A$ and g is convex. Based on numerical results by solving performance estimation problems including (9.13) we conjecture, under the assumptions of Theorem 9.4, that if g is c_2-strongly convex relative to $\|.\|_B$, Algorithm 9.1 enjoys the following convergence rates

$$D(\lambda^\star) - D(\lambda^N) \leq \frac{\|\lambda^0 - \lambda^\star\|^2 + t^2\|z^0 - z^\star\|_B^2}{4Nt + \frac{2c_1c_2}{c_1+c_2}},$$
$$\left\|Ax^N + Bz^N - b\right\| \leq \frac{\sqrt{\|\lambda^0 - \lambda^\star\|^2 + t^2\|z^0 - z^\star\|_B^2}}{Nt + \frac{c_1c_2}{c_1+c_2}}.$$

We have verified these conjectures numerically for many specific values of the parameters. Nevertheless, we could not manage to guess a closed-form formula for the residual dual in this case.

9.4 Linear convergence of ADMM

In this section we study the linear convergence of ADMM. The linear convergence of ADMM has been addressed by some authors and some conditions for linear convergence have been proposed, see [DY16, LYZZ18, Han22, HSZ18, HL17, NLR$^+$15, YZZ20]. Two common types of assumptions employed for proving the linear convergence of ADMM are error bound property and L-smoothness. To the best knowledge of authors, most scholars investigated the linear convergence of the sequence $\{(x^k, z^k, \lambda^k)\}$ to a saddle point and there is no result in terms of dual objective value for ADMM. In line with the previous section, we study the linear convergence in terms of dual objective value and we derive some formulas for linear convergence rate by using performance estimation. It is noteworthy to mention that the term "Q-linear convergence" is also employed to describe the linear convergence in the literature.

As mentioned earlier, the *error bound property* is used by scholars for establishing the linear convergence; see e.g. [LYZZ18, HSZ18, HL17, PVZ21, YZZ20]. Let

$$D^a(\lambda) := \min f(x) + g(z) + \langle \lambda, Ax + Bz - b \rangle + \tfrac{a}{2}\|Ax + Bz - b\|^2, \qquad (9.27)$$

where stands for augmented dual objective for the given $a > 0$ and $\Lambda^\star$ denotes the optimal solution set of the dual problem. Note that the function D^a is an $\frac{1}{a}$-smooth function on its domain without assuming strong convexity; see [HL17, Lemma 2.2].

Definition 9.10. The function D^a is said to satisfy the error bound property if we have

$$d_{\Lambda^\star}(\lambda) \le \tau \|\nabla D^a(\lambda)\|, \quad \lambda \in \mathbb{R}^r, \qquad (9.28)$$

for some $\tau > 0$.

Hong et al. [HL17] established the linear convergence by employing the *error bound* property (9.28).

Recently, some scholars established the linear convergence of gradient methods for L-smooth convex functions by replacing strong convexity with some mild conditions, see [NNG19, AdKZ23a, BNPS17] and references therein. Inspired by these results, we prove the linear convergence of ADMM by using the so-called PŁ inequality. It is worth noting that we employ the nonsmooth version of the PŁ inequality introduced in [BDL07]. Concerning differentiability of dual objective, by (9.5), we have

$$b - A\partial f^*(-A^T\lambda) - B\partial g^*(-B^T\lambda) \subseteq \partial\left(-D(\lambda)\right). \tag{9.29}$$

Note that the inclusion in (9.29) holds as an equality under some mild conditions, see e.g. [Bec17, Chapter 3].

Definition 9.11. The function D is said to satisfy the PŁ inequality if there exists an $L_p > 0$ such that for any $\lambda \in \mathbb{R}^r$ we have

$$D(\lambda^\star) - D(\lambda) \leq \tfrac{1}{2L_p}\|\xi\|^2, \quad \xi \in \partial\left(-D(\lambda)\right). \tag{9.30}$$

Note that if f and g are strongly convex, then $-D$ is an L-smooth convex function with $L \leq \frac{\lambda_{\max}(A^TA)}{\mu_1} + \frac{\lambda_{\max}(B^TB)}{\mu_2}$. In this setting, we have $L_p \leq \frac{\lambda_{\max}(A^TA)}{\mu_1} + \frac{\lambda_{\max}(B^TB)}{\mu_2}$. This follows from the duality between smoothness and strong convexity and

$$\begin{aligned}\|\nabla D(\lambda) - \nabla D(\nu)\| &\leq \left\|\nabla f^*(-A^T\lambda) - \nabla f^*(-A^T\nu)\right\|_A + \left\|\nabla g^*(-B^T\lambda) - \nabla g^*(-B^T\nu)\right\|_B \\ &\leq \tfrac{1}{\mu_1}\left\|A^T\lambda - A^T\nu\right\|_A + \tfrac{1}{\mu_2}\left\|B^T\lambda - B^T\nu\right\|_B \leq \left(\tfrac{\lambda_{\max}(A^TA)}{\mu_1} + \tfrac{\lambda_{\max}(B^TB)}{\mu_2}\right)\|\lambda - \nu\|.\end{aligned}$$

In the next proposition, we show that definitions (9.28) and (9.30) are equivalent.

Proposition 9.12. *Let $L_a = \frac{1}{a}$ denote the Lipschitz constant of ∇D^a, where D^a is given in* (9.27). *Suppose that* (9.29) *holds as equality.*

i) If D^a satisfies the error bound (9.28)*, then D satisfies the PŁ inequality* (9.30) *with $L_p = \frac{1}{L_a\tau^2}$.*

ii) If D satisfies the PŁ inequality (9.30)*, then D^a satisfies the error bound* (9.28) *with $\tau = \frac{L_p}{1+aL_p}$.*

Proof. First we prove $i)$. Suppose $\lambda \in \mathbb{R}^r$ and $\xi \in b - A\partial f^*(-A^T\lambda) - B\partial g^*(-B^T\lambda)$. By the identity (9.4), we have $\xi = b - A\bar{x} - B\bar{z}$ for some $(\bar{x}, \bar{z}) \in \operatorname{argmin} f(x) + g(z) + \langle \lambda, Ax + Bz - b\rangle$. Due to the smoothness of D^a and (9.28), we get

$$D^a(\lambda^\star) - D^a(\nu) \leq \tfrac{L_a\tau^2}{2}\|\nabla D^a(\nu)\|^2, \quad \nu \in \mathbb{R}^r, \tag{9.31}$$

where $\lambda^\star \in \Lambda^\star$ with $d_{\Lambda^\star} = \|\nu - \lambda^\star\|$. Suppose that $\bar{\nu} = \lambda - a(A\bar{x} + B\bar{z} - b)$. As we assume strong duality, we have $D^a(\lambda^\star) = D(\lambda^\star)$. By the definitions of $\bar{x}, \bar{y}$, we get

$$(\bar{x}, \bar{z}) \in \operatorname{argmin} f(x) + g(z) + \langle \bar{\nu}, Ax + Bz - b\rangle + \tfrac{a}{2}\|Ax + Bz - b\|^2.$$

By [HL17, Lemma 2.1], we have $\nabla D^a(\bar{\nu}) = A\bar{x} + B\bar{z} - b$. This equality with (9.31) imply

$$D(\lambda^\star) - D(\lambda) \leq D^a(\lambda^\star) - D^a(\bar{\nu}) \leq \tfrac{L_a\tau^2}{2}\|A\bar{x} + B\bar{z} - b\|^2,$$

and the proof of $i)$ is complete.
Now we establish $ii)$. Let λ be in the domain of ∇D^a. By [HL17, Lemma 2.1], we have $\nabla D^a(\lambda) = A\bar{x} + B\bar{z} - b$ for some $(\bar{x}, \bar{z}) \in \operatorname{argmin} f(x) + g(z) + \langle \lambda, Ax + Bz - b\rangle + \frac{a}{2}\|Ax + Bz - b\|^2$, which implies that

$$0 \in \partial f(\bar{x}) + A^T(\lambda + a(A\bar{x} + B\bar{z} - b)), 0 \in \partial g(\bar{z}) + B^T(\lambda + a(A\bar{x} + B\bar{z} - b)). \tag{9.32}$$

Supposing $\nu = \lambda + a(A\bar{x} + B\bar{z} - b)$. By (9.32), one can infer that $D(\nu) = f(\bar{x}) + g(\bar{z}) + \langle \nu, A\bar{x} + B\bar{z} - b\rangle$. In addition, (9.4) implies that $b - A\bar{x} - B\bar{z} \in b - A\partial f^*(-A^T\nu) - B\partial g^*(-B^T\nu)$. By the PŁ inequality, we have

$$\tfrac{1}{2L_p}\|A\bar{x} + B\bar{z} - b\|^2 \geq D(\lambda^\star) - D(\nu) = D^a(\lambda^\star) - D^a(\lambda) - \tfrac{a}{2}\|A\bar{x} + B\bar{z} - b\|^2,$$

where the equality follows from $D(\nu) = D^a(\lambda) + \frac{a}{2}\|A\bar{x} + B\bar{z} - b\|^2$ and $D^a(\lambda^\star) = D(\lambda^\star)$. Hence,

$$D^a(\lambda^\star) - D^a(\lambda) \leq \left(\tfrac{1}{2L_p} + \tfrac{a}{2}\right)\|\nabla D^a(\lambda)\|^2.$$

This inequality says that D^a satisfies the PŁ inequality. On the other hand, the PŁ inequality implies the error bound with the same constant, see [BNPS17], and the proof is complete. □

In what follows, we employ performance estimation to derive a linear convergence rate for ADMM in terms of dual objective when the PŁ inequality holds. To this end, we compare the value of dual problem in two consecutive iterations, that is, $\frac{D(\lambda^\star)-D(\lambda^2)}{D(\lambda^\star)-D(\lambda^1)}$. The following optimization problem gives the worst-case convergence rate,

$$
\begin{aligned}
\max\ & \tfrac{D(\lambda^\star)-D(\lambda^2)}{D(\lambda^\star)-D(\lambda^1)} \\
\text{s. t. } & \{x^2, z^2, \lambda^2\} \text{ is generated by Algorithm 9.1 w.r.t. } f, g, A, B, b, \lambda^1, z^1 \\
& (x^\star, z^\star) \text{ is an optimal solution and its Lagrangian multipliers is } \lambda^\star \\
& D \text{ satisfies the PŁ inequality} \\
& f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n), g \in \mathcal{F}^B_{c_2,\infty}(\mathbb{R}^n) \\
& \lambda^1 \in \mathbb{R}^r, z^1 \in \mathbb{R}^m, A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}, b \in \mathbb{R}^r.
\end{aligned}
\tag{9.33}
$$

Analogous to our discussion in Section 9.2, we may assume without loss of generality $b = 0$, $\lambda^1 = \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} x^\dagger \\ z^\dagger \end{pmatrix}$ and $\lambda^\star = \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix}$ for some $\bar{x}, x^\dagger, \bar{z}, z^\dagger$. In addition, we assume that $\hat{x}^1 \in \operatorname{argmin} f(x) + \langle \lambda^1, Ax\rangle$ and $\hat{x}^2 \in \operatorname{argmin} f(x) + \langle \lambda^2, Ax\rangle$. Hence,

$$
D(\lambda^1) = f(\hat{x}^1) + g(z^1) + \langle \lambda^1, A\hat{x}^1 + Bz^1\rangle, \;\; D(\lambda^2) = f(\hat{x}^2) + g(z^2) + \langle \lambda^2, A\hat{x}^2 + Bz^2\rangle,
$$

and

$$
\begin{aligned}
-A^T\lambda^1 &\in \partial f(\hat{x}^1), & -B^T\lambda^1 &\in \partial g(z^1), \\
-A^T\lambda^2 &\in \partial f(\hat{x}^2), & -B^T\lambda^2 &\in \partial g(z^2).
\end{aligned}
\tag{9.34}
$$

Moreover, by (9.34) and (9.29), we get

$$
-A\hat{x}^1 - Bz^1 \in \partial\left(-D(\lambda^1)\right), \quad -A\hat{x}^2 - Bz^2 \in \partial\left(-D(\lambda^2)\right).
$$

On the other hand, $\lambda^2 = \lambda^1 + tAx^2 + tBz^2$. Therefore, by using Theorem 2.40, problem (9.33) may be relaxed as follows,

$$
\begin{aligned}
\max\ & \frac{f^\star + g^\star - \hat{f}^2 - g^2 - \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\rangle}{f^\star + g^\star - \hat{f}^1 - g^1 - \langle Ax^\dagger + Bz^\dagger, A\hat{x}^1 + Bz^1\rangle}\\
\text{s.t.}\ & \Big\{\big(\hat{x}^1, -A^TAx^\dagger - A^TBz^\dagger, \hat{f}^1\big), \big(x^2, -A^TAx^\dagger - A^TBz^\dagger - tA^TAx^2 - tA^TBz^1, f^2\big),\\
& \big(\hat{x}^2, -A^TAx^\dagger - A^TBz^\dagger - tA^TAx^2 - tA^TBz^2, \hat{f}^2\big), \big(0, -A^TA\bar{x} - A^TB\bar{z}, f^\star\big)\Big\}\\
& \text{satisfy interpolation constraints (2.5)}\\
& \Big\{\big(z^1, -B^TAx^\dagger - B^TBz^\dagger, g^1\big), \big(z^2, -B^TAx^\dagger - B^TBz^\dagger - tB^TAx^2 - tB^TBz^2, g^2\big),\\
& \big(0, -B^TA\bar{z} - B^TB\bar{z}, g^\star\big)\Big\} \text{ satisfy interpolation constraints (2.5)}\\
& f^* + g^* - \hat{f}^1 - g^1 - \big\langle Ax^\dagger + Bz^\dagger, A\hat{x}^1 + Bz^1\big\rangle \le \tfrac{1}{2L_p}\left\|A\hat{x}^1 + Bz^1\right\|^2 \qquad (9.35)\\
& f^* + g^* - \hat{f}^2 - g^2 - \big\langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\big\rangle \le \tfrac{1}{2L_p}\left\|A\hat{x}^2 + Bz^2\right\|^2\\
& A \in \mathbb{R}^{r\times n}, B \in \mathbb{R}^{r\times m}.
\end{aligned}
$$

By deriving an upper bound for the optimal value of problem (9.35) in the next theorem, we establish the linear convergence of ADMM in the presence of the PŁ inequality.

Theorem 9.13. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}^B_{c_2,\infty}(\mathbb{R}^m)$ with $c_1, c_2 > 0$, and let D satisfies the PŁ inequality with L_p. Suppose that $t \le \sqrt{c_1c_2}$.*

(i) If $c_1 \ge c_2$, then

$$
\frac{D(\lambda^\star) - D(\lambda^2)}{D(\lambda^\star) - D(\lambda^1)} \le \frac{2c_1c_2 - t^2}{2c_1c_2 - t^2 + L_pt\,(4c_1c_2 - c_2t - 2t^2)}, \tag{9.36}
$$

in particular, if $t = \sqrt{c_1c_2}$,

$$
\frac{D(\lambda^\star) - D(\lambda^2)}{D(\lambda^\star) - D(\lambda^1)} \le \frac{1}{1 + L_p\left(2\sqrt{c_1c_2} - c_2\right)}.
$$

(ii) If $c_1 < c_2$, then

$$
\frac{D(\lambda^\star) - D(\lambda^2)}{D(\lambda^\star) - D(\lambda^1)} \le \tag{9.37}
$$

$$
\frac{4c_2^2 - 2c_2\sqrt{c_1c_2} - t^2}{4c_2^2 - 2c_2\sqrt{c_1c_2} - t^2 + L_pt\left(8c_2^2 + 5c_2t - 2\sqrt{c_1c_2}\left(1 + \frac{t}{c_1}\right)(2c_2 + t)\right)}.
$$

Proof. The argument is based on weak duality. Indeed, by introducing suitable Lagrangian multipliers, we establish that the given convergence rates are upper bounds for problem (9.35). First, we prove (i). Assume that α denotes the right hand side of inequality (9.36). As $2c_1c_2 - t^2 > 0$ and $4c_1c_2 - c_2t - 2t^2 > 0$, we have $0 < \alpha < 1$. With some algebra, one can show that

$$
\begin{aligned}
& f^\star + g^\star - \hat{f}^2 - g^2 - \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\rangle - \\
& \alpha\left(f^\star + g^\star - \hat{f}^1 - g^1 - \langle Ax^\dagger + Bz^\dagger, A\hat{x}^1 + Bz^1\rangle\right) + \\
& \alpha\left(\hat{f}^2 - \hat{f}^1 + \langle Ax^\dagger + Bz^\dagger, A\hat{x}^2 - A\hat{x}^1\rangle - \tfrac{c_1}{2}\left\|\hat{x}^2 - \hat{x}^1\right\|_A^2\right) + \\
& \alpha\left(f^2 - \hat{f}^2 + \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, Ax^2 - A\hat{x}^2\rangle - \tfrac{c_1}{2}\left\|x^2 - \hat{x}^2\right\|_A^2\right) + \\
& \alpha\left(\hat{f}^2 - f^2 + \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^1, A\hat{x}^2 - Ax^2\rangle - \tfrac{c_1}{2}\left\|\hat{x}^2 - x^2\right\|_A^2\right) + \\
& \alpha\left(g^2 - g^1 + \langle Ax^\dagger + Bz^\dagger, Bz^2 - Bz^1\rangle - \tfrac{c_2}{2}\left\|z^2 - z^1\right\|_B^2\right) + \\
& (1-\alpha)\Big(-f^\star - g^\star + \left\langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\right\rangle + \hat{f}^2 + g^2 + \\
& \tfrac{1}{2L_p}\left\|A\hat{x}^2 + Bz^2\right\|^2\Big) = \tfrac{-c_1\alpha}{2}\left\|\hat{x}^1 - \hat{x}^2\right\|_A^2 - \tfrac{c_2\alpha}{2}\left\|Bz^1 - Bz^2 + \tfrac{t}{c_2}Ax^2 - \tfrac{t}{c_2}A\hat{x}^2\right\|^2 - \\
& \alpha\left(c_1 - \tfrac{t^2}{2c_2}\right)\left\|Ax^2 + \tfrac{tc_2}{2c_1c_2 - t^2}Bz^2 - \tfrac{tc_2 - 2c_1c_2 + t^2}{t^2 - 2c_1c_2}A\hat{x}^2\right\|^2.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
& f^\star + g^\star - \hat{f}^2 - g^2 - \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\rangle \le \\
& \alpha\left(f^\star + g^\star - \hat{f}^1 - g^1 - \langle Ax^\dagger + Bz^\dagger, A\hat{x}^1 + Bz^1\rangle\right)
\end{aligned}
$$

for any feasible point of problem (9.33) and the proof of the first part is complete. For (ii), we proceed analogously to the proof of (i), but with different Lagrange multipliers. Let β denote the right hand side of inequality (9.37), i.e.

$$
\beta = \frac{4c_2^2 - 2c_2\sqrt{c_1c_2} - t^2}{4c_2^2 - 2c_2\sqrt{c_1c_2} - t^2 + L_p t\left(8c_2^2 + 5c_2t - 2\sqrt{c_1c_2}\left(1 + \frac{t}{c_1}\right)(2c_2 + t)\right)}.
$$

It is seen that $0 < \beta < 1$. By doing some calculations, we have

$$
\begin{aligned}
&f^\star + g^\star - \hat{f}^2 - g^2 - \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\rangle - \\
&\beta\left(f^\star + g^\star - \hat{f}^1 - g^1 - \langle Ax^\dagger + Bz^\dagger, A\hat{x}^1 + Bz^1\rangle\right) + \\
&\beta\left(\hat{f}^2 - \hat{f}^1 + \langle Ax^\dagger + Bz^\dagger, A\hat{x}^2 - A\hat{x}^1\rangle - \tfrac{c_1}{2}\left\|\hat{x}^2 - \hat{x}^1\right\|_A^2\right) + \\
&\sqrt{\tfrac{c_2}{c_1}}\beta\left(f^2 - \hat{f}^2 + \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, Ax^2 - A\hat{x}^2\rangle - \tfrac{c_1}{2}\left\|x^2 - \hat{x}^2\right\|_A^2\right) + \\
&\sqrt{\tfrac{c_2}{c_1}}\beta\left(\hat{f}^2 - f^2 + \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^1, A\hat{x}^2 - Ax^2\rangle - \tfrac{c_1}{2}\left\|\hat{x}^2 - x^2\right\|_A^2\right) + \\
&\sqrt{\tfrac{c_2}{c_1}}\beta\left(g^2 - g^1 + \langle Ax^\dagger + Bz^\dagger, Bz^2 - Bz^1\rangle - \tfrac{c_2}{2}\left\|z^2 - z^1\right\|_B^2\right) + \\
&\left(\sqrt{\tfrac{c_2}{c_1}} - 1\right)\beta\Big(g^1 - g^2 + \langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, Bz^1 - Bz^2\rangle - \\
&\tfrac{c_2}{2}\left\|z^1 - z^2\right\|_B^2\Big) + (1-\beta)\big(-f^\star - g^\star + \left\langle Ax^\dagger + Bz^\dagger + tAx^2 + tBz^2, A\hat{x}^2 + Bz^2\right\rangle + \\
&\hat{f}^2 + g^2 + \tfrac{1}{2L_p}\left\|A\hat{x}^2 + Bz^2\right\|^2\big) \\
&= -\tfrac{c_1\beta}{2}\left\|\hat{x}^1 - \hat{x}^2\right\|_A^2 - \left(\sqrt{c_1c_2}\beta\right)\left\|Ax^2 - \left(1 - \frac{t}{2\sqrt{c_1c_2}}\right)A\hat{x}^2 + \frac{t}{2\sqrt{c_1c_2}}Bz^1\right\|^2 - \\
&\left(\frac{\beta-1}{2L_p} + \beta t\left(1 - \frac{t}{4\sqrt{c_1c_2}}\right)\right)\left\|A\hat{x}^2 - \left(\frac{\beta L_p\left(-2c_2\sqrt{c_1c_2} + 4c_2^2 - t^2\right)}{-\beta L_p t^2 + 2\sqrt{c_1c_2}(2\beta L_p t + \beta - 1)}\right)^{\frac{1}{2}} Bz^1 + \right. \\
&\left.\left(\frac{2\left(2\beta c_2 L_p\left(t + c_2\right) + \sqrt{c_1c_2}\left(\beta - \beta L_p c_2 - 1\right)\right)}{-\beta L_p t^2 + 2\sqrt{c_1c_2}(2\beta L_p t + \beta - 1)}\right)^{\frac{1}{2}} Bz^2\right\|^2.
\end{aligned}
$$

The rest of the proof is similar to that of the former case. □

We computed the bounds in Theorem 9.13 by selecting suitable Lagrangian multipliers and solving the semidefinite formulation of problem (9.35) by hand. The semidefinite formulation is formed analogously to problem (9.14). Note that the optimal value of problem (9.35) may be smaller than the bounds introduced in Theorem 9.13. Indeed, our aim was to provide a concrete mathematical proof for the linear convergence rate. However, the linear convergence rate factor is not necessarily tight. Needless to say that the optimal value of problem (9.35) also does not necessarily give the tight convergence factor as it is just a relaxation of problem (9.33).

Recently the authors showed that the PŁ inequality is necessary and sufficient conditions for the linear convergence of the gradient method with constant step lengths for L-smooth function; see[AdKZ23a, Theorem 5]. In what follows, we establish that the PŁ inequality is a necessary condition for the linear convergence of ADMM. Firstly, we present a lemma that is very useful for our proof.

Lemma 9.14. *Let $f \in \mathcal{F}^{A}_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}^{B}_{c_2,\infty}(\mathbb{R}^m)$. Consider Algorithm 9.1. If $(\hat{x}^1, z^1) \in \operatorname{argmin} f(x) + g(z) + \langle \lambda^1, Ax + Bz - b\rangle$, then*

$$\langle A\hat{x}^1 + Bz^1 - b, Ax^2 + Bz^2 - b\rangle \leq \left\|A\hat{x}^1 + Bz^1 - b\right\|^2. \tag{9.38}$$

Proof. Without loss of generality we assume that $c_1 = c_2 = 0$. By optimality conditions, we have

$$\begin{aligned}
&f(\hat{x}^1) - \langle \lambda^1, Ax^2 - A\hat{x}^1\rangle \leq f(x^2), \quad g(z^1) - \langle \lambda^1, Bz^2 - Bz^1\rangle \leq g(z^2),\\
&f(x^2) - \langle \lambda^1 + t(Ax^2 + Bz^1 - b), A\hat{x}^1 - Ax^2\rangle \leq f(\hat{x}^1),\\
&g(z^2) - \langle \lambda^1 + t(Ax^2 + Bz^2 - b), Bz^1 - Bz^2\rangle \leq g(z^1).
\end{aligned}$$

By using these inequities, we get

$$\begin{aligned}
0 \leq\, &\tfrac{1}{t}\left(f(x^2) - f(\hat{x}^1) + \left\langle \lambda^1, Ax^2 - A\hat{x}^1\right\rangle\right) + \tfrac{1}{t}\left(g(z^2) - g(z^1) + \left\langle \lambda^1, Bz^2 - Bz^1\right\rangle\right) +\\
&\tfrac{1}{t}\left(f(\hat{x}^1) - f(x^2) + \left\langle \lambda^1 + t(Ax^2 + Bz^1 - b), A\hat{x}^1 - Ax^2\right\rangle\right) +\\
&\tfrac{1}{t}\left(g(z^1) - g(z^2) + \left\langle \lambda^1 + t(Ax^2 + Bz^2 - b), Bz^1 - Bz^2\right\rangle\right)\\
=\, &\left\|A\hat{x}^1 + Bz^1 - b\right\|^2 - \left\langle A\hat{x}^1 + Bz^1 - b, Ax^2 + Bz^2 - b\right\rangle - \tfrac{3}{4}\left\|B\left(z^1 - z^2\right)\right\|^2 -\\
&\left\|A\left(\hat{x}^1 - x^2\right) + \tfrac{1}{2}B\left(z^1 - z^2\right)\right\|^2.
\end{aligned}$$

Hence, we have

$$\frac{\langle A\hat{x}^1 + Bz^1 - b, Ax^2 + Bz^2 - b\rangle}{\|A\hat{x}^1 + Bz^1 - b\|^2} \leq 1,$$

which completes the proof. □

The next theorem establishes that the PŁ inequality is a necessary condition for the linear convergence of ADMM.

Theorem 9.15. *Let $f \in \mathcal{F}^{A}_{c_1,\infty}(\mathbb{R}^n)$ and $g \in \mathcal{F}^{B}_{c_2,\infty}(\mathbb{R}^m)$. If Algorithm 9.1 is linearly convergent with respect to the dual objective value, then D satisfies the PŁ inequality.*

Proof. Consider $\lambda^1 \in \mathbb{R}^r$ and $\xi \in b - A\partial f^*(-A^T\lambda^1) - B\partial g^*(-B^T\lambda^1)$. Hence, $\xi = b - A\hat{x}^1 - Bz^1$ for some $(\hat{x}^1, z^1) \in \operatorname{argmin} f(x) + g(z) + \langle \lambda, Ax + Bz - b\rangle$. If one sets $z^0 = z^1$ and $\lambda^0 = \lambda^1 - t(A\hat{x}^1 + Bz^1 - b)$ in Algorithm 9.1, the algorithm may generate λ^1. As Algorithm 9.1 is linearly convergent, there exist $\gamma \in [0,1)$ with

$$D(\lambda^\star) - D(\lambda^2) \leq \gamma\left(D(\lambda^\star) - D(\lambda^1)\right).$$

So, we have

$$(1-\gamma)\left(D(\lambda^\star) - D(\lambda^1)\right) \leq D(\lambda^2) - D(\lambda^1) \leq \left\langle A\hat{x}^1 + Bz^1 - b, \lambda^2 - \lambda^1\right\rangle,$$

where the last inequality follows from the concavity of the function D. Since $\lambda^2 - \lambda^1 = t(Ax^2 + Bz^2 - b)$, Lemma 9.14 implies that

$$D(\lambda^\star) - D(\lambda^1) \le \tfrac{t}{1-\gamma} \|\xi\|^2,$$

so D satisfies the PŁ inequality. □

Another assumption used in the literature for establishing linear convergence is L-smoothness; see for example [NLR+15, DY16, GB16, DY17]. Deng et al. [DY16] show that the sequence $\{(x^k, z^k, \lambda^k)\}$ is convergent linearly to a saddle point under Scenario 1 and 2 given in Table 9.1.

Table 9.1: Scenarios leading to linear convergence rates

Scenario	Strong convexity	Lipschitz continuity	Full row rank
1	f, g	∇f	A
2	f, g	$\nabla f, \nabla g$	-
3	f	$\nabla f, \nabla g$	B^T

It is worth mentioning that Scenario 1 or Scenario 2 implies strong convexity of the dual objective function and therefore the PŁ inequality is implied, see [AdKZ23a]. Hence, Theorem 9.13 implies the linear convergence in terms of dual value under Scenario 1 or Scenario 2. Deng et al. [DY16] studied the linear convergence under Scenario 3, but they just proved the linear convergence of the sequence $\{(x^k, Bz^k, \lambda^k)\}$. In the next section, we investigate the R-linear convergence without assuming L-smoothness of f. Indeed, we establish the R-linear convergence when f is strongly convex, g is L-smooth and B has full row rank.

Note that the PŁ inequality does not imply necessarily Scenario 1 or Scenario 2. Indeed, consider the following optimization problem,

$$\begin{aligned} \min\ & f(x) + g(z), \\ \text{s.t.}\ & x + z = 0, \\ & x, z \in \mathbb{R}^n, \end{aligned}$$

where $f(x) = \frac{1}{2}\|x\|^2 + \|x\|_1$ and $g(z) = \frac{1}{2}\|z\|^2 + \|z\|_1$. With some algebra, one may show that $D(\lambda) = \sum_{i=1}^n h(\lambda_i)$ with

$$h(s) = \begin{cases} -(s-1)^2, & s > 1 \\ 0, & |s| \le 1 \\ -(s+1)^2, & s < -1. \end{cases}$$

Hence, the PŁ inequality holds for $L_p = \frac{1}{2}$ while neither f nor g is L-smooth.

As mentioned earlier the performance estimation problem including the PŁ inequality at finite set of points is a relaxation for computing the worst-case convergence rate. Contrary to Theorem 9.13, we could not manage to prove the linear convergence of primal and dual residuals under the assumptions of Theorem 9.13 by employing performance estimation.

9.5 *R*-linear convergence of ADMM

This section focuses on examining the linear convergence rate for ADMM from a weaker convergence rate perspective than Q-linear which is already studied in Section 9.4. This concept is known as R-linear convergence where R stands for root [NW06]. Recall that ADMM enjoys R-linear convergence in terms of dual objective value if there exists sequence $\{s_k\} \subseteq \mathbb{R}_+$ such that

$$D(\lambda^\star) - D(\lambda^N) \leq s_k,$$

and s_k tends Q-linearly to zero. It is easily seen that the linear convergence implies R-linear convergence. For an extensive discussion of convergence rates see [NW06, Section A.2] or [BGLS06, Section 1.5] and Section 1.4.

We investigate the R-linear convergence under the following scenarios:

- (S1): $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ is L-smooth with $c_1 > 0$ and A has full row rank;
- (S2): $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ with $c_1 > 0$, g is L-smooth and B has full row rank.

Under these scenarios, we could not manage to find a value of q within the range $[0, 1)$ that satisfies the inequality:

$$D(\lambda^\star) - D(\lambda^{N+1}) \leq q\left(D(\lambda^\star) - D(\lambda^N)\right).$$

As a result, we turn our attention towards studying the R-linear convergence.

Our technique for proving the R-linear convergence is based on establishing the linear convergence of the sequence $\{V^k\}$ given by

$$V^k = \|\lambda^k - \lambda^\star\|^2 + t^2 \left\|z^k - z^\star\right\|^2_B. \tag{9.39}$$

Note that V^k is called *Lyapunov function* for ADMM and it decreases in each iteration; see [BPC+11]. It is worth noting Q-linear and R-linear convergence of

ADMM have been studied under similar scenarios for some performance measures, see e.g. [DY17, GB16, NLR+15]. However, to the best of knowledge, no existing results in the literature address the dual objective and V^k under Scenario (S1) and (S2).

First we consider the case that f is L-smooth and c_1-strongly convex relative to A. The following proposition establishes the linear convergence of $\{V^k\}$.

Proposition 9.16. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ be L-smooth with $c_1 > 0$, $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ and let A has full row rank. If $t < \sqrt{\frac{c_1 L}{\lambda_{\min}(AA^T)}}$, then*

$$V^{k+1} \leq \left(1 - \frac{2c_1 t}{c_1 d + 2c_1 t + t^2}\right) V^k, \tag{9.40}$$

where $d = \frac{L}{\lambda_{\min}(AA^T)}$.

Proof. We may assume without loss of generality that $x^\star, z^\star$ and b are zero; see our discussion in Section 9.2. By optimality conditions, we have

$$\begin{aligned} \nabla f(x^{k+1}) &= -A^T\left(\lambda^k + tAx^{k+1} + tBz^k\right), & \eta^k &= -B^T\lambda^{k+1}, \\ \nabla f(x^\star) &= -A^T\lambda^\star, & \eta^\star &= -B^T\lambda^\star, \end{aligned}$$

for some $\eta^k \in \partial g(z^{k+1})$ and $\eta^\star \in \partial g(z^\star)$. Let $\alpha = \frac{2t}{c_1^2 d^2 + 2c_1 d t^2 - 4c_1^2 t^2 + t^4}$. By Theorem 2.5, we get

$$\begin{aligned} &\alpha\left(t^2 + c_1 d\right)^2\left(f(x^{k+1}) - f^\star + \left\langle \lambda^\star, Ax^{k+1}\right\rangle - \tfrac{1}{2L}\left\|A^T\left(\lambda^k + tAx^{k+1} + tBz^k - \lambda^\star\right)\right\|^2\right) + \\ &2\alpha t^2\left(c_1 d + t^2\right)\left(f^\star - f(x^{k+1}) - \tfrac{c_1}{2}\left\|x^{k+1}\right\|_A^2 - \left\langle \lambda^k + tAx^{k+1} + tBz^k, Ax^{k+1}\right\rangle\right) + \\ &2t\left(g(z^{k+1}) - g^\star + \left\langle \lambda^\star, Bz^{k+1}\right\rangle\right) + 2t\left(g^\star - g(z^{k+1}) - \left\langle \lambda^{k+1}, Bz^{k+1}\right\rangle\right) + \\ &\alpha\left(c_1^2 d^2 - t^4\right)\Bigg(f^\star - f(x^{k+1}) - \left\langle \lambda^k + tAx^{k+1} + tBz^k, Ax^{k+1}\right\rangle - \\ &\tfrac{1}{2L}\left\|A^T\left(\lambda^k + tAx^{k+1} + tBz^k - \lambda^\star\right)\right\|^2\Bigg) \geq 0. \end{aligned}$$

As $\|A^T\lambda\|^2 \geq \frac{L}{d}\|\lambda\|^2$ and $\lambda^{k+1} = \lambda^k + tAx^{k+1} + tBz^{k+1}$, we obtain the following inequality after performing some algebraic manipulations

$$\begin{aligned} &\left(1 - \tfrac{2ct}{cd + 2ct + t^2}\right)\left(\left\|\lambda^k - \lambda^\star\right\|^2 + t^2\left\|Bz^k\right\|^2\right) - \left(\left\|\lambda^{k+1} - \lambda^\star\right\|^2 + t^2\left\|Bz^{k+1}\right\|^2\right) - \\ &2\alpha c_1^2 t\left\|\lambda^k - \lambda^\star + \tfrac{t^2 + 2c_1 t + c_1 d}{2c_1}Ax^{k+1} + \tfrac{t^2 + c_1 d}{2c_1}Bz^k\right\|^2 \geq 0. \end{aligned}$$

The above inequality implies that

$$V^{k+1} \leq \left(1 - \frac{2c_1 t}{c_1 d + 2c_1 t + t^2}\right) V^k,$$

and the proof is complete. □

Note that one can improve bound (9.40) under the assumptions of Proposition 9.16 and the μ-strong convexity of f by employing the following known inequality

$$\frac{1}{2(1-\frac{\mu}{L})}\left(\tfrac{1}{L}\|\nabla f(x)-\nabla f(y)\|^2+\mu\|x-y\|^2-\tfrac{2\mu}{L}\langle\nabla f(x)-\nabla f(y),x-y\rangle\right)$$
$$\leq f(y)-f(x)-\langle\nabla f(x),y-x\rangle\,.$$

Indeed, we employed the given inequality but we could not manage to obtain a closed form formula for the convergence rate. The next theorem establishes the R-linear convergence of ADMM in terms of dual objective value under the assumptions of Proposition 9.16.

Theorem 9.17. *Let $N\geq 4$ and let A has full row rank. Suppose that $f\in\mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ is L-smooth with $c_1>0$ and $g\in\mathcal{F}_{0,\infty}(\mathbb{R}^m)$. If $t<\min\{c_1,\sqrt{\frac{c_1L}{\lambda_{\min}(AA^T)}}\}$, then*

$$D(\lambda^\star)-D(\lambda^N)\leq\rho\left(1-\tfrac{2c_1t}{c_1d+2c_1t+t^2}\right)^N,$$

where $d=\frac{L}{\lambda_{\min}(AA^T)}$ and $\rho=\frac{V^0}{16t}\left(1-\frac{2c_1t}{c_1d+2c_1t+t^2}\right)^{-4}$.

Proof. By Theorem 9.4 and Proposition 9.16, one can infer the following inequalities,

$$\begin{aligned}D(\lambda^\star)-D(\lambda^N)&\leq\tfrac{V^{N-4}}{16t}\\&\leq\tfrac{V^0}{16t}\left(1-\tfrac{2c_1t}{c_1d+2c_1t+t^2}\right)^{N-4},\end{aligned}$$

which shows the desired inequality. □

In the sequel, we investigate the R-linear convergence under the hypotheses of scenario (S2). The next proposition shows the linear convergence of $\{V^k\}$.

Proposition 9.18. *Let $f\in\mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ with $c_1>0$ and let $g\in\mathcal{F}_{0,\infty}(\mathbb{R}^m)$ be L-smooth. Suppose that B has full row rank and $k\geq 1$. If $t\leq\min\{\frac{c_1}{2},\frac{L}{2\lambda_{\min}(BB^T)}\}$, then*

$$V^{k+1}\leq\left(\tfrac{L}{L+t\lambda_{\min}(BB^T)}\right)^2V^k.\tag{9.41}$$

Proof. Analogous to the proof of Proposition 9.16, we assume that $x^\star=0$, $z^\star=0$ and $b=0$. Due to the optimality conditions, we have

$$\begin{aligned}&\xi^{k+1}=-A^T\left(\lambda^k+tAx^{k+1}+tBz^k\right), && && \xi^\star=-A^T\lambda^\star,\\&\nabla g(z^k)=-B^T\lambda^k, && \nabla g(z^{k+1})=-B^T\lambda^{k+1}, && \nabla g(z^\star)=-B^T\lambda^\star,\end{aligned}$$

for some $\xi^{k+1} \in \partial f(x^{k+1})$ and $\xi^\star \in \partial f(x^\star)$. Suppose that $d = \frac{L}{\lambda_{\min}(BB^T)}$ and $\alpha = \frac{2dt}{d+t}$. By Theorem 2.5, we obtain

$$
\begin{aligned}
&\frac{\alpha\left(d^2+t^2\right)}{d^2-t^2}\left(f^\star - f(x^{k+1}) - \left\langle \lambda^k + tAx^{k+1} + tBz^k, Ax^{k+1}\right\rangle - \tfrac{c_1}{2}\left\|x^{k+1}\right\|_A^2\right) + \\
&\frac{\alpha\left(d^2+t^2\right)}{d^2-t^2}\left(f(x^{k+1}) - f(x^\star) + \left\langle \lambda^\star, Ax^{k+1}\right\rangle - \tfrac{c_1}{2}\left\|x^{k+1}\right\|_A^2\right) + \\
&\alpha\left(g(z^{k+1}) - g^\star + \left\langle \lambda^\star, Bz^{k+1}\right\rangle - \tfrac{1}{2L}\left\|B^T\left(\lambda^\star - \lambda^{k+1}\right)\right\|^2\right) + \\
&\alpha\left(g^\star - g(z^{k+1}) - \left\langle \lambda^{k+1}, Bz^{k+1}\right\rangle - \tfrac{1}{2L}\left\|B^T\left(\lambda^\star - \lambda^{k+1}\right)\right\|^2\right) + \\
&\alpha\left(g(z^k) - g(z^{k+1}) + \left\langle \lambda^{k+1}, Bz^k - Bz^{k+1}\right\rangle - \tfrac{1}{2L}\left\|B^T\left(\lambda^{k+1} - \lambda^k\right)\right\|^2\right) + \\
&\alpha\left(g(z^{k+1}) - g(z^k) + \left\langle \lambda^k, Bz^{k+1} - Bz^k\right\rangle - \tfrac{1}{2L}\left\|B^T\left(\lambda^{k+1} - \lambda^k\right)\right\|^2\right) \geq 0.
\end{aligned}
$$

By employing $\|B^T\lambda\|^2 \geq \frac{L}{d}\|\lambda\|^2$ and $\lambda^{k+1} = \lambda^k + tAx^{k+1} + tBz^{k+1}$, the aforementioned inequality can be expressed as follows after some algebraic manipulation,

$$
\begin{aligned}
&\tfrac{-\alpha^2}{4}\left\|\left(\frac{2t^2}{d^2-dt}\right)Ax^{k+1} + Bz^k - \left(1+\tfrac{t}{d}\right)Bz^{k+1}\right\|^2 - \frac{2t\left(d^2+t^2\right)\left(cd^2 - dt(c+t) - t^3\right)}{(d^2-t^2)^2} \\
&\left\|Ax^{k+1}\right\|^2 - \tfrac{\alpha^2}{4d^2}\left\|\lambda^k - \lambda^\star + \left(\frac{2d^2-(d-t)^2}{d-t}\right)Ax^{k+1} + (d+t)Bz^{k+1}\right\|^2 \\
&+\left(\tfrac{d}{d+t}\right)^2\left(\left\|\lambda^k - \lambda^\star\right\|^2 + t^2\left\|Bz^k\right\|^2\right) - \left(\left\|\lambda^{k+1} - \lambda^\star\right\|^2 + t^2\left\|Bz^{k+1}\right\|^2\right) \geq 0.
\end{aligned}
$$

Hence, we have

$$V^{k+1} \leq \left(\tfrac{d}{d+t}\right)^2 V^k,$$

and the proof is complete. □

As the sequence $\{V^k\}$ is not increasing [BPC⁺11, Convergence Proof], we have $V^1 \leq V^0$. Thus, by using Theorem 9.4 and Proposition 9.18, one can infer the following theorem.

Theorem 9.19. *Let $f \in \mathcal{F}^A_{c_1,\infty}(\mathbb{R}^n)$ with $c_1 > 0$ and let $g \in \mathcal{F}_{0,\infty}(\mathbb{R}^m)$ be L-smooth. Assume that $N \geq 5$ and B has full row rank. If $t < \min\{\frac{c_1}{2}, \frac{L}{2\lambda_{\min}(BB^T)}\}$, then*

$$D(\lambda^\star) - D(\lambda^N) \leq \rho\left(\tfrac{L}{L+t\lambda_{\min}(BB^T)}\right)^{2N}, \tag{9.42}$$

where $\rho = \frac{V^0}{16t}\left(\frac{L}{L+t\lambda_{\min}(BB^T)}\right)^{-10}$.

In the same line, one can infer the R-linear convergence in terms of primal and dual residuals under the assumptions of Theorem 9.17 and Theorem 9.19. In this section, we proved the linear convergence of $\{V^k\}$ under two scenarios (S1) and (S2). By (9.5), it is readily seen that function $-D$ is strongly convex under the hypotheses of both scenarios (S1) and (S2). Therefore, both scenarios imply the PŁ inequality. One may wonder that if the PŁ inequality and the strong convexity of f imply the linear of $\{V^k\}$. By using performance estimation, we could not establish such an implication.

As mentioned above, function $-D$ under both scenarios are μ-strongly convex. Hence, the optimal solution set of the dual problem is unique and one can infer the R-linear convergence of λ^N by using Theorem 9.17 (Theorem 9.19) and the known inequality,

$$\tfrac{\mu}{2}\left\|\lambda^N-\lambda^\star\right\|^2 \leq D(\lambda^\star)-D(\lambda^N).$$

9.6 Concluding remarks

In this chapter we developed a performance estimation framework to handle dual-based methods. Thanks to this framework, we could obtain some tight convergence rates for ADMM. This framework may be exploited for the analysis of other variants of ADMM in the ergodic and non-ergodic sense. Moreover, similarly to [KF16], one can apply this framework for introducing and analyzing new accelerated ADMM variants. Moreover, most results hold for any arbitrary positive step length, t, but we managed to get closed form formulas for some interval of positive numbers.

It is worth mentioning that $\mathcal{F}^A_{c_1,\infty}=\mathcal{F}^{\tau A}_{\tau^{-2}c_1,\infty}$. However, strong convexity relative to $\|.\|_A$ is assumed throughout the chapter when A is the first block of the constraint matrix, which is fixed for a given problem. If one considers strong convexity relative to $\|.\|_{\tau A}$, they need to modify constraints accordingly. In what follows, we investigate two possible cases, and show that these transformations have no influence.

Case 1. Consider the following problem

$$\begin{aligned}&\min f(x)+g(z)\\&\text{s.t. } \tau A(\tau^{-1}x)+Bz=b.\end{aligned}$$

To be consistent with problem (1), we define new variable $y=\tau^{-1}x$, and the problem is formulated as

$$\begin{aligned} &\min\ h(y)+g(z)\\ &\text{s.t. } \tau Ay+Bz=b, \end{aligned}$$

where h is given by $h(y)=f(\tau y)$. It is seen that f is c-strongly convex relative to $\|.\|_A$ if and only if h is c-strongly convex relative to $\|.\|_{\tau A}$. In addition, we have the same initial condition as $(\tau^{-1}x^\star, z^\star, \lambda^\star)$ is a saddle point for the new problem. Hence, we get same results for both formulations.

Case 2. Consider the following case,

$$\begin{aligned} &\min\ f(x)+g(z)\\ &\text{s.t. } \tau Ax+\tau Bz=\tau b. \end{aligned}$$

For this problem $(x^\star, z^\star, \tau^{-1}\lambda^\star)$ is a saddle point and $f\in\mathcal{F}^{\tau A}_{\tau^{-2}c_1,\infty}$. Let $\{(x^k, z^k, \lambda^k)\}$ be the generated points via ADMM for solving Problem (9.1) with initial point (z^0, λ^0) and step length t. It is seen that ADMM can generate $\{(x^k, z^k, \tau^{-1}\lambda^k)\}$ for solving the new problem with initial point $(z^0, \tau^{-1}\lambda^0)$ and step length $\tau^{-2}t$. In this case, we also get the same results as

$$\tfrac{1}{t}\|\lambda^0-\lambda^\star\|^2+t\left\|z^0-z^\star\right\|_B^2=\tfrac{1}{t\tau^{-2}}\|\tau^{-1}\lambda^0-\tau^{-1}\lambda^\star\|^2+t\tau^{-2}\left\|z^0-z^\star\right\|_{\tau B}^2.$$

Not what we have, but what we enjoy, constitutes our abundance.

Epicurus

10 Conclusion and outlook

The primary focus of this thesis has been the complexity analysis of first-order methods. This line of investigation has far-reaching implications for researchers and practitioners across various fields, particularly in the realm of machine learning, as first-order methods are used extensively to solve the optimization problems that arise in real-world applications. The use of performance estimation methods has allowed us to study the worst-case behavior of some first-order methods.

Using this methodology, we started with one of the most famous first-order methods, the gradient descent method with fixed step length. We studied the complexity of this algorithm over L-smooth functions and by means of Proposition 4.4 showed that the presented bound is in fact tight for some step lengths. We show that in this case the bound is tight for step-lengths in the interval $(0, \frac{1}{L}]$. It is known that the gradient descent method is convergent if the step-length lies in $(0, \frac{2}{L})$.

Open problem 10.1. *Consider the class of L-smooth functions. Given the norm of the gradient as the stopping criterion for the gradient descent method, what is the tight bound when all step-lengths are within the interval $\left(\frac{1}{L}, \frac{2}{L}\right)$?*

We continued to study the gradient method by providing necessary and sufficient conditions so that the algorithm enjoys linear convergence, namely if the function class under the study satisfies PŁ inequality; see Theorem 5.8. As the given bound under the PŁ inequality is not tight, an important question to be answered is the following.

Open problem 10.2. *When may a function that satisfies the PŁ inequality at a given, finite set of points be extended to a function that satisfies the same PŁ inequality on some open set containing these points?*

We also studied the relation between this class of functions with other class of functions; see Section 5.3.

We continued by studying the coordinate descent method, a variant of the gradient descent method. We showed that the worst-case bounds given in the literature are not tight for some class of functions. As the bounds we obtained via SDP are only numerical, see Sections 6.2 and 6.3, it is important to provide a mathematical proof as well as a closed-form formula for the given bounds. We also studied worst case behavior of the non-linear Gauss–Seidel method as well as the related weighed Jacobi method; see Sections 6.3.1 and 6.3.2, respectively. Another important question might be to find interpolation constraints to find a better bound for the randomized coordinate descent method. These constraints not only contribute to refining the bounds for the randomized coordinate method but also pave the way for effectively addressing other stochastic methods, such as the widely used stochastic gradient descent method in the machine learning community, and answers several open questions in this area.

Proceeding with the examination of gradient-based algorithms, we studied the gradient descent-ascent method which is mainly used to find saddle points of minimax problems. In this study, we presented a tight convergence rate for one iteration of the algorithm for some class of functions as well as an optimal step length based on the given bound; see Section 7.2. Moreover, we studied necessary and sufficient conditions that the algorithm enjoys linear convergence under those conditions; see Section 7.3. As it is mentioned, the given bound is tight only for one iteration, this might be because of the lack of an interpolation constraint for minimax problems.

Open problem 10.3. *What are the condition(s) that a saddle-point function $F(x, y)$, with properties provided in Section 7.1, must satisfy for a finite set of given points to be extended to a function with the same properties on some open set containing these points? In other words, what is the interpolation constraint for this function class?*

In Chapter 8, we delved into the Difference of Convex Functions Algorithm

(DCA). A large class of functions may be written as the difference of two convex functions. In our study, we provided some worst-case convergence rate for some class of functions. We provided the worst-case convergence rate for the case that at least one of the two convex functions is L-smooth. We showed that the order of the given bound is tight by providing an example; see Section 8.3.1. Moreover, we studied the convergence of DCA when both functions are non-smooth. Also, we studied the relation between DCA and the gradient descent method in Section 8.3.3, and DCA with the proximal gradient method in Section 8.3.4. In addition, we provided necessary conditions that DCA enjoys linear convergence.

Open problem 10.4. *Is the PŁ inequality a sufficient condition for the DCA to have linear convergence?*

The convergence of DCA with regularization also is studied in this chapter. As a side result of our analysis, we derived a convergence rate for proximal gradient method. Regarding this algorithm the following important questions arise.

Open problem 10.5. *What conditions must a function satisfy in order to belong to the function class* $\{f \mid f := f_1 - f_2,\ f_1 \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n),\ f_2 \in \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)\}$*, for some* $0 \le \mu_1 < L_1$ *and* $0 \le \mu_2 < L_2$*? In the words, what is a (easily verifiable) sufficient condition for* $f \in \mathcal{F}_{\mu_1,L_1}(\mathbb{R}^n) - \mathcal{F}_{\mu_2,L_2}(\mathbb{R}^n)$*?*

Another open problem would be the following.

Open problem 10.6. *Assume a function* f *with its gradient vector is only specified on a finite set of points in* $\mathbb{R}^n$*. Under what condition(s) can the function* f *be decomposed as* $f_1 - f_2$ *on some open convex set containing the points, where* f_1 *and* f_2 *are convex? In other words, what is the interpolation theorem for the class of difference of two convex functions? In particular, are the usual interpolation conditions for* f_1 *and* f_2 *enough?*

In the last chapter of the thesis, we studied convergence rate for the alternating direction method of multipliers (ADMM). We studied the convergence rate of the algorithm for different class of functions. We derive new non-ergodic convergence rates for ADMM and showed that the given bound are tight by providing a worst-case example; see Section 9.3. In a similar vein, we also examined the conditions that the algorithm enjoys linear convergence under those conditions, namely PŁ inequality that is shown as necessary and sufficient condition for linear convergence of the algorithm; see Section 9.4. Moreover, we study the R-linear convergence of ADMM under two new scenarios; see Section 9.5. It is important for future research to study the convergence rate of ADMM for long-step sizes and find the optimal step-length by the provided bound.

Open problem 10.7. *Is ADMM convergent for long step-sizes? If so, what are the optimal step-sizes for ADMM?*

To the best of our knowledge, the accelerated version of ADMM is not studied very deeply. Most of the researchers just use the same step-length as the step-length given for the accelerated gradient method.

Open problem 10.8. *What is the optimal step-length for the accelerated ADMM?*

It is worth mentioning that, as our analysis does not depend on the dimension of the problem n, the most of our results hold on any Hilbert space equipped with an appropriate inner-product.

Likewise, to offer a viewpoint for subsequent investigations, besides the possible research topics that are mentioned earlier in this chapter or referenced in the final segments of each chapter for the algorithm under the study, additional areas of study that may hold potential for future exploration. As we derived convergence rate of the proximal gradient method from the convergence rate of the DCA, studying the relation of the algorithms and deriving convergence rate of algorithms using other algorithms is very interesting research topic. As we studied the convergence rate of the algorithms with fixed step length for a bounded step length, it is shown in practice and, in some cases in theory, that the convergence rate can be improved for considering a periodic long step regime. Therefore, for algorithms like the descent ascent method it can be very interesting topic as well as studying accelerated version of this algorithm. Moreover, there are some algorithms that are not studied using performance estimation method or the known bound is not tight, e.g., the Chambolle-Pock algorithm [CP11], studying their convergence rate also can be interesting. At the end, in Table 10.1, we present a summary of the main results presented in the thesis.

Table 10.1: The summary of the main results on the convergence rates presented in the thesis

Method	Function class	Analytical	Rate of convergence	Proven tightness
Gradient descent method	L-smooth	Yes (Thm. 4.3)	Sub-linear	For some cases (Prop. 4.4)
	L-smooth satisfying PŁ inequality	Yes (Thm. 5.4)	Linear	No
Coordinate descent method	L-smooth	No	-	No
	Quadratic	No	-	No
Randomized coordinate descent method	L-smooth	No	-	No
	μ-strongly convex L-smooth	No	-	No
	Convex quadratic	No	-	No
Gradient descent-ascent method	Strongly convex-strongly concave smooth	Yes (Thm. 7.2)	Linear	For some cases (Prop. 7.4)
	Convex-concave smooth satisfying quadratic gradient growth	Yes (Thm. 7.9)	Linear	No
Difference of convex algorithm (DCA)	smooth	Yes (Thm. 8.6)	Sub-linear	For some cases (Ex. 8.8)
	Nonsmooth	Yes (Thm. 8.12)	Sub-linear	No
	Smooth satisfying PŁ inequality	Yes (Thm. 8.20)	Linear	No
ADMM	Strongly convex	Yes (Thm. 9.4, 9.6, 9.8)	Sub-linear	Yes (Ex. 9.5, 9.7, 9.9)
	Strongly convex with dual satisfying PŁ	Yes (Thm. 9.13)	Linear	No
	One function is smooth and strongly convex the other one is convex	Yes (Thm. 9.17)	R-linear	No
	One function is strongly convex the other one is smooth	Yes (Thm. 9.19)	R-linear	No

A

Appendices

A.1 Nonnegativity of multipliers in Theorem 7.2

Recall that, in the proof of Theorem 7.2, γ_1 is defined by

$$\gamma_1 = \frac{t\Big(t^2\big(2+L^2+L\mu\big)-t(3L+\mu)+(Lt-1)\sqrt{(Lt+\mu t-2)^2+4t^2}+2\Big)}{\sqrt{(Lt+\mu t-2)^2+4t^2}}.$$

Since t is nonnegative, we only need to prove that

$$\hat{\gamma}_1 := 2t^2 - 3Lt - \mu t + (Lt-1)\sqrt{(Lt+\mu t-2)^2+4t^2} + L^2t^2 + L\mu t^2 + 2$$

is nonnegative. We show that the following optimization problem is lower bounded by zero,

$$\begin{aligned} \min_{L,t,\mu} \;& \hat{\gamma}_1 \\ \text{s.t.} \;& L \geq \mu, \mu \geq 0, t \geq 0, \end{aligned}$$

where L, t, μ are decision variables. First we consider the case that $Lt - 1 \leq 0$. We have the following optimization problem

$$\begin{aligned} &\min_{L,t}\left(\min_{0\leq\mu\leq L} \;\; 2t^2 + L^2t^2 + L\mu t^2 - 3Lt - \mu t + (Lt-1)\sqrt{(Lt+\mu t-2)^2+4t^2} + 2\right) \\ &\text{s.t.} \;\; Lt \leq 1, L \geq \mu, t \geq 0. \qquad\qquad (A.1) \end{aligned}$$

The function $\hat{\gamma}_1$ is concave in μ, therefore, we just consider $\mu = 0$ and $\mu = L$. First we consider the case that $\mu = 0$. By substituting $\mu = 0$ in $\hat{\gamma}_1$ we have

$$\hat{\gamma}_1 = 2t^2 + (Lt-1)\left(Lt-2+\sqrt{(Lt-2)^2+4t^2}\right).$$

We argue that the above function is nonnegative on the feasible set of problem (A.1). By a conjugate multiplication of $Lt-2+\sqrt{(Lt-2)^2+4t^2}$ one has

$$\hat{\gamma}_1 = 2t^2\left(1-\frac{2(1-Lt)}{(2-Lt)+\sqrt{(Lt-2)^2+4t^2}}\right),$$

since $(2-Lt)+\sqrt{(Lt-2)^2+4t^2} \geq 2(2-Lt)$ we conclude that $0 \leq \frac{2(Lt-1)}{(Lt-2)+\sqrt{(Lt-2)^2+4t^2}} \leq 1$ which proves $\hat{\gamma}_1$ is nonnegative.

Now we consider the case that $\mu = L$. By substituting $\mu = L$ we have

$$\hat{\gamma}_1 = 2t^2 + 2L^2t^2 - 4Lt + 2(Lt-1)\sqrt{(Lt-1)^2+t^2} + 2.$$

Now we show that $\frac{1}{2}\hat{\gamma}_1 = t^2 + (Lt-1)\left((Lt-1)+\sqrt{(Lt-1)^2+t^2}\right)$ is nonnegative on the given set. Note that, again by conjugate multiplication,

$$\frac{1}{2}\hat{\gamma}_1 = t^2\left(1-\frac{(1-Lt)}{(1-Lt)+\sqrt{(Lt-1)^2+t^2}}\right),$$

which always is nonnegative due to the nonnegativity of $(1-Lt)$.

Now we consider the case that $tL-1>0$. We have

$$\begin{aligned}\hat{\gamma}_1 =& 2t^2 + L^2t^2 + L\mu t^2 - 3Lt - \mu t + (Lt-1)\sqrt{(Lt+\mu t-2)^2+4t^2} + 2\\ \geq& 2t^2 + L^2t^2 + L\mu t^2 - 3Lt - \mu t + (Lt-1)|Lt+\mu t-2| + 2.\end{aligned}$$

Here, we need to consider two sub-cases. Firstly, when $2-Lt-\mu t \geq 0$, we have

$$2t^2 + L^2t^2 + L\mu t^2 - 3Lt - \mu t + (Lt-1)(2-Lt-\mu t) + 2 -= 2t^2 \geq 0.$$

If $Lt+\mu t-2 \geq 0$, we have

$$\begin{aligned}&2t^2 + L^2t^2 + L\mu t^2 - 3Lt - \mu t + (Lt-1)(Lt\mu t-2) + 2\\ =&(Lt-2)^2 + (L-\mu)t + t^2 + L\mu t^2 \geq 0,\end{aligned}$$

which completes the proof.

To show that γ_2 is nonnegative we follow the same procedure. Recall the definition of γ_2

$$\gamma_2 = \frac{t\left(t^2\left(2+\mu^2+L\mu\right)-t(3\mu+L)+(1-\mu t)\sqrt{(Lt+\mu t-2)^2+4t^2}+2\right)}{\sqrt{(Lt+\mu t-2)^2+4t^2}}.$$

We define $\hat{\gamma}_2$ as

$$\hat{\gamma}_2 = t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\sqrt{\left(Lt+\mu t-2\right)^2+4t^2}+2.$$

Due to $t \geq 0$, we only need to show that $\hat{\gamma}_2$ is nonnegative. To this end, we show that the following optimization problem is lower bounded by zero.

$$\begin{aligned}\min_{L,t,\mu}\ & \hat{\gamma}_2 = t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\sqrt{\left(Lt+\mu t-2\right)^2+4t^2}+2\\ \text{s.t.}\ & L \geq \mu, \mu \geq 0, t \geq 0.\end{aligned}$$

First we consider the case that $1-\mu t \geq 0$. We have

$$\hat{\gamma}_2 \geq t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)|Lt+\mu t-2|+2.$$

We consider two sub-cases. Firstly, $Lt+\mu t-2 \geq 0$:

$$\begin{aligned}\hat{\gamma}_2 =& t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\sqrt{\left(Lt+\mu t-2\right)^2+4t^2}+2\\ \geq& t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\left(Lt+\mu t-2\right)+2 = 2t^2 \geq 0.\end{aligned}$$

Now assume that $Lt+\mu t-2 \leq 0$.

$$\begin{aligned}\hat{\gamma}_2 =& t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\sqrt{\left(Lt+\mu t-2\right)^2+4t^2}+2\\ \geq& t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\left(2-Lt-\mu t\right)+2\\ =& 2\left(\left(\mu t-1\right)^2+\left(2-\mu t-Lt\right)+\mu Lt^2+t^2\right) \geq 0.\end{aligned}$$

Now we consider the case that $1-\mu t \leq 0$.

$$\begin{aligned}\min_{t,\mu} & \left(\min_{\mu \leq L \leq \frac{2\mu-t}{\mu t}} \ \hat{\gamma}_2 = t^2\left(2+\mu^2+L\mu\right) - t\left(3\mu+L\right) + \left(1-\mu t\right)\sqrt{\left(Lt+\mu t-2\right)^2+4t^2}+2\right)\\ \text{s.t.}\ & \mu t \geq 1, \mu \geq 0, t \geq 0.\end{aligned}$$

Note that $\hat{\gamma}_2$ is concave with respect to the variable L. Therefore, we should study the boundaries of L. If we set $L = \mu$ we have

$$\begin{aligned}\hat{\gamma}_2 =& 2(t^2+\mu^2 t^2 - 2\mu t + (1-\mu t)\sqrt{(\mu t-1)^2+t^2}+1)\\ =& 2t^2 + 2(\mu t-1)((\mu t-1) - \sqrt{(\mu t-1)^2+t^2}).\end{aligned}$$

By conjugate multiplication, we have

$$\hat{\gamma}_2 = 2t^2\left(1 - \frac{(\mu t-1)}{\mu t - 1 + \sqrt{(\mu t-1)^2+t^2}}\right) \geq 0.$$

By $L \leq \frac{2\mu - t}{\mu t}$ one can see that $L \leq \frac{2}{t}$. Setting $L = \frac{2}{t}$:

$$\begin{aligned}\hat{\gamma}_2 =& -\mu t + 2t^2 + \mu^2 t^2 + (1 - \mu t)\sqrt{4t^2 + \mu^2 t^2} \\ =& 2t^2 \left(1 - \frac{2(\mu t - 1)}{\mu t + \sqrt{\mu^2 t^2 + 4t^2}}\right) \geq 0.\end{aligned}$$

This completes the proof.

A.2 Identity used in the proof of Theorem 7.2

The proof of Theorem 7.2 requires the following identity, that may be verified through direct (symbolic) calculation:

$$\begin{aligned}&\left\|x^0 - tG_x^{0,0}\right\|^2 + \left\|y^0 + tG_y^{0,0}\right\|^2 - \bar{\alpha}\left(\left\|x^0\right\|^2 + \left\|y^0\right\|^2\right) + \gamma_1\Big(F^{0,0} - F^{\star,0} - \left\langle G_x^{\star,0}, x^0\right\rangle - \\ &\frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_x^{0,0} - G_x^{\star,0}\right\|^2 + \mu\left\|x^0\right\|^2 - \frac{2\mu}{L}\left\langle G_x^{\star,0} - G_x^{0,0}, -x^0\right\rangle\right)\Big) + \gamma_2\Big(F^{\star,0} - F^{0,0} + \\ &\left\langle G_x^{0,0}, x^0\right\rangle - \frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_x^{\star,0} - G_x^{0,0}\right\|^2 + \mu\left\|x^0\right\|^2 - \frac{2\mu}{L}\left\langle G_x^{0,0} - G_x^{\star,0}, x^0\right\rangle\right)\Big) + \gamma_2\Big(F^{0,\star} - \\ &F^{\star,\star} - \frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_x^{0,\star}\right\|^2 + \mu\left\|x^0\right\|^2 - \frac{2\mu}{L}\left\langle G_x^{0,\star}, x^0\right\rangle\right)\Big) + \gamma_1\Big(F^{\star,\star} - F^{0,\star} + \left\langle G_x^{0,\star}, x^0\right\rangle - \\ &\frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_x^{0,\star}\right\|^2 + \mu\left\|x^0\right\|^2 - \frac{2\mu}{L}\left\langle G_x^{0,\star}, x^0\right\rangle\right)\Big) + \gamma_1\Big(F^{0,\star} - F^{0,0} + \left\langle G_y^{0,\star}, y^0\right\rangle - \frac{L}{2(L-\mu)} \\ &\left(\frac{1}{L}\left\|G_y^{0,0} - G_y^{0,\star}\right\|^2 + \mu\left\|y^0\right\|^2 - \frac{2\mu}{L}\left\langle G_y^{0,\star} - G_y^{0,0}, y^0\right\rangle\right)\Big) + \gamma_2\Big(F^{0,0} - F^{0,\star} - \left\langle G_y^{0,0}, y^0\right\rangle - \\ &\frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_y^{0,\star} - G_y^{0,0}\right\|^2 + \mu\left\|y^0\right\|^2 - \frac{2\mu}{L}\left\langle -G_y^{0,0} + G_y^{0,\star}, y^0\right\rangle\right)\Big) + \gamma_2\Big(-F^{\star,0} + F^{\star,\star} - \\ &\frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_y^{\star,0}\right\|^2 + \mu\left\|y^0\right\|^2 - \frac{2\mu}{L}\left\langle G_y^{\star,0}, -y^0\right\rangle\right)\Big) + \gamma_1\Big(-F^{\star,\star} + F^{\star,0} + \left\langle G_y^{\star,0}, -y^0\right\rangle - \\ &\frac{L}{2(L-\mu)}\left(\frac{1}{L}\left\|G_y^{\star,0}\right\|^2 + \mu\left\|y^0\right\|^2 - \frac{2\mu}{L}\left\langle -G_y^{\star,0}, y^0\right\rangle\right)\Big) + \gamma_3\left(\left\|x^0\right\|^2 - \left\|G_y^{0,0} - G_y^{\star,0}\right\|^2\right) + \\ &\gamma_3\left(\left\|x^0\right\|^2 - \left\|G_y^{0,\star}\right\|^2\right) + \gamma_3\left(\left\|y^0\right\|^2 - \left\|G_x^{0,0} - G_x^{0,\star}\right\|^2\right) + \gamma_3\left(\left\|y^0\right\|^2 - \left\|G_x^{\star,0}\right\|^2\right) \\ =& -\zeta_1\left\|x^0 - \zeta_2 G_x^{0,0} - \zeta_3(G_x^{0,\star} - G_x^{\star,0})\right\|^2 - \zeta_4\left\|G_x^{0,0} - G_x^{0,\star} - G_x^{\star,0}\right\|^2 - \\ &\zeta_1\left\|y^0 + \zeta_2 G_y^{0,0} - \zeta_3(G_y^{0,\star} - G_y^{\star,0})\right\|^2 - \zeta_4\left\|G_y^{0,0} - G_y^{\star,0} - G_y^{0,\star}\right\|^2,\end{aligned}$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are given by

$$\zeta_1 = \tfrac{1}{2} t \left(\tfrac{\left(L^2+\mu^2\right)\beta}{L-\mu} - \tfrac{2t^2(L-\mu)}{\beta} + (L+\mu)(t(L+\mu)-2) \right),$$
$$\zeta_2 = -\tfrac{\left(L^2 t - L - \mu^2 t + \mu\right)\beta - L^2 t(Lt+\mu t-3) - (L+\mu)\left(\mu^2 t^2 - 2\mu t + 2t^2 + 2\right) + \mu^2 t}{2t^2(L+\mu)^2(L\mu+1) - 8L\mu t(L+\mu) + 8L\mu},$$
$$\zeta_3 = -\tfrac{t\left(L^2+6L\mu+\mu^2\right) - 2t^2(L+\mu)(L\mu+1) - (L-\mu)\beta - 2(L+\mu)}{2t^2(L+\mu)^2(L\mu+1) - 8L\mu t(L+\mu) + 8L\mu},$$
$$\zeta_4 = \tfrac{t(\beta+Lt-\mu t)^2}{4(L-\mu)\beta}.$$

Note that $\zeta_1, \zeta_4 \geq 0$, as required.

A.3 Proof for positive semidefniteness of the matrix presented in Theorem 9.6

Lemma A.1. *Let $N \geq 4$ and $t, c_1 \in \mathbb{R}$. Let $D(t, c_1)$ be $N \times N$ symmetric matrix given in Theorem 9.6. If $c_1 > 0$ is given, then*

$$[0, c_1] \subseteq \{t : D(t, c_1) \succeq 0\}.$$

Proof. The argument proceeds in the same manner as in Lemma 9.2. Due to the convexity of $\{t : D(t, c_1) \succeq 0\}$, is sufficient to establish the positive semidefiniteness of $D(0, c_1)$ and $D(c_1, c_1)$. As $D(0, c_1)$ is diagonally dominant, it is positive semidefinite. Next, we proceed to demonstrate the positive definiteness of the matrix $K = D(1, 1)$ by computing its leading principal minors. One can show that the claim holds for $N = 4$. So we investigate $N \geq 5$. To accomplish this, we perform the following elementary row operations on matrix D:

i) Add the second row to the third row;

ii) Add the second row to the last row;

iii) Add the third row to the forth row;

iv) For $i = 4 : N - 2$

- Add $i - th$ row to $(i+1) - th$ row;
- Add $\frac{3-i}{2i^2-3i-1}$ times of $i - th$ row to the last row;

v) Add $\frac{2N^2-8N+9}{2N^2-7N+4}$ times of $(N-1) - th$ row to $N - th$ row.

By executing these operations, we transform K into an upper triangular matirx J with diagonal

$$J_{k,k} = \begin{cases} 2, & k=1 \\ 2k^2-3k-1, & 2 \le k \le N-1 \\ 2N^2-7N+8-\frac{N^2}{(N-1)^2}-\frac{(2N^2-8N+9)^2}{2N^2-7N+4}-\sum_{i=4}^{N-2}\frac{(i-3)^2}{2i^2-3i-1}, & k=N. \end{cases}$$

It is seen all first $(N-1)$ diagonal elements of J are positive. We show that $J_{N,N}$ is also positive. By using inequality (9.15), we get

$$\begin{aligned} &2N^2-7N+8-\frac{N^2}{(N-1)^2}-\frac{(2N^2-8N+9)^2}{2N^2-7N+4}-\sum_{i=4}^{N-2}\frac{(i-3)^2}{2i^2-3i-1} \ge \\ &2N^2-7N+8-\tfrac{25}{16}-(2N^2-8N+9)-\tfrac{N-5}{2}-1+\tfrac{2}{N-3} \ge \tfrac{N}{2}-\tfrac{17}{16} > 0, \end{aligned}$$

for $N \ge 5$, which implies $J_{N,N} > 0$. Hence, $D(c_1, c_1) \succeq 0$ and the proof is complete. □

Summary

Academic Summary

This thesis presents a comprehensive analysis of the convergence rates and performance of iterative optimization algorithms used in machine learning. Focusing on gradient descent and other first-order methods, the study introduces novel theoretical insights on convergence rates of several methods by utilizing Performance Estimation Problems (PEPs) formulated as semidefinite programming problems. Key contributions include the establishment of exact worst-case convergence rates and conditions under which these methods achieve linear convergence. This work not only deepens the theoretical understanding of several optimization algorithms, but also guides practical applications in machine learning where efficient data processing is crucial.

Summary for Non-Experts

This thesis investigates how certain mathematical methods, known as iterative optimization algorithms, where the best option must be chosen among feasible alternatives. Specifically, it explores how quickly these methods can reach reliable conclusions. The study introduces and improves mathematical tools and theories that help predict the performance of these algorithms, focusing on those that use a technique called gradient descent. This research is important because it helps improve the efficiency of algorithms that process vast amounts of data, making them faster and more effective in applications like image recognition, voice recognition, and other AI technologies.

Academische Samenvatting

Dit proefschrift presenteert een uitgebreide analyse van de convergentiesnelheden en prestaties van iteratieve optimalisatie-algoritmen die gebruikt worden in machine learning. De studie richt zich op gradiëntafdaling en andere methoden van de eerste orde, en introduceert nieuwe theoretische inzichten in de convergentietempo van verschillende methoden door gebruik te maken van prestatieschattingsproblemen (PSP's), geformuleerd als semidefiniete optimalisatieproblemen. Belangrijke bijdragen omvatten het vaststellen van exacte convergentietempo's in het slechtste geval en de omstandigheden waaronder deze methoden lineaire convergentie bereiken. Dit werk verdiept niet alleen het theoretische begrip van verschillende optimalisatie-algoritmen, maar faciliteert ook praktische toepassingen in machine learning waarbij efficiënte dataverwerking cruciaal is.

Samenvatting voor Niet-Experts

Dit proefschrift onderzoekt bepaalde wiskundige methoden, bekend als iteratieve optimalisatie-algoritmen, die zich bezighouden met het kiezen van de beste optie uit haalbare alternatieven. Het onderzoekt wijst hoe snel deze methoden betrouwbare conclusies kunnen bereiken. De studie introduceert en verbetert wiskundige hulpmiddelen en theorieën die de prestaties van deze algoritmen helpen voorspellen, met een nadruk op technieken zoals gradiëntafdaling. Dit onderzoek is belangrijk omdat het de efficiëntie van algoritmen verbetert die grote hoeveelheden gegevens verwerken, waardoor ze sneller en effectiever worden in toepassingen zoals beeldherkenning, spraakherkenning en andere AI-technologieën.

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Hadi Abbaszadehpeivasti (Tabriz, Iran, 1989) earned his bachelor's degree in Applied Mathematics from the University of Zanjan. Subsequently, he pursued and obtained his Master's degrees in Industrial Engineering from both Sharif University of Technology and Sabancı University. Following this, he continued as a Ph.D. candidate in the Department of Econometrics and Operations Research at Tilburg University, funded by the NWO ENW-GROOT project Optimization for and with Machine Learning (OPTIMAL).

Nowadays, complex real-world optimization problems, especially in Machine Learning, are gaining importance. Understanding which algorithm has the best performance for a specific problem is crucial. To answer this, it is important to understand how different algorithms behave. This dissertation focuses on studying the behaviors of certain algorithms widely used in optimization. We utilized performance estimation methodology that relies on semidefinite programming. In this thesis, we thoroughly studied five different algorithms.

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